Convergence to fractional kinetics for random walks associated with unbounded conductances

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Abstract

We consider a random walk among unbounded random conductances whose distribution has infinite expectation and polynomial tail. We prove, that the scaling limit of this process is a Fractional-Kinetics process – that is the time change of a d-dimensional Brownian motion by the inverse of an independent α-stable subordinator. We further show, that the same process appears in the scaling limit of the non-symmetric Bouchaud's trap model.

1 Introduction and main results

In this paper we establish a quenched non-Gaussian scaling limit theorem for two processes in a random environment in $\mathbb{Z}^d$ (with $d \geq 3$); these are the simple random walk among random conductances (often called the *Random Conductance Model*), and the non-symmetric Bouchaud's trap model on $\mathbb{Z}^d$. We will show that if the distribution of the environment is sufficiently heavy-tailed (and regular) then both these models, suitably normalised, converge to the Fractional Kinetics process, which is the non-Markovian, self-similar continuous process defined as the time change of a standard d-dimensional Brownian motion by the inverse of a stable subordinator.

Since the Fractional Kinetics process is sub-diffusive, that is its mean-square increment increases sub-linearly with time, our results prove that these models have anomalous diffusion. In [BČ07] an analogous scaling limit theorem was established for the much simpler symmetric Bouchaud's trap model,

Before discussing our results we describe the models more precisely. We begin by defining continuous time random walks associated with a family of (non-random) conductances on $\mathbb{Z}^d$. Let E^d be the set of all nearest-neighbour edges in $\mathbb{Z}^d$, and let μ_e, $e \in E^d$, be strictly positive. Write $x \sim y$ if x, y are neighbours in $\mathbb{Z}^d$. Set

$$\mu_x = \sum_{e \ni x} \mu_e \qquad \text{for } x \in \mathbb{Z}^d, \tag{1.1}$$

$$p_{xy} = \mu_{(xy)}/\mu_x \qquad \text{if } x \sim y; \tag{1.2}$$

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here $(xy) \in E^d$ is the edge connecting x and y. We consider random walks which jump from x to a neighbour y according to the transition probabilities p_{xy}. Let ν be a measure on $\mathbb{Z}^d$, and write $\nu_x = \nu(\{x\})$, $x \in \mathbb{Z}^d$. We study the continuous-time Markov chain on $\mathbb{Z}^d$ with transition rate from x to y given by μ_{xy}/ν_x. This random walk is reversible, and ν_x is its reversible measure.

There are two choices of ν which will concern us. If we take $\nu_x = \mu_x$, then the random walk has transition rates p_{xy}. We use $X = (X(t), t \geq 0)$ and $P_x^{\boldsymbol{\mu}}$ to denote this Markov chain and its law on $D^d := D([0,\infty), \mathbb{R}^d)$, and call this the *constant-speed* random walk (CSRW) in the configuration of conductances $\boldsymbol{\mu} = \{\mu_e : e \in E^d\}$. The term 'constant-speed' refers to the fact that the total jump rate out of any point x is independent of x and is equal to one.

The second walk, which we call the *variable-speed* random walk (VSRW) among the random conductances $\boldsymbol{\mu}$, is given by taking $\nu_x \equiv 1$. We use $Y = (Y(t), t \geq 0)$ and (with a slight abuse of notation) $P_x^{\boldsymbol{\mu}}$ to denote this process on D^d. Since the CSRW and VSRW have the same jump probabilities, they are time-changes of each other, and this will play an important role in our proofs.

To obtain a random walk in random environment we take μ_e to be random. The first model we study is obtained by taking $\Omega = (0,\infty)^{E^d}$ to be the set of configurations of conductances, and let $\mathbb{P}$ be the product measure on Ω under which the canonical coordinates μ_e, $e \in E^d$, are i.i.d. random variables. This gives us the *Random Conductance Model* (RCM).

We are interested in obtaining a $\mathbb{P}$-a.s. limit for the CSRW X under the law $P_0^{\boldsymbol{\mu}}$. In order to state our principal result we need to introduce the limiting Fractional-Kinetics (FK) process.

Definition 1.1. Let $\mathsf{BM}_d(t)$ be a standard d-dimensional Brownian motion started at 0, and let V_α be an α-stable subordinator independent of BM_d, which is determined by $\mathbb{E}[e^{-\lambda V_\alpha(t)}] = e^{-t\lambda^\alpha}$. Let $V_\alpha^{-1}(s) := \inf\{t : V_\alpha(t) > s\}$ be the generalised right-continuous inverse of $V_\alpha(t)$. We define the *fractional-kinetics process* $\mathsf{FK}_{d,\alpha}$ by

$$\mathsf{FK}_{d,\alpha}(s) = \mathsf{BM}_d(V_\alpha^{-1}(s)), \quad s \in [0,\infty). \tag{1.3}$$

The FK process is non-Markovian process, which is γ-Hölder continuous for all $\gamma < \alpha/2$. It is self-similar: $\mathsf{FK}_{d,\alpha}(\cdot) \overset{\text{law}}{=} \lambda^{-\alpha/2}\,\mathsf{FK}_{d,\alpha}(\lambda\cdot)$, $\lambda > 0$, and the density of its fixed time distribution $p(t,x)$ satisfies the fractional-kinetics equation

$$\frac{\partial^\alpha}{\partial t^\alpha} p(t,x) = \frac{1}{2}\Delta p(t,x) + \delta_0(x)\frac{t^{-\alpha}}{\Gamma(1-\alpha)}. \tag{1.4}$$

This process is well known in the physics literature. See the broad survey by G. Zaslavsky [Zas02], the recent book [Zas05] about the relevance of this process for chaotic deterministic systems, and also [GM03, Hil00, MK00, MS84, SZK93] for more on this class of processes and further references.

Our main result is the following quenched functional limit theorem for the CSRW in a heavy-tailed environment.

Theorem 1.2. *Assume that $d \geq 3$,*

$$\mathbb{P}[\mu_e \geq u] = C_1 u^{-\alpha}(1 + o(1)), \quad u \to \infty, \tag{1.5}$$

for some $\alpha \in (0,1)$, $C_1 \in (0,\infty)$, and that $\mathbb{P}[\mu_e > \underline{c}] = 1$ for some $\underline{c} \in (0,\infty)$. Let

$$X_n(t) = n^{-1}X(tn^{2/\alpha}), \qquad t \in [0,\infty), n \in \mathbb{N}, \tag{1.6}$$

be the rescaled CSRW. Then there exists a constant $\mathcal{C} \in (0,\infty)$ such that $\mathbb{P}$-a.s., under $P_0^{\boldsymbol{\mu}}$, the sequence of processes X_n converges in law to a multiple of the fractional-kinetics process $\mathcal{C}\,\mathsf{FK}_{d,\alpha}$ on D^d equipped with the topology of the uniform convergence on compact subsets of $[0,\infty)$.

Theorem 1.2 contrasts with a long line of Gaussian functional central limit theorems for both the CSRW, the VSRW and also for the discrete-time walk corresponding to the CSRW. Let

$$X_n'(\cdot) = n^{-1}X(n^2\cdot), \qquad Y_n(\cdot) = n^{-1}Y(n^2\cdot). \tag{1.7}$$

In the 1980s it was proved that, provided $\mathbb{E}\mu_e < \infty$, the processes X_n' and Y_n converge to a multiple of a standard Brownian motion, $\sigma\,\mathsf{BM}_d$, in law under semi-direct product measure $\mathbb{P} \times P_0^{\boldsymbol{\mu}}$, with the possibility that $\sigma = 0$ in some cases — see [KV86, Koz85, DFGW89].

These 'annealed' or averaged invariance principles were greatly improved in [SS04], where, under the ellipticity assumption $\mathbb{P}[\mu_e \in (a,b)] = 1$ for some $0 < a \le b < \infty$, it was shown that for all $d \ge 1$, the rescaled discrete-time walk converges to $\sigma\,\mathsf{BM}_d$ with $\sigma > 0$, almost surely with respect to $\mathbb{P}$. The ellipticity assumption was relaxed to boundedness from above (that is $\mathbb{P}\big[\mu_e \in [0,b]\big] = 1$) in [BP07, Mat08], with the same non-trivial Brownian limit. (These papers impose the condition $\mathbb{P}(\mu_e = 0) < p_c(d)$, where $p_c(d)$ is the critical probability for bond percolation in $\mathbb{Z}^d$, in the cases when $\mathbb{P}(\mu_e = 0) > 0$. Note also the papers [Bar04, BB07, MP07] for the percolation case $\mu_e \in \{0,1\}$.) When the conductances are bounded from above, the CSRW, the VSRW, and the discrete-time walk with jump probabilities p_{xy} are related by time changes that are asymptotically linear. Hence there is little difference in the asymptotic behaviour of these processes.

This is no longer the case for unbounded conductances. This situation was studied in [BD08] where it was proved that, under the assumption that $\mathbb{P}[\mu_e > a] = 1$, both X_n' and Y_n converge $\mathbb{P}$-a.s. to multiples of Brownian motion: $\sigma_X\,\mathsf{BM}_d$ and $\sigma_Y\,\mathsf{BM}_d$ respectively. While $\sigma_Y > 0$ always, it was shown that the constant for the CSRW satisfies $\sigma_X > 0$ if and only if $\mathbb{E}[\mu_e] < \infty$. The results of [BD08] therefore only give that X_n' converges to 0; Theorem 1.2 above identifies the right scaling and gives a non-trivial limiting process.

We now describe our second random environment, *Bouchaud's trap model*. This was introduced in the physics literature [Bou92, BD95], on the complete graph, to explain some strange dynamical properties of complex disordered systems, in particular aging. The version on $\mathbb{Z}^d$ was first studied (using physics arguments) in [RMB01].

It is defined as follows. Let $\tilde{\Omega} = \mathbb{R}^{\mathbb{Z}^d}$ and let $\tilde{\mathbb{P}}$ be a product measure on $\tilde{\Omega}$ under which the canonical coordinates $\mathcal{E}_x$, $x \in \mathbb{Z}^d$, interpreted as energies, are i.i.d. variables. We define $\tau_x = e^{\beta\mathcal{E}_x}$ to be the non-normalised Gibbs measure on $\mathbb{Z}^d$, and write $\boldsymbol{\tau} = (\tau_x : x \in \mathbb{Z}^d)$. Let $a \in [0,1]$ be a parameter, and define random conductances $\tilde{\mu}_e$ by

$$\tilde{\mu}_{(xy)} = \tau_x^a \tau_y^a \qquad \text{if } x \sim y, \tag{1.8}$$

Let $\nu_x = \tau_x$; then Bouchaud's trap model (BTM) is the continuous-time Markov chain on $\mathbb{Z}^d$ whose transition rates w_{xy} are given by

$$w_{xy} = \frac{\tilde{\mu}_{(xy)}}{\tau_x} = \tau_x^{a-1}\tau_y^a = e^{-\beta((1-a)\mathcal{E}_x - a\mathcal{E}_y)}, \qquad \text{for } x \sim y. \tag{1.9}$$

We use $\tilde{X} = (\tilde{X}(t), t \geq 0)$ to denote the BTM, and we write $\tilde{P}_x^{\boldsymbol{\tau}}$ for the law of $\tilde{X}$ on D^d. Note that for any $a \in [0,1]$ the Gibbs measure τ_x is reversible for the BTM. If $a = 0$, then $\tilde{\mu}_e = 1$ for all e, and the BTM is a time change of the simple random walk on $\mathbb{Z}^d$. This case is sometimes called *symmetric* BTM, while *non-symmetric* refers to the general case $a \neq 0$.

We can define the VSRW associated with the conductances $\tilde{\mu}_e$ in the same way as before; this process has jump rates given by (1.8) and counting measure as its reversible measure. We write $\tilde{Y} = (\tilde{Y}(t), t \geq 0)$ for this process, and $\tilde{P}_x^{\boldsymbol{\tau}}$ for its law on D^d.

The following theorem, analogical to Theorem 1.2, is our main result on the BTM.

Theorem 1.3. *Let $d \geq 3$, $a \in [0,1]$, and suppose that*

$$\tilde{\mathbb{P}}[\tau_x \geq u] = C_1 u^{-\alpha}(1 + o(1)), \quad u \to \infty, \tag{1.10}$$

for some $\alpha \in (0,1)$, $C_1 \in (0,\infty)$, and $\tilde{\mathbb{P}}[\tau_x > \underline{c}] = 1$ for some $\underline{c} \in (0,\infty)$. Let

$$\tilde{X}_n(t) = n^{-1}\tilde{X}(tn^{2/\alpha}), \qquad t \in [0,\infty). \tag{1.11}$$

Then there exists a constant $\tilde{\mathcal{C}} \in (0,\infty)$ such that $\tilde{\mathbb{P}}$-a.s., under $\tilde{P}_0^{\boldsymbol{\tau}}$, the sequence of processes $\tilde{X}_n$ converges in law to a multiple of the fractional-kinetics process $\tilde{\mathcal{C}}\,\mathsf{FK}_{d,\alpha}$ on D^d.

Let us recall previous rigorous results on the BTM. The first papers on this model concentrated on its aging behaviour. Scaling limit statements, if present, were used as technical tools. Moreover, with the exception of [BČ05], only the symmetric BTM is explored in these papers. This include [BBG03] considering the BTM on a n-dimensional hypercube, and [FIN02] where aging of the BTM on $\mathbb{Z}$ is proved. In [FIN02] it is also shown that the scaling limit (in the sense of finite-dimensional distributions) of the one-dimensional symmetric BTM is a singular diffusion in a random environment, and so quite different from the FK process. The result of [FIN02] was extended to the non-symmetric case in [BČ05]. It was proved there that the scaling limit is independent of a, giving an indication that this parameter has a little influence on the asymptotic behaviour of the BTM. The techniques of [FIN02] and [BČ05] used strongly the fact that $d = 1$.

Aging for the symmetric BTM on $\mathbb{Z}^d$ was shown in [BČM06] ($d = 2$) and [Čer03] ($d \geq 3$). The scaling limit approach to studying the BTM was put forward in [BČ08]. In [BČ07], a theorem analogous to our Theorem 1.3 was shown for $d \geq 2$, but in the symmetric case only. Theorem 1.3 confirms the fact that the influence of a on the asymptotic behaviour of the BTM is small.

There is another natural Markov chain which has the Gibbs measure τ_x as the reversible measure, namely the Metropolis chain. Our techniques can be used to prove the same result as Theorem 1.3 for this chain too.

The assumptions of Theorems 1.2 and 1.3 are not optimal. The assumptions (1.5), (1.10) on the tail of $\mathbb{P}$ and $\tilde{\mathbb{P}}$ could be replaced by a weaker condition $\mathbb{P}[\mu_e \ge u] = u^{-\alpha}L(u)$, where $L(u)$ is a function slowly varying at infinity. We decided to use (1.5) and (1.10) only to avoid unnecessary technical complications. It is clear that some tail regularity of $\mathbb{P}$ and $\tilde{\mathbb{P}}$ is necessary if we are to obtain convergence to the $\mathsf{FK}_{d,\alpha}$ process.

We also believe that both Theorems 1.2 and 1.3 hold in $d = 2$. It should be possible to adapt the approach of this paper to that case, but since the underlying random walks on $\mathbb{Z}^2$ are recurrent, the situation is rather more delicate.

It is also likely that the assumptions in the Theorems 1.2 and 1.3 that μ_e and τ_x are bounded from below, could be avoided. However, this would require combining and extending the arguments of [BP07, Mat08] and [BD08] to cover the general RCM with positive conductances, and this is out of the scope of this paper.

The method that we use to prove both main theorems is based on the coarse-graining procedure developed in [BČM06, BČ07] to control the symmetric BTM. In those papers, however, the proofs exploit heavily the fact that the symmetric BTM is the time change of the simple random walk. They use the local limit theorems and precise estimates on hitting probabilities and Green functions that are available for the simple random walk. In addition, they use the fact that all these asymptotic results hold *uniformly* in the starting position.

In this paper, however, the CSRW and the (non-symmetric) BTM are not time changes of the simple random walk. The process that corresponds to the simple random walk, that is the process with the flat reversible measure, is the VSRW. We do not have such precise estimates for the VSRW as for the simple random walk – and it is clear that any estimates that do hold cannot do so uniformly. In fact, for the RCM, we have available only the quenched FCLT and Gaussian heat kernel bounds for the VSRW proved in [BD08], and results (such as Harnack inequalities) that follow from them. Although these do yield a local limit theorem (see [BD08, Theorem 5.13]), this local limit theorem is obtained for at most a *finite number* of starting points simultaneously. (The ergodic theorems used to prove the FCLT for the VSRW do not give any information on the rate of convergence and its dependence on the starting position.) This proves to be a significant obstacle to using the coarse graining procedure of [BČM06, BČ07].

In order to overcome this difficulty, we had to improve considerably the original coarse graining. The coarse graining used in this paper requires as input only the FCLT for the process started at the origin and Gaussian heat-kernel estimates. It is therefore much more robust than the original one. These inputs are known for the VSRW for the RCM, and can be easily obtained for the VSRW $\tilde{Y}$ associated with the BTM by checking that the BTM satisfies the conditions of [BD08, Theorem 6.1].

We close the introduction by a short discussion related to the sub-diffusivity of the limiting FK process. Anomalous diffusions has been studied extensively in the physics literature since 1960's (see e.g. [MK00] for an extensive list of references). It has been recognised that there are essentially two mechanisms leading to it. First, the presence of obstacles at a broad range of size scales can slow down the process. Mathematically this leads to studying processes on fractal sets that are nowadays quite well understood ([Bar98, Kig01]). Second, the distribution of some local characteristic of the environment can be very broad and the process can be trapped at places where this characteristic is exceptionally large. To our knowledge, mathematical studies of this second mechanism are sparse and address to some extent trivial situations: either processes on the one-dimensional lattice $\mathbb{Z}$ ([FIN02, BČ05]), or processes that can be expressed as a time change of a simple random walk on $\mathbb{Z}^d$, $d \geq 1$, ([SZ97, MS04] where the time change is independent of the trajectory and [BČ07] where the time change depends on the environment). Our result is thus an important step in broadening our understanding of this second mechanism.

2 Time changes and other results

We have stated in Introduction that the CSRW and the VSRW are related by a time change. Since, as in [BČ07], this time change will be an important subject of our study we define it precisely here. We further state two auxiliary theorems, one for the VSRW and one for the BTM, that will be used to prove Theorems 1.2 and 1.3.

Consider the VSRW Y and define the *clock process* $S : [0, \infty) \to [0, \infty)$ by

$$S(t) = \int_0^t \mu_{Y(t)} \, \mathrm{d}t. \tag{2.1}$$

Since $\mu_x \geq 2d\underline{c}$ (see below (1.5) for definition of $\underline{c}$), S is strictly increasing and its inverse S^{-1} is well defined. It is easy to see that the process $X(t) = Y(S^{-1}(t))$ has the same law as the CSRW. Hence, Y and X can be constructed on the same probability space and we always suppose that they are.

While the clock process S is continuous, its scaling limit is not, as we will see soon. It follows that the clock process does not converge in the usual Skorokhod J_1-topology, and that the right topology to consider on D^1 is the weaker Skorokhod M_1-topology. To distinguish which topology we use, we let $D^d(U)$, resp. $D^d(M_1)$, stand for the space D^d equipped with the topology of uniform convergence on compacts, resp. with the M_1-topology.

Theorem 1.2 is a consequence of the following more detailed result, which gives the joint convergence of the clock process and of the position of the embedded VSRW Y to constant multiples of an independent stable subordinator V_α and Brownian motion BM_d.

Theorem 2.1. *Let, for $t \geq 0$, $n \in \mathbb{N}$,*

$$Y_n(t) = n^{-1} Y(n^2 t) \qquad \text{and} \qquad S_n(t) = n^{-2/\alpha} S(n^2 t). \tag{2.2}$$

Under the assumptions of Theorem 1.2, there exist constants $\mathcal{C}_Y$, $\mathcal{C}_S \in (0, \infty)$ such that $\mathbb{P}$-a.s., under $P_0^{\boldsymbol{\mu}}$, the joint distribution of (S_n, Y_n) converges to the distribution of $(\mathcal{C}_S V_\alpha, \mathcal{C}_Y \, \mathsf{BM}_d)$ weakly on the space $D^1(M_1) \times D^d(U)$.

Theorem 1.3 is a consequence of an analogous statement. Recall that $\tilde{Y}$ is the VSRW associated with the conductances $\tilde{\mu}_e$. Defining the clock process $\tilde{S}$ analogously to (2.1),

$$\tilde{S}(t) = \int_0^t \tau_{\tilde{Y}(t)} \, \mathrm{d}t, \tag{2.3}$$

is easy to see that the BTM can be written as $\tilde{X}(t) = \tilde{Y}(\tilde{S}^{-1}(t))$.

Theorem 2.2. *Under the hypotheses of Theorem 1.3, the conclusions of Theorem 2.1 hold after the replacement of* S, Y, $\mathbb{P}$, $P_0^{\boldsymbol{\mu}}$ *by* $\tilde{S}$, $\tilde{Y}$, $\tilde{\mathbb{P}}$ *and* $\tilde{P}_0^{\boldsymbol{\tau}}$.

The rest of the paper is organised as follows. We prove Theorems 1.2 and 2.1 for the CSRW in Sections 3–8. In Section 3 we recall the results of [BD08] on the VSRW and prove some preliminary facts. In Section 4 we obtain some estimates on the Green function of the VSRW that replace the precise estimates for the simple random walk used in [BČM06, BČ07]. In the next three sections we study the contribution to $S_n(t)$ of different sizes of conductances. In Section 5 we show that edges with conductance less than $\varepsilon n^{2/\alpha}$ make little contribution. The main work is in Section 6, where we study edges with conductance between $\varepsilon n^{2/\alpha}$ an $\varepsilon^{-1} n^{2/\alpha}$. Here we introduce the improved coarse-graining procedure used to control the time that the CSRW spends in the vicinity of these edges. Section 7 treats the remaining corrections: edges with conductivity greater than $\varepsilon^{-1} n^{2/\alpha}$, or closely spaced edges with conductivity greater than $\varepsilon n^{2/\alpha}$. Finally in Section 8, Theorem 2.1 and then Theorem 1.2 are proved.

Theorems 1.3 and 2.2 for the BTM can be proved analogously to those for the RCM, and so we do not give a detailed proof. In Section 9 we prove that this process satisfies the conditions of [BD08, Theorem 6.1], so that Gaussian heat kernels and the FCLT hold for the VSRW $\tilde{Y}$ associated with the BTM. We then indicate the places where the proofs of Sections 3–8 need to be modified, and give more details only at places where a different argument is necessary.

3 Preliminaries

We begin by introducing some further notation. Let $B(x, R)$ be the Euclidean ball centred at x of radius R and let $Q(x, R)$ be a cube centred at x with side length R whose edges are parallel to the coordinate axes. Both balls and cubes can be understood either as subsets of $\mathbb{R}^d$, $\mathbb{Z}^d$ or of E^d (an edge is in $B(x, R)$ if both its vertices are), depending on the context. For $A \subset \mathbb{Z}^d$ we write $\partial A = \{y \notin A \exists x \in A, (xy) \in E^d\}$ and $\bar{A} = A \cup \partial A$. For $A, B \subset \mathbb{Z}^d$ we set $d(A, B) = \inf\{|x - y| : x \in A, y \in B\}$, where $|x - y|$ stands for the Euclidean distance of x and y.

Sometimes, we identify a set of edges $G \subset E^d$ with the set of their vertices. It allows us to write, e.g., $x \in G$, meaning that $x \in \mathbb{Z}^d$ is a vertex of an edge in G. For a set $A \subset \mathbb{Z}^d$ we write $B(A, R) = \bigcup_{x \in A} B(x, R)$.

We use the convention that all large values appearing in the proofs are rounded above to the closest integer, if necessary. It allows us to write that, e.g., $\varepsilon n \mathbb{Z}^d \subset \mathbb{Z}^d$ for $\varepsilon \in (0, 1)$ and n large.

The following quenched FCLT for the VSRW for the RCM was recently obtained in [BD08].

Proposition 3.1 (Theorem 1.1 of [BD08]). *Let* Y_n *be as in* (2.2) *and* $d \geq 2$. *Then there exists* $\mathcal{C}_Y \in (0, \infty)$ *such that* $\mathbb{P}$*-a.s., under* $P_0^{\boldsymbol{\mu}}$, *the sequence* Y_n *converges in law on* $D^d(U)$ *to a multiple of a standard d-dimensional Brownian motion,* $\mathcal{C}_Y \,\mathsf{BM}_d$.

Let Z be a process or path in $\mathbb{R}^d$. For $A \subset \mathbb{R}^d$ we define its hitting and exit times $\sigma_A(Z) = \inf\{t \geq 0 : Z(t) \in A\}$, $\tau_A(Z) = \inf\{t \geq 0 : Z(t) \notin A\}$. When Z is the

VSRW Y we will usually write σ_A and τ_A for $\sigma_A(Y)$ and $\tau_A(Y)$. We write $g^{\boldsymbol{\mu}}_A(x,y)$ for the Green function of Y killed on exiting from A:

$$g^{\boldsymbol{\mu}}_A(x,y) = E^{\boldsymbol{\mu}}_x \int_0^{\tau_A} \mathbf{1}\{Y_s = y\}\, \mathrm{d}t. \tag{3.1}$$

If $A = \mathbb{Z}^d$, we omit it from the notation, and we sometimes also omit $\boldsymbol{\mu}$ and write $g(x,y) = g^{\boldsymbol{\mu}}_{\mathbb{Z}^d}(x,y)$.

Recall the definition of p_{xy} in (1.2). We say that a function h is harmonic on $A \subset \mathbb{Z}^d$ if h is defined on $\bar{A}$ and

$$h(x) = \sum_{y\sim x} p_{xy}(h(y) - h(x)) \text{ for all } x \in A. \tag{3.2}$$

We say that the elliptic Harnack inequality (EHI) with constant C_H holds in a ball $B(x,r)$ if whenever $h \geq 0$ is harmonic in $B(x,r)$, then

$$h(y) \leq C_H h(y') \quad \text{ for all } y, y' \in B(x, r/2). \tag{3.3}$$

From [BD08] we obtain the following – see Theorem 1.3 for the bounds on $g^{\boldsymbol{\mu}}$ and Theorem 4.7 for the Harnack inequality. (Note that the EHI follows immediately from the parabolic Harnack inequality.)

Proposition 3.2. *Let $d \geq 3$. There exists a collection of random variables $(V_x, x \in \mathbb{Z}^d)$ on Ω and positive constants c_1, c_2, C_H, $\eta = 1/3$ with the following properties.*
(a) For all $x \in \mathbb{Z}^d$,

$$\mathbb{P}[V_x \geq n] \leq c_2 \exp(-c_1 n^{\eta}). \tag{3.4}$$

(b) If $x, y \in \mathbb{Z}^d$,

$$\frac{c_1}{|x-y|^{d-2}} \leq g^{\boldsymbol{\mu}}(x,y) \leq \frac{c_2}{|x-y|^{d-2}} \quad \text{if } |x-y| \geq V_x \wedge V_y. \tag{3.5}$$

(c) If $R \geq V_x$, then EHI holds with constant C_H for $B(x,R)$.
(d) Let $C_0 = \Gamma(\frac{d}{2}-1)/(2\pi^{d/2}\mathcal{C}_Y^2)$. For any $\varepsilon > 0$ there exists a random variable M_ε on Ω with $\mathbb{P}(M_\varepsilon < \infty) = 1$ such that

$$\frac{(1-\varepsilon)C_0}{|x|^{d-2}} \leq g^{\boldsymbol{\mu}}(0,x) \leq \frac{(1+\varepsilon)C_0}{|x|^{d-2}} \quad \text{for } |x| > M_\varepsilon. \tag{3.6}$$

For all $f \in \ell^2(\mathbb{Z}^d, \mu_x)$ we set

$$\mathcal{E}^{\boldsymbol{\mu}}(f) = \frac{1}{2} \sum_{x,y\in\mathbb{Z}^d} \mu_{xy}(f(y) - f(x))^2, \tag{3.7}$$

and we define the conductance between two disjoint subsets A, B of $\mathbb{Z}^d$ as

$$C_{\text{eff}}[A,B] = \inf\{\mathcal{E}^{\boldsymbol{\mu}}(f) : f|_A = 0, f|_B = 1\}. \tag{3.8}$$

We use $C_{\text{eff}}[A,\infty]$ to denote $\lim_{n\to\infty} C_{\text{eff}}[A, B(A,n)^c]$, and $R_{\text{eff}}[A,B] = C_{\text{eff}}[A,B]^{-1}$ to denote the resistance between A and B. We recall the well-known fact

$$g_A(x,y) \leq g_A(x,x) = R_{\text{eff}}[x, A^c], \qquad \forall x, y \in A. \tag{3.9}$$

We now prove several auxiliary lemmas. Let $K \geq 1$ be a fixed large number.

Lemma 3.3. *There exist $c_3 > 0$ and $N_0 = N_0(\boldsymbol{\mu})$ with $\mathbb{P}(N_0 < \infty) = 1$ such that on $\{\boldsymbol{\mu} : n \geq N_0(\boldsymbol{\mu})\}$,*

$$\sup\{V_x : x \in B(0, Kn)\} \leq c_3(\log n)^{1/\eta} =: b_1(n). \tag{3.10}$$

Proof. Using (3.4), the probability that (3.10) fails is at most $C(Kn)^d e^{(-c_1 c_3^\eta \log n)}$, which is summable if $c_1 c_3^\eta$ is large enough. □

Lemma 3.4. *(a) There exists a constant c_4 such that the Green function satisfies*

$$g^{\boldsymbol{\mu}}(x, y) \leq c_4 \qquad \text{for all } x, y \in \mathbb{Z}^d. \tag{3.11}$$

(b) Further, for any $x \in \mathbb{Z}^d$, $g^{\boldsymbol{\mu}}(x, x) \geq c_1 V_x^{2-d}$. In particular, $\mathbb{P}$-a.s. for all but finitely many $n \in \mathbb{N}$,

$$\inf_{x \in B(0,Kn)} g^{\boldsymbol{\mu}}(x, x) \geq c_1 b_1(n)^{2-d} = c_1 c_3^{2-d} (\log n)^{(2-d)/\eta}. \tag{3.12}$$

Proof. (a) is immediate from (3.9) and the comparison of the random conductances network $\boldsymbol{\mu}$ with a homogeneous network where every conductance equals $\underline{c}$ (see (1.5)). The first claim in (b) follows from (3.5), and the fact that $g^{\boldsymbol{\mu}}(x, \cdot)$ attains its maximum at x. Using (3.10) then gives (3.12). □

We need estimates on resistance to boundaries of large balls.

Lemma 3.5. *There exists a constant c_5 such that for every $\varepsilon \in (0, 1)$, $\mathbb{P}$-a.s. for all but finitely many $n \in \mathbb{N}$,*

$$R_{\text{eff}}[x, \infty] \geq R_{\text{eff}}[x, B(x, b_2(n, \varepsilon))^c] \geq (1 - \varepsilon) R_{\text{eff}}[x, \infty] \quad \forall x \in B(0, Kn), \tag{3.13}$$

where $b_2(n, \varepsilon) = c_5 \varepsilon^{\frac{1}{2-d}} b_1(n)$.

Proof. The first inequality in (3.13) is obvious. To prove the second, set $r = b_2(n, \varepsilon)$. Observe that the function $f_x(y) := g(x, y) - g_{B(x,r)}(x, y)$ is non-negative and harmonic on $B(x, r)$. By (3.5) and Lemma 3.3, $f_x(y) \leq c_2 r^{2-d}$ for $x \in B(0, Kn)$ and $y \in \partial B(x, r)$, $\mathbb{P}$-a.s. for all n large. Hence, by the maximum principle, $f_x(y) \leq c_2 r^{2-d}$ on $B(x, r)$. On the other hand, $\mathbb{P}$-a.s., $g(x, x) \geq c(\log n)^{(2-d)/\eta}$. Combining these two bounds (3.9), and taking c_5 large enough, the second inequality follows easily. □

We now fix the constant C_H in EHI to be as in Proposition 3.2(b). Let N_0 be as in Lemma 3.3. We will require the Harnack inequality to hold for many balls simultaneously.

Lemma 3.6. *(a) If $n \geq N_0$, $x \in B(0, Kn)$ and $r \geq b_1(n)$ then EHI holds for $B(x, r)$.*
(b) Let $n \geq N_0$. If $z \in B(0, Kn)$, $r \geq b_1(n)$, $m \geq 1$, and $h \geq 0$ is harmonic on $B(z, 2^m r)$, then, writing $\rho = (2C_H)^{-1}$,

$$\frac{\sup_{B(z,r)} h}{\inf_{B(z,r)} h} \leq 1 + (1 - \rho)^{m-1} C_H. \tag{3.14}$$

Proof. (a) This is immediate from Proposition 3.2(b) and Lemma 3.3.
(b) We consider the chain of balls $B_i = B(z, 2^i r)$, $i = 0, \dots m$. By (a) EHI holds for each B_i. Write $\mathrm{Osc}(f, A) = \sup_A f - \inf_A f$. A standard argument (see e.g. [BH09, Proposition 3.2]) gives that

$$\mathrm{Osc}(h, B_{i-1}) \le (1-\rho)\mathrm{Osc}(h, B_i), \quad 1 \le i \le m. \tag{3.15}$$

Further, using EHI in B_m,

$$\mathrm{Osc}(h, B_{m-1}) \le \sup_{B_{m-1}} h \le C_H \inf_{B_{m-1}} h \le C_H \inf_{B_0} h. \tag{3.16}$$

So,

$$\sup_{B_0} h - \inf_{B_0} h = \mathrm{Osc}(h, B_0) \le (1-\rho)^{m-1}\mathrm{Osc}(h, B_{m-1}) \le C_H(1-\rho)^{m-1} \inf_{B_0} h. \tag{3.17}$$

This finishes the proof. □

4 Behaviour of Green functions

In this section we prove the estimates on Green functions that we use to replace the standard asymptotic formulae on the Green function of the simple random walk used in [BČM06, BČ07]. Before we present our results, the following approximate calculations on a simplified model may prove useful.

Let $d \ge 3$, and consider $\mathbb{Z}^d$ with $\mu_e \equiv 1$, except in a ball $B(x, r)$, where $\mu_e \equiv A \gg 1$. Write $\tilde{g}(y, z)$ for the Greens function for this graph. If $|x - y| > 2r$ then $\tilde{g}(y, y) = R_{\mathrm{eff}}(y, \infty) = O(1)$, while $\tilde{g}(x, x) \simeq R_{\mathrm{eff}}(B(x, r)^c, \infty) \simeq cr^{2-d}$. Since $|x - y| > 2r$ the anomalous region around x has little effect on the hitting probability of y starting from $z \in B(x, 3r/2)^c$, and so $P_x(\sigma_y < \infty) \simeq |x - y|^{2-d}$. Thus we have

$$\tilde{g}(x, y) = P_x(\sigma_y < \infty)\tilde{g}(y, y) \simeq c|x - y|^{2-d}, \quad \text{ if } |x - y| > 2r. \tag{4.1}$$

Thus the anomalous region in $B(x, r)$ has little effect on $\tilde{g}(x, \cdot)$ outside $B(x, cr)$.

Proposition 4.1. *Let $0 < 2\varepsilon_o < \varepsilon_g \le \frac{1}{2}$, and $\delta > 0$. Then there exists $\varepsilon_b \in (0, \varepsilon_o)$ with $\lim_{\delta\to 0} \varepsilon_b = 0$ such that if $n \ge N_0$, $\varepsilon_b n \ge b_1(n)$, $x \in B(0, (K - \varepsilon_g)n)$ and $A \subset \mathbb{Z}^d$ with $B(x, \varepsilon_g n) \subset A$ then*

$$\frac{\sup_{y\in B(z,\varepsilon_b n)} g^{\boldsymbol{\mu}}_A(x, y)}{\inf_{y\in B(z,\varepsilon_b n)} g^{\boldsymbol{\mu}}_A(x, y)} \le 1 + \delta \quad \textit{whenever } \varepsilon_o n \le |x - z| \le (\varepsilon_g - \varepsilon_o)n, \tag{4.2}$$

$$\frac{\sup_{x'\in B(x,\varepsilon_b n)} g^{\boldsymbol{\mu}}_A(x', y)}{\inf_{x'\in B(x,\varepsilon_b n)} g^{\boldsymbol{\mu}}_A(x', y)} \le 1 + \delta \quad \textit{whenever } \varepsilon_o n < |y - x| < \varepsilon_g n. \tag{4.3}$$

Proof. Choose m so that $C_H(1-\rho)^{m-1} < \delta$, and write $h = g^{\boldsymbol{\mu}}_A(x, \cdot)$. Fix $\varepsilon_b \le 2^{-m}\varepsilon_o$. If z is as in (4.2), then $z \in B(0, Kn)$ and $B(z, 2^m\varepsilon_b n) \subset B(z, \varepsilon_o n) \subset A \setminus \{x\}$, and therefore h is harmonic on $B(z, 2^m\varepsilon_b n)$. Hence, for n satisfying the assumptions of the proposition, using Lemma 3.6,

$$\frac{\sup_{B(z,\varepsilon_b n)} h}{\inf_{B(z,\varepsilon_b n)} h} \le 1 + (1-\rho)^{m-1}C_H \le 1 + \delta. \tag{4.4}$$

This proves (4.2). The proof of (4.3) is similar. □

Write $g^*_A(x,y)$ for the Green function of $W := \mathcal{C}_Y\,\mathsf{BM}_d$ killed on exit from $A \subset \mathbb{R}^d$ (see Proposition 3.1 for $\mathcal{C}_Y$). We have

$$g^*(x,y) = C_0|x-y|^{2-d}, \qquad g^*_{B(x,r)}(x,y) = C_0\big(|x-y|^{2-d} - r^{2-d}\big), \tag{4.5}$$

where the constant C_0 is as in Proposition 3.2. Using the FCLT we now show that an estimate similar to the second part of (4.5) holds for $g^{\boldsymbol{\mu}}_{B(nx,nr)}(nx,ny)$, provided that $x,y \in \mathbb{R}^d$ and r are fixed, and y is not too close either to x or to $B(x,r)^c$.

Lemma 4.2. *Let $0 < 3\varepsilon_o < \varepsilon_g < \frac{1}{2}$, $\delta > 0$. Let $r \in (\varepsilon_g - \varepsilon_o, \varepsilon_g + \varepsilon_o)$ and let $x,y \in \mathbb{R}^d$, $x \in B(0, K-\varepsilon_g)$, $y \in B(x, r-\varepsilon_o) \setminus B(x,\varepsilon_o)$. Then there exists $c_0(\delta)$ with $\lim_{\delta\to 0} c_0(\delta) = 0$ such that if*

$$A^r_n(x,y) = \left\{\boldsymbol{\mu} : 1 - c_0(\delta) \le \frac{g^{\boldsymbol{\mu}}_{B(nx,rn)}(nx,ny)}{C_0 n^{2-d}\big(|x-y|^{2-d} - r^{2-d}\big)} \le 1 + c_0(\delta)\right\}, \tag{4.6}$$

then $A^r_n(x,y)$ holds for all sufficiently large n, $\mathbb{P}$-a.s.

Proof. We use the notation $c_i(\delta)$ to denote functions of δ such that $\lim_{\delta\to 0} c_i(\delta) = 0$. Let $\varepsilon_b < \varepsilon_o$. For any path $\gamma \in D^d$ write $\sigma_1(\gamma) = \sigma_{B(x,\varepsilon_b)}(\gamma)$, and $\tau_1(\gamma) = \inf\{t \ge \sigma_1 : \gamma \notin B\}$ where $B = B(x,r)$, and let

$$F(\gamma) = \int_{\sigma_1(\gamma)}^{\tau_1(\gamma)} \mathbf{1}\{\gamma(s) \in B(y,\varepsilon_b)\}\mathrm{d}s. \tag{4.7}$$

By the FCLT (Proposition 3.1) for Y, we have,

$$\lim_{n\to\infty} E^{\boldsymbol{\mu}}_0 F(Y_n) = E_0 F(W). \tag{4.8}$$

The right hand side of (4.8) is

$$E_0 \mathbf{1}\{\sigma_1(W) < \infty\} \int_{B(y,\varepsilon_b)} g^*_B(W_{\sigma_1}, y)\mathrm{d}y. \tag{4.9}$$

Using the uniform results for g^*_B analogous to Proposition 4.1, and writing $v_d = |B(0,1)|$ for the volume of the ball in $\mathbb{R}^d$, it follows that

$$E_0 F(W) \ge (1 - c_1(\delta)) P_0[\sigma_1(W) < \infty] v_d \varepsilon_b^d g^*_B(x,y). \tag{4.10}$$

Similarly, we have, writing $\sigma_n = \sigma_{B(nx,n\varepsilon_b)}(Y)$, $nB = B(nx,nr)$,

$$E^{\boldsymbol{\mu}}_0 F(Y_n) = E^{\boldsymbol{\mu}}_0\Big[n^{-2}\mathbf{1}\{\sigma_n < \infty\} \sum_{z \in B(ny,n\varepsilon_b)} g^{\boldsymbol{\mu}}_{nB}(Y_{\sigma_n}, z)\Big]. \tag{4.11}$$

Using Proposition 4.1 (with $\varepsilon_o/2$ on place of ε_o and $\varepsilon_g - \varepsilon_o$ on place of ε_g) we can choose ε_b small such that, for n sufficiently large and $z \in B(ny, n\varepsilon_b)$

$$g^{\boldsymbol{\mu}}_{nB}(Y_{\sigma_n}, z) \le (1+\delta) g^{\boldsymbol{\mu}}_{nB}(nx,z) \le (1+\delta)^2 g^{\boldsymbol{\mu}}_{nB}(nx,ny), \tag{4.12}$$

with a similar lower bound. Therefore,

$$E_0^{\boldsymbol{\mu}} F(Y_n) \le (1 + c_2(\delta)) P_0^{\boldsymbol{\mu}}[\sigma_n < \infty] n^{d-2} v_d \varepsilon_1^d g_{nB}^{\boldsymbol{\mu}}(nx, ny), \tag{4.13}$$

where the additional factor n^{-2} comes from the scaling of Y to Y_n. The functional central limit theorem also implies that

$$P_0^{\boldsymbol{\mu}}[\sigma_n < \infty] \xrightarrow{n\to\infty} P_0[\sigma_1(W) < \infty]. \tag{4.14}$$

Combining this with (4.8), (4.10) and (4.13), we obtain that

$$n^{d-2} g_{nB}^{\boldsymbol{\mu}}(nx, ny) \ge (1 - c_0(\delta)) g_B^*(x, y), \tag{4.15}$$

provided n is sufficiently large. The upper bound on $g_{nB}^{\boldsymbol{\mu}}(nx, ny)$ is proved in the same way. From these bounds and (4.5) we deduce that A_n^r holds for all large n. □

Finally, we show that the last lemma holds uniformly over $B(0, Kn)$.

Proposition 4.3. *Let $0 < 3\varepsilon_o < \varepsilon_g < \frac{1}{2}$, and $\delta > 0$. Let $r \in (\varepsilon_g, \varepsilon_g + \varepsilon_o/2)$. Then there exists $c_1(\delta)$ with $\lim_{\delta\to 0} c_1(\delta) = 0$ such that, for all but finitely many n, $\mathbb{P}$-a.s., for all $x \in B(0, (K - \varepsilon_g)n)$, $y \in B(x, (\varepsilon_g - \varepsilon_o)n) \setminus B(x, \varepsilon_o n)$,*

$$1 - c_1(\delta) \le \frac{g_{B(x,rn)}^{\boldsymbol{\mu}}(x, y)}{C_0\big(|x - y|^{2-d} - (nr)^{2-d}\big)} \le 1 + c_1(\delta). \tag{4.16}$$

Proof. Let ε_b be as in the proof Lemma 4.2. Let $\{x_1, \ldots x_m\} \subset B(0, K - \varepsilon_g)$ be chosen so that $B(0, K - \varepsilon_g) \subset \cup_i B(x_i, \varepsilon_b)$. Write $J = \{(i, j) : r - \varepsilon_o/2 \ge |x_i - x_j| \ge \varepsilon_o/2\}$. Lemma 4.2 implies that there exists $N' = N'(\boldsymbol{\mu}) \ge N_0$ such that $A_n^{r+\varepsilon_b}(x_i, x_j)$ holds for all $(i, j) \in J$ and $n \ge N'$.

Now let $n \ge N'$ and let x, y be as required. Then there exists $(i, j) \in J$ such that $|x - nx_i| < n\varepsilon_b$, $|y - nx_j| < n\varepsilon_b$. By Proposition 4.1, since $A_n^{r+\varepsilon_b}(x_i, x_j)$ holds,

$$\begin{aligned} g_{B(x,nr)}^{\boldsymbol{\mu}}(x, y) &\le (1 + \delta)^2 g_{B(x,nr)}^{\boldsymbol{\mu}}(nx_i, nx_j) \\ &\le (1 + \delta)^2 g_{B(nx_i,n(r+\varepsilon_b))}^{\boldsymbol{\mu}}(nx_i, nx_j) \\ &\le (1 + \delta)^2 C_0 n^{2-d}(|x_i - x_j|^{2-d} - (r + \varepsilon_b)^{2-d}) \\ &\le (1 + c(\delta)) C_0\big(|x - y|^{2-d} - (nr)^{2-d}\big). \end{aligned} \tag{4.17}$$

The lower bound in (4.16) is proved in the same way. □

5 Edges with small conductance

The next three sections extend the methods of [BČM06, BČ07] to the CSRW. We begin with one simplification, and replace the hypothesis (1.5) by

$$\mathbb{P}[\mu_e \ge u] = u^{-\alpha}(1 + o(1)), \quad u \to \infty; \tag{5.1}$$

that is, we assume $C_1 = 1$ in (1.5). An easy rescaling argument recovers the general case.

We aim at proving that the rescaled clock process S_n (see Theorem 2.1) converges to a stable subordinator. Since the stable subordinator at time t is well approximated by a large but finite number of its largest jumps before t, we will, in the next section, control the (suitably coarse-grained) large jumps of S_n and prove that their distribution converge to the distribution of the large jumps of the subordinator. As we will see these large jumps are due to the visits of the VSRW to edges with conductance of order $n^{2/\alpha}$.

Before studying the large jumps we prove in this section that the contribution of less conducting edges to the clock process can be neglected. More precisely, we show that the contribution of the edges with conductances smaller than $\varepsilon n^{2/\alpha}$ to the rescaled clock process S_n at time T a is very likely to be smaller than δT, for suitably chosen ε, δ. This is the content of the next proposition.

Proposition 5.1. *Let*

$$\tilde{T}_n(0,\varepsilon) = \{x \in \mathbb{Z}^d : \mu_{xy} \le \varepsilon n^{2/\alpha} \ \forall y \sim x\}. \tag{5.2}$$

Then, for every $\delta > 0$ there exists ε such that for all $T > 0$, $\mathbb{P}$-a.s. for all but finitely many n,

$$P_0^{\boldsymbol{\mu}}\Big[T^{-1}n^{-2/\alpha}\int_0^{Tn^2} \mu_{Y(t)}\mathbf{1}\{Y(t) \in \tilde{T}_n(0,\varepsilon)\}\,\mathrm{d}t \ge \delta\Big] \le \delta. \tag{5.3}$$

Proof. Let $\mathbb{B} = B(0, Kn)$, and write $\tilde{\mathbb{B}}$ for the set of edges with at least one vertex in $\mathbb{B}$. We first insert the trivial term $\mathbf{1}\{Y(t) \in \mathbb{B}\} + \mathbf{1}\{Y(t) \notin \mathbb{B}\}$ in the integral in (5.3), and write the resulting integral as $I_{\mathbb{B}}(T) + I_{\mathbb{B}^c}(T)$. By the FCLT (Proposition 3.1), for each $T > 0$ it is possible to choose $K = K(T)$ large such that, $\mathbb{P}$-a.s., $P_0^{\boldsymbol{\mu}}[I_{\mathbb{B}^c}(T) > 0] < \delta/2$.

To bound the contribution of the first term we show that for every δ there exists ε such that, $\mathbb{P}$-a.s. for large n,

$$E_0^{\boldsymbol{\mu}}\Big[\int_0^{Tn^2} \mu_{Y(t)}\mathbf{1}\{Y(t) \in \tilde{T}_n(0,\varepsilon) \cap \mathbb{B}\}\,\mathrm{d}t\Big] \le \frac{1}{2}Tn^{2/\alpha}\delta^2. \tag{5.4}$$

The claim (5.3) then follows using the Chebyshev inequality.

To show (5.4), we set $i_{\max} = \min\{i \in \mathbb{N} : 2^{-i}\varepsilon n^{2/\alpha} \le \underline{c}\} = O(\log n)$, and

$$H_n(i) = \{e \in \bar{\mathbb{B}} : \mu_e \in \varepsilon n^{2/\alpha}(2^{-i}, 2^{-i+1}]\}, \quad i \in \{1, \dots, i_{\max}\}. \tag{5.5}$$

Using Proposition 3.2, Lemmas 3.3, 3.4, and the notation $g^{\boldsymbol{\mu}}(0,(xy)) = g^{\boldsymbol{\mu}}(0,x) + g^{\boldsymbol{\mu}}(0,y)$, the left-hand side of (5.4) can bounded from above by

$$\sum_{(xy)\in E^d\cap\bar{\mathbb{B}}} \mu_{(xy)}g^{\boldsymbol{\mu}}(0,(xy))\mathbf{1}\{\mu_{(xy)} \le \varepsilon n^{2/\alpha}\} \le \sum_{i=1}^{i_{\max}} 2^{-i+1}\varepsilon n^{2/\alpha} \sum_{e\in H_n(i)} \bar{g}_n(0,e), \tag{5.6}$$

where

$$\bar{g}_n(0,e) = \begin{cases} c_4 & \text{if } d(0,e) \le b_1(n), \\ c_2 d(0,e)^{2-d} & \text{otherwise.} \end{cases} \tag{5.7}$$

Let $p_{n,i} = \mathbb{P}\big[\mu_e \in \varepsilon n^{2/\alpha}(2^{-i}, 2^{-i+1}]\big] \le c\varepsilon^{-\alpha}2^{i\alpha}n^{-2}$ and $\lambda_n > 0$. Let $C \ge 1$. For fixed i, using the i.i.d. property of the environment, we get

$$\begin{aligned}\mathbb{P}\Big[2^{-i+1}\varepsilon n^{2/\alpha} \sum_{e\in H_n(i)} \bar{g}_n(0,e) \ge C\varepsilon^{1-\alpha}2^{i(\alpha-1)}Tn^{2/\alpha}\Big] \\ \le e^{-\lambda_n cC\varepsilon^{-\alpha}2^{i\alpha}T} \prod_{e\in\bar{\mathbb{B}}} \big(1 + p_{n,i}(e^{\lambda_n \bar{g}_n(0,e)} - 1)\big).\end{aligned} \tag{5.8}$$

The logarithm of the product above is bounded by

$$\log \prod_{e\in\bar{\mathbb{B}}} \big(1 + p_{n,i}(e^{\lambda_n\bar{g}_n(0,e)} - 1)\big) \le \sum_{e\in\bar{\mathbb{B}}} c\varepsilon^{-\alpha}2^{i\alpha}n^{-2}\big(e^{\lambda_n\bar{g}_n(0,e)} - 1\big). \tag{5.9}$$

Taking $\lambda_n = (\log n)/c_4$ and dividing the sum according to $d(0,e)$ being smaller or larger than $b_1(n)$, we get

$$\sum_{e:d(0,e)\le b_1(n)} c\varepsilon^{-\alpha}2^{i\alpha}n^{-2}\big(e^{\lambda_n\bar{g}_n(0,e)} - 1\big) \le c2^{i\alpha}n^{-2}b_1(n)^d\varepsilon^{-\alpha}n \xrightarrow{n\to\infty} 0, \tag{5.10}$$

and, using $\lambda_n b_1(n)^{2-d} = O((\log n)^{1+2\eta^{-1}(2-d)}) \to 0$,

$$\begin{aligned}\sum_{e\in\mathbb{B}:d(0,e)>b_1(n)} c\varepsilon^{-\alpha}2^{i\alpha}n^{-2}\big(e^{\lambda_n\bar{g}_n(0,e)} - 1\big) &\le \sum_{e\in\mathbb{B}:d(0,e)>b_1(n)} c\varepsilon^{-\alpha}2^{i\alpha}n^{-2}\lambda_n d(0,e)^{2-d} \\ &\le c'2^{i\alpha}\varepsilon^{-\alpha}TK^2\lambda_n.\end{aligned} \tag{5.11}$$

Hence, (5.8) is smaller than $\exp(\varepsilon^{-\alpha}2^{i\alpha}T\lambda_n(-cC + c'K^2))$. Let $c_5 = (cC - c'K^2)\log 2$, and choose C large enough so that $T\varepsilon^{-\alpha}c_5/c_4 \ge 2$. The probability that (5.4) fails is thus bounded by

$$\begin{aligned}&\mathbb{P}\Big[\bigcup_{i=1}^{i_{\max}}\Big\{c2^{-i}\varepsilon n^{2/\alpha} \sum_{e\in H_n(i)} g^{\boldsymbol{\mu}}(0,e) \ge C\varepsilon^{1-\alpha}2^{i(\alpha-1)}Tn^{2/\alpha}\Big\}\Big] \\ &\quad\le \sum_{i=1}^{i_{\max}} \exp\{\varepsilon^{-\alpha}2^{i\alpha}T\lambda_n(-cC + c'K^2)\}, \\ &\quad\le \sum_{i=1}^{\infty} \exp\{-c_5\varepsilon^{-\alpha}iT\lambda_n\} \le cn^{-c_5T\varepsilon^{-\alpha}/c_4}.\end{aligned}$$

Therefore, for all large n (5.6) is $\mathbb{P}$-a.s. smaller than $\sum_{i=1}^{i_{\max}} C\varepsilon^{1-\alpha}2^{i(\alpha-1)}Tn^{2/\alpha}$, which is smaller than $Tn^{2/\alpha}\delta^2/2$ if ε is small enough. This completes the proof of (5.4) and thus of Proposition 5.1. □

6 Coarse graining

In this section we use a coarse-graining inspired by [BČM06] and [BČ07] to control the contribution of edges with conductances between $\varepsilon_s n^{2/\alpha}$ and $\varepsilon_s^{-1}n^{2/\alpha}$ (we call these deep edges) to the rescaled clock process. We will show that these edges create jumps of S_n in the limit and that the distribution of these jumps converges to the distribution of the jumps of a stable subordinator. Since the construction is quite technical we find it useful to give a short verbal description here.

Note that caution is necessary when speaking about the jumps, since the clock process S_n is continuous by definition. It has however some very steep pieces at instants when the VSRW visits a deep edge. Moreover, the visits to deep edges occurs in clusters, since after visiting a deep edge, the VSRW has a reasonable chance of returning there soon. It is thus suitable to identify the jump of S_n with the total contribution to S_n of one cluster of visits to one deep edge.

In the coarse-graining construction we observe the VSRW only before the exit from a large ball $B(0, Kn)$. The VSRW spends a time of order K^2n^2 in this ball. We show that before exiting this ball, only finitely many (depending on ε_s and K) clusters of visits occur and that they are well separated. Moreover, to prove the convergence to the stable subordinator, we need to know that the times that the VSRW spends to in going from one cluster to another are asymptotically independent and exponentially distributed. To this end we cut the trajectory of the VSRW into small pieces of spacial size $\varepsilon_g n$ (and temporal size $\sim \varepsilon_g^2 n^2$). We show that the probability that any such piece contains a cluster of visits to a deep edge is proportional to ε_g^2 if ε_g is small. Further, we show that (asymptotically as $n \to \infty$) no more than one cluster occurs during each piece of the path before exiting $B(0, Kn)$. We further control the distribution of the contribution of one cluster to S_n. All these results are contained in Proposition 6.7 below.

The rough strategy is thus quite similar to [BČM06, BČ07]. There are however several important technical differences, which we would like to point out.

First, in previous papers the process was cut into pieces using balls of mesoscopopic radius, i.e. of radius much smaller than n. Due to the imprecision of our estimates on the Green function, we are forced to use macroscopic ball-like sets of radius $\varepsilon_g n$ here. These introduces some additional, mainly technical, difficulties.

Second, since, unlike the simple random walk, the VSRW is a process in a non-uniform environment, some additional random variables appear in the argument. As an example consider the total time spent by the CSRW in a site x given $X(0) = x$. This time has mean $\mu_x g^{\boldsymbol{\mu}}(x, x)$, where $g^{\boldsymbol{\mu}}(x, x)$ is the usual Green function of the VSRW Y. The same was of course also true in the context of [BČM06], except that there Y was a simple random walk, and so the Green function $g(x, x)$, $x \in \mathbb{Z}^d$, was constant. Hence, we need, in addition, to deal with the randomness of $g^{\boldsymbol{\mu}}(x, x) = C_{\mathrm{eff}}[x, \infty]^{-1}$, $x \in \mathbb{Z}^d$, which are not independent. To recover, at least partially, the independence we use Lemma 3.5 to approximate the diagonal Green functions by conductances to balls of size $O((\log n)^{1/\eta})$ – see Lemma 6.2 below.

We now start the construction. Let $\mathbb{B} = B(0, Kn)$, where K is as in the previous section. Recall that we identify a set of edges $G \subset E^d$ with the set of their vertices. We define

$$\begin{aligned} \tilde{E}_n(u, w) &= \{e \in E^d : \mu_e \in [u, w)n^{2/\alpha}\}, \\ \tilde{T}_n(u, w) &= \{x \in \mathbb{Z}^d : x \in \tilde{E}_n(u, w), x \notin \tilde{E}_n(w, \infty)\}. \end{aligned} \tag{6.1}$$

For the arguments of this section we need that the deep edges that we observe are well separated from each other. Therefore we introduce a set of *bad* sites, that is the set of sites in $\tilde{T}_n(u,w)$ which are close to more than one edge in $\tilde{E}_n(u,\infty)$. We write

$$\begin{aligned} \mathcal{B}_n(u,w) = \big\{x \in \tilde{T}_n(u,w) : \exists y \text{ s.t. } (xy) \in \tilde{E}_n(u,w), \\ B(\{x,y\},\nu) \cap \tilde{T}_n(n^{-\iota},\infty) \neq \{x,y\}\big\}, \end{aligned} \tag{6.2}$$

where $\nu = n^{\omega}$, $\omega \in (0,1/d)$, is a mesoscopic scale, and $\iota \in (0,2/\alpha)$ an arbitrary fixed small constant. We show in the next section that the bad sites are never hit before exiting $\mathbb{B}$, with a large probability. Finally, we set

$$\begin{aligned} T_n(u,w) &= \tilde{T}_n(u,w) \setminus \mathcal{B}_n(u,w), \\ E_n(u,w) &= \{e = (xy) \in E^d : x,y \in T_n(u,w)\}. \end{aligned} \tag{6.3}$$

We begin with an easy bound on the distance between the origin and $\tilde{E}_n(u,\infty)$.

Lemma 6.1. *For any $u > 0$, $\mathbb{P}$-a.s., for all but finitely many n,*

$$B(0,\nu) \cap \tilde{E}_n(u,\infty) = \emptyset. \tag{6.4}$$

Proof. For any edge e

$$\mathbb{P}[e \in E_n(u,\infty)] \leq \mathbb{P}[e \in \tilde{E}_n(u,\infty)] = \mathbb{P}[\mu > n^{2/\alpha}u] = n^{-2}u^{-\alpha}(1+o(1)). \tag{6.5}$$

So

$$\mathbb{P}[B(0,\nu) \cap \tilde{E}_n(u,\infty) \neq \emptyset] \leq c n^{\omega d - 2} u^{-\alpha}, \tag{6.6}$$

and using Borel Cantelli completes the proof. □

We start by introducing the random variables that we use to approximate the diagonal Green function. We fix $\varepsilon_c > 0$ and set $\bar{b}(n) = b_2(n, \varepsilon_c/2)$ (see Lemma 3.5). For any $e \in E^d$ and $z \in \mathbb{Z}^d$ we set

$$\gamma_n(e) = C_{\text{eff}}[e, B(e,\bar{b}(n))^c], \tag{6.7}$$

$$\gamma_n(z) = C_{\text{eff}}[z, B(z,\bar{b}(n)+1)^c]. \tag{6.8}$$

We write F_C for the law of $C_{\text{eff}}[e,\infty]$.

Lemma 6.2. *(i) For all e, $\gamma_n(e)$ is independent of μ_e.*
(ii) If $z \in e$ then $\gamma_n(z) \leq \gamma_n(e)$.
(iii) $\mathbb{P}$-a.s. for all but finitely many n, for all $e \in E_n(u,v) \cap \mathbb{B}$ and $z \in e$,

$$(1+\varepsilon_c)\gamma_n(z) \geq \gamma_n(e). \tag{6.9}$$

(iv) For any $k > 0$ there exists n_0 such that for all $n \geq n_0$

$$\mathbb{E}[\gamma_n(z)^k] \leq \mathbb{E}[\gamma_n(e)^k] < \infty. \tag{6.10}$$

(v) The law of $\gamma_n(e)$ converges weakly as $n \to \infty$ to F_C.

Proof. The claims (i), (ii) and (v) are obvious. To prove (iii) let $h(x) = P_x^{\boldsymbol{\mu}}[\sigma_z < \tau_{B(z,\bar{b}(n))}]$, so that $C_{\mathrm{eff}}[z, B(z, \bar{b}(n))^c] = \mathcal{E}(h, h)$. Let $y \in e$, $y \neq z$ be the second vertex of e. Since $e \in E_n(u, v)$ we know that $\mu_e \geq un^{2/\alpha}$ and that the conductances of all edges attached to y different from e are at most $n^{2/\alpha} n^{-\iota}$. Therefore, $h(y) \geq 1 - cn^{-\iota}$. Let $\bar{h}(\cdot) = h(y)^{-1} h(\cdot) \wedge 1$. Then $\bar{h} = 0$ on $B(z, \bar{b}(n))^c \supset B(e, \bar{b}(n))^c$ and $\bar{h} = 1$ on e. Hence,

$$C_{\mathrm{eff}}[z, B(z, \bar{b}(n))^c] = \mathcal{E}(h, h) \geq h(y)^2 \mathcal{E}(\bar{h}, \bar{h}) \geq (1 - cn^{-\iota})^2 \gamma_n(e). \tag{6.11}$$

and taking n large enough gives (6.9).

To prove (iv) we set, without lost of generality, $z = 0$. We define $k' = \alpha^{-1}(k+1)$ and take n_0 such that $\bar{b}(n_0) > k'$. Consider a new electric network, where all edges (xy) such that $\|x\|_\infty = \|y\|_\infty$ are short-cut, that is their conductance is set to be infinite. Recall the notation $Q(x, r)$ for cubes centre x and side r introduced at the start of Section 3. If $C'[\cdot, \cdot]$ denotes the conductance in this new network we have, for $n \geq n_0$,

$$\mathbb{P}[\gamma_n(z) \geq u] \leq \mathbb{P}\big[C_{\mathrm{eff}}[z, Q(z, k')^c] \geq u\big] \leq \mathbb{P}\big[C'[z, Q(z, k')^c] \geq u\big]. \tag{6.12}$$

Let $L_i = \{(xy) \in E^d : \|x\|_\infty = i, \|y\|_\infty = i - 1\}$. Then

$$C'[z, Q(z, k')^c] = \Big\{ \sum_{i=1}^{k'} \Big(\sum_{e \in L_i} \mu_e \Big)^{-1} \Big\}^{-1} =: \Big(\sum_{i=1}^{k'} (C_{L_i})^{-1} \Big)^{-1}. \tag{6.13}$$

Hence,

$$\begin{aligned} &\mathbb{P}[C'[z, Q(z, k')^c] \geq u] \leq \mathbb{P}[\min\{C_{L_i} : i = 1, \dots, k'\} \geq u] \\ &\quad = \prod_{i=1}^{k'} \mathbb{P}[C_{L_i} \geq u] \leq \prod_{i=1}^{k'} c(d, i) u^{-\alpha} \leq c(d, k') u^{-\alpha k'} \leq cu^{-k-1}. \end{aligned} \tag{6.14}$$

This proves that $\mathbb{E}[\gamma_n(z)^k] < \infty$. If z is replaced by e, the proof is analogous. $\square$

We will split the sets $E_n(u, v)$ according to the value of $\gamma_n(e)$. To this end we define

$$\begin{aligned} E_n(u, v, w, w') &= \{e \in E_n(u, w) : \gamma_n(e) \in [v, w')\}, \\ T_n(u, v, w, w') &= \mathbb{Z}^d \cap E_n(u, v, w, w'). \end{aligned} \tag{6.15}$$

We need the next estimate on the size of these sets and their spacial distribution.

Lemma 6.3. *Let $u, v, w, w' > 0$ and $\delta, \varepsilon_b > 0$ be fixed. Then $\mathbb{P}$-a.s. for all but finitely many n,*

$$\begin{aligned} &n^d \varepsilon_b^d d(1 - \delta) p_n(u, v, w, w') \leq \inf_{x \in \varepsilon_b n \mathbb{Z}^d \cap \mathbb{B}} |Q(x, \varepsilon_b n) \cap E_n(u, v, w, w')| \\ &\leq \sup_{x \in \varepsilon_b n \mathbb{Z}^d \cap \mathbb{B}} |Q(x, \varepsilon_b n) \cap E_n(u, v, w, w')| \leq n^d \varepsilon_b^d d(1 + \delta) p_n(u, v, w, w'), \end{aligned} \tag{6.16}$$

where $p_n(u, v, w, w') = n^{-2}(u^{-\alpha} - w^{-\alpha}) F_C([v, w'))$.

Proof. The number of points in $\varepsilon_b n\mathbb{Z} \cap \mathbb{B}$ is bounded uniformly in n. It is hence sufficient to show that

$$\sum_{n=1}^{\infty} \mathbb{P}\Big[\frac{|Q(0,\varepsilon_b n) \cap E_n(u,v,w,w')|}{n^d \varepsilon_b^d d p_n(u,v,w,w')} \notin (1-\delta, 1+\delta)\Big] < \infty. \tag{6.17}$$

The lemma will then follow from the Borel-Cantelli lemma and the translation invariance of $\boldsymbol{\mu}$.

To prove (6.17) we set $A_n = E^d \cap Q(0, 3\nu)$ and for $e \in A_n$ we define $G_n(e) = Q(0, \varepsilon_b n) \cap (e + 3\nu\mathbb{Z})$. Hence

$$|A_n| = d(3\nu)^d(1+o(1)) \qquad \text{and} \qquad |G_n(e)| = (n\varepsilon_b/3\nu)^d(1+o(1)). \tag{6.18}$$

The series (6.17) is bounded by

$$\sum_{n=1}^{\infty}\sum_{e\in A_n} \mathbb{P}\Big[\frac{|G_n(e) \cap E_n(u,v,w,w')|}{|G_n(e)| p_n(u,v,w,w')} \notin (1-\delta, 1+\delta)\Big]. \tag{6.19}$$

From the definitions of $E_n(u,v,w,w')$ and $\gamma_n(e)$, and the independence of μ_e and $\gamma_n(e)$ given by Lemma 6.2(i) it follows that $\{\mathbf{1}_{e'\in E_n(u,v,w,w')} : e' \in G_n(e)\}$ is an i.i.d. family of Bernoulli random variables with success probability

$$\mathbb{P}\big[\mu_e \in n^{2/\alpha}[u,w)\big]\mathbb{P}\big[\mu_{e'} \le n^{-\iota} n^{2/\alpha} \forall e' \in B(e,\nu)\setminus e, \gamma_n(e) \in [v,w')\big]. \tag{6.20}$$

The first probability in (6.20) is $n^{-2}(u^{-\alpha} - w^{-\alpha})(1+o(1))$. To control the second term in (6.20) note that if ι is sufficiently small, then $\mathbb{P}\big[\mu_{e'} \le n^{-\iota} n^{2/\alpha} \forall e' \in B(e,\nu)\setminus e\big] \xrightarrow{n\to\infty} 1$. Therefore it follows from Lemma 6.2(v) that the second probability in (6.20) converges to $F_C([v,w'))$. Thus (6.20) equals $p_n(u,v,w,w')(1+o(1))$.

It follows that $|G_n(e) \cap E_n(u,v,w,w')|$ has binomial distribution with parameters $|G_n(e)|$ and $p_n(u,v,w,w')(1+o(1))$. Standard estimates on the binomial distribution yield that for any $\delta > 0$ there is $c > 0$ such that all summands in (6.19) are bounded by $\exp(-c|G_n(e)|p_n(u,v,w,w'))$. Since $|A_n| = O(\nu^d)$, the series (6.17) is summable and the proof is finished. ☐

The papers [BČM06, BČ07] used a family of (large but mesoscopic) Euclidean balls to coarse grain the trajectory. They then used the fact that the probability that the simple random walk exits a large ball $B(x,R)$ at a given site $x' \in \partial B(x,R)$ can be bounded from above by cR^{1-d}. This fact was used to control the probability that the simple random walk does not exit a large ball too close to a site in $T_n(\varepsilon_s, \infty)$.

It is not clear that bounds of this type hold for the VSRW Y_t, since large values of μ_e for edges close to a boundary point x' might have a substantial effect on the probability of Y exiting at x'. We therefore replace balls by ball-like sets whose boundary avoids $\tilde{T}_n(\varepsilon, \infty)$ and $\tilde{E}_n(\varepsilon, \infty)$. This construction will then allow us to apply the coarse-graining without requiring such precise information on the exit distribution of Y from large balls.

Lemma 6.4. *Let $\varepsilon_s > 0$ and $K \ge 1$. Then $\mathbb{P}$-a.s. for all but finitely many n, for all $x \in \mathbb{Z}^d$ and $r \in (0, Kn)$ there exist sets $\mathfrak{B}_n(x,r)$ which satisfy*
(i) $\mathfrak{B}_n(x,r)$ is simply connected in $\mathbb{Z}^d$.
(ii) For all $x \in B(0,Kn)$, $B(x,r) \subset \mathfrak{B}_n(x,r) \subset B(x, r + 3\nu\log n)$.
(iii) For all $x \in B(0,Kn)$, $\partial\mathfrak{B}_n(x,r) \cap \bigcup_{e\in\tilde{E}_n(\varepsilon_s,\infty)} B(e,\nu) = \emptyset$.

Proof. If $x \notin B(0, Kn)$ we just take $\mathfrak{B}_n(x,r) = B(x,r)$.

Let $\mathbb{B} = B(0, Kn)$ and let $\mathcal{O} = \bigcup_{e\in\tilde{E}_n(\varepsilon_s,\infty)} B(e,\nu) \cap \mathbb{Z}^d$, and let $\mathcal{C}(z)$ be the $\mathbb{Z}^d$-connected component of $\mathcal{O}$ containing z. Finally, let $K(z) = \big|\{(xy) \in \tilde{E}_n(\varepsilon_s,\infty) : \{x,y\} \subset \mathcal{C}(z)\}\big|$. Using (6.6) we have

$$\begin{aligned}\mathbb{P}[\operatorname{diam}\mathcal{C}(z) \geq 2k\nu] &\leq \mathbb{P}[K(z) \geq k] \\ &\leq (c\nu^d n^{-2})(2c\nu^d n^{-2})\dots(kc\nu^d n^{-2}) \leq ck!(n^{-2}\nu^d)^k.\end{aligned} \tag{6.21}$$

There are, $\mathbb{P}$-a.s., for all but finitely many n, at most $cn^{-2}n^d$ sites in $\tilde{T}_n(\varepsilon_s,\infty) \cap 2\mathbb{B}$. Hence,

$$\mathbb{P}[\max_{z\in 2\mathbb{B}} \operatorname{diam}\mathcal{C}(z) \geq 2\nu\log n] \leq cn^{d-2}\lceil\log n\rceil!(n^{-2}\nu^d)^{\lceil\log n\rceil}, \tag{6.22}$$

which is summable if $\nu \leq (n/\log n)^{2/d}$. The Borel-Cantelli lemma then implies that (for all n large) the largest component of $\mathcal{O} \cap 2\mathbb{B}$ has diameter smaller than $2\nu\log n$. The claim of the lemma then follows easily, by taking $\mathfrak{B}_n(x,r)$ to be the union of $B(x,r)$ and every connected component of $\mathcal{O}$ which intersects $B(x,r)$. ☐

This lemma does not uniquely specify the family $(\mathfrak{B}_n(x,r))$. In what follows we will take $\mathfrak{B}_n(x,r)$ to be the set given by the procedure in the proof of Lemma 6.4.

We finally have all the ingredients that we need to start the coarse-graining construction. Let $\varepsilon_g > 0$, $t_n(0) = 0$, $y_n(0) = 0$ and for $i \geq 1$ let

$$\begin{aligned} t_n(i) &= \inf\{t > t_n(i-1) : Y(t) \notin \mathfrak{B}_n(y_n(i-1), \varepsilon_g n)\}, \\ y_n(i) &= Y(t_n(i)). \end{aligned} \tag{6.23}$$

We denote by $Y[i] = \{Y(s) : t_n(i) \leq s < t_n(i+1)\}$ the i^{th} a piece of the trajectory of Y.

Observe that, by the definition of $\mathfrak{B}_n(x,r)$, $d(y_n(i), \tilde{E}_n(\varepsilon_s,\infty)) \geq \nu$ for all $i \geq 1$. The same is $\mathbb{P}$-a.s. true also for $0 = y_n(0)$, by Lemma 6.1.

The functional central limit theorem (Proposition 3.1) and the construction of the sets $\mathfrak{B}_n(x,r)$ imply the following lemma.

Lemma 6.5. *Let BM_d be the standard Brownian motion in $\mathbb{R}^d$ and let $W(\cdot) = \mathcal{C}_Y \mathsf{BM}_d(\cdot)$ (see Proposition 3.1). Define $t_\infty(0) = 0$ and, for $i \geq 1$, $t_\infty(i) = \inf\{t \geq t_\infty(i-1) : W \notin B(W(t_\infty(i-1)), \varepsilon_g)\}$. Then, $\mathbb{P}$-a.s., under $P_0^{\boldsymbol{\mu}}$, the law of the sequence $(n^{-2}t_n(i) : i \in \mathbb{N})$ converges as $n \to \infty$ to the law of $(t_\infty(i) : i \in \mathbb{N})$. In particular, for every $s > 0$ and $\delta > 0$ there exists $\varepsilon_g > 0$ such that $\mathbb{P}$-a.s. for all but finitely many n,*

$$P_0^{\boldsymbol{\mu}}[n^{-2}t_n(\lfloor d\mathcal{C}_Y^2\varepsilon_g^{-2}s\rfloor) \notin (s-\delta, s+\delta)] < \delta. \tag{6.24}$$

Proof. The first claim is a direct consequence of the functional limit theorem. From standard properties of Brownian motion, we have that $(t_\infty(i) - t_\infty(i-1) : i \geq 1)$ is an i.i.d. sequence, the expectation of $t_\infty(1) - t_\infty(0) = t_\infty(1)$ equals $\varepsilon_g^2\mathcal{C}_Y^{-2}d^{-1}$, and its variance is finite. Hence, by the weak law of large numbers (for $\varepsilon_g \to 0$), there exists ε_g small such that $\mathbb{P}[t_\infty(d\mathcal{C}_Y^2\varepsilon_g^{-2}) \notin (1-\delta, 1+\delta)] \leq \delta/2$. Since, as $n \to \infty$, $\mathbb{P}$-a.s., $n^{-2}t_n(d\mathcal{C}_Y\varepsilon_g^{-2})$ converges in law to $t_\infty(d\mathcal{C}_Y\varepsilon_g^{-2})$, the proof is finished. ☐

We define

$$s_n(i;u,v,w,w') = n^{-2/\alpha}\int_{t_n(i)}^{t_n(i+1)} \mu_{Y(t)}\mathbf{1}\{Y(t)\in T_n(u,v,w,w')\}\mathrm{d}t; \tag{6.25}$$

this is the increment of the clock process between times $t_n(i)$ and $t_n(i+1)$ caused by sites in $T_n(u,v,w,w')$.

We now wish to calculate the distribution of these random variables, and begin with the following lemma.

Lemma 6.6. *Let $z\in\mathbb{B}_0$, $e=(xy)\in E_n(\varepsilon_s)\cap\mathfrak{B}_n(z,\varepsilon_g n)$ be such that $\mu_e = un^{2/\alpha}$, $\gamma_n(e)=v$. Then, $\mathbb{P}$-a.s., the distribution of*

$$n^{-2/\alpha}\int_0^{\tau_{\mathfrak{B}_n(z,\varepsilon_g n)}} \mathbf{1}\{Y(t)\in\{x,y\}\}\mu_{Y(t)}\mathrm{d}t \tag{6.26}$$

under both $P_x^{\boldsymbol{\mu}}$ and $P_y^{\boldsymbol{\mu}}$ converges as $n\to\infty$ to the exponential distribution with mean $2u/v$.

Proof. Since, $e\in E_n(\varepsilon_s)$ (and thus all edges incident to e have conductance smaller than $n^{2/\alpha}n^{-\iota}$), it is easy to see that $\mu_x,\mu_y=\mu_e(1+o(1))$. Moreover, Lemma 6.2 implies that $\gamma_n(x),\gamma_n(y)=\gamma_n(e)(1+O(\varepsilon_c))$. Let N_x, N_y be the number of visits to x and y by Y before exiting $\mathfrak{B}_n(z,\varepsilon_g n)$ and let T_x, T_y be the total time spend there, so that $T_x=\int_0^{\tau_{\mathfrak{B}_n(z,\varepsilon_g n)}}\mathbf{1}\{Y(t)=x\}\mathrm{d}t$. It is well known fact that under $P_x^{\boldsymbol{\mu}}$, T_x has exponential distribution with mean $R_{\mathrm{eff}}[x,\partial\mathfrak{B}_n(z,\varepsilon_g n)]$. By the definition of $\mathfrak{B}_n(z,\varepsilon_g n)$, $d(\{x,y\},\partial\mathfrak{B}_n(z,\varepsilon_g n))\ge\nu$. Therefore, $\gamma_n(x)^{-1}\le R_{\mathrm{eff}}[x,\partial\mathfrak{B}_n(z,\varepsilon_g n)]\le\gamma_n(x)^{-1}(1+\varepsilon_c)$. Hence, the mean of T_x is equal to $1/\gamma_n(x)(1+O(\varepsilon_c))=v^{-1}(1+O(\varepsilon_c))$. Hence, the random variable N_x has geometrical distribution with mean $\mu_x v^{-1}(1+O(\varepsilon_c))$. Moreover, since $p_{xy}=1-O(n^{-\iota})$, it follows that with probability converging to one, $N_y\ge(1-o(n^{-\iota/2}))N_x$. Similar argument gives also $N_x\ge(1-o(n^{-\iota/2}))N_y$ with a large probability. Therefore, T_x+T_y is asymptotically exponentially distributed with mean $2v^{-1}(1+O(\varepsilon_c))$. Taking ε_c arbitrarily small, the distribution of (6.26), which is $n^{-2/\alpha}(T_x\mu_x+T_y\mu_y)$ under $P_x^{\boldsymbol{\mu}}$, converges to the exponential distribution with mean $2u/v$. ☐

The following proposition, which is the main result of this section, gives the distribution of $s_n(i,\varepsilon_s,\varepsilon_s,\varepsilon_s^{-1},\varepsilon_s^{-1})$.

Proposition 6.7. *Let T, $\varepsilon_s,\varepsilon_g>0$. Define $s_n(i)=s_n(i,\varepsilon_s,\varepsilon_s,\varepsilon_s^{-1},\varepsilon_s^{-1})$. Then, $\mathbb{P}$-a.s., under $P_0^{\boldsymbol{\mu}}$, the sequence $(s_n(i),i\in\{1,\dots,\varepsilon_g^{-2}T\})$ converges as $n\to\infty$ to an i.i.d. sequence $(s_\infty(i):i\in\{1,\dots,\varepsilon_g^{-2}T\})$. Moreover, as $\varepsilon_g\to 0$,*

$$P_0^{\boldsymbol{\mu}}[s_\infty(i)=0]=1-c_{\varepsilon_s}\varepsilon_g^2+o(\varepsilon_g^2), \tag{6.27}$$

$$P_0^{\boldsymbol{\mu}}[s_\infty(i)\in A]=\varepsilon_g^2\nu_{\varepsilon_s}(A)+o(\varepsilon_g^2),\qquad A\subset(0,\infty), \tag{6.28}$$

where

$$c_{\varepsilon_s}=\mathcal{C}_Y^{-2}\int_{\varepsilon_s}^{\varepsilon_s^{-1}}\int_{\varepsilon_s}^{\varepsilon_s^{-1}}\alpha v u^{-\alpha-1}\mathrm{d}uF_C(\mathrm{d}v), \tag{6.29}$$

and

$$\nu_{\varepsilon_s}(\mathrm{d}x)=\mathcal{C}_Y^{-2}\int_{\varepsilon_s}^{\varepsilon_s^{-1}}\int_{\varepsilon_s}^{\varepsilon_s^{-1}}\frac{v}{2u}\exp\Big\{-\frac{xv}{2u}\Big\}\alpha v u^{-\alpha-1}\mathrm{d}uF_C(\mathrm{d}v)\mathrm{d}x. \tag{6.30}$$

For the reader's convenience, before starting the proof we summarise the role of various small and large parameters appearing there and their dependence. The parameter ε_s in the definition of a deep edge is kept constant in this section: it will be chosen to be small in Section 8. The same holds for ε_g, which determines the scale of the coarse-graining. In the proof we will choose $\delta > 0$ which will eventually tend to zero. All others ε's appearing in the proof are chosen so that our approximations have a multiplicative error $1 + O(\delta)$. These include ε_o, which gives the size of the zones close to the centre and the border of $\mathcal{B}_n(x, \varepsilon_g n)$ where we do not have precise estimates on the Green function (see Proposition 4.3), ε_b, the parameter giving the size of boxes where we apply the homogeneity estimates of Lemma 6.3, and ε_c, which controls the approximation of the diagonal Green function by $\gamma_n(z)$. These three ε's are mutually independent. Finally K is chosen so, that the VSRW does not exit $\mathbb{B}$ before $t_n(\varepsilon_g^{-2}T)$ with a large probability depending on ε_g.

Proof. We first prove (6.27) and (6.28) for a fixed $i \le \varepsilon_g^{-2}T$. We comment on the asymptotic independence of $s_n(i)$'s at the end of the proof.

We choose K as function of ε_g, such that $P_0^{\boldsymbol{\mu}}[\tau_{\mathbb{B}_0} \le t_n(\varepsilon_g^{-2}T)] = o(\varepsilon_g^2)$, where $\mathbb{B}_0 = B(0, (K - \varepsilon_g)n)$. We write, as usual, $\mathbb{B} = B(0, Kn)$. Let $E_n(\varepsilon_s) = E_n(\varepsilon_s, \varepsilon_s, \varepsilon_s^{-1}, \varepsilon_s^{-1})$, and let

$$W_n(i) = \sum_{e \in E_n(\varepsilon_s)} P_0^{\boldsymbol{\mu}}\big[Y[i] \cap e \neq \emptyset\big]. \tag{6.31}$$

We first show that, $\mathbb{P}$-a.s.,

$$\lim_{n\to\infty} W_n(i) = c_{\varepsilon_s}\varepsilon_g^2 + o(\varepsilon_g^2). \tag{6.32}$$

For any edge $e \in E_n(\varepsilon_s)$ we choose one of its vertices as its 'representative'. Let $H_n(\varepsilon_s) = H_n(\varepsilon_s, \varepsilon_s, \varepsilon_s^{-1}, \varepsilon_s^{-1})$ stands for the set of these representatives. Since, due to the definition of $E_n(\varepsilon_s)$, all edges incident to $E_n(\varepsilon_s)$ have conductances smaller than $n^{2/\alpha}n^{-\iota}$, the event that $Y[i]$ intersects an edge $e \in E_n(\varepsilon_s)$ but not its representative has probability $O(n^{-\iota})$. Therefore,

$$W_n(i) = (1 + O(n^{-\iota})) \sum_{x \in H_n(\varepsilon_s)} P_0^{\boldsymbol{\mu}}[x \in Y[i]]. \tag{6.33}$$

Let $\varepsilon_m > 0$ small and let $\varepsilon_s = u_0 < u_1 < \cdots < u_q = \varepsilon_s^{-1}$ be such that $u_i - u_{i-1} \in (\varepsilon_m, 2\varepsilon_m)$, $1 \le i \le q$. Using the notation $H_n(j,k) = H_n(u_j, u_k, u_{j+1}, u_{k+1})$, we get

$$W_n(i) = (1 + O(n^{-\iota})) \sum_{j=0}^{q-1}\sum_{k=0}^{q-1} \sum_{x \in H_n(j,k)} P_0^{\boldsymbol{\mu}}[x \in Y[i]]. \tag{6.34}$$

Decomposing on the value of $y_n(i)$, the interior sum equals

$$\begin{aligned}\sum_{x \in H_n(j,k)} P_0^{\boldsymbol{\mu}}\big[x \in Y[i]\big] &= \sum_{z \in \mathbb{Z}^d} P_0^{\boldsymbol{\mu}}[y_n(i) = z] \sum_{x \in H_n(j,k)} P_z^{\boldsymbol{\mu}}[\sigma_x < \tau_{\mathfrak{B}_n(z,\varepsilon_g n)}] \\ &= \sum_{z \in \mathbb{B}_0} P_0^{\boldsymbol{\mu}}[y_n(i) = z] \sum_{x \in H_n(j,k)} \frac{g_{\mathfrak{B}_n(z,\varepsilon_g n)}(z,x)}{g_{\mathfrak{B}_n(z,\varepsilon_g n)}(x,x)} + o(\varepsilon_g^2),\end{aligned} \tag{6.35}$$

where the error results from restricting the summation to $\mathbb{B}_0$. By (3.9), Lemmas 3.5, 6.2, and the definition of $\mathfrak{B}_n(z, \varepsilon_g n)$, for $x \in H_n(k,j) \cap \mathbb{B}$,

$$u_k(1 + 2\varepsilon_m \varepsilon_s^{-1}) \geq u_{k+1} \geq \gamma_n(x) \geq g_{\mathfrak{B}_n(z,\varepsilon_g n)}(x,x)^{-1} \geq (1-\varepsilon_c)\gamma_n(x) \geq (1-\varepsilon_c)u_k.$$

Hence, up to an multiplicative error $1 + O(\varepsilon_m/\varepsilon_s) + O(\varepsilon_c)$, we can replace the Green function in the denominator of (6.35) by u_k^{-1}.

To apply Proposition 4.3, we choose $\delta > 0$ and $\varepsilon_o \in (0, \varepsilon_g/100)$, say. Then, for this δ, ε_o and K, ε_g as above, we fix $\varepsilon_b \in (0, \varepsilon_o/2)$ as in Lemma 4.1, and we split the sum in (6.35) again,

$$\begin{aligned} &o(\varepsilon_g^2) + u_k(1 + O(\varepsilon_c, \varepsilon_m/\varepsilon_s)) \sum_{z \in \mathbb{B}_0} P_0^{\boldsymbol{\mu}}[y_n(i) = z] \\ &\times \Big\{ \sum_{\substack{y \in \varepsilon_b n\mathbb{Z} \\ |z-y| < 2\varepsilon_o n}} + \sum_{\substack{y \in \varepsilon_b n\mathbb{Z} \\ 2\varepsilon_o n \leq |z-y| \leq (\varepsilon_g - 2\varepsilon_o)n}} + \sum_{\substack{y \in \varepsilon_b n\mathbb{Z} \\ |z-y| > (\varepsilon_g - 2\varepsilon_o)n}} \Big\} \sum_{x \in H_n(j,k) \cap Q(y, \varepsilon_b n)} g_{\mathfrak{B}_n(z,\varepsilon_g n)}(z,x). \end{aligned} \tag{6.36}$$

We first estimate the second sum over y, since the other two sums contribute to the error only. By Lemma 6.3, any square $Q(y, \varepsilon_b n)$ contains $d\varepsilon_b^d n^d p_n(u_j, u_k, u_{j+1}, u_{k+1})(1 + O(\delta))$ points in $H_n(j,k)$. On the other hand, since $|y - x| > 2\varepsilon_o n$, by Proposition 4.1 the Green function $g_{\mathfrak{B}_n(z,\varepsilon_g n)}(z,x)$ is almost constant in this square and can be approximated by $g_{\mathfrak{B}_n(z,\varepsilon_g n)}(z,y)(1 + O(\delta))$. Hence, the second sum over y in (6.36) equals

$$\begin{aligned} &d\varepsilon_b^d n^d p_n(u_j, u_k, u_{j+1}, u_{k+1}) \sum_{\substack{y \in \varepsilon_b n\mathbb{Z}^d \\ |z-y| \in [2\varepsilon_o n, (\varepsilon_g - 2\varepsilon_o)n]}} g_{\mathfrak{B}_n(z,\varepsilon_g n)}(z,y)(1 + O(\delta)) \\ &= dp_n(u_j, u_k, u_{j+1}, u_{k+1}) \sum_{\substack{y \in \varepsilon_b n\mathbb{Z}^d \\ |z-y| \in [2\varepsilon_o n, (\varepsilon_g - 2\varepsilon_o)n]}} \sum_{x \in Q(y, n\varepsilon_b n)} g_{\mathfrak{B}_n(z,\varepsilon_g n)}(z,x)(1 + O(\delta)), \end{aligned} \tag{6.37}$$

where we once more used the regularity of the Green function. Using Proposition 4.3, the second summation (i.e. over x) in (6.37) can be estimated, and equals

$$n^2 \frac{\varepsilon_g^2}{d\mathcal{C}_Y} \big(1 + O(\varepsilon_o/\varepsilon_g) + c_1(\delta)\big). \tag{6.38}$$

Inserting these estimates back into (6.34), and replacing the summation over j and k by an integration, we get that the contribution of the second sum over y in (6.36) to (6.34) equals $c_{\varepsilon_s}\varepsilon_g^2(1 + O(\delta, c_1(\delta), \varepsilon_m/\varepsilon_s, \varepsilon_c)) + o(\varepsilon_g^2)$. This can be made arbitrarily close to the right-hand side of (6.32) by choosing δ, ε_o, ε_m and ε_c small.

It remains to show that the other two summations over y in (6.36) contribute to the error term only. For $|y - z| \geq (\varepsilon_g - 2\varepsilon_o)n$, observe first that

$$\Bigg| H_n(\varepsilon_s) \cap \mathfrak{B}_n(z, \varepsilon_g n) \cap \bigcup_{\substack{y \in \varepsilon_b n\mathbb{Z} \\ |z-y| > (\varepsilon_g - 2\varepsilon_o)n}} Q(y, \varepsilon_b n) \Bigg| \leq C n^d \varepsilon_o \varepsilon_g^{d-1} n^{-2} \varepsilon_s^{-\alpha}. \tag{6.39}$$

For such y, using the global upper bound on the Green function (3.11),

$$g^{\boldsymbol{\mu}}_{\mathfrak{B}_n(z,\varepsilon_g n)}(z,x) \le g^{\boldsymbol{\mu}}(z,x) \le C(\varepsilon_g - 3\varepsilon_o)^{2-d} n^{2-d}. \tag{6.40}$$

Inserting these two estimates into (6.36), it is easy to see that the contribution of y with $|y-z| > (\varepsilon_g - 2\varepsilon_o)n$ is bounded from above by $C(\varepsilon_g - 3\varepsilon_o)^{2-d}\varepsilon_g^{d-1}\varepsilon_s^{-\alpha}\varepsilon_o$, which can be made arbitrarily small by choosing ε_o small.

It is slightly more delicate to bound the contribution of $|y-z| < 2\varepsilon_o n$. We use similar argument as in [BČM06]. We need to improve our homogeneity estimates (Lemma 6.3) first: Let $j_{\max}$ be the smallest integer satisfying $2^i\nu \ge 2\varepsilon_o n$, that is $j_{\max} = O(\log n)$. Then there exists $K > 0$ such that $\mathbb{P}$-a.s. for all large n, all $j \in \{0,\dots,j_{\max}\}$ and all $x \in \mathbb{B}$,

$$|Q(x,2^j\nu) \cap H_n(\varepsilon_s)| \le K(\log n \vee 2^{jd}\nu^d n^{-2}\varepsilon_s^{-\alpha}). \tag{6.41}$$

Indeed, by definition $|Q(x,2^j\nu)\cap H_n(\varepsilon_s)| \le |Q(x,2^j\nu)\cap \tilde T_n(\varepsilon_s,\varepsilon_s,\varepsilon_s^{-1},\varepsilon_s^{-1})|$. Moreover, if x_1, x_2 be such that $|x_1 - x_2| \ge 2\bar b$, then the events that x_1, resp. x_2, is in $T_n(\varepsilon_s,\varepsilon_s,\varepsilon_s^{-1},\varepsilon_s^{-1})$ are independent. Hence, the probability of the complement of (6.41) is bounded, using a similar decomposition to summations of i.i.d. random variables as in Lemma 6.3, by

$$c\sum_{x\in\mathbb{B}}\sum_{j=0}^{j_{\max}} d\bar b^d e^{-K\lambda(\log n \vee 2^{jd}\nu^d n^{-2}\varepsilon_s^{-\alpha})/d\bar b^d}\left(1 + c\varepsilon_s^{-\alpha}(e^\lambda - 1)n^{-2}\right)^{2^{jd}\nu^d/\bar b^d} \le cn^d j_{\max}\bar b^d n^{-K'}, \tag{6.42}$$

where K' can be made arbitrarily large by choosing K large. This proves (6.41).

Let $E = \{-1,0,1\}^d \setminus \{(0,0,0)\}$ and let $\mathcal{O}_j$ be the union of $3^d - 1$ cubes of size $2^j\nu$ centred at $y_n(i) + 2^j\nu E$,

$$\mathcal{O}_j = \bigcup_{x\in E} Q(y_n(i) + x2^j\nu, 2^j\nu). \tag{6.43}$$

We cover $B(y_n(i), 2\varepsilon_o n) \setminus B(y_n(i),\nu)$ by $\bigcup_{j=0}^{j_{\max}} \mathcal{O}_j$. It is not necessary to cover the interior ball $B(y_n(i),\nu)$, since there are not any sites from $H_n(\varepsilon_s)$ by the definition of sets $\mathfrak{B}_n(x,r)$. The contribution of the sum over $y < 2\varepsilon_o n$ in (6.36) to (6.34) is bounded from above, using the global upper estimate on the Green function (3.11) only, by

$$C\varepsilon_s^{-1}\sum_{j=0}^{j_{\max}}(\log n \vee 2^{jd}\nu^d n^{-2}\varepsilon_s^{-\alpha})(2^j\nu)^{2-d} \le C\varepsilon_s^{-1}\sum_{j=0}^{j_{\max}}\{(2^j\nu)^{2-d}\log n + 2^{2j}\nu^2 n^{-2}\varepsilon_s^{-\alpha}\}, \tag{6.44}$$

where the ε_s^{-1} before the summation comes from the bound on the Green function in the denominator of (6.35). The last display is bounded by $C\varepsilon_s^{-1}\big(\nu^{2-d}\log n + \varepsilon_o^2\varepsilon_s^{-\alpha}\big)$, which can be made arbitrarily small by choosing n large and ε_o small enough. This finishes the proof of (6.32).

We now show that the probability that two different edges from $E_n(\varepsilon_s)$ are visited during one part of the trajectory is small: that is $\mathbb{P}$-a.s.,

$$\lim_{n\to\infty} P_0^{\boldsymbol{\mu}}[\exists i \le \varepsilon_g^{-2}T, Y[i] \cap E_n(\varepsilon_s) \ge 2] = o(\varepsilon_g^2). \tag{6.45}$$

(6.32) implies that the probability that $Y[i]$ hits at least one edge in $E_n(\varepsilon_s)$ is $O(\varepsilon_g^2)$. Given that one such edge is visited, one can prove, using the just the global upper bound on the Green function, that the probability that $Y[i]$ visits a second such edge is $O(\varepsilon_g^2)$ again. From this (6.45) follows immediately.

Fix $u, v > \varepsilon_s$. By the same reasoning as for (6.32) and (6.45), $\mathbb{P}$-a.s.,

$$\lim_{n\to\infty} P_0^{\boldsymbol{\mu}}[|Y[i] \cap E_n(u, v, \varepsilon_s^{-1}, \varepsilon_s^{-1})| = 1] = \frac{\varepsilon_g^2}{\mathcal{C}_Y^2} \int_v^{\varepsilon_s^{-1}} \int_u^{\varepsilon_s^{-1}} \alpha v' u'^{-\alpha-1} \mathrm{d}u' F_C(\mathrm{d}v') + o(\varepsilon_g^2). \tag{6.46}$$

Combining this formula with Lemma 6.6, we obtain (6.28). (6.27) is then a consequence of (6.32), (6.45) and Bonferroni's inequalities.

To show the asymptotic independence of $s_n(i)$'s, it is sufficient to inspect more carefully the above argument. If we replace the distribution $P_0^{\boldsymbol{\mu}}[\cdot]$ by $P_0^{\boldsymbol{\mu}}[\cdot | s_n(0) \in A_0, \dots, s_n(i-1) \in A_{n-1}]$, the only object that changes in the above computations is the distribution of $y_n(i)$. However, the only property of this distribution we used is $\sum_{z\in\mathbb{Z}^d} P_0^{\boldsymbol{\mu}}[y_n(i) = z] = 1$, which remains valid when we consider the conditional distribution. Hence, given $s_n(0), \dots, s_n(i-1)$, the distribution of $s_n(i)$ satisfies (6.27), which implies the asymptotic independence. □

As a consequence of Proposition 6.7 we get the following lemma, which we need to show the asymptotic independence of the clock process and the trajectory. We define

$$r_n(i) = n^{-1}(y_n(i) - y_n(i-1)). \tag{6.47}$$

Lemma 6.8. *For all $\varepsilon_g, \varepsilon_s > 0$, $\ell \in \mathbb{N}$, $\lambda_1, \dots, \lambda_\ell > 0$, $\xi_1, \dots, \xi_\ell \in \mathbb{R}^n$, and $i_1 < \dots < i_\ell \le T\varepsilon_g^{-2}$, $\mathbb{P}$-a.s.,*

$$\lim_{n\to\infty} E_0^{\boldsymbol{\mu}}\Big[\exp\Big\{-\sum_{j=1}^{\ell}\big[\lambda_j s_n(i_j) + \xi_j \cdot r_n(i_j)\big]\Big\}\Big] = \prod_{j=1}^{\ell}\Big[1 + \varepsilon_g^2\Big(\frac{|\xi_j|^2}{2d} - c_{\varepsilon_s} + G(\lambda_j)\Big) + o(\varepsilon_g^2)\Big], \tag{6.48}$$

where

$$G(\lambda) = G^{\varepsilon_s}(\lambda) = \int_{\varepsilon_s}^{\varepsilon_s^{-1}} \int_{\varepsilon_s}^{\varepsilon_s^{-1}} \frac{\alpha v^2 u^{-\alpha-1}}{v + 2u\lambda} \mathrm{d}u F_C(\mathrm{d}v). \tag{6.49}$$

Proof. In Proposition 6.7 we have proved that $s_n(i)$'s converge as $n \to \infty$ weakly to an i.i.d. sequence. The same is true for $r_n(i)$'s. Moreover, the same reasoning as at the end of the proof of Proposition 6.7 can be used to show that $s_n(i)$ is asymptotically independent of $r_n(j)$ for $i \neq j$. This implies the product structure of (6.48). It remains to compute the joint Laplace transform of one pair $(s_n(i), r_n(i))$,

$$\begin{aligned} &E_0^{\boldsymbol{\mu}}\big[e^{-\lambda s_n(i) - \xi\cdot r_n(i)}\big] \\ &\quad = E_0^{\boldsymbol{\mu}}\big[e^{-\xi\cdot r_n(i)} \mathbf{1}\{s_n(i) = 0\}\big] + E_0^{\boldsymbol{\mu}}\big[e^{-\lambda s_n(i) - \xi\cdot r_n(i)} \mathbf{1}\{s_n(i) \neq 0\}\big] \\ &\quad = E_0^{\boldsymbol{\mu}}\big[e^{-\xi\cdot r_n(i)} \mathbf{1}\{s_n(i) = 0\}\big] + E_0^{\boldsymbol{\mu}}\big[e^{-\lambda s_n(i)} \mathbf{1}\{s_n(i) \neq 0\}\big]\mathcal{R}(n), \end{aligned} \tag{6.50}$$

where, since $|r_n(i)| \le 2\varepsilon_g$, the error term satisfies $e^{-2\varepsilon_g|\xi|} \le \mathcal{R}(n) \le e^{2\varepsilon_g|\xi|}$, and thus

$\mathcal{R}(n) = 1 + O(\varepsilon_g)$. The first term in (6.50) can be written as

$$E_0^{\boldsymbol{\mu}}\big[e^{-\xi\cdot r_n(i)}\big] + E_0^{\boldsymbol{\mu}}\big[e^{-\xi\cdot r_n(i)}\mathbf{1}\{s_n(i)\neq 0\}\big] = 1 + \frac{\varepsilon_g^2|\xi|^2}{2d} + o(\varepsilon_g^2) - \mathcal{R}'(n)P_0^{\boldsymbol{\mu}}[s_n(i)\neq 0],$$

where we used the fact that, by the functional central limit theorem, the distribution of $r_n(i)$ converges to the uniform distribution on the sphere with radius ε_g. The error term $\mathcal{R}'(n)$ has the same asymptotics as $\mathcal{R}(n)$. The second term in (6.50) can be computed using Proposition 6.7. As $n\to\infty$ it converges to

$$\varepsilon_g^2\int_0^\infty e^{-\lambda x}\nu_{\varepsilon_s}(\mathrm{d}x) = \varepsilon_g^2 G^{\varepsilon_s}(\lambda). \tag{6.51}$$

The sum of the last two displays is exactly equal to the square bracket on the right-hand side of (6.48). □

7 Remaining corrections

In this section we prove that the contribution to the clock process S_n of those edges that were not considered in the previous two sections is zero with a high probability. We should control edges that have either $\gamma_n(e)\geq\varepsilon_s^{-1}$, or $\mu_e\geq n^{2/\alpha}\varepsilon_s^{-1}$, or are in the set $\mathcal{B}_n(\varepsilon_s,\varepsilon_s^{-1})$ (see (6.2)).

The following lemma treats the first case.

Lemma 7.1. *For every $\delta, m>0$ there exists $\varepsilon_s>0$ such that, $\mathbb{P}$-a.s., for all but finitely many n,*

$$P_0^{\boldsymbol{\mu}}\big[\sigma_{\tilde{E}_n(\varepsilon_s,\varepsilon_s^{-1},\infty,\infty)}<\tau_{\mathbb{B}}\big]\leq\delta \tag{7.1}$$

Proof. By Lemma 6.2(iv), $F_C([u,\infty))\leq\mathbb{P}[\gamma(e)\geq u]\leq cu^{-4}$. This implies that for a large constant C, $\mathbb{P}$-a.s., $\max_{e\in\mathbb{B}}\mu_e\leq Cn^{d/4}\log n$. Let $i_{\min}$ be the largest integer such that $2^{i_{\min}}\leq\varepsilon_s^{-1}$ and let $i_{\max}$ be the smallest integer such that $2^{i_{\max}+1}\geq Cn^{d/4}\log n$. By very similar arguments as for (6.41), using the inequality $F_C([2^i,2^{i+1}))\leq c2^{-4i}$, one can easily check that, $\mathbb{P}$-a.s., for large n, all $x\in\mathbb{B}$ and all $i\in\{i_{\min},\dots,i_{\max}\}$

$$|Q(x,\varepsilon_b n)\cap\tilde{E}(\varepsilon_s,2^i,\infty,2^{i+1})|\leq C\varepsilon_b^d\big(\varepsilon_s^{-\alpha}n^{d-2}2^{-4i}+\log n\big). \tag{7.2}$$

Then arguments analogous to those leading to (6.32) yield that

$$\begin{aligned}\sum_{i=i_{\min}}^{i_{\max}}P_0^{\boldsymbol{\mu}}\big[\sigma_{\tilde{E}_n(\varepsilon_s,2^i,\infty,2^{i+1})}<\tau_{\mathbb{B}}\big] &\leq\sum_{i=i_{\min}}^{i_{\max}}cn^{2-d}2^iC\big(\varepsilon_s^{-\alpha}n^{d-2}2^{-4i}+\log n\big)\\ &\leq c\varepsilon_s^{3-\alpha}+c'n^{2-d+(d/4)}(\log n)^2,\end{aligned} \tag{7.3}$$

which can be made arbitrarily small by choosing ε_s small and n large. □

The remaining corrections are treated in the next lemma. Its proof is almost analogous to the previous one (actually simpler), and we do not present it here.

Lemma 7.2. *For every $\delta, m>0$ there exists $\varepsilon_s>0$ such that, $\mathbb{P}$-a.s., for all but finitely many n,*

$$P_0^{\boldsymbol{\mu}}\big[\sigma_{\tilde{E}_n(\varepsilon_s^{-1},\varepsilon_s,\infty,\varepsilon_s^{-1})}<\tau_{\mathbb{B}}\big]\leq\delta\qquad\text{and}\qquad P_0^{\boldsymbol{\mu}}\big[\sigma_{\mathcal{B}_n(\varepsilon_s,\varepsilon_s^{-1})}<\tau_{\mathbb{B}}\big]\leq\delta. \tag{7.4}$$

8 Proof of the main theorems

Proof of Theorem 2.1. We first show that $S_n(t)$ converges as $n \to \infty$ to $\mathcal{C}_S V$ in the M_1-topology. To check the convergence of finite dimensional distribution we fix $\ell \in \mathbb{N}$ and $0 < s_1 < \cdots < s_\ell < \infty$. By Lemma 6.5, for every $\delta > 0$ it is possible to choose ε_g small enough, such that except on a set of probability smaller than δ, for all $j \le \ell$

$$t_n(\lfloor (1-\delta) d\mathcal{C}_Y^2 \varepsilon_g^{-2} s_j \rfloor) \le n^2 s_n(j) \le t_n(\lfloor (1+\delta) d\mathcal{C}_Y^2 \varepsilon_g^{-2} s_j \rfloor). \tag{8.1}$$

Further, it follows from Proposition 5.1 and Lemmas 7.1 and 7.2 that by choosing ε_s small enough, except on a set of probability smaller than δ, for all $i \le T\varepsilon_g^{-2}$ and n large enough,

$$0 \le S_n(t_n(i)) - \sum_{j=1}^{i} s_n(i) \le \delta. \tag{8.2}$$

Hence, for $u_1, \dots, u_\ell \in [0, \infty)$,

$$\begin{aligned} \lim_{n\to\infty} P_0^{\boldsymbol{\mu}}[S_n(s_j) \ge u_j \forall j \le \ell] &= \lim_{\varepsilon_s,\varepsilon_g \to 0} \lim_{n\to\infty} P_0^{\boldsymbol{\mu}}[S_n(s_j) \ge u_j \forall j \le \ell] \\ &\le \lim_{\varepsilon_s,\varepsilon_g \to 0} \lim_{n\to\infty} P_0^{\boldsymbol{\mu}}[S_n(n^{-2} t_n(\lfloor (1+\delta) d\mathcal{C}_Y^2 \varepsilon_g^{-2} s_j \rfloor)) \ge u_j \forall j \le \ell] \\ &\le \lim_{\varepsilon_s,\varepsilon_g \to 0} \lim_{n\to\infty} P_0^{\boldsymbol{\mu}}\Big[\sum_{i=1}^{\lfloor (1+\delta) d\mathcal{C}_Y^2 \varepsilon_g^{-2} s_j \rfloor} s_n(i) \le u_j \forall j \le \ell \Big]. \end{aligned} \tag{8.3}$$

The lower bound can be obtained analogously by changing δ to $-\delta$. The distribution of the sum of $s_n(i)$ is given in Proposition 6.7. Sending first $n \to \infty$ and then $\varepsilon_g \to 0$ we get that the distribution of the process $\sum_{i=1}^{\lfloor \cdot \varepsilon_g^{-2} \rfloor} s_n(i)$ converges to a compound Poisson process with intensity measure ν_{ε_s}. As $\varepsilon_s \to 0$, ν_{ε_s} converges to the measure

$$x^{-1-\alpha} \alpha 2^\alpha \Gamma(\alpha+1) \mathcal{K}_C \mathrm{d}x, \tag{8.4}$$

where $\mathcal{K}_C = \int_0^\infty v^{1-\alpha} F_C(\mathrm{d}v)\mathrm{d}x$. Therefore, as $n \to \infty$ and then $\varepsilon_s, \varepsilon_g \to 0$, the sum $\sum_{i=1}^{s\varepsilon_g^{-2}} s_n(i)$ converges (in the sense of finite-dimensional distributions) to a stable subordinator V' whose log-Laplace transform at time s, $\log \mathbb{E}[e^{-\lambda V'(s)}]$, is given by

$$- s\mathcal{K}_C \int_0^\infty (1 - \varepsilon^{-\lambda x}) x^{-1-\alpha} \alpha 2^\alpha \Gamma(\alpha+1) = -s\mathcal{K}_C 2^\alpha \pi\alpha \csc(\pi\alpha) \lambda^\alpha. \tag{8.5}$$

Putting this together with the estimate (8.3), we obtain that $S_n(\cdot)$ converges, in the sense of the finite-dimensional distributions, to $\mathcal{C}_S V_\alpha(\cdot)$ where

$$\mathcal{C}_S = \big(d\mathcal{C}_Y^2 \mathcal{K}_C 2^\alpha \pi\alpha \csc(\pi\alpha) \big)^{1/\alpha}. \tag{8.6}$$

To check the asymptotic independence of Y_n and S_n we use Lemma 6.8. As follows from the previous discussion $Y_n(s)$ and $S_n(s)$ are well approximated by $\sum_{i=0}^{d\mathcal{C}_Y^2 \varepsilon_g^{-2} s} r_n(i)$ and $\sum_{i=0}^{d\mathcal{C}_Y^2 \varepsilon_g^{-2} s} s_n(i)$. The joint Laplace transform of these two sums

$$E_0^{\boldsymbol{\mu}}\Big[\exp\Big\{ - \sum_{i=0}^{d\mathcal{C}_Y^2 \varepsilon_g^{-2} s} [\xi \cdot r_n(i) + \lambda s_n(i)] \Big\} \Big] \xrightarrow{n\to\infty} \Big(1 + \varepsilon_g^2 \Big(\frac{|\xi|^2}{2d} - c_{\varepsilon_s} + G(\lambda) \Big) + o(\varepsilon_g^2) \Big)^{d\mathcal{C}_Y \varepsilon_g^{-2} s}, \tag{8.7}$$

which converges, as $\varepsilon_g \to 0$, to

$$\exp\Big\{d\mathcal{C}_Y s\Big(\frac{|\xi|^2}{2d} - c_{\varepsilon_s} + G(\lambda)\Big)\Big\}. \tag{8.8}$$

The same calculation applies for the higher dimensional distributions, which implies that the sums of s_n and r_n (as processes) are independent in the limit.

We further prove that the sequence S_n is tight on $D^1(M_1)$, that is on the space $D = D^1 = D([0,\infty),\mathbb{R})$ equipped with the Skorokhod M_1-topology. We recall the criterion for the tightness on on this space (see, e.g., [Whi02, Theorem 12.12.3])

Lemma 8.1. *(a) The sequence of probability measures $\{P_n\}$ on $D([0,T],\mathbb{R})$ is tight in the M_1-topology if*

(i) For each positive ε there exists c such that

$$P_n[f : \|f\|_\infty > c] \le \varepsilon, \qquad n \ge 1. \tag{8.9}$$

(ii) For each $\varepsilon > 0$ and $\eta > 0$, there exist a δ, $0 < \delta < T$, and an integer n_0 such that

$$P_n[f : w_f(\delta) \ge \eta] \le \varepsilon, \qquad n \ge n_0, \tag{8.10}$$

and

$$P_n[f : v_f(0,\delta) \ge \eta] \le \varepsilon \text{ and } P_n[f : v_f(T,\delta) \ge \eta] \le \varepsilon, \qquad n \ge n_0. \tag{8.11}$$

Here, $w_f(\delta)$ and $v_f(t,\delta)$ stands for

$$\begin{aligned} w_f(\delta) &= \sup\Big\{\inf_{\alpha\in[0,1]} |f(t) - \alpha f(t') - (1-\alpha)f(t'')| : t' \le t \le t'' \le T, t'' - t' \le \delta\Big\}, \\ v_f(t,\delta) &= \sup\big\{|f(t') - f(t'')| : t', t'' \in [0,T] \cap (t-\delta, t+\delta)\big\}. \end{aligned} \tag{8.12}$$

(b) The sequence of probability measures $\{P_n\}$ on $D^1 = D([0,\infty),\mathbb{R})$ is tight in the M_1-topology if for every $T > 0$ its natural projection to $D([0,T],\infty)$ is tight.

Returning back to S_n, we note that since S_n are increasing, the condition (i) of Lemma 8.1 is equivalent to the tightness of $S_n(T)$ which can be easily checked from the convergence of finite dimensional distributions. In order to check condition (ii) of Lemma 8.1 remark that for increasing functions the oscillation function $w_f(\delta)$ is equal to zero. So checking (ii) reduces to controlling the boundary oscillations $v_{S_n}(0,\delta)$ and $v_{S_n}(T,\delta)$. For the first quantity (using again the monotonicity of S_n) this amounts to checking that for any $\varepsilon, \eta > 0$ there is δ such that $\mathbb{P}[S_n(\delta) \ge \eta] < \varepsilon$, which follows again from the convergence of the finite-dimensional distributions. The reasoning for $v_{S_n}(T,\delta)$ is analogous. The sequence of distributions of S_n is thus M_1-tight on $D([0,T],\mathbb{R})$ for all $T > 0$, and therefore, by Lemma 8.1, on $D^1(M_1)$.

The tightness of Y_n in $D^d(U)$ follows from Proposition 3.1. The tightness of both components implies the tightness of the pair (S_n, Y_n) in the product topology on $D^1(M_1) \times D^d(U)$. □

Proof of Theorem 1.2. This follows from the description of X_n as $X_n(\cdot) = Y_n(S_n^{-1}(\cdot))$. Let $D_{u,\uparrow}$ denote the subset of D^1 consisting of unbounded increasing functions. By Corollary 13.6.4 of [Whi02] the inverse map from $D_{u,\uparrow}(M_1)$ to $D_{u,\uparrow}(U)$ is continuous at strictly increasing functions. Since the stable subordinator V_α (the limit of S_n in $(D_{u,\uparrow}, M_1)$) is a.s. strictly increasing, the distribution of S_n^{-1} converges to the distribution of V_α^{-1} weakly on $D_{u,\uparrow}(U)$ and the limit is a.s. continuous. It is easy to check that the composition $(f,g) \mapsto f \circ g$ as a mapping from $D^d(U) \times D_{u,\uparrow}(U)$ to $D^d(U)$ is continuous on $C^d \times C$ (here C is the space of continuous function). The weak convergence of X_n on $D^d(U)$ then follows. □

9 The proof for the BTM

In this Section we discuss the proofs of Theorems 1.3 and 2.2 for the BTM. More precisely, we will explain how the proofs from [BD08], and from Sections 3–8 should be modified for the BTM.

Let $\tilde{Y}$ be the VSRW associated with the conductances $\tilde{\mu}_{xy} = \tau_x^a \tau_y^a$, and let

$$\tilde{g}_A^{\boldsymbol{\tau}}(x,y) = \tilde{E}_x^{\boldsymbol{\tau}} \int_0^{\tau_A} \mathbf{1}\{\tilde{Y}_s = y\}\, \mathrm{d}t \tag{9.1}$$

be the associated Greens function. The first step is to obtain the FCLT, Proposition 3.1 and the Greens function bounds Proposition 3.2 for $\tilde{Y}$ and $\tilde{g}_A^{\boldsymbol{\tau}}(x,y)$.

Note that $\tilde{\mu}_e$ are not i.i.d., but that this is still a stationary ergodic process which is bounded below. We can therefore use [BD08, Theorem 6.1], provided we verify the condition (9.4) on the metric $\tilde{d}(x,y)$ given there. This metric is defined as follows. For all edges $(xy) \in \mathbb{E}^d$, let

$$t(xy) = \min\{\tilde{\mu}_{xy}^{-1/2}, C\} \tag{9.2}$$

for some $C > 0$. The new metric $\tilde{d}$ is given by the first-passage percolation distance,

$$\tilde{d}(x,y) = \inf\Big\{\sum_{i=1}^n t(x_{i-1}x_i)\Big\}, \tag{9.3}$$

where the infimum is taken over all nearest-neighbours paths, $x = x_0 \sim x_1 \sim \cdots \sim x_n = y$, connecting x and y. In the case of the VSRW for the RCM [BD08] uses first-passage percolation results from [Kes86] to show that the metric $\tilde{d}$ is equivalent with the Euclidean one, with a high probability. Since these results are only available when t_e are i.i.d., we need an equivalent of Lemma 4.2 of [BD08] to prove the necessary estimates on $\tilde{g}^{\boldsymbol{\tau}}$.

Lemma 9.1. *Let $\tilde{B}(x,r) = \{y : \tilde{d}(x,y) \le r\}$ be the balls in the $\tilde{d}$-metric. Then there exists $c_1, \dots c_4 \in (0,\infty)$ such that*

$$\tilde{\mathbb{P}}[\tilde{B}(0,c_1 r) \subset B(0,r) \subset \tilde{B}(0,c_2 r)] \ge 1 - c_3 e^{-c_4 r}. \tag{9.4}$$

Consequently, the conclusions of [BD08, Theorem 6.1] hold for $\tilde{Y}$ and $g_A^{\boldsymbol{\tau}}(x,y)$, and in particular Proposition 3.1 and the Proposition 3.2 hold for $\tilde{Y}$ and $\tilde{g}_A^{\boldsymbol{\tau}}(x,y)$.

Proof. As $t(xy)$ is bounded from above by C, there exists c_2 such that the right hand inclusion always holds. To show the left one define, for some small $\delta > 0$ which will be fixed later,

$$\bar{t}(xy) = \delta \mathbf{1}\{\tilde{t}(xy) \geq \delta\}, \tag{9.5}$$

and $\bar{d}(x,y)$ analogously to $\tilde{d}(x,y)$. Obviously $\tilde{d}(x,y) \geq \bar{d}(x,y)$ and thus $\tilde{B}(x,y) \subset \bar{B}(x,y)$, where $\bar{B}$ is ball in the $\bar{d}$ metric. We have $\tilde{\mathbb{P}}(\bar{t}(xy) = \delta) = \psi(\delta)$ with $\psi(\delta) \to 1$ as $\delta \to 0$. Moreover, $\bar{t}(xy)$ and $\bar{t}(x'y')$ are independent if $\{x,y\}$ and $\{x',y'\}$ are disjoint. We can now use coupling with independent percolation as in [LSS97]. According to it, it is possible to construct an i.i.d. family $(t'(e) : e \in E^d)$ of Bernoulli random variables on $\{0,\delta\}$ on the same probability space as $\bar{t}(e)$ such that $\bar{t}(e) \geq t'(e)$, $\tilde{\mathbb{P}}$-a.s., and $\tilde{\mathbb{P}}[t'(e) = \delta] = \phi(\psi(\delta))$ and $\phi(u) \to 1$ as $u \to 1$. We now fix δ such that $\phi(\psi(\delta))$ is larger that the percolation threshold on $\mathbb{Z}^d$. Since $t'(e)$ are independent, by the same argument as in [BD08], $\mathbb{P}[B'(0, c_1' r) \subset B(0,r)] \geq 1 - c_3 e^{-c_4 r}$, where B' is the ball corresponding to the metric t'. Since $t'(e) \leq \bar{t}(e)$ and thus $B'(0,r) \supset \bar{B}(0,r)$, the proof is finished. □

Finally, we mention the modifications that are necessary in Sections 3–8. The proof for the BTM is actually simpler, since the 'sites with large equilibrium measure' do not come in pairs as in the CSRW, but are typically isolated.

The only changes in Section 3 are the obvious replacement of $\boldsymbol{\mu}$ by $\boldsymbol{\tau}$ and adding tildas everywhere. Section 5 can also be easily adapted after replacing the definition (5.2) by

$$\tilde{T}_n(0,\varepsilon) = \{x \in \mathbb{Z}^d : \tau_x \leq \varepsilon n^{2/\alpha}\}. \tag{9.6}$$

Similarly, in Section 6 one should define

$$\begin{aligned} \tilde{T}_n(u,w) &= \{x \in \mathbb{Z}^d : \tau_x \in [u,w) n^{2/\alpha}\}, \\ \mathcal{B}_n(u,w) &= \big\{x \in \tilde{T}_n(u,w) : B(x,\nu) \cap \tilde{T}_n(n^{-\iota},\infty) > 1\big\}, \\ T_n(u,v,w,w') &= \{x \in T_n(u,w) : \gamma_n(x) \in [v,w)\}. \end{aligned} \tag{9.7}$$

It is not necessary to define $E(u,w)$ and $E(u,v,w,w')$. In Lemma 6.2 only $\gamma_n(z)$ should be considered. In the proof of this lemma, in (6.14), the minimum and products should be taken only over even integers to recover the independence. The remaining parts of Section 6 are essentially unchanged. Note however that the values of the constants in (6.29) and (6.30) will change. Similar changes apply also for Sections 7 and 8.

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