Convergence Rates for Nonholonomic Systems in Power Form

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Abstract

This paper investigates the convergence rates of several controllers for low dimensional nonholonomic systems in power form. The method of multiple scales is found to be effective in determining the asymptotic form of the solutions. The general form of the perturbation solutions indicates how parameters in the control laws may be chosen to achieve a desired convergence rate. A detailed analysis of controllers exhibiting exponential convergence is included.

1 Introduction

This paper focuses on the problem of determining the rates of convergence of control systems of the form

$$\dot{x} = \sum_{i=1}^{m} g_i(x) u_i(x,t) \qquad x \in \mathbf{R}^n, \qquad (1)$$

where each g_i is a smooth vector field on $\mathbf{R}^n$ and the controls, $u_i(x,t)$, are continuous functions of x. Systems of this form arise in the study of mechanical systems with velocity constraints and have received renewed attention as an example of strongly nonlinear systems. See [9] for an introduction and more detailed motivation. For such systems, control methods based on linearization cannot be applied and nonlinear techniques must be utilized. Convergence rates of solutions are of practical importance and we attempt to shed some light on the connection between convergence rates and the smoothness of the feedback $u_i(x,t)$.

A control law $u = k(x,t)$ globally stabilizes a point $x_0 \in \mathbf{R}^n$ if $x(t) \to x_0$ as $t \to \infty$ for all initial conditions of the system. For a nonholonomic control system, the dependence of a stabilizing control law on time is essential since the system (1) does not satisfy Brockett's necessary condition for smooth or even continuous stabilization [1]. Hence there does not exist a smooth static state feedback law which stabilizes the system to a point. Recent work by Coron has shown that it is possible to stabilize a nonholonomic system using time-varying feedback [2]. Constructive approaches have been presented by Samson [11], Pomet [10], and Coron and Pomet [3]. In this paper we analyze some specific nonholonomic systems in so called power form [13]. The structure of the stabilizing control laws are taken from [13]. These control laws are based on earlier work using sinusoids for open-loop planning and have connections with the recent work of Sussmann and Liu [12] and Gurvits [4].

Nonholonomic systems in power form are represented by

the following set of equations

$$\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_2 \\
\dot{x}_3 &= x_1 u_2 \\
\dot{x}_4 &= \tfrac{1}{2} x_1^2 u_2 \\
&\vdots \\
\dot{x}_n &= \tfrac{1}{(n-2)!} x_1^{n-2} u_2.
\end{aligned} \qquad (2)$$

The control Lie algebra for this system is spanned by the input vector fields and Lie products of the form $\mathrm{ad}_{g_1}^k g_2$. It is worthwhile to note that this form satisfies some of the simplifying assumptions used by Pomet to generate controllers for more general nonholonomic control systems [10].

Systems in power form characterize the fundamental difficulties of nonholonomic systems in a very simple and useful form. By understanding the geometry of controllers applied to power form, we hope to understand the geometry of controllers applied to more general nonholonomic systems. This point of view has been used very successfully by Sussmann, who has shown how results applied to a "symbolic" representation of the control system can be used to understand systems with a compatible control Lie algebra [8].

The control laws studied here are based on those presented by Teel et al. in which sinusoids at integrally related frequencies locally stabilize power form systems to a point [13]. Necessary and sufficient conditions for conversion to power form are given in this reference. Teel's controllers have the form

$$\begin{aligned}
u_1 &= -x_1 - \sum_{j=1}^{n-2} x_{j+2}^2 (\sin t - \cos t) \\
u_2 &= -x_2 - \sum_{j=1}^{n-2} c_j x_{j+2} \cos jt,
\end{aligned} \qquad (3)$$

with $c_j > 0$. The proof of the stability relies on the interaction between sinusoids at integrally related frequencies to produce motion in the appropriate directions. We use essentially the same structure as (3) except that the x_j's are replaced by general functions of these variables. The reader is referred to [13] for more details.

We explore the convergence properties of these controls laws using multi-time scale analysis techniques and concentrating on low-dimensional cases. We are optimistic that the stabilizing controllers presented here can be extended to the more general case and that by understanding their action on a canonical system, we can understand their extension to systems with a similar Lie algebraic structure.

2 3-Dimensional Power System

The three dimensional nonholonomic system in power form is represented by the following system of equations:

$$\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_2 \\
\dot{x}_3 &= x_2 u_1.
\end{aligned} \qquad (4)$$

We will study the behavior of the closed loop system under the following feedback:

$$u_1 = -x_1 + F_1(x_3)\cos t$$
$$u_2 = -x_2 + F_2(x_3)\sin t,$$
(5)

where $F_1(\cdot)$ and $F_2(\cdot)$ are smooth functions but otherwise unspecified. Center manifold theory may be used to choose the F_i's such that the origin is locally asymptotically stable in the closed loop system [13]. In order to study the *rates* of convergence of the solutions, it is useful to employ a two-timing perturbation procedure [7]. The method of two-timing will be used to develop the structure of some approximate solutions of system (4) with the feedback (5). Two-timing relies upon a small system parameter which causes the flow of the system to evolve with two different time scales. Assume for the moment that the x_3 equation is replaced with

$$\dot{x}_3 = \epsilon x_2 u_1 = \epsilon x_2(-x_1 + F_1(x_3)c_1),$$

where $0 < \epsilon \ll 1$ and $c_i = \cos it$ and $s_i = \sin it$. This notation will be used in the sequel. A two-timing perturbation expansion may be constructed by assuming that the solutions of the closed loop system may be written in the following form,

$$x_i(t) = \bar{x}_i(t,\tau) + \epsilon \tilde{x}_i(t,\tau) + \dots$$
(6)

where $\tau = \epsilon t$ is interpreted as slow time and t as fast time. The overbar denotes the leading order terms in the expansion and the tilde denotes higher order terms. τ is treated as another independent variable in this expansion.

The difficult step in the two-timing procedure is obtaining the form of the expansion. Once the structure of the expansion is known (or assumed, as we have) it is relatively simple, although tedious, to obtain the equations for the individual terms. Substituting the expansion into the closed loop equations and equating coefficients of powers of ϵ yields the following leading order terms for x_1, x_2 and x_3,

$$\dot{\bar{x}}_1 = -\bar{x}_1 + F_1(\bar{x}_3)c_1,$$
(7)
$$\dot{\bar{x}}_2 = -\bar{x}_2 + F_2(\bar{x}_3)s_1,$$
(8)
$$\dot{\bar{x}}_3 = 0,$$
(9)
$$\bar{x}_3' + \dot{\tilde{x}}_3 = -\bar{x}_2(\bar{x}_1 + F_1(\bar{x}_3)c_1).$$
(10)

Here $\dot{x}_i$ denotes differentiation with respect to t and x_i' denotes differentiation with respect to τ. Equation (9) implies that $\bar{x}_3$ is a function of τ only,

$$\bar{x}_3(t,\tau) = A_3(\tau).$$

In terms of $A_3(\tau)$ equations (7) and (8) may be rewritten as,

$$\dot{\bar{x}}_1 = -\bar{x}_1 + F_1(A_3(\tau))c_1$$
$$\dot{\bar{x}}_2 = -\bar{x}_2 + F_2(A_3(\tau))s_1.$$

These equations represent first-order linear systems driven by sinusoids. Since asymptotic solutions are desired, the transient behavior of the solutions is not of interest. The *steady state* solutions of these equations are,

$$\bar{x}_1(t,\tau) = \frac{1}{2}F_1(A_3(\tau))(c_1 + s_1)$$
$$\bar{x}_2(t,\tau) = \frac{1}{2}F_2(A_3(\tau))(-c_1 + s_1).$$
(11)

$A_3(\tau)$ is determined from equation (10),

$$A_3(\tau)' + \dot{\tilde{x}}_3 = \bar{x}_2(-\bar{x}_1 + F_1(A_3)c_1).$$

Substituting the expressions in (11) into equation (10) yields,

$$A_3' + \dot{\tilde{x}}_3 = -\frac{1}{4}F_1(A_3(\tau))F_2(A_3(\tau))(c_1 - s_1)^2.$$

$$= -\frac{1}{4}F_1(A_3(\tau))F_2(A_3(\tau))(1 - 2c_1s_1)$$

The slow and fast time scales are considered unrelated as far as these equations are concerned. Hence A_3' is equated to

$$A_3' = -\frac{1}{4}F_1(A_3)F_2(A_3).$$
(12)

$\dot{\tilde{x}}_3$ is equal to the remaining term

$$\dot{\tilde{x}}_3 = \frac{1}{2}F_1(A_3)F_2(A_3)c_1s_1.$$

The presence of the $F_i(\tau)$'s in this equation present no problems since these functions remain constant compared to the fast time, t. The initial conditions have been ignored so these approximate solutions will be valid after the transient behavior has died away. The magnitudes of $\bar{x}_1$ and $\bar{x}_2$, after the solution has settled down to its steady state, only depends on $\bar{x}_3$. It is evident from equation (12) that the rate of convergence of the x_3 variable is controlled by the product of F_1 and F_2. The following example will illustrate these ideas.

Example 1. Suppose F_1 and F_2 are chosen such that,

$$F_1(x_3) = x_3 \qquad F_2(x_3) = x_3^2,$$
(13)

then the origin is locally asymptotically stable [13]. We introduce the parameter ϵ by scaling the dependent variables

$$(x_1, x_2, x_3) \to (\epsilon^{\frac{1}{2}}x_1, \epsilon x_2, \epsilon^{\frac{1}{2}}x_3).$$

The closed loop equations transform to,

$$\dot{x}_1 = -x_1 + x_3c_1$$
$$\dot{x}_2 = -x_2 + x_3^2s_1$$
$$\dot{x}_3 = \epsilon x_2(-x_1 + x_3c_1).$$

The equation (12) with (13) implies

$$x_3(t) \sim \frac{x_3(0)}{\sqrt{\frac{1}{2}x_3{}^2(0)\epsilon t + 1}}.$$

Equations (11) yield the following asymptotic solution for $x_1(t)$ and $x_2(t)$,

$$x_1(t) \sim \frac{1}{2}\frac{x_3(0)}{\sqrt{\frac{1}{2}x_3{}^2(0)\epsilon t + 1}}(c_1 + s_1)$$

$$x_2(t) \sim \frac{1}{2}\frac{x_3{}^2(0)}{\frac{1}{2}x_3{}^2(0)\epsilon t + 1}(-c_1 + s_1).$$

These approximate methods usually yield estimates which are valid only on finite time intervals. Hence, no conclusion can be made concerning the rate of convergence of the solutions without some extra analysis. However, a simple calculation shows that the leading order multiscale procedure gives results equivalent to the center manifold approach when F_1 and F_2 are smooth. The perturbation solution is merely the *averaged* center manifold equation. Thus, the perturbation results are asymptotic in this case. Figure 1 compares the numerical solution of system to the asymptotic results developed above. The initial conditions where chosen to minimize the transients.

3   Control Laws for Exponential Convergence

The form of the asymptotic solutions is very suggestive. It was noted earlier that the product of F_1 and F_2 controlled the convergence rate of the three state system cf. equation (12).

Example 1, continued. We now explore choices of F_1 and F_2 which give exponential reates of stabilization. The preceding analysis requires the differentiability of F_1 and F_2. Forget for
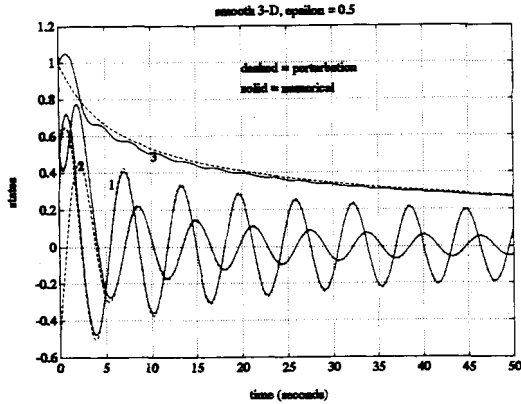
Figure 1: Comparison of numerical and perturbation solutions

the moment these restrictions and consider the 3-D closed loop system when the F_i's are chosen as,

$$F_1(x_3) = \text{sgn}(x_3)\sqrt{|x_3|} \qquad F_2(x_3) = \sqrt{|x_3|}, \qquad (14)$$

where $\text{sgn}(\cdot)$ is the signum function. Once again the dependent variables may be scaled so that the closed loop equations are in the proper form,

$$\dot{x}_1 = -x_1 + \text{sgn}(x_3)\sqrt{|x_3|}c_1$$
$$\dot{x}_2 = -x_2 + \sqrt{|x_3|}s_1 \qquad (15)$$
$$\dot{x}_3 = \epsilon x_2(-x_1 + \text{sgn}(x_3)\sqrt{|x_3|}c_1).$$

Note that the equations are Hölder continuous but not Lipschitz at $x_3 = 0$. Continuity of the vector field ensures existence but not uniqueness of solutions. We will proceed on the assumption that the solutions are unique. Away from the set $x_3 = 0$ the vector field is Lipschitz and the perturbation analysis indicates that for properly chosen initial conditions (where the transients are minimal) x_3 only approaches the "bad" set in the limit as $t \to \infty$.

Equation (12) implies that $\bar{x}_3$, and hence $\bar{x}_1$ and $\bar{x}_2$, converge exponentially. One could argue that the asymptotic solution is still valid with this C^0 control law since the only region in phase space where the control is *not* smooth is the plane $x_3 = 0$ and numerical simulations indicate that after an initial period of transient behavior the x_3 variable maintains constant sign, never passing through $x_3 = 0$. The perturbation solution is given by

$$\bar{x}_3(t) = x_3(0)\exp\left(-\frac{\epsilon}{4}t\right)$$

and $\bar{x}_2(t)$ and $\bar{x}_2(t)$ are given by the expressions (11). Figure 2 compares the perturbation solution with numerical integration for this nonsmooth system. Figure 3 is a semilog plot of the Euclidean norm of the states from Figure 2 and the exponential convergence rate bounds provided from the analysis. The fact the trajectories are always bounded above by a line of negative slope indicates exponential convergence. The perturbation solution places the exponential convergence rate bound for x_3 at $-\frac{\epsilon}{4}$ while the x_1 and x_2 variables are half this rate. The plots show that these bounds are tight. ●

The availability of more rigorous theory to study non-Lipschitz vector fields is required even though these heuristic arguments and simulations are convincing. This is discussed in the following paragraphs.

The number of tools for analyzing the properties of differential equations dwindles as we pass from differentiable to
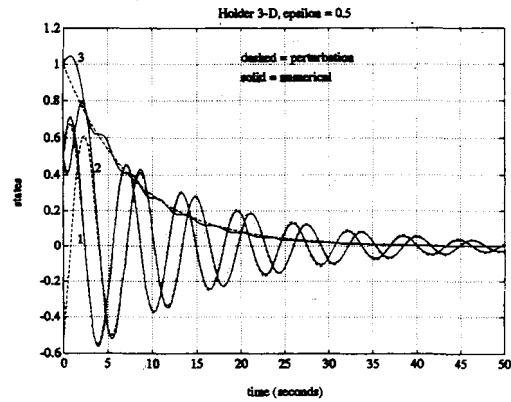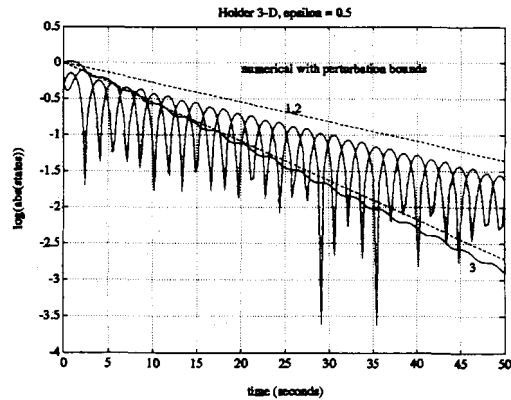

Figure 2: 3-D System with Nonsmooth Feedback

Figure 3: Exponential Rates of Convergence for 3-D System

continuous vector fields. However, Kawski has developed some useful theory for a special class of vector fields [6]. In the interest of conserving space only those details necessary for the study of this example will be introduced.

Example 1, continued. Recall the original unscaled vector field,

$$\dot{x}_1 = -x_1 + \text{sgn}(x_3)\sqrt{|x_3|}c_1$$
$$\dot{x}_2 = -x_2 + \sqrt{|x_3|}s_1 \qquad (16)$$
$$\dot{x}_3 = x_2(-x_1 + \text{sgn}(x_3)\sqrt{|x_3|}c_1).$$

The system of equations is *invariant* under the following scaling,

$$(x_1, x_2, x_3) \to (\lambda x_1, \lambda x_2, \lambda^2 x_3) \quad \lambda > 0 \qquad (17)$$

This scaling, called a *dilation*, is symbolically represented as the operator, $\delta_\lambda x = (\lambda x_1, \lambda x_2, \lambda^2 x_3)$. Differential equations which are invariant under the dilation are termed *homogeneous of order zero with respect to* δ_λ. It is useful to define a *homogeneous norm*, ρ, that satisfies the property $\rho(\delta_\lambda x) = \lambda\rho(x)$, $\lambda > 0$. In this context, a different notion of exponential stability may be introduced [6].

Definition 1 *A vector field which is homogeneous of order zero with respect to the dilation δ_λ is exponentially stable if there exist constants $M > 0$ and $\alpha > 0$ such that,*

$$\rho(x(t)) \le Me^{-\alpha t}\rho(x(0)),$$

where $\rho(\cdot)$ is a corresponding homogeneous norm.

The homogeneous norm is not equivalent to the standard p-norms when the variables are scaled by different factors. In this case it is impossible to substitute the usual definition of exponential stability for the one given above. Hence, based on our simulations, it is likely that the system is exponentially stable with respect to the appropriate homogeneous norm. A particular choice which satisfies the scaling property in the example is

$$\rho(x) = (x_1^4 + x_2^4 + x_3^2)^{\frac{1}{4}}.$$

This norm is useful in applications since it is smooth except at the origin. Another result by Kawski [6] states that exponential stability with respect to the homogeneous norm is equivalent to uniform asymptotic stability in the norm (just like the linear case). Since our example is time-periodic we need only show asymptotic stability. Proving asymptotic stability for a non-Lipcshitz continuous vector field is still a daunting task. However, the dilation specifies a transformation group on the phase space which facilitates the study of the equation by considering a vector field on the quotient manifold specified by the group. The quotient manifold is given by $\rho(x) = 1$ which is just a warped sphere, S^2, naturally embedded in R^3. The projection map $\pi : \mathsf{R}^2 \backslash \{0\} \to S^2$ onto the sphere is

$$\pi(x) = \left(\frac{x_1}{\rho(x)}, \frac{x_2}{\rho(x)}, \frac{x_3}{\rho^2(x)} \right).$$

Pushing forward the vector field with this map defines a unique vector field on the sphere by virtue of the transformation group invariance. Once the flow on the sphere is known the flow of the original vector field is determined by lifting the sphere flow with an addition scalar equation. The scalar equation is merely the differential equation for ρ written in the sphere coordinates. Determining the asymptotic behavior of the full set of equations amounts to computing the *nonwandering set*, Ω, on the sphere and observing the behavior of ρ on Ω. Full details for the time invariant case are found in [6]. However, we may simulate the equations on the sphere and characterize the attractive invariant sets for the time-periodic system in the extended phase space $S^2 \times S^1$. This is carried out below.

The equations on the sphere $\rho(x) = 1$ are,

$$\dot{y}_1 = -y_1 + \operatorname{sgn}(y_3)\sqrt{|y_3|}c_1 - y_1 Q(y,t)$$
$$\dot{y}_2 = -y_2 + \sqrt{|y_3|}s_1 - y_2 Q(y,t) \qquad (18)$$
$$\dot{y}_3 = y_2(-y_1 + \operatorname{sgn}(y_3)\sqrt{|y_3|}c_1) - 2y_3 Q(y,t),$$

where $Q(y,t)$ is given by,

$$Q(y,t) = y_1^3(-y_1 + \operatorname{sgn}(y_3)\sqrt{|y_3|}c_1) + y_2^3(-y_2 + \sqrt{|y_3|}s_1)$$
$$+ \frac{1}{2}y_2 y_3(-y_1 + \operatorname{sgn}(y_3)\sqrt{|y_3|}c_1).$$

The scalar "radial" equation is,

$$\dot{\rho} = Q(y,t)\rho.$$

Simulations of the sphere equations (18) imply the existence of two *stable* sets on the sphere (Figure 4). The stability of the ρ equation is determined by what $Q(y,t)$ does on these sets. Figure 5 shows that on the sets identified in Figure 4 that $Q(y,t)$ is periodic with a nonzero negative average. Hence ρ is exponentially stable with the average rate of convergence equal to the average of $Q(y,t)$. This analysis is not complete until all components of Ω have been identified. Other limit sets on the sphere that aren't observable in the simulations, because they are unstable, are the points $(\pm1,0,0)$ and $(0,\pm1,0)$. It is obvious from the original equations that trajectories starting here converge exponentially. Hence for the six sets identified
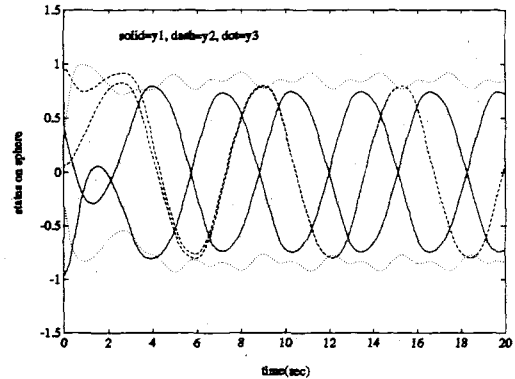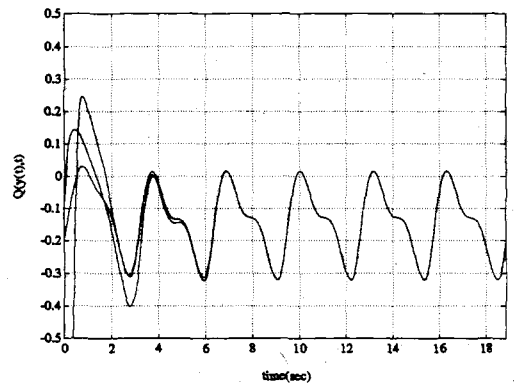

Figure 4: Stable sets on the sphere $\rho(x) = 1$

Figure 5: ρ equation coefficient

on the sphere, exponential stability of the ρ equation has been demonstrated. ●

This analysis reveals the relationship between regularity of the control law and the corresponding achievable convergence rate. We have not exhausted all of the possibilities for choosing the functional dependence of the F_i's for exponential stability. As long as their *product* is z_3 we would expect exponential convergence. However, our choice is the most regular in terms of the Hölder continuity of the control law. Similarly, suppose a Lipschitz feedback is desired for the 3-D system. The perturbation result indicates that the fastest convergence rate possible corresponds to,

$$\dot{\bar{x}}_3 = -\frac{\epsilon}{4}\operatorname{sgn}(\bar{x}_3)\bar{x}_3^2,$$

with $F_1 = \bar{x}_3$ and $F_2 = |\bar{x}_3|$. A faster rate would require a continuous but not Lipschitz control.

4 Higher Dimensional Power Systems

Is the two-timing procedure generalizable to any system in power form? Obtaining the rates of convergence for power systems requires solving a simplified form the the closed loop equations. Since each power form system with its corresponding control law is a distinct dynamical system, one could not expect to exhibit a general algorithm for computing asymptotic solutions valid for all time. Averaging guarantees that solutions of the approximate system and original system remain close for finite time and hence does not reveal the long term behavior, even for asymptotically stable systems. Hence

the analysis must be carried out on a case-by-case basis. We now look at the 4-dimensional power system.

Refer to the 4-dimensional nonholonomic power form system defined by the general expression of Equation 2. The structure of the control law is,

$$u_1 = -x_1 + F_1(x_3)c_1 + G_1(x_4)c_2$$
$$u_2 = -x_2 + F_2(x_3)s_1 + G_2(x_4)s_1. \tag{19}$$

The F_i's and G_i's are smooth functions of their arguments. To justify a multiscale perturbation expansion we artificially introduce a parameter ϵ again. Suppose the closed loop equations are in the form,

$$\dot{x}_1 = u_1 \qquad \dot{x}_3 = \epsilon x_2 u_1$$
$$\dot{x}_2 = u_2 \qquad \dot{x}_4 = \epsilon^2 \frac{1}{2} x_2^2 u_1, \tag{20}$$

with u_1 and u_2 defined above. Applying the method of multiple scales once again requires defining the form of the expansion for each variable. It is necessary to use three time scales t, $\tau = \epsilon t$ and $\sigma = \epsilon^2 t$,

$$x_i(t) = \bar{x}_i(t,\tau,\sigma) + \epsilon \tilde{x}_i(t,\tau,\sigma) + \ldots .$$

The $\bar{x}_i$'s are the leading order terms in the expansion and the $\tilde{x}_i$'s represent higher order terms. The computations are carried out in an analogous manner to the three state system. The leading order terms of x_1 and x_2 are merely the steady state solutions of the $\dot{x}_1$ and $\dot{x}_2$ equations when treating x_3 and x_4 as constants. These computations are suppressed to conserve space. It can be shown that $\bar{x}_3$ is not a function of t and $\bar{x}_4$ is not a function of t and τ, hence we define,

$$\bar{x}_3(t,\tau,\sigma) = A_3(\tau) \qquad \bar{x}_4(t,\tau,\sigma) = A_4(\sigma).$$

$\bar{x}_3$ will depend on σ but we aren't concerned with solving the higher order equations which exhibit the proper dependence since we are interested in the leading order terms only. The differential equations governing A_3 and A_4 are obtained by substituting the leading order expression for x_1 and x_2 into the $\dot{x}_3$ and $\dot{x}_4$ equations. This yields

$$\frac{\partial A_3}{\partial \tau} = -\frac{1}{4}(F_2(A_3) + G_2(A_4))F_1(A_3)(c_1 - s_1)^2$$
$$\qquad - \frac{1}{10}(F_2(A_3) + G_2(A_4))G_1(A_4)(c_1 - s_1)(4c_2 - 2s_2)$$
$$\frac{\partial A_4}{\partial \sigma} = \frac{1}{16}(F_2(A_3) + G_2(A_4))^2(c_1 - s_1)^2 F_1(A_3) \tag{21}$$
$$\qquad + \frac{1}{40}(F_2(A_3) + G_2(A_4))^2 G_1(A_4)(4c_2 - 2s_2).$$

The quantities which are trigonometric functions of t vary rapidly compared to those which depend on τ or σ so it is tempting to average equations (21). However, the solutions of the averaged equations do not uniformly approximate those of the original system. For example, the expression containing $G_1(\bar{x}_4)(c_1 - s_1)(4c_2 - 2s_2)$ in the equation for A_3 would average to zero but this term is important for the long term behavior of A_3. It is impossible to solve equations (21) for general F_i's and G_i's so we consider a specific example again.

Example 2. Suppose we make the following choices [13],

$$F_1(\bar{x}_3) = x_3 \qquad G_1(\bar{x}_4) = -x_4$$
$$F_2(\bar{x}_3) = x_3^2 \qquad G_2(\bar{x}_4) = x_4^2.$$

The dependent variables may be rescaled to place the closed loop equations into the form of equation (20),

$$(x_1, x_2, x_3, x_4) \rightarrow (\epsilon^{\frac{1}{2}} x_1, \epsilon x_2, \epsilon^{\frac{1}{2}} x_3, \epsilon^{\frac{1}{2}} x_4). \tag{22}$$
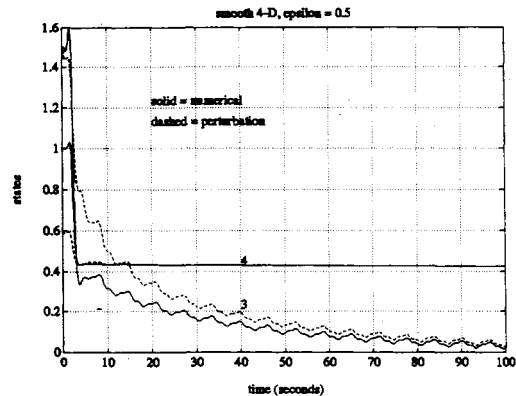

Figure 6: Smooth 4-D System, states x_3 and x_4

The differential equations for A_3 and A_4 are,

$$\frac{\partial A_3}{\partial \tau} = -\frac{1}{4}(c_1 - s_1)^2(A_3^2 + A_4^2)A_3$$
$$\qquad - \frac{1}{10}(A_3^2 + A_4^2)A_4(c_1 - s_1)(-4c_2 + 2s_2)$$
$$\frac{\partial A_4}{\partial \sigma} = \frac{1}{16}(A_3^2 + A_4^2)^2(c_1 - s_1)^3 \tag{23}$$
$$\qquad + \frac{1}{40}(A_3^2 + A_4^2)^2 A_4(c_1 - s_1)^2(-4\cos(2t) + 2s_2).$$

One could attempt an asymptotic solution of system (23). For example the averaged equations would yield information on the initial behavior of the solutions. The long term behavior and hence the actual rate of decay of the solutions is provided by another asymptotic solution if the analyst has enough fortitude to complete the computations. Instead it is more instructive to solve equations (23) numerically to verify that the leading order multiscale perturbation solution does in fact provide solutions that uniformly approximate the true solutions. Once this subsystem of equations has been approximated or solved the expressions for A_3 and A_4 may be substituted back into the leading order expression of $\bar{x}_1$ and $\bar{x}_2$. Figure 6 compares the complete numerical integration of the system with the integration of the partial system of equations (23) obtained from the multiscale analysis. Only x_3 and x_4 are shown in Figure 6 since the perturbation solutions for x_1 and x_2 are just algebraic functions of x_3 and x_4.

Exponential stability of the 4-dimensional system is now considered. Analysis of the smooth 3-D example illustrated how the convergence rates of x_1 and x_2 where determined by the convergence rate of x_3. Similarly, analysis of the smooth 4-D equations showed the explicit dependence of x_3 on x_4 for long time and that x_3 and x_4 essentially control the convergence of x_1 and x_2. Hence, our concern is with the subsystem formed by equations (21). Exponentially stability of the closed loop equations implies exponential stability of the averaged equations. Thus, it is reasonable to consider the averaged version of equations (21) and try to determine the F_i's and G_i's which make the averaged system exponentially stable. The averaged subsystem is,

$$\frac{dA_3}{d\tau} = -\frac{1}{4}(F_2(A_3) + G_2(A_4))F_1(A_3)$$
$$\frac{dA_4}{d\sigma} = \frac{1}{40}(F_2(A_3) + G_2(A_4))^2 G_1(A_4). \tag{24}$$

We have abused notation by using A_3 and A_4 to represent the dependent variables in the averaged equations. To guarantee exponential stability we need to chose the highest order terms
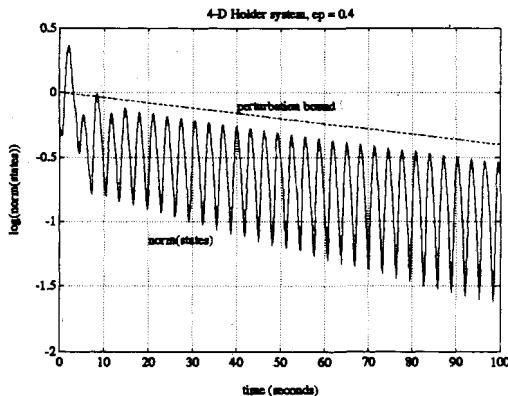
Figure 7: Exponential Rates of Convergence of 4-D System

in (24) to be linear. This implies,

$$F_2(A_3)F_1(A_3) = A_3, \qquad G_2(A_4)^2 G_1(A_4) = -A_4. \quad (25)$$

One possible choice is,

$$F_1(A_3) = \text{sgn}(A_3)\sqrt{|A_3|} \qquad G_1(A_4) = -(A_4)^{\frac{1}{2}}$$
$$F_2(A_3) = \sqrt{|A_3|} \qquad G_2(A_4) = (A_4)^{\frac{1}{4}}.$$

The 4-D equations cannot be rescaled to the form (20). However, the analysis may still be valid as far as the stability type is concerned even though the relative rates of exponential convergence may not be correct. The simulations support this supposition (see Figure 7).

It is interesting to note that the 4-D equations with (4) are invariant under the following scaling,

$$(x_1, x_2, x_3, x_4) \to (\epsilon x_1, \epsilon x_2, \epsilon^2 x_3, \epsilon^3 x_4).$$

This transformation is a dilation like the one described for the three dimensional example. Hence, the numerical analysis suggests that this system is exponentially stable with respect to the corresponding homogeneous norm defined by,

$$\rho(x) = (x_1^{12} + x_2^{12} + x_3^6 + x_4^4)^{\frac{1}{12}}.$$

Computing the vector field on the sphere is carried out in an analogous manner to the 3-D case. These results are not presented. Figure 7 is a numerical simulation of the 4-dimensional nonsmooth system with the logarithm of the norm of the states plotted with respect to time. The slowest exponential convergence rate predicted by the perturbation analysis is the rate corresponding to the x_4 variable and is given by $-\frac{\epsilon^2}{40}$ cf. equations (24) and (4). This bound is plotted and provides a very tight estimate. •

5 Conclusions

The analysis presented in this paper is far from complete. The 3- and 4-D systems provide some very illuminating examples to guide our thinking for higher dimensional power form and general nonholonomic control systems. Gurvitz and Li have made some progress in establishing a connection between the desired rate of convergence of closed loop solutions and the maximal smoothness of the feedback [5]. In the context of exponential stabilization, their result states that the inverse of the degree of nonholonomy (the smallest order of brackets required for controllability) is an upper bound for the Hölder continuity of the control law. It is interesting to note that our control laws achieve these bounds. The 3-dimensional system has degree of nonholonomy 2 and our control laws are Hölder

continuous with constant $\frac{1}{2}$. Analogous results hold for the 4-dimensional system and our proposed exponentially stabilizing control laws.

The multi-time scale analysis is effective in providing the proper structure of the solutions. These perturbation solutions guided our choices in specifying the control parameters that determined certain convergence rates for the approximate systems.

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