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CONVERGENCE RATES OF APPROXIMATE SUMS OF THE AREAS OF SURFACES OF REVOLUTION

By

Yuriko Gorai, Hiroyuki Tasaki, and Mio Yamakawa

Abstract. We represent the convergence rates of approximate sums of the areas of surfaces of revolution as limits of their expanded error terms and estimate them. In the case of convex surfaces of revolution we represent the convergence rates of them by the integral of certain functions.

1. Introduction

Let $[a, b]$ be a bounded closed interval and f be a function of class C^1 defined on $[a, b]$. We assume $f > 0$ and consider the surface of revolution defined by f:

$$\{(x, f(x)\cos\theta, f(x)\sin\theta) \mid a \le x \le b, 0 \le \theta \le 2\pi\}.$$

The area S of this surface of revolution is given by

$$S = 2\pi \int_a^b f(x)\{f'(x)^2 + 1\}^{1/2}\, dx.$$

We take an n-division Δ_n of $[a, b]$ defined by

$$\Delta_n : a = s_0 \le s_1 \le \cdots \le s_{n-1} \le s_n = b.$$

We join $(s_{i-1}, f(s_{i-1}))$ and $(s_i, f(s_i))$ by segment and revolve the obtained polygon around the x-axis. At this point we obtain a union of frustums, whose

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area $S(\Delta_n)$ is given by

$$S(\Delta_n) = 2\pi \sum_{i=1}^{n} \frac{f(s_{i-1}) + f(s_i)}{2} \{(f(s_i) - f(s_{i-1}))^2 + (s_i - s_{i-1})^2\}^{1/2}.$$

We regard $S(\Delta_n)$ as an approximate sum of the area S.

In the present paper we consider the limits of the expanded error terms

$$(*) \qquad n^2|S - S(\Delta_n)| \quad \text{and} \quad n^2(S - S(\Delta_n)),$$

which represent the convergence rates of the approximate sums $S(\Delta_n)$ of S. Our purpose is to estimate the limits of $(*)$, stated in Theorems 1.2, 1.3, and 1.4.

In the case of the lengths of curves Gleason [2] has treated convergence rates of some approximate sums of the lengths of curves and obtained the following theorem.

THEOREM 1.1 (Gleason). *Let A be a curve in euclidean space of class C^2 and length L. For each positive integer n, let P_n be the longest polygon of n edges properly inscribed in A. Then*

$$\lim_{n\to\infty} n^2(L - L(P_n)) = \frac{1}{24}\left(\int_A \kappa^{2/3}\, ds\right)^3,$$

where κ is the curvature of A.

In this case for any polygon P inscribed in A we have $L \geq L(P)$, however in our case we do not know which of S and $S(\Delta)$ is greater for a division Δ of $[a, b]$. So we cannot use Theorem 1.1 directly to the case of the areas of surfaces of revolution, but some of arguments and a lemma in Gleason [2] are still useful in our case.

Schwarz showed an example of polyhedra inscribed in a cylinder whose areas did not converge to the area of the cylinder in [3], which is now called the lantern of Schwarz. So we have to be careful in treating approximate sums of the areas of surfaces.

Approximate sums of Riemann integrals are the most fundamental approximate sums. Chui [1] and the second named author [4] obtained some results on the approximate sums of Riemann integrals. It is possible to apply their results to the function $f(x)\{f'(x)^2 + 1\}^{1/2}$ of our case, however with the approximate sums consisting the derivatives of $f(x)$. Since our approximate sums can be obtained directly from the surface of f without the derivatives, our approximation is useful.

For an n-division Δ_n of $[a,b]$, we define $d(\Delta_n)$ by

$$d(\Delta_n) = \max\{s_i - s_{i-1} \mid 1 \le i \le n\}.$$

For a sequence $\{\Delta_n\}_{n=1}^{\infty}$ of n-divisions of $[a,b]$, we define $e(\{\Delta_n\}_{n=1}^{\infty})$ by

$$e(\{\Delta_n\}_{n=1}^{\infty}) = \limsup_{n\to\infty} nd(\Delta_n).$$

This is useful for estimate of convergence rate of approximate sums.

From now on we assume that the function f is of class C^4, because we use the fourth derivative of f in the proofs of Theorems 1.2, 1.3, and 1.4. In order to state the theorems we define a function φ by

$$\varphi(x) = -\frac{1}{2}f''(x)\{f'(x)^2+1\}^{-1/2} + \frac{1}{4}f(x)f''(x)^2\{f'(x)^2+1\}^{-3/2}.$$

This function φ is the coefficient of the third term of the local error term, which we show in Corollary 2.3.

Theorem 1.2. *For a sequence* $\{\Delta_n\}_{n=1}^{\infty}$ *of n-divisions of* $[a,b]$, *we have*

$$\limsup_{n\to\infty} n^2|S - S(\Delta_n)| \le \frac{\pi}{3}e(\{\Delta_n\}_{n=1}^{\infty})^3 \max_{[a,b]}|\varphi|.$$

The left hand side of the inequality in the above theorem is a limit of the expanded error terms mentioned in the abstract. In the case where φ is positive we can get a sharper estimate of expanded error terms for some divisions defined as follows: The set of all n-divisions of $[a,b]$ is compact and

$$\Delta \mapsto S(\Delta)$$

is continuous, so there exists an n-division $\Delta_n^{\#}$ at which $|S - S(\Delta)|$ attains its minimum. This n-division $\Delta_n^{\#}$ is optimal for the approximate sum $S(\Delta)$. It may not be unique, but the error term $|S - S(\Delta_n^{\#})|$ is unique. Thus we can consider $|S - S(\Delta_n^{\#})|$.

Theorem 1.3. *If* $\varphi > 0$ *holds, then we have*

$$\limsup_{n\to\infty} n^2|S - S(\Delta_n^{\#})| \le \frac{\pi}{3}\left(\int_a^b \varphi(x)^{1/3}\,dx\right)^3.$$

If the surface of revolution is convex, that is, if $f'' < 0$, then $\varphi > 0$ and $S - S(\Delta) \geq 0$ for any division Δ of $[a, b]$. In this case we can represent the limit of expanded error terms by the same integral in the above theorem.

THEOREM 1.4. *If the surface of revolution is convex, that is, $f'' < 0$, then we have*

$$\lim_{n \to \infty} n^2 (S - S(\Delta_n^{\#})) = \frac{\pi}{3} \left(\int_a^b \varphi(x)^{1/3} \, dx \right)^3.$$

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2. Local Error Terms

For a subinterval $[p, q] \subset [a, b]$ we define the local error term $F(p, q)$ by

$$F(p, q) = \int_p^q f(x)\{f'(x)^2 + 1\}^{1/2} \, dx - \frac{f(p) + f(q)}{2} \{(f(q) - f(p))^2 + (q - p)^2\}^{1/2}.$$

The first term of the right hand side of the definition of $F(p, q)$ multiplied by 2π is equal to the local area of the surface of revolution and the second term multiplied by 2π is equal to the area of the corresponding frustum. We express $F(p, q)$ by the linear combination of $(q - p)^i$ $(0 \leq i \leq 4)$ in Corollary 2.4. Using the expression we will prove our main theorems in Sections 3, 4, and 5.

In order to make the calculation simpler we change variables and functions by

$$u = q - p, \quad g(u) = f(q) - f(p).$$

The new function g satisfies

$$g(0) = 0, \quad g^{(i)}(u) = f^{(i)}(q) \quad (i \geq 1).$$

We set $I = [0, b - a]$ and

$$h(u) = (g(u)^2 + u^2)^{1/2}.$$

I is the range of the variable u. Using h we can write $F(p,q)$ as follows:

$$F(p,q) = \int_p^{p+u} f(t)(f'(t)^2+1)^{1/2}\,dt - \frac{1}{2}(2f(p)+g(u))h(u).$$

By the definition of h, we note that if g is of class C^r, then h is also of class C^r.

We need the following lemma to calculate the higher derivatives of F.

Lemma 2.1. *If g is of class C^4, then we obtain the following equalities.*

$$h'(u) = h(u)^{-1}(g(u)g'(u)+u) \quad (u>0),$$

$$h'(0) = (g'(0)^2+1)^{1/2},$$

$$h''(u) = h(u)^{-1}(g'(u)^2+g(u)g''(u)+1-h'(u)^2) \quad (u>0),$$

$$h''(0) = h'(0)^{-1}g'(0)g''(0),$$

$$h'''(u) = h(u)^{-1}(3g'(u)g''(u)+g(u)g'''(u)-3h'(u)h''(u)) \quad (u>0),$$

$$h'''(0) = \frac{3}{4}h'(0)^{-3}g''(0)^2 + h'(0)^{-1}g'(0)g'''(0),$$

$$\begin{aligned}h^{(4)}(u) = h(u)^{-1}(&3g''(u)^2+4g'(u)g'''(u)+g(u)g^{(4)}(u)\\ &-3h''(u)^2-4h'(u)h'''(u)) \quad (u>0),\end{aligned}$$

$$h^{(4)}(0) = -\frac{3}{8}h'(0)^{-5}g'(0)g''(0)^3 + \frac{1}{2}h'(0)^{-3}g''(0)g'''(0) + h'(0)^{-1}g'(0)g^{(4)}(0).$$

Proof. By the definition of h we obtain the following equalities.

$$h(u)h'(u) = g(u)g'(u)+u \quad (u \in I),$$

$$h'(u) = h(u)^{-1}(g(u)g'(u)+u) \quad (u>0),$$

$$h'(0) = \lim_{u\to 0} h'(u) = (g'(0)^2+1)^{1/2}.$$

Differentiating the first equality of the above we obtain

$$h'(u)^2 + h(u)h''(u) = g'(u)^2 + g(u)g''(u) + 1 \tag{2.1}$$

and

$$h''(u) = h(u)^{-1}(g'(u)^2+g(u)g''(u)+1-h'(u)^2) \quad (u>0).$$

Using this we consider the limit

$$h''(0) = \lim_{u \to 0} h''(u) = 3h'(0)^{-1}g'(0)g''(0) - 2h''(0)$$

and get

$$h''(0) = h'(0)^{-1}g'(0)g''(0) = (g'(0)^2 + 1)^{-1/2}g'(0)g''(0).$$

Differentiating (2.1) we obtain

$$3h'(u)h''(u) + h(u)h'''(u) = 3g'(u)g''(u) + g(u)g'''(u) \tag{2.2}$$

and

$$h'''(u) = h(u)^{-1}(3g'(u)g''(u) + g(u)g'''(u) - 3h'(u)h''(u)) \quad (u > 0).$$

Using this we consider the limit

$$\begin{aligned} h'''(0) &= \lim_{x \to 0} h'''(x) \\ &= 3h'(0)^{-1}g''(0)^2 + 4h'(0)^{-1}g'(0)g'''(0) - 3h'(0)^{-1}h''(0)^2 - 3h'''(0) \end{aligned}$$

and get

$$h'''(0) = \frac{3}{4}h'(0)^{-3}g''(0)^2 + h'(0)^{-1}g'(0)g'''(0).$$

Differentiating (2.2) we obtain

$$3h'(u)h''(u) + h(u)h'''(u) = 3g'(u)g''(u) + g(u)g'''(u)$$

and

$$h^{(4)}(u) = h(u)^{-1}(3g''(u)^2 + 4g'(u)g'''(u) + g(u)g^{(4)}(u) - 3h''(u)^2 - 4h'(u)h'''(u)).$$

Using this we consider the limit

$$\begin{aligned} h^{(4)}(0) &= \lim_{u \to 0} h^{(4)}(u) \\ &= 10h'(0)^{-1}g''(0)g'''(0) + 5h'(0)^{-1}g'(0)g^{(4)}(0) \\ &\quad - 10h'(0)^{-1}h''(0)h'''(0) - 4h^{(4)}(0). \end{aligned}$$

and get

$$h^{(4)}(0) = -\frac{3}{8}h'(0)^{-5}g'(0)g''(0)^3 + \frac{1}{2}h'(0)^{-3}g''(0)g'''(0) + h'(0)^{-1}g'(0)g^{(4)}(0). \quad \square$$

LEMMA 2.2. *If f is of class C^4, then we obtain the following equalities.*

$$F(p,p)=0,\quad \left.\frac{\partial F(p,q)}{\partial q}\right|_{q=p}=0,\quad \left.\frac{\partial^2 F(p,q)}{\partial q^2}\right|_{q=p}=0,$$

$$\left.\frac{\partial^3 F(p,q)}{\partial q^3}\right|_{q=p}=-\frac{1}{2}h'(0)^{-1}g''(0)+\frac{1}{4}f(p)h'(0)^{-3}g''(0)^2,$$

$$\begin{aligned}\frac{\partial^4 F(p,q)}{\partial q^4}&=(g'(u)^2+1)^{1/2}g'''(u)+3(g'(u)^2+1)^{-1/2}g'(u)g''(u)^2\\&+3(g'(u)^2+1)^{-3/2}g'(u)g''(u)^2+3(g'(u)^2+1)^{-1/2}g'(u)^2g'''(u)\\&+(f(p)+g(u))\{-3(g'(u)^2+1)^{-5/2}g'(u)g''(u)^3\\&+3(g'(u)^2+1)^{-3/2}g''(u)g'''(u)+(g'(u)^2+1)^{-1/2}g'(u)g^{(4)}(u)\}\\&-\frac{1}{2}\{g^{(4)}(u)h(u)+4g'''(u)h'(u)+6g''(u)h''(u)+4g'(u)h'''(u)\\&+(2f(p)+g(u))h^{(4)}(u)\}.\end{aligned}$$

PROOF. The definition of F implies $F(p,p)=0$. We calculate the higher derivatives of F.

First we get

$$\frac{\partial F(p,q)}{\partial q}=(f(p)+g(u))(g'(u)^2+1)^{1/2}-\frac{1}{2}g'(u)h(u)-\frac{1}{2}(2f(p)+g(u))h'(u)$$

and

$$\left.\frac{\partial F(p,q)}{\partial q}\right|_{q=p}=f(p)(g'(0)^2+1)^{1/2}-f(p)h'(0)=0.$$

Secondly we get

$$\begin{aligned}\frac{\partial^2 F(p,q)}{\partial q^2}&=g'(u)(g'(u)^2+1)^{1/2}+(f(p)+g(u))(g'(u)^2+1)^{-1/2}g'(u)g''(u)\\&-\frac{1}{2}\{g''(u)h(u)+2g'(u)h'(u)+(2f(p)+g(u))h''(u)\}\end{aligned}$$

and

$$\left.\frac{\partial^2 F(p,q)}{\partial q^2}\right|_{q=p} = g'(0)(g'(0)^2+1)^{1/2} + f(p)(g'(0)^2+1)^{-1/2}g'(0)g''(0)$$
$$- g'(0)h'(0) - f(p)h''(0)$$
$$= 0.$$

Using the equality

$$\{(g'(u)^2+1)^{1/2}\}'' = (g'(u)^2+1)^{-3/2}g''(u)^2 + (g'(u)^2+1)^{-1/2}g'(u)g'''(u)$$

and the Leibniz rule we obtain

$$\frac{\partial^3 F(p,q)}{\partial q^3} = (g'(u)^2+1)^{1/2}g''(u) + 2(g'(u)^2+1)^{-1/2}g'(u)^2 g''(u)$$
$$+ (f(p)+g(u))\{(g'(u)^2+1)^{-3/2}g''(u)^2 + (g'(u)^2+1)^{-1/2}g'(u)g'''(u)\}$$
$$-\frac{1}{2}\{g'''(u)h(x) + 3g''(u)h'(u) + 3g'(u)h''(u) + (2f(p)+g(u))h'''(u)\}$$

and

$$\left.\frac{\partial^3 F(p,q)}{\partial q^3}\right|_{q=p} = -\frac{1}{2}h'(0)^{-1}g''(0) + \frac{1}{4}f(p)h'(0)^{-3}g''(0)^2.$$

Using the equality

$$\{(g'(u)^2+1)^{1/2}\}''' = -3(g'(u)^2+1)^{-5/2}g'(u)g''(u)^3 + 3(g'(u)^2+1)^{-3/2}g''(u)g'''(u)$$
$$+ (g'(u)^2+1)^{-1/2}g'(u)g^{(4)}(u)$$

and the Leibniz rule we obtain

$$\frac{\partial^4 F(p,q)}{\partial q^4} = (g'(u)^2+1)^{1/2}g'''(u) + 3(g'(u)^2+1)^{-1/2}g'(u)g''(u)^2$$
$$+ 3(g'(u)^2+1)^{-3/2}g'(u)g''(u)^2 + 3(g'(u)^2+1)^{-1/2}g'(u)^2 g'''(u)$$
$$+ (f(p)+g(u))\{-3(g'(u)^2+1)^{-5/2}g'(u)g''(u)^3$$
$$+ 3(g'(u)^2+1)^{-3/2}g''(u)g'''(u) + (g'(u)^2+1)^{-1/2}g'(u)g^{(4)}(u)\}$$
$$-\frac{1}{2}\{g^{(4)}(u)h(u) + 4g'''(u)h'(u) + 6g''(u)h''(u) + 4g'(u)h'''(u)$$
$$+ (2f(p)+g(u))h^{(4)}(u)\}. \qquad \square$$

Lemmas 2.1 and 2.2 imply the following corollary.

Corollary 2.3.

$$\left.\frac{\partial^3 F(p,q)}{\partial q^3}\right|_{q=p} = -\frac{1}{2}(f'(p)^2+1)^{-1/2}f''(p) + \frac{1}{4}f(p)(f'(p)^2+1)^{-3/2}f''(p)^2.$$

The right hand side of the above equality is nothing but the definition of φ in Section 1. This is the reason why we consider the function φ.

Using Taylor's theorem, Lemma 2.2, and Corollary 2.3 we obtain an expression of F as follows:

Corollary 2.4. *For a subinterval $[p,q] \subset [a,b]$ there exists $p < r < q$ with property that*

$$F(p,q) = \frac{1}{3!}\varphi(p)(q-p)^3 + \frac{1}{4!}\left.\frac{\partial^4 F(p,q)}{\partial q^4}\right|_{q=r}(q-p)^4.$$

The explicit expression of the fourth derivative of F is given in Lemma 2.2.

3. Proof of Theorem 1.2

Before the proof of Theorem 1.2 we mention several fundamental properties of $e(\{\Delta_n\}_{n=1}^{\infty})$ defined in Section 1.

Proposition 3.1. *For a sequence $\{\Delta_n\}_{n=1}^{\infty}$ of n-divisions of $[a,b]$, we have*

$$b-a \le e(\{\Delta_n\}_{n=1}^{\infty}) \le \infty.$$

Moreover for any α satisfying $b-a \le \alpha \le \infty$ there exists a sequence $\{D_n\}_{n=1}^{\infty}$ of n-divisions of $[a,b]$ with property that $e(\{D_n\}_{n=1}^{\infty}) = \alpha$.

Proof. For any positive integer n we have $(b-a)/n \le d(\Delta_n)$, so we get $b-a \le e(\{\Delta_n\}_{n=1}^{\infty}) \le \infty$.

We take α satisfying $b-a \le \alpha \le \infty$. If $\alpha < \infty$, then we define n-divisions D_n for $n > \alpha/(b-a)$ by

$$s_1 = a + \frac{\alpha}{n}, \quad s_i = a + \frac{\alpha}{n} + \frac{i-1}{n-1}(b-s_1) \quad (2 \le i \le n-1).$$

We can see $e(\{D_n\}_{n=1}^{\infty}) = \alpha$. If $\alpha = \infty$, then we define n-divisions D_n by

$$s_1 = \frac{a+b}{2}, \quad s_i = \frac{a+b}{2} + \frac{i-1}{n-1}(b-s_1) \quad (2 \le i \le n-1).$$

We can see $e(\{D_n\}_{n=1}^{\infty}) = \infty$. □

Roughly speaking $e(\{\Delta_n\}_{n=1}^{\infty})$ measures the difference between $\{\Delta_n\}_{n=1}^{\infty}$ and the sequence of regular n-divisions defined by

$$s_i = a + \frac{i}{n}(b-a) \quad (1 \le i \le n).$$

REMARK 3.2. There exists a sequence $\{\Delta_n\}_{n=1}^{\infty}$ of n-divisions of $[a,b]$ which satisfies $e(\{\Delta_n\}_{n=1}^{\infty}) = \infty$ and $\lim_{n\to\infty} d(\Delta_n) = 0$. We define n-divisions Δ_n for $n \ge 2$ by

$$s_1 = a + \frac{b-a}{\log n}, \quad s_i = a + \frac{b-a}{\log n} + \frac{i-1}{n-1}(b-s_1) \quad (2 \le i \le n-1).$$

We can see $e(\{\Delta_n\}_{n=1}^{\infty}) = \infty$ and $\lim_{n\to\infty} d(\Delta_n) = 0$.

PROOF OF THEOREM 1.2. If $e(\{\Delta_n\}_{n=1}^{\infty}) = \infty$, then there is nothing to prove. So we assume that $e(\{\Delta_n\}_{n=1}^{\infty}) < \infty$. In this case $\lim_{n\to\infty} d(\Delta_n) = 0$ holds. According to the expression of the fourth derivative of F described in Lemma 2.2 there exists a positive constant M with property that

$$\left| \left. \frac{\partial^4 F(p,q)}{\partial q^4} \right|_{q=r} \right| \le M \tag{3.1}$$

for any $a \le p < r < q \le b$. This inequality and Corollary 2.4 imply

$$\begin{aligned} n^2|S - S(\Delta_n)| &= 2\pi n^2 \left| \sum_{i=1}^{n} F(s_{i-1}, s_i) \right| \\ &\le 2\pi n^2 \sum_{i=1}^{n} \left(\left| \frac{1}{3!}\varphi(s_{i-1})(s_i - s_{i-1})^3 \right| + \left| \frac{1}{4!} \frac{\partial^4 F}{\partial q^4}(s_{i-1}, r_i)(s_i - s_{i-1})^4 \right| \right) \\ &\le \frac{\pi}{3} \max_{[a,b]} |\varphi| n^2 \cdot n d(\Delta_n)^3 + \frac{\pi}{12} M n^2 \cdot n d(\Delta_n)^4 \\ &= \frac{\pi}{3} \max_{[a,b]} |\varphi| \{n d(\Delta_n)\}^3 + \frac{\pi}{12} M \{n d(\Delta_n)\}^3 \cdot d(\Delta_n), \end{aligned}$$

where $s_{i-1} < r_i < s_i$. Considering the limits of the above inequality we obtain

$$\limsup_{n\to\infty} n^2 |S - S(\Delta_n)| \le \frac{\pi}{3} e(\{\Delta_n\}_{n=1}^{\infty})^3 \max_{[a,b]} |\varphi|,$$

which completes the proof of Theorem 1.2. □

Corollary 3.3. *If* $\{\Delta_n\}_{n=1}^{\infty}$ *is a sequence of n-divisions of* $[a,b]$ *with property that* $e(\{\Delta_n\}_{n=1}^{\infty}) < \infty$, *then we have*

$$\lim_{n\to\infty} S(\Delta_n) = S.$$

4. Proof of Theorem 1.3

Theorem 1.2 holds for any sequence of n-divisions. On the other hand, Theorem 1.3 holds for a sequence of optimal n-divisions and the estimate in it is sharper than that in Theorem 1.2. In order to consider approximate sums of optimal divisions we need the following lemma obtained by Gleason [2].

Lemma 4.1 (Gleason). *Let* $\Phi(t)$ *be a nonnegative continuous function defined on* $[a,b]$. *For any positive integer n there exists a division of* $[a,b]$:

$$a = s_0 < s_1 < \cdots < s_{n-1} < s_n = b$$

such that all of

$$(s_i - s_{i-1}) \max_{[s_{i-1},s_i]} \Phi(t) \quad (1 \le i \le n)$$

are equal to each other. We denote by J_n *the equal value. Then we obtain*

$$\lim_{n\to\infty} nJ_n = \int_a^b \Phi(t)\,dt.$$

Proof of Theorem 1.3. According to the mean value theorem there exists $p < r_1 < q$ which satisfies

$$h(u) = (q-p)\{f'(r_1)^2 + 1\}^{1/2}.$$

This implies

$$|h(u)| \le (b-a) \max_{[p,q]}\{(f')^2 + 1\}^{1/2} \quad (0 \le u \le q-p).$$

Using Lemma 2.1 we can see that there exist positive constants M_i $(1 \le i \le 4)$ which satisfy

$$|h^{(i)}(u)| \le M_i \max_{[p,q]}\{(f')^2 + 1\}^{1/2} \quad (0 \le u \le q-p).$$

From these estimates and the expression of the fourth derivative of F in Lemma 2.2 there exits a positive constant M which satisfies

$$\left|\frac{\partial^4 F(p,q)}{\partial q^4}\Big|_{q=r}\right| \le M \max_{[p,q]}\{(f')^2 + 1\}^{1/2}$$

for any $p < r < q$. We define ψ by

$$\psi(x) = -\frac{1}{2} f''(x)\{(f'(x))^2 + 1\}^{-1} + \frac{1}{4} f(x)(f''(x))^2\{(f'(x))^2 + 1\}^{-2}.$$

By the definitions of φ and ψ we have

$$\varphi(x) = \{(f'(x))^2 + 1\}^{1/2}\psi(x).$$

The assumption that $\varphi > 0$ implies $\psi > 0$. We set

$$m = \min_{[a,b]} \psi > 0.$$

We multiply $m \le \psi(x)$ by $\{(f'(x))^2 + 1\}^{1/2}$ and obtain

$$m\{(f'(x))^2 + 1\}^{1/2} \le \varphi(x).$$

We apply Lemma 4.1 to the function $\varphi^{1/3}$ and get an n-division Δ_n^G of $[a,b]$ which satisfies all of

$$(s_i - s_{i-1}) \max_{[s_{i-1}, s_i]} \varphi^{1/3} \quad (1 \le i \le n)$$

are equal to each other. We denote by J_n the equal value. Then we have

$$\lim_{n\to\infty} nJ_n = \int_a^b \varphi(t)^{1/3}\, dt.$$

Corollary 2.4 implies

$$\begin{aligned}
&n^2|S - S(\Delta_n^G)| \\
&\quad \le 2\pi n^2 \sum_{i=1}^n \left(\left| \frac{1}{3!}\varphi(s_{i-1})(s_i - s_{i-1})^3 \right| + \left| \frac{1}{4!} \frac{\partial^4 F}{\partial q^4}(s_i, r_i)(s_i - s_{i-1})^4 \right| \right) \\
&\quad \le 2\pi n^2 \sum_{i=1}^n \left(\frac{1}{3!} \left(\max_{[s_{i-1}, s_i]} \varphi^{1/3}(s_i - s_{i-1}) \right)^3 + \frac{1}{4!} \max_{[s_{i-1}, s_i]} \{(f')^2 + 1\}^{1/2} M(s_i - s_{i-1})^4 \right) \\
&\quad \le 2\pi n^2 \sum_{i=1}^n \left(\frac{1}{3!} J_n^3 + \frac{M}{4!m} \max_{[s_{i-1}, s_i]} \varphi(s_i - s_{i-1})^4 \right) \\
&\quad = \frac{\pi}{3}(nJ_n)^3 + \frac{2\pi M}{4!m} \cdot \frac{1}{n}(nJ_n)^3(b - a),
\end{aligned}$$

where $s_{i-1} < r_i < s_i$. Considering the limits of the above inequality we obtain

$$\limsup_{n\to\infty} n^2|S - S(\Delta_n^G)| \le \frac{\pi}{3}\left(\int_a^b \varphi(t)^{1/3}\,dt\right)^3.$$

Since $|S - S(\Delta_n^{\#})| \le |S - S(\Delta_n^G)|$, we complete the proof of Theorem 1.3. $\square$

5. Proof of Theorem 1.4

The assumption that $f'' < 0$ implies that $F(p,q) > 0$ for any subinterval $[p,q] \subset [a,b]$, because the orthogonal projection of the surface determined by $[p,q]$ to the corresponding frustum is area-decreasing. The condition $f'' < 0$ implies $\varphi > 0$, so we have already obtained the estimate of $n^2(S - S(\Delta_n))$ from above. In this section we estimate $n^2(S - S(\Delta_n))$ from bellow.

In order to prove Theorem 1.4 it is sufficient to prove the following lemmas.

Lemma 5.1. *We assume $f'' < 0$. For a sequence Δ_n of n-divisions of $[a,b]$, we have*

$$\liminf_{n\to\infty} n^2(S - S(\Delta_n)) \ge \frac{\pi}{3}\left(\int_a^b \varphi(x)^{1/3}\,dx\right)^3.$$

Proof. First we prove that there exists a positive constant J independent of p, q which satisfies

$$\left|F(p,q) - \frac{1}{6}\varphi(\xi)(q-p)^3\right| \le J(q-p)^4 \tag{5.1}$$

for any interval $[p,q] \subset [a,b]$ and any $\xi \in [p,q]$. Corollary 2.4, (3.1), and the mean value theorem imply

$$\begin{aligned}\left|F(p,q) - \frac{1}{6}\varphi(\xi)(q-p)^3\right| &= \frac{1}{6}(q-p)^3\left|(\varphi(p) - \varphi(\xi)) + \frac{1}{4}\frac{\partial^4 F(p,q)}{\partial q^4}\bigg|_{q=c}(q-p)\right| \\ &\le J(q-p)^4.\end{aligned}$$

Next we prove the following lemma, which implies Lemma 5.1. $\square$

Lemma 5.2. *For any $\varepsilon > 0$ there exists a positive integer N such that for any $n \ge N$ and any n-division Δ of $[a,b]$ the following inequality holds.*

$$n^2(S - S(\Delta)) \ge \frac{\pi}{3}\left(\int_a^b \varphi(x)^{1/3}\,dx\right)^3 - \varepsilon.$$

Proof. We have

$$\left|\frac{F(p,q)}{(q-p)^3} - \frac{1}{6}\varphi(\xi)\right| \le J(q-p)$$

by (5.1). Since the function $x \mapsto x^{1/3}$ is uniformly continuous on $[0,\infty)$, for any $\delta > 0$ there exists $\delta_1 > 0$ such that $|x-y| \le \delta_1$ implies $|x^{1/3} - y^{1/3}| \le \delta$. We take $\eta > 0$ with property that $q - p \le \eta$ implies $J(q-p) \le \delta_1$. If we take $p < q$ which satisfy $q - p \le \eta$, then we have

$$\left|\frac{F(p,q)^{1/3}}{(q-p)} - \left(\frac{1}{6}\varphi(\xi)\right)^{1/3}\right| \le \delta,$$

that is,

$$\left|F(p,q)^{1/3} - \left(\frac{1}{6}\varphi(\xi)\right)^{1/3}(q-p)\right| \le \delta(q-p). \tag{5.2}$$

We take a positive integer r with property that $(b-a)/r \le \eta$. For any n-division Δ of $[a,b]$ we can add at most r points to Δ such that the width of each subinterval is less than or equal to η. We denote the new division by

$$\Delta' : s_0 = a \le s_1 \le \cdots \le s_t = b,$$

where $t \le n + r$. According to the first mean value theorem for integration we can take s_i' in $[s_{i-1}, s_i]$ satisfying

$$\int_{s_{i-1}}^{s_i} \varphi(x)^{1/3}\,dx = \varphi(s_i')^{1/3}(s_i - s_{i-1}).$$

By the estimate (5.2) we get

$$\frac{1}{6^{1/3}}\varphi(s_i')^{1/3}(s_i - s_{i-1}) \le F(s_{i-1}, s_i)^{1/3} + \delta(s_i - s_{i-1}).$$

We add the above inequalities for $i = 1, \dots, t$ and get

$$\begin{aligned}\frac{1}{6^{1/3}}\int_a^b \varphi(x)^{1/3}\,dx &= \sum_{i=1}^{t} \frac{1}{6^{1/3}}\varphi(s_i')^{1/3}(s_i - s_{i-1}) \\ &\le \sum_{i=1}^{t} F(s_{i-1}, s_i)^{1/3} + \delta(b-a). \qquad (*)\end{aligned}$$

We apply the Hölder inequality to the first term of $(*)$ and get

$$(2\pi)^{1/3}\sum_{i=1}^{t} F(s_{i-1},s_i)^{1/3} \le t^{2/3}\left(\sum_{i=1}^{t} 2\pi F(s_{i-1},s_i)\right)^{1/3}$$
$$= t^{2/3}(S - S(\Delta'))^{1/3}.$$

From these we have

$$\left(\frac{\pi}{3}\right)^{1/3}\int_a^b \varphi(x)^{1/3}\,dx \le t^{2/3}(S - S(\Delta'))^{1/3} + \delta(2\pi)^{1/3}(b-a).$$

The inequality $S - S(\Delta') \le S - S(\Delta)$, the estimate obtained above and $t \le n + r$ imply

$$\left(\frac{\pi}{3}\right)^{1/3}\int_a^b \varphi(x)^{1/3}\,dx \le (n+r)^{2/3}(S - S(\Delta))^{1/3} + \delta(2\pi)^{1/3}(b-a).$$

Using the result obtained above, we prove Lemma 5.1. Since the function $x \mapsto x^3$ is continuous, for any $\varepsilon > 0$ there exists $\rho > 0$ such that if

$$\rho \ge \left(\frac{\pi}{3}\right)^{1/3}\int_a^b \varphi(x)^{1/3}\,dx - x,$$

then we have

$$\frac{\varepsilon}{2} \ge \frac{\pi}{3}\left(\int_a^b \varphi(x)^{1/3}\,dx\right)^3 - x^3.$$

So we take $\delta > 0$ which satisfies $\delta(2\pi)^{1/3}(b-a) \le \rho$. We can apply the result obtained above and get a positive integer r with property that for any n-division Δ of $[a,b]$

$$\rho \ge \delta(2\pi)^{1/3}(b-a) \ge \left(\frac{\pi}{3}\right)^{1/3}\int_a^b \varphi(x)^{1/3}\,dx - (n+r)^{2/3}(S - S(\Delta))^{1/3},$$

which implies

$$\frac{\varepsilon}{2} \ge \frac{\pi}{3}\left(\int_a^b \varphi(x)^{1/3}\,dx\right)^3 - (n+r)^2(S - S(\Delta)).$$

We can substitute the optimal division $\Delta_n^{\#}$ for Δ in the above inequality and get

$$(n+r)^2(S - S(\Delta^{\#})) \ge \frac{\pi}{3}\left(\int_a^b \varphi(x)^{1/3}\,dx\right)^3 - \frac{\varepsilon}{2}.$$

Since

$$\limsup_{n\to\infty}(2nr + r^2)(S - S(\Delta^{\#})) = 0,$$

we can choose a positive integer N with property that for $n \geq N$

$$0 \leq (2nr + r^2)(S - S(\Delta^{\#})) \leq \frac{\varepsilon}{2}$$

holds. Thus for $n \geq N$ we have

$$\begin{aligned} n^2(S - S(\Delta^{\#})) &\geq (n + r)^2(S - S(\Delta^{\#})) - \frac{\varepsilon}{2} \\ &\geq \frac{\pi}{3}\left(\int_a^b \varphi(x)^{1/3}\, dx\right)^3 - \varepsilon. \end{aligned}$$

Therefore for any n-division Δ of $[a, b]$ we have

$$n^2(S - S(\Delta)) \geq n^2(S - S(\Delta^{\#})) \geq \frac{\pi}{3}\left(\int_a^b \varphi(x)^{1/3}\, dx\right)^3 - \varepsilon,$$

which completes the proof of Lemma 5.1. □

PROOF OF THEOREM 1.4. Since the condition $f'' < 0$ implies $\varphi > 0$, by Theorem 1.3 and Lemma 5.1 we have

$$\begin{aligned} \frac{\pi}{3}\left(\int_a^b \varphi(x)^{1/3}\, dx\right)^3 &\leq \liminf_{n\to\infty} n^2(S - S(\Delta_n^{\#})) \leq \limsup_{n\to\infty} n^2(S - S(\Delta_n^{\#})) \\ &\leq \frac{\pi}{3}\left(\int_a^b \varphi(x)^{1/3}\, dx\right)^3, \end{aligned}$$

which completes the proof of Theorem 1.4. □

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Ose Highschool, Hitachiohmiya
Ibaraki, 319-2401 Japan

Graduate School of Pure and Applied Science
University of Tsukuba, Tsukuba
Ibaraki, 305-8571 Japan
E-mail address: tasaki@math.tsukuba.ac.jp

Graduate School of Pure and Applied Science
University of Tsukuba, Tsukuba
Ibaraki, 305-8571 Japan
E-mail address: m.yamakawa@math.tsukuba.ac.jp