

## Research Article

# Convergence Rates in Homogenization of the Mixed Boundary Value Problems

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We will study the convergence rates of solutions for homogenization of the mixed boundary value problems. By utilizing the smoothing operator as well as duality argument, we deal with the mixed boundary conditions in a uniform fashion. As a consequence, we establish the sharp rate of convergence in  $H^1$  and  $L^2$ , with no smoothness assumption on the coefficients.

## 1. Introduction

Convergence rates estimates of solutions are one of the main questions in homogenization theory. There are many papers about convergence of solutions for elliptic homogenization problems. Assume that all of functions are smooth enough, the  $O(\varepsilon)$  error estimate in  $L^\infty$  was presented by Bensoussan, Lions, and Papanicolaou [1]. In 1987, Avellaneda and Lin [2] proved  $L^p$  convergence by the method of maximum principle. At the same year, they [3] also obtained  $L^\infty$  error estimate when  $f$  is less regular than Bensoussan, Lions, and Papanicolaou's. Recently, there were many activities in the theory of homogenization with error estimates. In 2012, Kenig, Lin, and Shen [4] obtained convergence of solutions in  $L^2$  and  $H^{1/2}$  in Lipschitz domains with Dirichlet or Neumann boundary conditions. In 2014, they [5] have also studied the asymptotic behavior of the Green and Neumann functions obtaining some error estimates of solutions. In 2015, the first author [6] obtained the pointwise as well as  $W^{1,p}$  convergence results, which is based on Fourier analysis. In 2016, Shen [7] proved the  $H^1$  convergence rates with Dirichlet or Neumann conditions.

The problems of changing type of boundary conditions in homogenization have been studied extensively in various settings in the past years. In [8] the  $H^1$  weak convergence of solutions was obtained in homogenization problems of multi-level-junction type. In 2011, Cardone [9] considered

the homogenization with mixed boundary value problems in a thin periodically perforated plate and obtained the logarithmic rate of convergence of solutions. In the monograph [10], the  $H^1$  convergence rate was shown by the method of potentials for the solutions to the Dirichlet-Fourier mixed boundary value problem in the perforated domain. The  $L^2$  convergence rate of Steklov-type problems was studied in [11]. In 2017, Shen [12] also obtained the  $L^2$  convergence rate with Dirichlet-Neumann mixed boundary value problem.

In homogenization problems for the Poisson equations in a domain with oscillating boundary, the  $H^1$  convergence rate have been studied in [13–15], while work [16] deals with the multilevel oscillation of the boundary with different conflicting boundary conditions. See also [17–19] for more results on the asymptotic behavior of eigenvalues for the boundary value problems in domains with oscillating boundaries or interfaces.

In this paper, we shall establish the sharp rates of convergence in  $H^1$  and  $L^2$  for oscillating operators with the mixed Dirichlet-Robin boundary condition. In particular, the  $L^2$  estimate was proved by Griso in [20, 21] for Dirichlet or Neumann boundary conditions, using the periodic unfolding method. Our results, on one hand, extend the classical Laplace operator to oscillating operator; on the other hand they extend the classical boundary value problems to a broader mixed boundary conditions settings in homogenization. Meanwhile, our approach is utilizing the smoothing

operator which is much more simple and direct than periodic unfolding method.

More precisely, let  $\Omega$  be a bounded  $C^{1,1}$  domain in  $\mathbb{R}^n$ . Suppose that  $\partial\Omega = \Gamma_1 \cup \Gamma_2$ , where  $\Gamma_1$  and  $\Gamma_2$  are two disjoint closed sets of  $\partial\Omega$ . Let  $u_\varepsilon \in H^1(\Omega)$  be a weak solution to the following problem:

$$\begin{aligned} L_\varepsilon u_\varepsilon &= -\operatorname{div} \left( A \left( \frac{x}{\varepsilon} \right) \nabla u_\varepsilon \right) \\ &= -\frac{\partial}{\partial x_i} \left( a_{ij} \left( \frac{x}{\varepsilon} \right) \frac{\partial u_\varepsilon}{\partial x_j} \right) = f \quad \text{in } \Omega, \end{aligned} \quad (1)$$

$$u_\varepsilon = g \quad \text{on } \Gamma_1,$$

$$\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} + k u_\varepsilon = h \quad \text{on } \Gamma_2,$$

where  $k \geq 0$  is a number. Here  $\partial u_\varepsilon / \partial \nu_\varepsilon = n_i a_{ij} (\partial u_\varepsilon / \partial x_j)$  denotes the conormal derivative with  $L_\varepsilon$  and  $n(x)$  is the outward unit normal to  $\partial\Omega$  at the point  $x$ .

Throughout this paper, the summation convention is used. We assume that the matrix  $A(y) = (a_{ij}(y))$  with  $1 \leq i, j \leq n$  is real symmetric and satisfies the ellipticity condition, i.e.,

$$\begin{aligned} a_{ij}(y) &= a_{ji}(y), \\ \lambda |\xi|^2 &\leq a_{ij}(y) \xi_i \xi_j \leq \frac{1}{\lambda} |\xi|^2, \end{aligned} \quad (2)$$

for  $y \in \mathbb{R}^n$  and  $\xi = (\xi_i) \in \mathbb{R}^n$ ,

where  $\lambda > 0$ , and the periodicity condition

$$A(y+z) = A(y) \quad \text{for } y \in \mathbb{R}^n \text{ and } h \in \mathbb{Z}^n. \quad (3)$$

We impose the smoothness condition

$$\begin{aligned} A(y) \text{ is bounded measurable, } f \in L^2(\Omega), g \\ \in H^1(\partial\Omega), h \in H^{-1/2}(\partial\Omega). \end{aligned} \quad (4)$$

Without loss of generality, we also assume the compatibility condition

$$\int_{\partial\Omega} u_\varepsilon(x) d\sigma(x) = \int_{\Omega} f(x) dx + \int_{\Gamma_2} h(x) d\sigma(x) = 0. \quad (5)$$

Associated with (1) is the homogenized problem

$$L_0 u_0 = -q_{ij} \frac{\partial^2 u_0}{\partial x_i \partial x_j} = f \quad \text{in } \Omega,$$

$$u_0 = g \quad \text{on } \Gamma_1, \quad (6)$$

$$\frac{\partial u_0}{\partial \nu_0} + k u_0 = h \quad \text{on } \Gamma_2,$$

where the constant matrix  $q_{ij}$  is known as the homogenized matrix of  $a_{ij}(y)$  and  $\partial u_0 / \partial \nu_0 = n_i q_{ij} (\partial u_0 / \partial x_j)$ .

Recall that  $u_\varepsilon$  is called the weak solution of (1), if for any  $\phi \in H^1(\Omega)$ , function  $u_\varepsilon - g \in H^1(\Omega)$  holds

$$\begin{aligned} \int_{\Omega} A_\varepsilon \nabla u_\varepsilon \cdot \nabla \phi dx + k \int_{\partial\Omega} u_\varepsilon \phi d\sigma(x) \\ = \int_{\Omega} f \phi dx + \langle h, \phi \rangle_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)}. \end{aligned} \quad (7)$$

The existence and uniqueness of the weak solution to the mixed boundary value problem (1) follow from Lax-Milgram theorem. It is well known that the solution  $u_\varepsilon$  converges to  $u_0$  weakly in  $H^1(\Omega)$  and strongly in  $L^2(\Omega)$ , as  $\varepsilon \rightarrow 0$ .

For the Dirichlet or Neumann boundary value problems, the regularity estimates of solutions in quantitative homogenization have been studied extensively. By the compactness method, interior and boundary Hölder's estimates,  $W^{1,p}$  estimates, and Lipschitz estimates, the regularity of solutions for second-order elliptic systems or equations was established by Avellaneda, Kenig, Lin, Shen, and Suslina in a series of papers [2, 3, 22–26]. For the case of homogenization with mixed boundary value problems, the uniform interior estimates and boundary Hölder's estimates have already been established in [27], and the sharp boundary regularity estimates have obtained in [28]. See also [29–32] for more related results on uniform regularity estimates.

The novelty of this paper lies in the fact that it deals with the mixed Dirichlet-Robin boundary condition which is a more general settings, for instance, in the case of Dirichlet problem when  $\Gamma_2 = \emptyset$ , in the case of Robin problem when  $\Gamma_1 = \emptyset$ , and for the Neumann problem when  $\Gamma_1 = \emptyset$  and  $k = 0$ . As far as the author knows, very few convergence rates results are known for (1) of such mixed boundary value problems.

The following are the main results of this paper.

**Theorem 1.** *Suppose that  $u_\varepsilon \in H^1(\Omega)$  and  $u_0 \in H^2(\Omega)$  are the weak solutions of the mixed boundary value problems (1) and (6), respectively. Then, under the assumptions (2)–(5), there exists a constant  $C$  such that*

$$\|u_\varepsilon - u_0 - \varepsilon \chi T_\varepsilon(\nabla u_0)\|_{H^1(\Omega)} \leq C\varepsilon \|u_0\|_{H^2(\Omega)}, \quad (8)$$

where  $T_\varepsilon$  is the smoothing operator and  $\chi$  is the solution of the cell problem.

**Theorem 2.** *Under the conditions as Theorem 1, then there exists a constant  $C$  such that*

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C\varepsilon \|u_0\|_{H^2(\Omega)}. \quad (9)$$

The rest of the paper is organized as follows. Section 2 contains some basic formulas and useful propositions which play important roles to get convergence rates. In Section 3, we show that the solution  $u_\varepsilon$  of partial differential equation with mixed boundary value problems  $H^1(\Omega)$  and  $L^2(\Omega)$  convergence to the solutions of the corresponding homogenized problems is based on using of smoothing operator. Finally, we summarize our results and discuss possible further development in Section 4.

## 2. Preliminaries

We begin by specifying our notations.

Let  $B_r(x) = \{y \in \mathbb{R}^n : |y-x| < r\}$  denote an open ball with center  $x$  and radius  $r$  and  $\bar{\Omega}_\varepsilon = \{x \in \mathbb{R}^n : \text{dist}(x, \partial\Omega) \leq \varepsilon\}$ . Since  $\Omega$  is Lipschitz, then there exists a bounded extension operator  $E : H^2(\Omega) \rightarrow H^2(\mathbb{R}^n)$ , such that  $\tilde{u}_0 = E(u_0)$  is an extension of  $u_0$  and  $\|\tilde{u}_0\|_{H^2(\mathbb{R}^n)} \leq C\|u_0\|_{H^2(\Omega)}$ . We set  $\varphi \in C_0^\infty(\mathbb{R}^n)$  to be a smooth function and  $\|\varphi\|_{H^1(\mathbb{R}^n)} \leq C\|\varphi\|_{H^1(\Omega)}$ . We will also use  $C$  to denote positive constant which may vary in different formulas.

Associated with operator  $L_\varepsilon$  in (1), the homogenized operator is

$$L_0 = -q_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \quad \text{in } \Omega, \quad (10)$$

where  $L_0$  is a constant coefficient operator which is also called homogenized operator. The constant matrix  $Q = (q_{ij})$  is given by

$$q_{ij} = \int_Y \left[ a_{ij}(y) + a_{ik}(y) \frac{\partial \chi^j(y)}{\partial y_k} \right] dy, \quad (11)$$

where  $Y = [0, 1)^n \simeq \mathbb{R}^n / \mathbb{Z}^n$ . Function  $\chi(y) = (\chi^j(y))$  is a solution of the following cell problem:

$$\begin{aligned} -\frac{\partial}{\partial y_i} \left[ a_{ik}(y) \frac{\partial \chi^j(y)}{\partial y_k} + a_{ij}(y) \right] &= 0, \quad \text{in } Y, \\ \chi^j(y+h) &= \chi^j(y), \\ &\text{for } y \in \mathbb{R}^n, h \in \mathbb{Y}^n, \end{aligned} \quad (12)$$

$$\int_Y \chi^j(y) dy = 0,$$

Fix  $\psi \in C_0^\infty(B_1(0))$  such that  $\psi \geq 0$  and  $\int_{\mathbb{R}^n} \psi dx = 1$ . Define operator  $T_\varepsilon$  on  $L^2$  as

$$T_\varepsilon(u)(x) = u * \psi_\varepsilon = \int_{\mathbb{R}^n} u(x-y) \psi_\varepsilon(y) dy, \quad (13)$$

where  $\psi_\varepsilon(x) = \varepsilon^{-n} \psi(x/\varepsilon)$ . We also call it the smoothing operator.

**Proposition 3.** *If  $u_0 \in H^2(\mathbb{R}^n)$ , then*

$$\|\nabla u_0 - T_\varepsilon(\nabla u_0)\|_{L^2(\mathbb{R}^n)} \leq C\varepsilon \|\nabla^2 u_0\|_{L^2(\mathbb{R}^n)} \quad (14)$$

and

$$\|T_\varepsilon(\nabla^2 u_0)\|_{L^2(\mathbb{R}^n)} \leq C \|\nabla^2 u_0\|_{L^2(\mathbb{R}^n)} \quad (15)$$

*Proof.* These estimates have proved by Parseval's Theorem and Hölder's inequality, which may be found in [7].  $\square$

**Proposition 4.** *Let  $F_{ij}(y) \in L^2(Y)$  be a periodic function,  $Y = [0, 1)^n$ . Suppose that  $\int_Y F_{ij}(y) dy = 0$  and  $(\partial/\partial y_i)(F_{ij}(y)) = 0$ . Then there exists  $\Phi_{kij} \in H^1(Y)$  such that  $F_{ij} = \partial\Phi_{kij}/\partial y_k$  and  $\Phi_{kij} = -\Phi_{ikj}$ .*

*Proof.* This proposition had been proved by Kenig, Lin, and Shen [4].  $\square$

**Remark 5.** Let

$$F_{ij}(y) = q_{ij} - a_{ij}(y) - a_{ik}(y) \frac{\partial \chi^j(y)}{\partial y_k}. \quad (16)$$

Note that periodic function  $F_{ij}(y)$  satisfies  $\int_Y F_{ij}(y) dy = 0$  and  $(\partial/\partial y_i)(F_{ij}) = 0$ . It follows from Proposition 4 that there exists a function  $\Phi_{kij}(y)$ , such that  $\Phi_{kij} = -\Phi_{ikj}$  and  $F_{ij} = \partial\Phi_{kij}/\partial y_k$ .

**Remark 6.** Under the assumption  $A(y)$  is bounded measurable in  $\mathbb{R}^n$ , it is known that  $\chi(y) \in C^{0,\alpha}(\mathbb{R}^n)$ . This implies that  $\Phi(y) \in C^{0,\alpha}(\mathbb{R}^n)$ . In particular,  $\|\chi\|_{L^\infty(\mathbb{R}^n)} + \|\Phi\|_{L^\infty(\mathbb{R}^n)} \leq C$ .

**Proposition 7.** *If  $u_0 \in H^2(\mathbb{R}^n)$ , then*

$$\|T_\varepsilon(\nabla u_0)\|_{L^2(\bar{\Omega}_\varepsilon)} \leq C\varepsilon^{1/2} \|u_0\|_{H^2(\mathbb{R}^n)}. \quad (17)$$

*Proof.* By Fubini's Theorem,

$$\begin{aligned} &\int_{\bar{\Omega}_\varepsilon} |T_\varepsilon(\nabla u_0)|^2 dx \\ &\leq C \int_{\bar{\Omega}_\varepsilon} \int_{B_1(0)} |\nabla u_0(x-\varepsilon y)|^2 dy dx \\ &\leq C \int_{B_1(0)} dy \int_{\bar{\Omega}_{2\varepsilon}} |\nabla u_0(x)|^2 dx \leq C\varepsilon \|u_0\|_{H^2(\mathbb{R}^n)}^2, \end{aligned} \quad (18)$$

where we have used the well-known estimate for the last inequality. See [33] or [12] for the proof.  $\square$

## 3. Proofs Theorems

The goal of this section is to establish  $H^1$  and  $L^2$  convergence rates of solutions.

Let

$$M_\varepsilon = u_\varepsilon - u_0 - \varepsilon \chi T_\varepsilon(\nabla \tilde{u}_0). \quad (19)$$

In order to prove Theorem 1, it suffices to show that  $\|M_\varepsilon\|_{H^1(\Omega)} \leq C\varepsilon \|u_0\|_{H^2(\Omega)}$ .

By the represented formula of  $M_\varepsilon$ , then  $M_\varepsilon$  satisfies the following boundary value problem:

$$L_\varepsilon M_\varepsilon = L_0 u_0 - L_\varepsilon u_0 - L_\varepsilon(\varepsilon \chi T_\varepsilon(\nabla \tilde{u}_0))$$

in  $\Omega$ ,

$$M_\varepsilon = -\varepsilon \chi T_\varepsilon(\nabla \tilde{u}_0) \quad \text{on } \Gamma_1, \quad (20)$$

$$\frac{\partial M_\varepsilon}{\partial \nu_\varepsilon} + k M_\varepsilon = \frac{\partial u_0}{\partial \nu_0} - \frac{\partial u_0}{\partial \nu_\varepsilon} - \frac{\partial(\varepsilon \chi T_\varepsilon(\nabla \tilde{u}_0))}{\partial \nu_\varepsilon}$$

$$-k\varepsilon \chi T_\varepsilon(\nabla \tilde{u}_0) \quad \text{on } \Gamma_2,$$

where we have used (1) and (6) satisfied by  $u_\varepsilon$  and  $u_0$ , respectively.

In view of the fact that

$$\begin{aligned} & \int_{\Omega} A_{\varepsilon} \nabla u_{\varepsilon} \cdot \nabla \varphi dx + k \int_{\partial\Omega} u_{\varepsilon} \varphi d\sigma(x) \\ &= \int_{\Omega} Q \nabla u_0 \cdot \nabla \varphi dx + k \int_{\partial\Omega} u_0 \varphi d\sigma(x), \end{aligned} \quad (21)$$

we obtain

$$\begin{aligned} & \int_{\Omega} A_{\varepsilon} \nabla M_{\varepsilon} \cdot \nabla \varphi dx + k \int_{\partial\Omega} M_{\varepsilon} \varphi d\sigma(x) \\ &= \int_{\Omega} [Q \nabla u_0 - A_{\varepsilon} \nabla u_0 - \varepsilon A_{\varepsilon} \nabla (\chi T_{\varepsilon} (\nabla \tilde{u}_0))] \cdot \nabla \varphi dx \\ & \quad - k \int_{\partial\Omega} \varepsilon A_{\varepsilon} \chi T_{\varepsilon} (\nabla \tilde{u}_0) \varphi d\sigma(x). \end{aligned} \quad (22)$$

It is easy to calculate that

$$\begin{aligned} & Q \nabla u_0 - A_{\varepsilon} \nabla u_0 - \varepsilon A_{\varepsilon} \nabla (\chi T_{\varepsilon} (\nabla \tilde{u}_0)) \\ &= [Q (\nabla u_0 - T_{\varepsilon} (\nabla \tilde{u}_0)) - A_{\varepsilon} (\nabla u_0 - T_{\varepsilon} (\nabla \tilde{u}_0))] \\ & \quad - \varepsilon A_{\varepsilon} \chi T_{\varepsilon} (\nabla^2 \tilde{u}_0) + [Q - A_{\varepsilon} - A_{\varepsilon} \nabla \chi] T_{\varepsilon} (\nabla \tilde{u}_0). \end{aligned} \quad (23)$$

Then, it follows from the bilinear form that

$$\begin{aligned} & \int_{\Omega} A_{\varepsilon} \nabla M_{\varepsilon} \cdot \nabla \varphi dx + k \int_{\partial\Omega} M_{\varepsilon} \varphi d\sigma(x) \\ &= \int_{\Omega} [Q (\nabla u_0 - T_{\varepsilon} (\nabla \tilde{u}_0)) - A_{\varepsilon} (\nabla u_0 - T_{\varepsilon} (\nabla \tilde{u}_0))] \\ & \quad - \varepsilon A_{\varepsilon} \chi T_{\varepsilon} (\nabla^2 \tilde{u}_0) \cdot \nabla \varphi dx + \int_{\Omega} [Q - A_{\varepsilon} - A_{\varepsilon} \nabla \chi] \\ & \quad \cdot T_{\varepsilon} (\nabla \tilde{u}_0) \cdot \nabla \varphi dx - k \int_{\partial\Omega} \varepsilon A_{\varepsilon} \chi T_{\varepsilon} (\nabla \tilde{u}_0) \varphi d\sigma(x) \\ & \doteq I_1 + I_2 + I_3. \end{aligned} \quad (24)$$

To estimate  $I_1$ , we note that, by Proposition 3,

$$|I_1| \leq C\varepsilon \|u_0\|_{H^2(\Omega)} \|\nabla \varphi\|_{L^2(\Omega)}. \quad (25)$$

Next, we shall estimate  $I_2$ . Let  $F_{ij} = q_{ij} - a_{ij}(y) - a_{ik}(y)(\partial \chi^j(y)/\partial y_k)$ . Note that  $F_{ij}$  is periodic and satisfies the conditions of Proposition 4. Then, in view of Remark 5, there exists a periodic function  $\Phi_{kij} \in H^1(Y)$ , such that  $\Phi_{kij} = -\Phi_{ikj}$ , and  $q_{ij} - a_{ij}(y) - a_{ik}(y)(\partial \chi^j(y)/\partial y_k) = \partial \Phi_{kij}/\partial y_k$ .

Thus, by the divergence theorem, it gives

$$\begin{aligned} I_2 &= \int_{\Omega} \frac{\partial}{\partial x_k} (\varepsilon \Phi_{kij}) T_{\varepsilon} \left( \frac{\partial \tilde{u}_0}{\partial x_j} \right) \cdot \frac{\partial \varphi}{\partial x_i} dx \\ &= - \int_{\Omega} \varepsilon \Phi_{kij} T_{\varepsilon} \left( \frac{\partial^2 \tilde{u}_0}{\partial x_k \partial x_j} \right) \frac{\partial \varphi}{\partial x_i} dx \\ & \quad - \int_{\Omega} \varepsilon \Phi_{kij} T_{\varepsilon} \left( \frac{\partial \tilde{u}_0}{\partial x_j} \right) \frac{\partial^2 \varphi}{\partial x_k \partial x_i} dx \\ &= - \int_{\Omega} \varepsilon \Phi_{kij} T_{\varepsilon} \left( \frac{\partial^2 \tilde{u}_0}{\partial x_k \partial x_j} \right) \frac{\partial \varphi}{\partial x_i} dx. \end{aligned} \quad (26)$$

Note that the second term vanishes in view of the antisymmetry of  $\Phi_{kij}$ .

As a result, using Proposition 3 and Remark 6, we get that

$$|I_2| \leq C\varepsilon \|u_0\|_{H^2(\Omega)} \|\nabla \varphi\|_{L^2(\Omega)}. \quad (27)$$

It remains to estimate  $I_3$ . It follows from Proposition 7 that

$$\begin{aligned} |I_3| &\leq k \int_{\partial\Omega} \varepsilon |A_{\varepsilon} \chi T_{\varepsilon} (\nabla \tilde{u}_0) \varphi| d\sigma(x) \\ &\leq C\varepsilon \int_{\overline{\Omega}_{\varepsilon}} |T_{\varepsilon} (\nabla \tilde{u}_0)| \cdot |\varphi| dx \\ &\leq C\varepsilon \|T_{\varepsilon} (\nabla \tilde{u}_0)\|_{L^2(\overline{\Omega}_{\varepsilon})} \|\varphi\|_{L^2(\Omega_{\varepsilon})} \\ &\leq C\varepsilon^{3/2} \|u_0\|_{H^2(\Omega)} \|\varphi\|_{L^2(\Omega_{\varepsilon})}. \end{aligned} \quad (28)$$

This, together with (25) and (27), gives that

$$\begin{aligned} & \left| \int_{\Omega} A_{\varepsilon} \nabla M_{\varepsilon} \cdot \nabla \varphi dx + k \int_{\partial\Omega} M_{\varepsilon} \varphi d\sigma(x) \right| \\ & \leq C\varepsilon \|u_0\|_{H^2(\Omega)} \|\nabla \varphi\|_{L^2(\Omega)}. \end{aligned} \quad (29)$$

By the coercive condition of bilinear form and duality argument, we get the desired result, which completes the proof of Theorem 1.

It follows from Theorem 1 and Proposition 3, by Minkowski's inequality, that

$$\begin{aligned} \|u_{\varepsilon} - u_0\|_{L^2(\Omega)} &\leq C\varepsilon \|u_0\|_{H^2(\Omega)} + \|\varepsilon \chi T_{\varepsilon} (\nabla u_0)\|_{L^2(\Omega)} \\ &\leq C\varepsilon \|u_0\|_{H^2(\Omega)}. \end{aligned} \quad (30)$$

This completes the proof of Theorem 2.

## 4. Conclusions and Perspectives

In this paper, we research the convergence rates of solutions for homogenization of the mixed Dirichlet-Robin boundary value problems. Our approach is utilizing the smoothing operator, which is much more simple and direct to deal with boundary discrepancies. As a consequence, we obtain the  $H^1$  and  $L^2$  convergence rates results, which extend the classical boundary value problems to a broader mixed boundary condition settings.

Indeed, it is expected that one could obtain the  $W^{1,p}$  convergence rates, for any  $1 \leq p < \infty$ . To the best of our knowledge, such estimates for the mixed boundary value problems in homogenization have not been reported so far in the literature. Hence, how to utilize the smoothing operator and avoid difficult of the terms from boundary discrepancies for such problems are an interesting problem. This is one further possible direction to be developed.

Generally, many other types of equations with the mixed boundary conditions settings could be considered by this method. One may naturally try to extend the classical second-order equations to  $2m$ -order higher-order equations or nonlinear elliptic equations. It is expected that the method of this work could contribute to a better solving of the mixed boundary value problem in homogenization.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

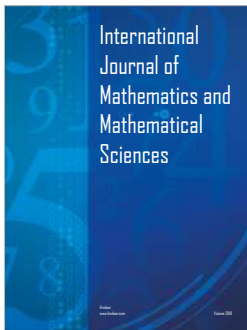
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