Convergence within a polyhedron: Controller design for time-delay systems with bounded disturbances

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Abstract: This paper considers linear systems with state/input time-varying delays and bounded disturbances. We study a new problem of designing a static output feedback controller which guarantees that the state vector of the closed-loop system converges within a pre-specified polyhedron. Based on the Lyapunov-Krasovskii method combining with the free-weighting matrix technique, a new sufficient condition for the existence of a static output feedback controller is derived. Our condition is expressed in terms of linear matrix inequalities with two parameters need to be tuned and therefore can be efficiently solved by using a two-dimensional search method combining with convex optimization algorithms. To be able to obtain directly an output feedback control matrix from the derived condition, we propose an appropriate combination between a state transformation with a choice of a special form of the free-weighting matrices. The feasibility and effectiveness of the derived results are illustrated through five numerical examples.

1 Introduction

Within recent years, the stabilization problem of linear systems with time-delays in the state and the input has received much considerable attention from researchers [1–8]. Based on the Lyapunov-Krasovskii method combining with the free-weighting matrix technique, a state feedback stabilization condition was first reported [2] for linear systems with two time-varying delays in the state and the input. This condition was given in terms of linear matrix inequalities with four parameters need to be tuned. For the case where there is a constant time delay in both the state and the input, by eliminating some free-weighting matrices, the authors [3–5] derived some simpler state feedback stabilization criteria which are given in terms of linear matrix inequalities and require only one tuned parameter. In practice, the assumption of full state information is a limiting one and it is more practical if only output information is used for the controller design purpose [7–11]. To our knowledge, there are few results dealing with output feedback stabilization problem for linear systems with time delays in the state and the input [7,8]. By using the Lyapunov method and delay-decomposition technique, the authors [7] proposed two methods for designing static and integral output feedback controllers for linear systems with one unknown constant time delay in both the state and the input. By using the sliding mode control method, a static output feedback stabilization condition for linear systems with state and input time-varying delays was reported in [8].

On the other hand, disturbances are unavoidable in practical control systems due to modelling errors, linearization approximations, unknown disturbance signals, measurement errors, etc. For systems with bounded disturbances, a central concept that has received considerable attention is the so-called reachable set, which is the set of all the states starting from the origin by inputs with peak value [12, 13]. The exact shape of reachable sets of a perturbed system is, in general, very complex and hard to obtain. Hence, it is usually approximated by outer bounding simple convex shapes like balls or ellipsoids or boxes. So far, the problem of reachable set bounding for systems with time delays and bounded disturbances has been studied extensively [13–27]. Very recently, the authors [27] considered a new problem which deals with the design of a state feedback controller such that reachable sets of the closed-loop system are contained in a pre-specified ellipsoid. This is an interesting and meaningful problem since the pre-specified ellipsoid can be chosen according to practical situations or special design requirements. For instance, given a set of finite points in state space $D = \{\xi_i : i = 1, \cdots, r\}$ and it is required to design a controller such that reachable sets of the closed-loop system do not contain any point ξ_i. As pointed out in [27], one first finds an ellipsoid $\epsilon(P)$ (as large as possible) that does not contain any point ξ_i and then design a controller such that reachable sets of the resulting closed-loop system are contained in the ellipsoid $\epsilon(P)$. Note that, in such a situation, it is clear that a polyhedron, which is an intersection of halfspaces ([28], page 31) can express the above requirement better, i.e., there exists a polyhedron Ω, which is larger than the ellipsoid $\epsilon(P)$ and does not contain any point ξ_i (for a visual illustration, see Figure 1, where the rectangle $ABCD$ contains the ellipse $\epsilon(P)$ but does not contain any point $\xi_i, i = 1, \cdots, 8$). Hence, the controller design problem for the case where reachable sets are contained in a polyhedron Ω will be easier than for the case where reachable sets are contained in an ellipsoid $\epsilon(P)$

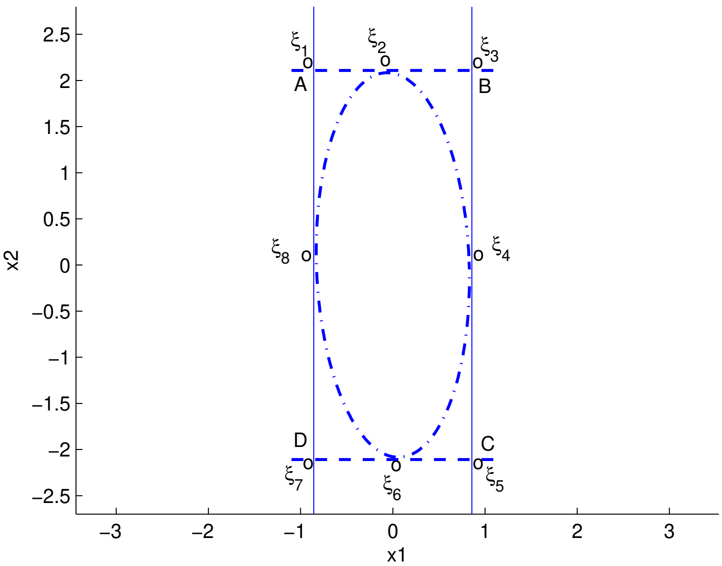

Figure 1: Rectangle $ABCD$ and ellipse $\epsilon(P)$

which is smaller than the polyhedron Ω.

Motivated by the above, in this paper, we consider linear systems with state/input time-varying delays and bounded disturbances. We solve a new problem of designing a static output feedback controller, which guarantees that the state vector of the closed-loop system converges within a pre-specified polyhedron. To solve this problem, we employ the Lyapunov-Krakovskii method and the free-weighting matrix technique [29, 30] with a choice of a special form of the free-weighting matrices. A new sufficient condition for the existence of a static output feedback controller is derived and expressed in terms of linear matrix inequalities with two parameters need to be tuned and can be efficiently solved by using a two-dimensional search method combining with convex optimization algorithms such as the Matlab's LMI toolbox [31]. Furthermore, to reduce the conservatism of our derived convergence condition, we use the recent effective techniques in stability analysis for time-delay systems, i.e. the Wirtinger-based integral inequality [32, 33] and the reciprocally convex combination inequality [34]. Also, for the case where disturbances are not present, the derived convergence condition is reduced to a static output feedback exponential stabilizability condition, which is shown to be less conservative than existing ones [2–5, 7, 35]. Lastly, the feasibility and effectiveness of the obtained results are illustrated through five numerical examples.

2 Problem statement and preliminaries

Consider the following linear system with state/input time-varying delays and bounded disturbances

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_1x(t-\tau_1(t)) + Bu(t) + B_2u(t-\tau_2(t)) + D\omega(t),\ t \geq 0, \\ y(t) &= Cx(t), \\ x(\theta) &= \phi(\theta),\ \theta \in [-h, 0], \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input vector, $y(t) \in \mathbb{R}^p$ is the measured output vector, $\omega(t) \in \mathbb{R}^k$ is the disturbance vector satisfying

$$\omega^T(t)\omega(t) \leq \overline{\omega}^2, \tag{2}$$

$\overline{\omega}$ is a given positive scalar, matrices $A \in \mathbb{R}^{n\times n}$, $A_1 \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $B_2 \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{n\times k}$ are given constant matrices, C is assumed to be a full-row rank matrix, $\tau_1(t)$ and $\tau_2(t)$ are time-varying delays satisfying

$$\begin{cases} 0 \leq \tau_1(t) \leq \tau_{1M},\ \dot{\tau}_1(t) \leq d_{1M} \leq 1, \\ 0 \leq \tau_2(t) \leq \tau_{2M},\ \dot{\tau}_2(t) \leq d_{2M} \leq 1, \end{cases}$$

where $\tau_{1M} \geq 0$, $\tau_{2M} \geq 0$, d_{1M} and d_{2M} are known constants, $h = \max\{\tau_{1M}, \tau_{2M}\}$, $\phi(\theta) \in C^1([-h, 0], \mathbb{R}^n)$ is an initial function with its norm defined as

$$\|\phi\|_c = \max\{\max_{t\in[-h,0]} ||\phi(t)||, \max_{t\in[-h,0]} ||\dot{\phi}(t)||\}. \tag{3}$$

With the following static output feedback control law

$$u(t) = Ky(t) \tag{4}$$

where $K \in \mathbb{R}^{m\times p}$, the closed-loop system is obtained as follows

$$\dot{x}(t) = (A + BKC)x(t) + A_1x(t-\tau_1(t)) + B_2KCx(t-\tau_2(t)) + D\omega(t). \tag{5}$$

Given q non-zero row matrices $L_j \in \mathbb{R}^{1\times n}$, $j = 1, \cdots, q$ and q positive scalars $b_j > 0$, $j = 1, \cdots, q$. It is easy to see that $\{x \in \mathbb{R}^n : L_jx = b_j\}$ and $\{x \in \mathbb{R}^n : L_jx = -b_j\}$ are two parallel $(n-1)$-planes in $\mathbb{R}^n$ and the set $\Omega_j = \{x \in \mathbb{R}^n : |L_jx| \leq b_j\}$ is the area between the two parallel planes. Then, the set $\Omega = \bigcap_{j=1}^{q} \Omega_j$ is a polyhedron [28] and the main problem of this paper is stated as follow:

Problem: Find a static output feedback controller (4) such that every solution $x(t, \phi)$ of the closed-loop system (5) satisfies

$$\limsup_{t\to\infty} |L_j x(t, \phi)| \leq b_j, \ j = 1, \cdots, q. \tag{6}$$

This means that the state vector of the closed-loop system (5) converges within the given polyhedron Ω as t tends to infinity. Note that if $\text{rank}([L_1^T \ L_2^T \ \cdots \ L_q^T]) = n$ then Ω is bounded and it is called a polytope [28] in $\mathbb{R}^n$.

The following lemmas are useful for our main results.

Lemma 1: For a given positive scalar δ, let $V(t)$ be a Lyapunov function for system (5). If $\dot{V}(t) + 2\delta V(t) - 2\frac{\delta}{\overline{\omega}^2}\omega^T(t)\omega(t) \leq 0, \ \forall t \geq 0$, then we have

$$\limsup_{t\to\infty} V(t) \leq 1.$$

Proof: Putting $v(s) = e^{2\delta s}V(s)$ and taking the derivative of $v(s)$ in s, we have

$$\begin{aligned}\dot{v}(s) &= e^{2\delta s}\Big(\dot{V}(s) + 2\delta V(s) - \frac{2\delta}{\overline{\omega}^2}\omega^T(s)\omega(s)\Big) + \frac{2\delta}{\overline{\omega}^2}\omega^T(s)\omega(s)e^{2\delta s} \\ &\leqslant 2\delta e^{2\delta s}.\end{aligned}$$

Integrating from 0 to t both sides of the above inequality, we obtain

$$v(t) - v(0) \leq e^{2\delta t} - 1, \ \forall t \geq 0,$$

and hence

$$V(t) \leq 1 + e^{-2\delta t}|V(0) - 1|, \ \forall t \geq 0.$$

This implies that $\limsup_{t\to\infty} V(t) \leq 1$. The proof of Lemma 1 is completed □

The Wirtinger-based integral inequality [32] and the reciprocally convex combination inequality [34], which has been reformulated by [32], are used in this paper.

Lemma 2: (The Wirtinger-based integral inequality [32]) For a given $n \times n$-matrix $R > 0$, any differentiable function $\varphi : [a, b] \to \mathbb{R}^n$, then the following inequality holds

$$\int_a^b \dot{\varphi}(u)R\dot{\varphi}(u)du \quad \geq \frac{1}{b-a}(\varphi(b) - \varphi(a))^T R(\varphi(b) - \varphi(a)) + \frac{12}{b-a}\Omega^T R\Omega,$$

where $\Omega = \frac{\varphi(b)+\varphi(a)}{2} - \frac{1}{b-a}\int_a^b \varphi(u)du.$

Lemma 3: (The reciprocally convex combination inequality [32, 34]) For given positive integers n, m, a scalar $\alpha \in (0,1)$, a $n \times n$-matrix $R > 0$, two $n \times m$-matrices W_1, W_2. Define, for all vector $\xi \in \mathbb{R}^m$, the function $\Theta(\alpha, R)$ given by

$$\Theta(\alpha, R) = \frac{1}{\alpha}\xi^T W_1^T R W_1 \xi + \frac{1}{1-\alpha}\xi^T W_2^T R W_2 \xi.$$

If there is a matrix $X \in \mathbb{R}^{n\times n}$ such that $\begin{bmatrix} R & X \\ \star & R \end{bmatrix} > 0$, then the following inequality holds

$$\min_{\alpha\in(0,1)} \Theta(\alpha, R) \geq \begin{bmatrix} W_1\xi \\ W_2\xi \end{bmatrix}^T \begin{bmatrix} R & X \\ \star & R \end{bmatrix} \begin{bmatrix} W_1\xi \\ W_2\xi \end{bmatrix}.$$

Lemma 4: (Schur Complement Lemma [31]) Let R be a symmetric positive definite matrix. For any matrices P, S with appropriate dimensions, where $P = P^T$, then

$$\begin{bmatrix} P & S \\ S^T & R \end{bmatrix} > 0$$

if and only if $P - SR^{-1}S^T > 0$.

3 Main results

To use conveniently the output information in designing a static output feedback controller, we first take the following state transformation to re-present the output matrix in a canonical form:

$$x(t) = Hz(t), \tag{7}$$

where $H = [C^+ \quad \text{null}(C)]$ is a nonsingular matrix, C^+ denotes the Moore-Penrose inverse of C, null(C) denotes an orthogonal basis for the null-space of C. Then, system (1) is transformed into the following system

$$\begin{aligned} \dot{z}(t) &= \overline{A}z(t) + \overline{A}_1 z(t-\tau_1(t)) + \overline{B}u(t) + \overline{B}_2 u(t-\tau_2(t)) + \overline{D}\omega(t), \ t \geq 0 \\ y(t) &= \overline{C}z(t), \end{aligned} \tag{8}$$

$$z(\theta) \;=\; H^{-1}\phi(\theta) := \varphi(\theta),\ \theta \in [-h, 0],$$

where $\overline{A} = H^{-1}AH$, $\overline{A}_1 = H^{-1}A_1H$, $\overline{B} = H^{-1}B$, $\overline{B}_2 = H^{-1}B_2$, $\overline{D} = H^{-1}D$ and $\overline{C} = CH$. With the choice of matrix H as above, the output matrix $\overline{C}$ is now in a canonical form, i.e. $\overline{C} = [I_p \quad 0]$. Note that condition (6) is equivalent to the following condition

$$\limsup_{t\to\infty} |L_jHz(t,\varphi)| \le b_j,\ j = 1, \cdots, q \tag{9}$$

and the polyhedron Ω corresponds to the polyhedron $\overline{\Omega} = \{z \in \mathbb{R}^n : |L_jHz| \le b_j,\ j = 1, \cdots, q\}$.

The following notations are needed in order to derive our main results. For two nonsingular matrices $Z_{11} \in \mathbb{R}^{p\times p}$ and $Z_{22} \in \mathbb{R}^{(n-p)\times(n-p)}$, matrices $Z_{21} \in \mathbb{R}^{(n-p)\times p}$, $G \in \mathbb{R}^{m\times p}$, $K \in \mathbb{R}^{m\times p}$, and two $n \times n$ positive-definite matrices R_1, R_2, we denote the following

$$Z = \begin{bmatrix} Z_{11} & 0_{p\times(n-p)} \\ Z_{21} & Z_{22} \end{bmatrix},\ \widetilde{G} = \begin{bmatrix} G & 0_{m\times(n-p)} \\ 0_{(n-m)\times p} & 0_{(n-m)\times(n-p)} \end{bmatrix},\ \ \widetilde{K} = \begin{bmatrix} K & 0_{m\times(n-p)} \\ 0_{(n-m)\times p} & 0_{(n-m)\times(n-p)} \end{bmatrix},$$

$$\widetilde{C} = \begin{bmatrix} \overline{C} \\ 0_{(n-p)\times n} \end{bmatrix},\ \mathcal{Z} = \mathrm{diag}\{Z, \cdots, Z, I_n\} \in \mathbb{R}^{11n\times 11n},\ \ \mathcal{G} = \mathrm{diag}\{\widetilde{G}, \cdots, \widetilde{G}\} \in \mathbb{R}^{11n\times 11n},$$

$$\widetilde{B} = [\overline{B} \ \ 0_{n\times(n-m)}] \in \mathbb{R}^{n\times n},\ \ \widetilde{B}_2 = [\overline{B}_2 \ \ 0_{n\times(n-m)}] \in \mathbb{R}^{n\times n},\ \ \ \widetilde{D} = [\overline{D} \ \ 0_{n\times(n-k)}] \in \mathbb{R}^{n\times n},$$

$$\mathcal{A}_c^T = [\overline{A} \ \ \overline{A}_1 \ \ 0_{n\times 7n} \ \ -I_n \ \ \widetilde{D}] \in \mathbb{R}^{n\times 11n},\ \ \ \mathcal{B}_c^T = [\widetilde{B} \ \ 0_{n\times 2n} \ \widetilde{B}_2 \ \ 0_{n\times 7n}] \in \mathbb{R}^{n\times 11n},$$

$$\mu_1(t) = \tfrac{1}{\tau_1(t)} \int_{t-\tau_1(t)}^{t} z^T(s)ds,\quad \mu_2(t) = \tfrac{1}{\tau_{1M}-\tau_1(t)} \int_{t-\tau_{1M}}^{t-\tau_1(t)} z^T(s)ds,$$

$$\mu_3(t) = \tfrac{1}{\tau_2(t)} \int_{t-\tau_2(t)}^{t} z^T(s)ds,\quad \mu_4(t) = \tfrac{1}{\tau_{2M}-\tau_2(t)} \int_{t-\tau_{2M}}^{t-\tau_2(t)} z^T(s)ds,$$

$$F = Z^{-1},\ \ \widetilde{\omega}(t) = [\omega^T(t) \ \ 0_{1\times(n-k)}]^T,$$

$$\begin{aligned}\xi^T(t) = \Big[&z^T(t)F^T \quad z^T(t-\tau_1(t))F^T \quad z^T(t-\tau_{1M})F^T \quad z^T(t-\tau_2(t))F^T \quad z^T(t-\tau_{2M})F^T \\ &\mu_1(t)F^T \ \ \mu_2(t)F^T \ \ \mu_3(t)F^T \ \ \ \mu_4(t)F^T \ \ \ \dot{z}(t)F^T \ \ \widetilde{\omega}^T(t)\Big] \in \mathbb{R}^{1\times 11n},\\ \zeta_0^T(t) = \Big[&z^T(t)F^T \quad \int_{t-\tau_{1M}}^{t} z^T(s)F^T ds \quad \int_{t-\tau_{2M}}^{t} z^T(s)F^T ds\Big] \in \mathbb{R}^{1\times 3n},\end{aligned}$$

$e_i = [0_{n\times(i-1)n} \quad I_n \quad 0_{n\times(11-i)n}]^T$, for $i = 1, \cdots, 11$,

$\rho(t) = [e_1 \quad \tau_1(t)e_6 + (\tau_{1M} - \tau_1(t))e_7 \quad \tau_2(t)e_8 + (\tau_{2M} - \tau_2(t))e_9] \in \mathbb{R}^{11n\times 3n}$,

$\Gamma_1 = [e_1 - e_2 \quad \sqrt{3}(e_1 + e_2 - 2e_6) \quad e_2 - e_3 \quad \sqrt{3}(e_2 + e_3 - 2e_7)] \in \mathbb{R}^{11n\times 4n}$,

$\Gamma_2 = [e_1 - e_4 \quad \sqrt{3}(e_1 + e_4 - 2e_8) \quad e_4 - e_5 \quad \sqrt{3}(e_4 + e_5 - 2e_9)] \in \mathbb{R}^{11n\times 4n}$,

$\widetilde{R}_1 = \begin{bmatrix} R_1 & 0 \\ 0 & R_1 \end{bmatrix}$, $\widetilde{R}_2 = \begin{bmatrix} R_2 & 0 \\ 0 & R_2 \end{bmatrix}$ and $\Upsilon_j = \frac{1}{b_j^2} H^T L_j^T L_j H,\ j = 1, \cdots, q$.

Note that, from the above notations and with some simple computations, we can verify that

$$\overline{B}K\overline{C}Z = \widetilde{B}\widetilde{K}\widetilde{C}Z = \widetilde{B} \times \begin{bmatrix} KZ_{11} & 0_{m\times(n-p)} \\ 0_{(n-m)\times p} & 0_{(n-m)\times(n-p)} \end{bmatrix} \in \mathbb{R}^{n\times n}.$$

By letting $G = KZ_{11}$, then system (8) with a static output feedback controller $u(t) = K\overline{C}z(t)$ is rewritten as follows:

$$[\mathcal{A}_c^T \mathcal{Z} + \mathcal{B}_c^T \mathcal{G}]\xi(t) = 0. \tag{10}$$

Now we are in a position to introduce the main result in the form of the following theorem.

Theorem 1: If there exist a positive scalar $\delta > 0$, a scalar λ, a positive-definite $3n \times 3n$-matrix P, six positive-definite $n \times n$-matrices Q_1, Q_2, S_1, S_2, R_1, R_2, q positive-definite $n \times n$-matrices $P_j, j = 1, \cdots, q$, two $2n \times 2n$-matrices X_1, X_2, two nonsingular matrices $Z_{11} \in \mathbb{R}^{p\times p}$, $Z_{22} \in \mathbb{R}^{(n-p)\times(n-p)}$, and two matrices $Z_{21} \in \mathbb{R}^{(n-p)\times p}$, $G \in \mathbb{R}^{m\times p}$ such that the following matrix inequalities hold

$$P - \begin{bmatrix} P_j & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} > 0,\ j = 1, \cdots, q, \tag{11}$$

$$P_j - Z^T \Upsilon_j Z > 0,\ j = 1, \cdots, q, \tag{12}$$

$$\Theta_i = \begin{bmatrix} \widetilde{R}_i & X_i \\ \star & \widetilde{R}_i \end{bmatrix} > 0,\ i = 1, 2, \tag{13}$$

$$\Sigma(\tau_1, \tau_2, \delta) < 0, \quad \forall(\tau_1, \tau_2) \in \{0, \tau_{1M}\} \times \{0, \tau_{2M}\}, \tag{14}$$

where

$$\begin{aligned}
\Sigma(\tau_1, \tau_2, \delta) \;=\;& 2\rho(t)P[e_{10} \quad e_1 - e_3 \quad e_1 - e_5]^T + 2\delta\rho(t)P\rho^T(t) \\
&+e_1(Q_1 + S_1 + Q_2 + S_2)e_1^T - e^{-2\delta\tau_{1M}}e_3Q_1e_3^T \\
&-e^{-2\delta\tau_{1M}}(1 - d_{1M})e_2S_1e_2^T - e^{-2\delta\tau_{2M}}e_5Q_2e_5^T \\
&-e^{-2\delta\tau_{2M}}(1 - d_{2M})e_4S_2e_4^T + e_{10}(\tau_{1M}^2R_1 + \tau_{2M}^2R_2)e_{10}^T \\
&-e^{-2\delta\tau_{1M}}\Gamma_1\Theta_1\Gamma_1^T - e^{-2\delta\tau_{2M}}\Gamma_2\Theta_2\Gamma_2^T \\
&+(e_1 + \lambda e_{10})(\mathcal{A}_c^T\mathcal{Z} + \mathcal{B}_c^T\mathcal{G} + \mathcal{Z}^T\mathcal{A}_c + \mathcal{G}^T\mathcal{B}_c) - 2\frac{\delta}{\overline{\omega}^2}e_{11}e_{11}^T,
\end{aligned} \tag{15}$$

then with the static output feedback controller $u(t) = GZ_{11}^{-1}y(t)$, every solution of the closed-loop system (8) converges within the given polyhedron $\overline{\Omega}$.

Proof: Consider the following Lyapunov-Krasovskii functional

$$V \;=\; V_1 + V_2 + V_3, \tag{16}$$

where

$$\begin{aligned}
V_1 =& \zeta_0^T(t)P\zeta_0(t), \\
V_2 =& \int_{t-\tau_{1M}}^{t} e^{2\delta(s-t)}z^T(s)F^TQ_1Fz(s)ds + \int_{t-\tau_1(t)}^{t} e^{2\delta(s-t)}z^T(s)F^TS_1Fz(s)ds \\
&+ \int_{t-\tau_{2M}}^{t} e^{2\delta(s-t)}z^T(s)F^TQ_2Fz(s)ds + \int_{t-\tau_2(t)}^{t} e^{2\delta(s-t)}z^T(s)F^TS_2Fz(s)ds, \\
V_3 =& \tau_{1M}\int_{-\tau_{1M}}^{0}\int_{v}^{0} e^{2\delta u}\dot{z}^T(t+u)F^TR_1F\dot{z}(t+u)dudv \\
&+ \tau_{2M}\int_{-\tau_{2M}}^{0}\int_{v}^{0} e^{2\delta u}\dot{z}^T(t+u)F^TR_2F\dot{z}(t+u)dudv.
\end{aligned}$$

Taking the derivatives of $V_i, i = 1, 2, 3$ in t, we have

$$\begin{aligned}
\dot{V}_1 + 2\delta V_1 \;=\;& 2\zeta_0^T(t)P\dot{\zeta}_0(t) + 2\delta\zeta_0^T(t)P\zeta_0(t) \\
=\;& \xi^T(t)\Big\{2\rho(t)P[e_{10} \quad e_1 - e_3 \quad e_1 - e_5]^T + 2\delta\rho(t)P\rho^T(t)\Big\}\xi(t),
\end{aligned} \tag{17}$$

$$\dot{V}_2 + 2\delta V_2 \;=\; z^T(t)F^T(Q_1 + S_1 + Q_2 + S_2)Fz(t) - e^{-2\delta\tau_{1M}}z^T(t - \tau_{1M})F^TQ_1Fz(t - \tau_{1M})$$

$$
\begin{aligned}
&\quad -e^{-2\delta\tau_1(t)}(1-\dot{\tau}_1(t))z^T(t-\tau_1(t))F^TS_1Fz(t-\tau_1(t))\\
&\quad -e^{-2\delta\tau_{2M}}z^T(t-\tau_{2M})F^TQ_2Fz(t-\tau_{2M})\\
&\quad -e^{-2\delta\tau_2(t)}(1-\dot{\tau}_2(t))z^T(t-\tau_2(t))F^TS_2Fz(t-\tau_2(t))\\
&\leq \xi^T(t)\Big\{e_1(Q_1+S_1+Q_2+S_2)e_1^T-e^{-2\delta\tau_{1M}}e_3Q_1e_3^T-e^{-2\delta\tau_{2M}}e_5Q_2e_5^T\\
&\qquad -e^{-2\delta\tau_{1M}}(1-d_{1M})e_2S_1e_2^T-e^{-2\delta\tau_{2M}}(1-d_{2M})e_4S_2e_4^T\Big\}\xi(t), \qquad (18)
\end{aligned}
$$

$$
\begin{aligned}
\dot{V}_3+2\delta V_3 &= \tau_{1M}^2\dot{z}^T(t)F^TR_1F\dot{z}(t)-\tau_{1M}\int_{t-\tau_{1M}}^{t}e^{2\delta(s-t)}\dot{z}^T(s)F^TR_1F\dot{z}(s)ds\\
&\quad +\tau_{2M}^2\dot{z}^T(t)F^TR_2F\dot{z}(t)-\tau_{2M}\int_{t-\tau_{2M}}^{t}e^{2\delta(s-t)}\dot{z}^T(s)F^TR_2F\dot{z}(s)ds\\
&\leq \xi^T(t)\Big\{e_{10}(\tau_{1M}^2R_1+\tau_{2M}^2R_2)e_{10}^T\Big\}\xi(t)\\
&\quad -\tau_{1M}e^{-2\delta\tau_{1M}}\Big\{\int_{t-\tau_1(t)}^{t}\dot{z}^T(s)F^TR_1F\dot{z}(s)ds+\int_{t-\tau_{1M}}^{t-\tau_1(t)}\dot{z}^T(s)F^TR_1F\dot{z}(s)ds\Big\}\\
&\quad -\tau_{2M}e^{-2\delta\tau_{2M}}\Big\{\int_{t-\tau_2(t)}^{t}\dot{z}^T(s)F^TR_2F\dot{z}(s)ds+\int_{t-\tau_{2M}}^{t-\tau_2(t)}\dot{z}^T(s)F^TR_2F\dot{z}(s)ds\Big\}.
\end{aligned}
\qquad (19)
$$

Using Lemma 2, we obtain the following estimation

$$
\begin{aligned}
&-\int_{t-\tau_1(t)}^{t}\dot{z}^T(s)F^TR_1F\dot{z}(s)ds\\
&\leq -\frac{1}{\tau_1(t)}\Big(z(t)-z(t-\tau_1(t))\Big)^TF^TR_1F\Big(z(t)-z(t-\tau_1(t))\Big)-\frac{12}{\tau_1(t)}\Big(\frac{z(t)}{2}+\frac{z(t-\tau_1(t))}{2}\\
&\quad -\frac{1}{\tau_1(t)}\int_{t-\tau_1(t)}^{t}z(s)ds\Big)^TF^TR_1F\Big(\frac{z(t)}{2}+\frac{z(t-\tau(t))}{2}-\frac{1}{\tau_1(t)}\int_{t-\tau_1(t)}^{t}z(s)ds\Big)\\
&= -\xi^T(t)\frac{1}{\tau_1(t)}\Big\{[e_1-e_2]R_1[e_1-e_2]^T+3[e_1+e_2-2e_6]R_1[e_1+e_2-2e_6]^T\Big\}\xi(t). \qquad (20)
\end{aligned}
$$

Similarly, we also obtain

$$
\begin{aligned}
&-\int_{t-\tau_{1M}}^{t-\tau_1(t)}\dot{z}^T(s)F^TR_1F\dot{z}(s)ds\\
&\leq -\xi^T(t)\frac{1}{\tau_{1M}-\tau_1(t)}\Big\{[e_2-e_3]R_1[e_2-e_3]^T+3[e_2+e_3-2e_7]R_1[e_2+e_3-2e_7]^T\Big\}\xi(t), \quad (21)
\end{aligned}
$$

$$
\begin{aligned}
&-\int_{t-\tau_2(t)}^{t}\dot{z}^T(s)F^TR_2F\dot{z}(s)ds\\
&\leq -\xi^T(t)\frac{1}{\tau_2(t)}\Big\{[e_1-e_4]R_2[e_1-e_4]^T+3[e_1+e_4-2e_8]R_2[e_1+e_4-2e_8]^T\Big\}\xi(t) \qquad (22)
\end{aligned}
$$

and

$$-\int_{t-\tau_{2M}}^{t-\tau_2(t)} \dot{z}^T(s)F^T R_2 F\dot{z}(s)ds$$
$$\leq -\xi^T(t)\frac{1}{\tau_{2M}-\tau_2(t)}\Big\{[e_4-e_5]R_2[e_4-e_5]^T + 3[e_4+e_5-2e_9]R_2[e_4+e_5-2e_9]^T\Big\}\xi(t). \quad (23)$$

Adding (19)-(23), using (13) and Lemma 3, we obtain

$$\dot{V}_3 + 2\delta V_3 \leq \xi^T(t)\Big\{e_{10}(\tau_{1M}^2 R_1 + \tau_{2M}^2 R_2)e_{10}^T - e^{-2\delta\tau_{1M}}\Gamma_1\Theta_1\Gamma_1^T - e^{-2\delta\tau_{2M}}\Gamma_2\Theta_2\Gamma_2^T\Big\}\xi(t). \quad (24)$$

Combining (10) with the free-weighting matrix technique [29, 30], we have

$$2\xi^T(t)(e_1 + \lambda e_{10})[\mathcal{A}_c^T\mathcal{Z} + \mathcal{B}_c^T\mathcal{G}]\xi(t) = 0. \quad (25)$$

By adding (17), (18), (24) and (25), we obtain

$$\dot{V}(t) + 2\delta V(t) - 2\frac{\delta}{\overline{\omega}^2}\widetilde{\omega}^T(t)\widetilde{\omega}(t) \leq \xi^T(t)\Sigma(\tau_1,\tau_2,\delta)\xi(t). \quad (26)$$

By some simple computations, we can verify that

$$\Big[\frac{\partial^2}{\partial\tau_i^2}\Sigma(\tau_1,\tau_2,\delta)\Big] \geq 0,\ i = 1, 2. \quad (27)$$

Consequently, $\Sigma(\tau_1,\tau_2,\delta)$ is convex with respect to τ_1 and τ_2. Hence, if condition (14) holds then we have

$$\dot{V}(t) + 2\delta V(t) - 2\frac{\delta}{\overline{\omega}^2}\widetilde{\omega}^T(t)\widetilde{\omega}(t) \leq 0,\ \forall t \geq 0. \quad (28)$$

This follows that $\limsup_{t\to\infty} V(t) \leq 1$ due to Lemma 1. On the other hand, using (11) and (12), we have

$$z^T(t)\Upsilon_j z(t) \leq z^T(t)F^T P_j F z(t) \leq V(t),\ j = 1, \cdots, q.$$

This implies that inequality (9) holds. The proof of Theorem 1 is completed. □

Remark 1: Note that for each $j = 1, \cdots, q$, matrix inequality (12) is a quadratic matrix inequality. By using singular value decomposition technique, we can reduce matrix inequalities (12) to linear but more conservative matrix inequalities. Indeed, for each $j = 1, \cdots, q$, assuming that $[U_j, Y_j, V_j]$ is a singular value decomposition of matrix $L_j H$, then $H^T L_j^T L_j H = V_j Y_j^T Y_j V_j^T$.

Since $L_j H \in \mathbb{R}^{1\times n}$ is a non-zero matrix, matrix Y_j has a form $Y_j = [s_j \;\; 0 \;\; \cdots \;\; 0] \in \mathbb{R}^{1\times n}$, where s_j is a non-zero scalar. This implies that

$$\Upsilon_j = V_j \begin{bmatrix} \frac{s_j^2}{b_j^2} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} V_j^T.$$

For a small positive scalar $\epsilon > 0$, we denote

$$\Upsilon_j^\epsilon = V_j \begin{bmatrix} \frac{s_j^2}{b_j^2} & 0 & 0 & \cdots & 0 \\ 0 & \epsilon & 0 & \cdots & 0 \\ 0 & 0 & \epsilon & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \epsilon \end{bmatrix} V_j^T.$$

Then Υ_j^ϵ is positive-definite matrix and $\Upsilon_j^\epsilon \geq \Upsilon_j$. Hence, matrix inequalities (12) can be replaced by more conservative matrix inequalities $P_j - Z^T \Upsilon_j^\epsilon Z > 0$, $j = 1, \cdots, q$. These matrix inequalities are equivalent to the following linear matrix inequalities due to Lemma 4

$$\begin{bmatrix} P_j & Z^T \\ Z & (\Upsilon_j^\epsilon)^{-1} \end{bmatrix} > 0, \;\; j = 1, \cdots, q. \tag{29}$$

Also note that matrix inequality (14) cannot be simplified into linear matrix inequality (LMI). However, when λ and δ are fixed, then (14) is reduced to LMI. Therefore, we can now use a two-dimensional search method combining with convex optimization algorithms such as the Matlab's LMI toolbox [31] to solve matrix inequalities (11), (29), (13) and (14). Note that the two parameters λ and δ are independent, hence in practice one can use parallel computing to find the two feasible parameters. Furthermore, parameter δ is the exponential rate, therefore it is positive and finite, i.e. it belongs to an interval. This helps to reduce partly the difficulty in searching for the two feasible parameters. On the other hand, the appropriate combination between a state transformation (7) with the choice of a special form of matrices $\widetilde{G}$ and Z allows us to obtain an output feedback control matrix $K = GZ_{11}^{-1}$.

Remark 2: Since the two time-varying delays considered in this paper are independent, the Lyapunov-Krasovskii functional (16) must be constructed by using different matrices for each delay. In this paper, we use the Wirtinger integral and the reciprocally convex combination inequality, which are known as recent effective techniques with moderate variables. Therefore, the number of variables in our derived conditions is moderate. However, the number of variables can be reduced for the following three special cases: (i) $\tau_1(t) \equiv \tau_2(t)$; (ii) $\tau_{1M} = \tau_{2M}$; and (iii) $\tau_1(t)$, $\tau_2(t)$ are non-differentiable or their derivatives are unknown. For case (i), we let $Q_1 = Q_2$, $R_1 = R_2$ and $S_1 = S_2$. For case (ii), we let $Q_1 = Q_2$, $R_1 = R_2$. Finally, for case (iii), we let $S_1 = S_2 = 0$.

Remark 3: Note that $\Sigma(\tau_1, \tau_2, \delta)$ is convex with respect to τ_1 and τ_2. It follows that if the condition (14) holds for $\forall(\tau_1, \tau_2) \in \{0, \tau_{1M}\} \times \{0, \tau_{2M}\}$ then it also holds for $\forall(\tau_1, \tau_2) \in \{0, \widetilde{\tau}_{1M}\} \times \{0, \widetilde{\tau}_{2M}\}$ where $\widetilde{\tau}_{1M} \leq \tau_{1M}$ and $\widetilde{\tau}_{2M} \leq \tau_{2M}$. This means that the condition (14) is monotonic increasing with respect to the delays' bounds τ_{1M} and τ_{2M}. So, we can use a two-dimensional search to calculate the maximum allowable values of the delays' bounds τ_{1M} and τ_{2M}.

Remark 4: The assumption that the derivatives of the time-varying delays are less than one is usually referred to as slow time-varying delays. For the case where the time-delays are non-differentiable or their derivatives are unknown, then this assumption is not needed and can be removed. By letting $S_1 = S_2 = 0$ and by following the same lines as in the proof of Theorem 1, we can obtain a similar result. Note that this result is more conservative than the one derived with the assumption that the derivatives of the time-varying delays are less than one.

Remark 5: For the case where the initial condition is zero, then $V(0) = 0$. Consequently, from the proof of Lemma 1, we have $V(t) \leq 1,\ \forall t \geq 0$. Similar to the proof of Theorem 1, we obtain

$$|L_j H z(t, 0)| \leq b_j, \forall t \geq 0, \forall j = 1, \cdots, q.$$

Hence, the condition stated in Theorem 1 guarantees that all reachable sets of the closed-loop system (8) are bounded by the polyhedron Ω for all time.

For the case where the system (1) does not have any disturbance, by setting $\omega(t) \equiv 0$ and $D = 0$, then the convergence condition in Theorem 1 is reduced to a static output feedback exponential stabilizability condition for system (1). Here, let us recall the definition of exponential stabilizability of system (1).

Definition 1: Given a positive scalar $\delta > 0$, system (1) without any disturbance is δ-*stabilizable* with a static output feedback controller (4) if every solution $x(t, \phi)$ of the closed-loop system (5) satisfies

$$\exists N > 0 : \quad \|x(t, \phi)\| \leq N\|\phi\|_c e^{-\delta t}, \quad \forall t \geq 0. \tag{30}$$

The positive scalars δ and N are called the convergence rate and the stability factor, respectively.

Remark 6: From the state transformation (7), it is easy to see that

$$\lambda_1||x(t)||^2 \leq ||z(t)||^2 \leq \lambda_2||x(t)||^2 \tag{31}$$

where $\lambda_1 = \lambda_{min}\Big((H^{-1})^T(H^{-1})\Big)$ and $\lambda_2 = \lambda_{max}\Big((H^{-1})^T(H^{-1})\Big)$. This implies that if system (8) is δ-stablizable with stability factor N then system (1) is also δ-stablizable with stability factor $\sqrt{\frac{\lambda_2}{\lambda_1}}||H^{-1}||N$. Therefore, to study δ-stabilizability for system (1), we only need to study δ-stabilizability for system (8).

Let us denote that

$$\begin{aligned}
\Xi(\tau_1, \tau_2, \delta) \;=\;& 2\rho(t)P[e_{10} \quad e_1 - e_3 \quad e_1 - e_5]^T + 2\delta\rho(t)P\rho^T(t) \\
&+e_1(Q_1 + S_1 + Q_2 + S_2)e_1^T - e^{-2\delta\tau_{1M}}e_3Q_1e_3^T \\
&-e^{-2\delta\tau_{1M}}(1 - d_{1M})e_2S_1e_2^T - e^{-2\delta\tau_{2M}}e_5Q_2e_5^T \\
&-e^{-2\delta\tau_{2M}}(1 - d_{2M})e_4S_2e_4^T + e_{10}(\tau_{1M}^2R_1 + \tau_{2M}^2R_2)e_{10}^T \\
&-e^{-2\delta\tau_{1M}}\Gamma_1\Theta_1\Gamma_1^T - e^{-2\delta\tau_{2M}}\Gamma_2\Theta_2\Gamma_2^T \\
&+(e_1 + \lambda e_{10})(\mathcal{A}_c^T\mathcal{Z} + \mathcal{B}_c^T\mathcal{G} + \mathcal{Z}^T\mathcal{A}_c + \mathcal{G}^T\mathcal{B}_c).
\end{aligned}$$

Similarly, we also get a sufficient condition for δ-stabilizability of system (8) via a static output feedback controller (4) as follow

Theorem 2: For a given positive scalar $\delta > 0$, if there exist a scalar λ, a positive-definite $3n\times 3n$-matrix P, six positive-definite $n \times n$-matrices Q_1, Q_2, S_1, S_2, R_1, R_2, two $2n \times 2n$-matrices X_1, X_2, two nonsingular matrices $Z_{11} \in \mathbb{R}^{p\times p}$, $Z_{22} \in \mathbb{R}^{(n-p)\times(n-p)}$, and two matrices $Z_{21} \in \mathbb{R}^{(n-p)\times p}$, $G \in \mathbb{R}^{m\times p}$ such that condition (13) and the following matrix inequality hold

$$\Xi(\tau_1, \tau_2, \delta) \leq 0, \;\; \forall(\tau_1, \tau_2) \in \{0, \tau_{1M}\} \times \{0, \tau_{2M}\}, \tag{32}$$

then system (8) without any disturbance is δ-stabilizable. The static output feedback controller is $u(t) = GZ_{11}^{-1}y(t)$. Moreover, every solution of the closed-loop system satisfies

$$\|z(t,\varphi)\| \leq \sqrt{\frac{\beta_2}{\beta_1}}\|\varphi\|_c e^{-\delta t}, \ \forall t \geq 0, \tag{33}$$

where

$$\begin{aligned}
\beta_1 =& \lambda_{min}(\text{diag}\{F^T, F^T, F^T\} \times P \times \text{diag}\{F, F, F\}), \\
\beta_2 =& (1+\tau_{1M}^2+\tau_{2M}^2)\lambda_{max}(\text{diag}\{F^T, F^T, F^T\} \times P \times \text{diag}\{F, F, F\}) \\
&+ \tau_{1M}\lambda_{max}\Big(F^T(Q_1+S_1)F\Big) + \tau_{2M}\lambda_{max}\Big(F^T(Q_2+S_2)F\Big) \\
&+ \frac{\tau_{1M}^3}{2}\lambda_{max}(F^T R_1 F) + \frac{\tau_{2M}^3}{2}\lambda_{max}(F^T R_2 F).
\end{aligned}$$

Proof: Also consider the Lyapunov-Krasovskii functional (16) and similarly, we obtain

$$V(t) \leq V(0)e^{-2\delta t}, \ \forall t \geq 0. \tag{34}$$

Denoting $\zeta_1^T(t) = [z^T(t) \quad \int_{t-\tau_{1M}}^t z^T(s)ds \quad \int_{t-\tau_{2M}}^t z^T(s)ds]$. Note that for all $s \in [-h, 0]$, we have $||z(t+s)||^2 \leq ||z_t||_c^2$ and $||\dot{z}(t+s)||^2 \leq ||z_t||_c^2$. By some computations, we have

$$\begin{aligned}
||\zeta_1(t)||^2 &\leq ||z(t)||^2 + \tau_{1M}\int_{t-\tau_{1M}}^t ||z(s)||^2 ds + \tau_{2M}\int_{t-\tau_{2M}}^t ||z(s)||^2 ds \\
&\leq (1+\tau_{1M}^2+\tau_{2M}^2)||z_t||_c^2
\end{aligned} \tag{35}$$

and

$$||z(t)||^2 \leq ||\zeta_1(t)||^2. \tag{36}$$

Combining (34), (35) with (36), we obtain the following inequality

$$\beta_1||z(t)||^2 \leq \beta_1||\zeta_1(t)||^2 \leq V(t) \leq \beta_2||z_t||_c^2, \ \forall t \geq 0, \tag{37}$$

which implies inequality (33). This completes the proof of Theorem 2. □

Remark 7: (Minimization stability factor) From (37), the stability factor is $N = \sqrt{\frac{\beta 2}{\beta 1}}$. When matrices P and F are found, we can further find a scalar $\alpha_1 \geq \beta_1$ such that $\alpha_1||z(t)||^2 \leq V(t)$. To find α_1, we can use an one-dimensional search method for the following inequality

$$\alpha_1\begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \leq \text{diag}\{F^T, F^T, F^T\} \times P \times \text{diag}\{F, F, F\}. \tag{38}$$

Hence, the stability factor N can be reduced to a minimal one $N_1 = \sqrt{\frac{\beta_2}{\alpha_1}}$. In Example 3 of the next section, we show that N_1 is smaller than N.

Remark 8: Consider an extended system of (1) as follows

$$\dot{x}(t) = Ax(t) + A_1x(t-\tau_1(t)) + A_2x(t-\tau_2(t)) + Bu(t) + B_2u(t-\tau_2(t)). \tag{39}$$

By re-notating $\mathcal{A}_c^T = [\overline{A}\ \ \overline{A}_1\ \ 0_{n\times n}\ \ \overline{A}_2\ \ 0_{n\times 5n} - I_n\ 0_{n\times n}] \in \mathbb{R}^{n\times 11n}$, with $\overline{A}_2 = H^{-1}A_2H$, then the result in Theorem 2 also gives a δ-stabilizability criterion for system (39) via a static output feedback controller (4). Note that the authors in [3–5, 7] only considered the case where in system (39), $A_1 = 0$ and τ_2 is a constant time delay (i.e, only one constant time delay in both the state and the control input).

Remark 9: Assume that the matrix inequality $\Xi(\tau_1, \tau_2, 0) < 0$ holds. Since $\rho(t)$ is bounded, we can choose a small enough scalar $\delta_0 > 0$ such that $\Xi(\tau_1, \tau_2, \delta_0) < 0$. Hence, we have an asymptotic stabilizability criterion for system (8) via static output feedback controller (4) as given in the following corollary.

Corollary 1: System (8) without any disturbance is asymptotically stabilizable via a static output feedback controller (4) if $\Xi(\tau_1, \tau_2, 0) < 0$ and (13) hold.

4 Numerical examples

In this section, we give five examples to illustrate the feasibility and effectiveness of our results on static output feedback control for two cases: (i) in the presence of bounded disturbances; and (ii) no disturbances. For the case (i), we will design a static output feedback controller, which guarantees the state vector of the closed-loop system converges within a pre-specified polyhedron Ω (Example 1 and Example 2). For the case (ii), we will design a static output feedback controller which guarantees δ-stability of the closed-loop system (Example 3, Example 4 and Example 5).

Example 1: (Convergence condition) Consider system (1) in the presence of disturbances $\omega(t)$, which is bounded by $\overline{\omega} = 0.3$, and

$$A = \begin{bmatrix} -1 & 0 & 0.3 & 0 \\ -0.1 & 0.2 & 1 & 0 \\ -0.3 & 0.1 & -2 & 0.2 \\ 0 & 0 & 0 & -1.2 \end{bmatrix}, \; A_1 = \begin{bmatrix} -2 & -0.1 & 0 & -0.2 \\ -0.2 & 0.3 & 0.3 & 0 \\ 0.1 & 0 & -2 & -0.2 \\ 0 & 0 & 0 & 0.1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, D = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, C = \begin{bmatrix} -1 & -1 & 2 & 0.2 \\ 0.2 & 1 & 0.3 & 1 \end{bmatrix}.$$

The two time-varying delays, $\tau_1(t)$ and $\tau_2(t)$ satisfying

$$\begin{cases} 0 \leq \tau_1(t) \leq 0.5, \; \dot{\tau}_1(t) \leq 0.05, \\ 0 \leq \tau_2(t) \leq 0.6, \; \dot{\tau}_2(t) \leq 0.05. \end{cases} \tag{40}$$

Given a polyhedron $\Omega = \{x \in \mathbb{R}^4 : |L_j x| \leq b_j, \; j = 1, 2\}$ where $L_1 = [1 \;\; 1 \;\; 0 \;\; 0]$, $L_2 = [0 \;\; 0 \;\; 1 \;\; 1]$, $b_1 = 0.2, b_2 = 0.1$. We design a static output feedback controller, which guarantees the state vector of the closed-loop system converges within the given polyhedron Ω.

By solving the linear matrix inequalities (11), (13), (14) and (29) with $\epsilon = 0.01$ and two parameters need to be turned δ and λ, we obtain $\delta = 0.1$, $\lambda = 0.48$ and a static output feedback control matrix $K = [0.3581 \;\; -0.7808]$. For a disturbance $\omega(t) = 0.3\sin(t)$, two time-varying delays $\tau_1(t) = 0.5\sin^2(\frac{t}{10})$ and $\tau_2(t) = 0.6\sin^2(\frac{t}{12})$, Figure 2 shows that the trajectory of $L_1 x(t) = x_1(t) + x_2(t)$ of the closed-loop system converges within the specified 0.2-bound, and $L_2 x(t) = x_3(t) + x_4(t)$ converges within the specified 0.1-bound. Also, Figure 3 shows that the vector $(L_1 x(t), L_2 x(t))$ converges within the rectangular with dimensions 0.4×0.2.

Example 2: (Convergence condition) Consider a three-dimensional system (1) with disturbances $\omega(t)$ is bounded by $\overline{\omega} = 0.2$, and

$$A = \begin{bmatrix} -0.8 & 0.1 & 0.5 \\ -0.1 & 0.2 & 1 \\ -0.3 & 0.1 & -2 \end{bmatrix}, \; A_1 = \begin{bmatrix} 0.1 & -0.1 & 0 \\ -0.2 & 0.3 & 0.3 \\ 0.1 & 0 & 1.5 \end{bmatrix},$$

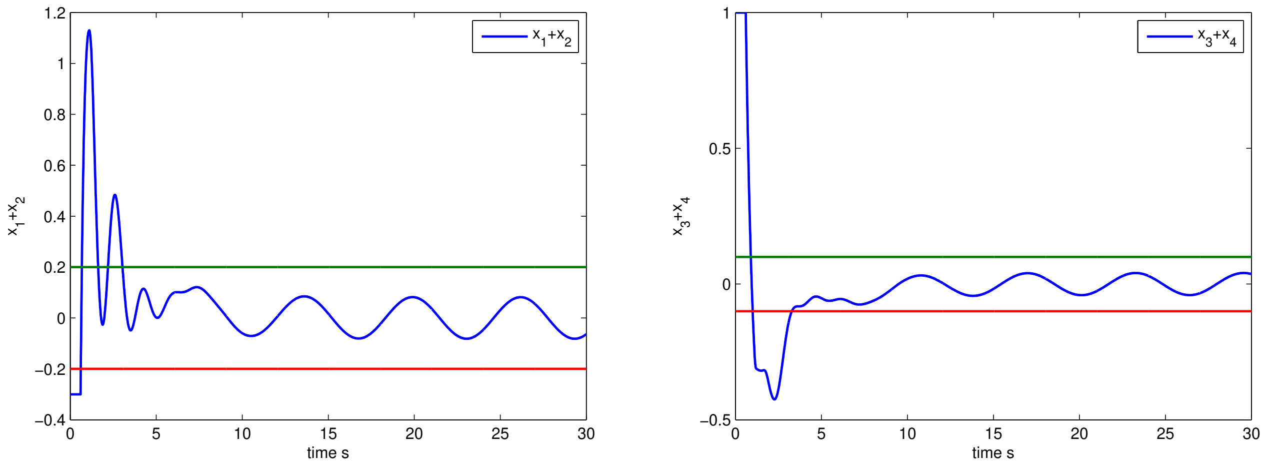

Figure 2: Trajectories of $x_1(t) + x_2(t)$ and $x_3(t) + x_4(t)$ of the closed-loop system.

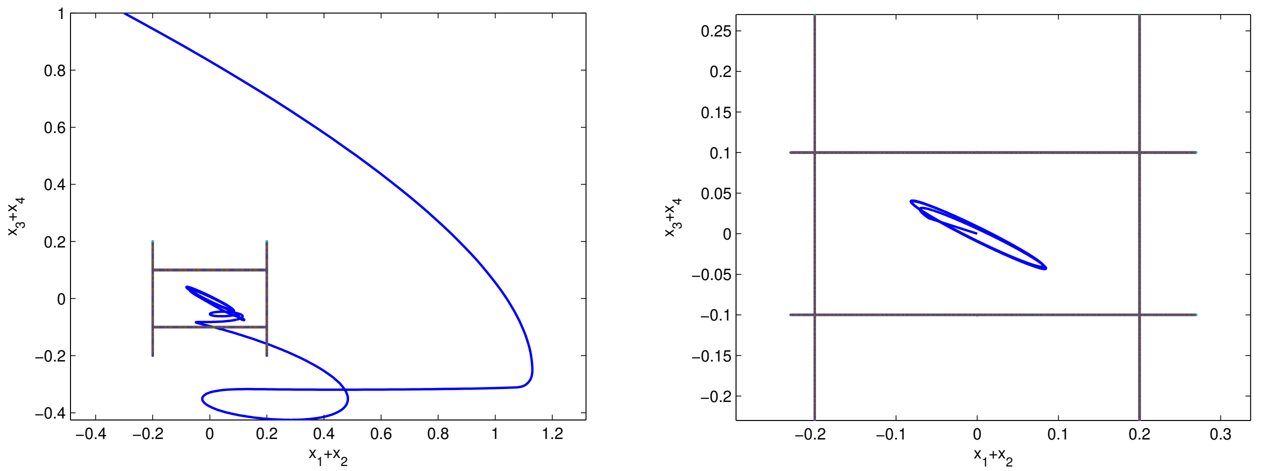

Figure 3: Trajectory of $(L_1x(t), L_2x(t))$ converges within a 0.4×0.2 rectangular.

$$B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, B_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, D = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The two time-varying delays $\tau_1(t)$ and $\tau_2(t)$ satisfying

$$\begin{cases} 0 \leq \tau_1(t) \leq 1, \ \dot{\tau}_1(t) \leq 0.1, \\ 0 \leq \tau_2(t) \leq 0.6, \ \dot{\tau}_2(t) \leq 0.1. \end{cases} \tag{41}$$

Given a box $\Omega = \{x \in \mathbb{R}^3 : |x_j| \leq b_j, \ j = 1, 2, 3\}$ where $b_1 = b_2 = 0.2, b_3 = 0.05$. We design a static output feedback controller which guarantees the state vector of the closed-loop system

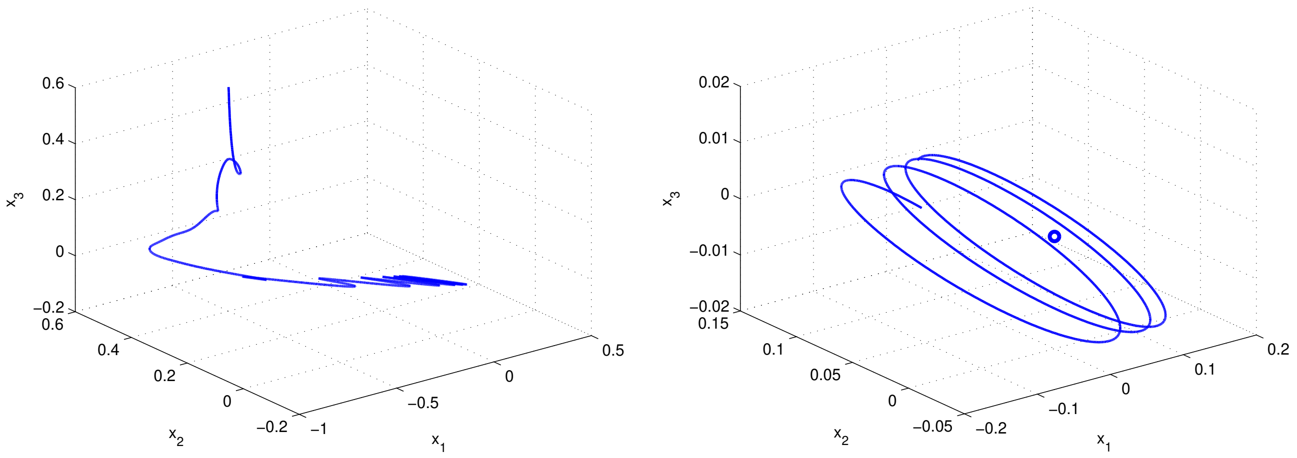

Figure 4: Trajectory of the closed-loop system, $x(t)$, converges within the given box $\Omega = \{x \in \mathbb{R}^3 : |x_j| \leq b_j,\ j = 1, 2, 3\}$.

converges within the given box Ω.

By solving the linear matrix inequalities (11), (13), (14) and (29) with $\epsilon = 0.01$ and two parameters need to be turned δ and λ, we obtain $\delta = 0.05$, $\lambda = 0.23$ and a static output feedback control matrix $K = [-0.8316 \quad -2.3089]$. For a disturbance $\omega(t) = 0.2\sin(t)$, Figure 4 shows that the trajectories of the closed-loop system converges within the given box Ω.

Example 3: (Static output feedback control) Consider the system in Example 1, where there are no disturbances, i.e., $\omega(t) \equiv 0$, and two time-varying delays in both the state and input satisfying

$$\begin{cases} 0 \leq \tau_1(t) \leq 0.5,\ \dot{\tau}_1(t) \leq 0.1, \\ 0 \leq \tau_2(t) \leq \tau_{2M},\ \dot{\tau}_2(t) \leq 0.1. \end{cases} \tag{42}$$

In this example, we find the maximal allowable delay τ_{2M} such that the system is 0.1-stabilizable via a static output feedback controller.

By using Theorem 2 with a pre-specified convergence rate $\delta = 0.1$, the allowable value of τ_{2M} is found to be 1.47. The output feedback control matrix and parameter are $K = [0.2117 \ \ -0.3928]$ and $\lambda = 1.14$, respectively. By Theorem 2 and Remark 7, the stability factor is $N = 44.6578$ and the minimal value is $N_1 = 13.7191$. Moreover, we have $\sqrt{\frac{\lambda 2}{\lambda_1}}||H^{-1}|| = 6.0805$,

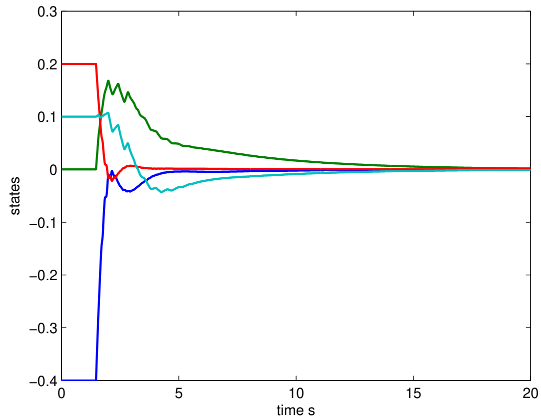

Figure 5: Trajectories of the closed-loop system in Example 3

which implies the following estimation

$$||x(t)|| \leq \left(\sqrt{\frac{\lambda 2}{\lambda_1}}||H^{-1}||\right)N_1||\varphi||_c e^{-0.1t} \leq 83.4191||\varphi||_c e^{-0.1t}, \quad \forall t \geq 0.$$

Figure 4 shows trajectories of the closed-loop system where two time-varying delays are chosen as $\tau_1(t) = 0.5\sin^2(\frac{t}{5})$ and $\tau_2(t) = 1.47\sin^2(\frac{t}{14.7})$.

Example 4: (State feedback control) Consider system (1), which was studied in [2], with two unknown constant delays in the state and input and

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ -0.5 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; A_1 = \begin{bmatrix} -2 & -0.5 & 0 & 0 \\ -0.2 & -1 & 0 & 0 \\ 0.5 & 0 & -2 & -0.5 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

By using Corollary 1 with $\tau_{2M} = 0.1$, the allowable value of τ_{1M}, which ensures system is asymptotically stabilizable, is 0.77, while Theorem 4 in [2] provided a smaller value, 0.56. The state feedback control matrix and parameter are obtained as $K = [-5.0329 \;\; -1.9171 \;\; 1.5028 \;\; -0.4175]$ and $\lambda = 1.42$. Note that the approaches in [3–5, 7] are available for linear systems with only one delay and the approach in [8] is available for linear systems without instantaneous input (i.e. $B = 0$). Therefore, the approaches [3–5, 7, 8] can not be applied to this example.

Example 5: (State feedback control) Consider a pendulum system (1), which was studied in [7], with time delay τ_2 is an unknown constant and $A_1 = A_2 = 0,\ B = 0$,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -21.54 & 0 & 14.96 & 0 \\ 0 & 0 & 0 & 1 \\ 65.28 & 0 & -15.59 & 0 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ 8.10 \\ 0 \\ -10.31 \end{bmatrix}.$$

In this example, the allowable values for τ_{2M} are derived in Table 1. The state feedback control matrix and parameter are $K = [-4.9687\ \ -1.4262\ \ -2.7016\ \ -0.7382]$ and $\lambda = 0.86$.

Table 1: Computed upper bounds, τ_{2M}, for Example 5

Methods	τ_{2M}	improvement (%)
Fridman *et al.* [35]	0.0384	100 (%)
Du *et al.* [7]	0.0768	200 (%)
Theorem 2	0.2130	554 (%)

5 Conclusion

The paper has considered the problem of designing a static output feedback controller for linear systems with state/input time-varying delays and bounded disturbances. A new sufficient condition for the existence of a static output feedback controller, which guarantees the state vector of the closed-loop system converges within a pre-specified polyhedron, has been derived. For the case where the disturbances are not present, the derived convergence condition is reduced to a static output feedback exponential stabilizability condition. Five numerical examples have been given to illustrate the feasibility and the effectiveness of the obtained results.

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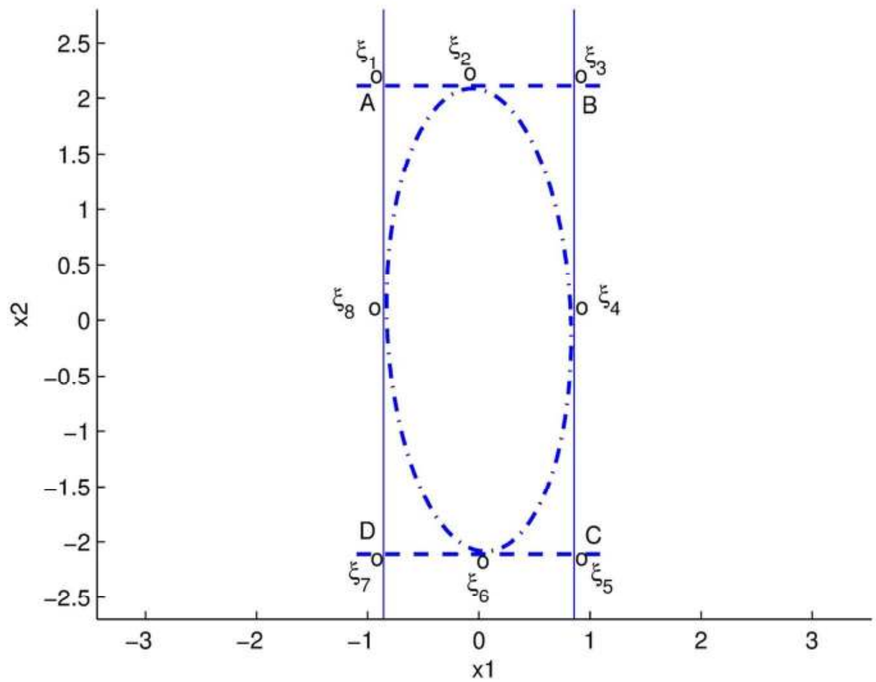

Figure 1: Rectangle $ABCD$ and ellipse $\epsilon(P)$
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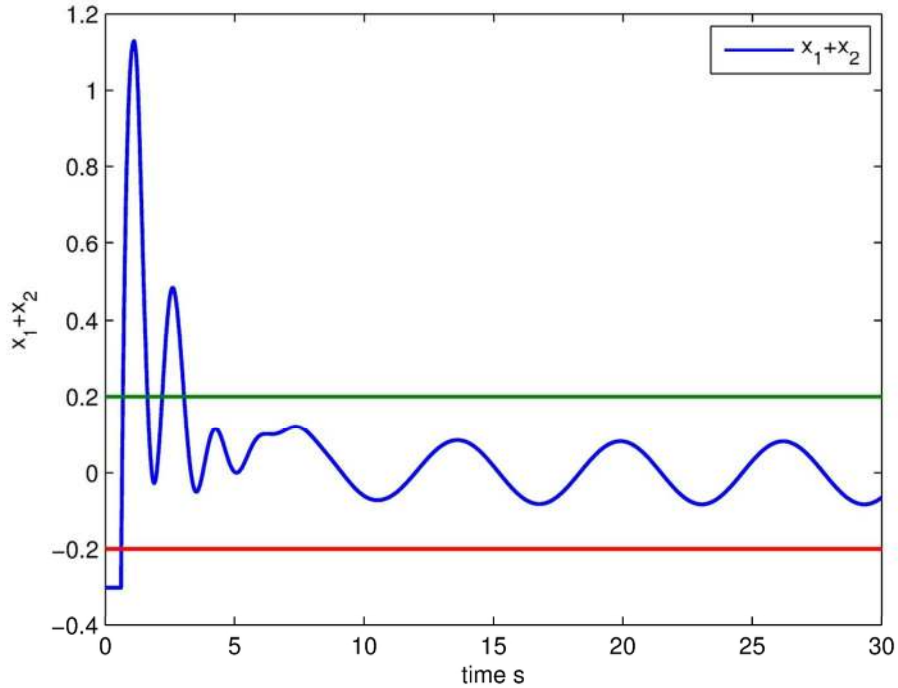

Figure 2: Trajectories of $x_1(t)+x_2(t)$ and $x_3(t)+x_4(t)$ of the closed-loop system.
111x83mm (300 x 300 DPI)

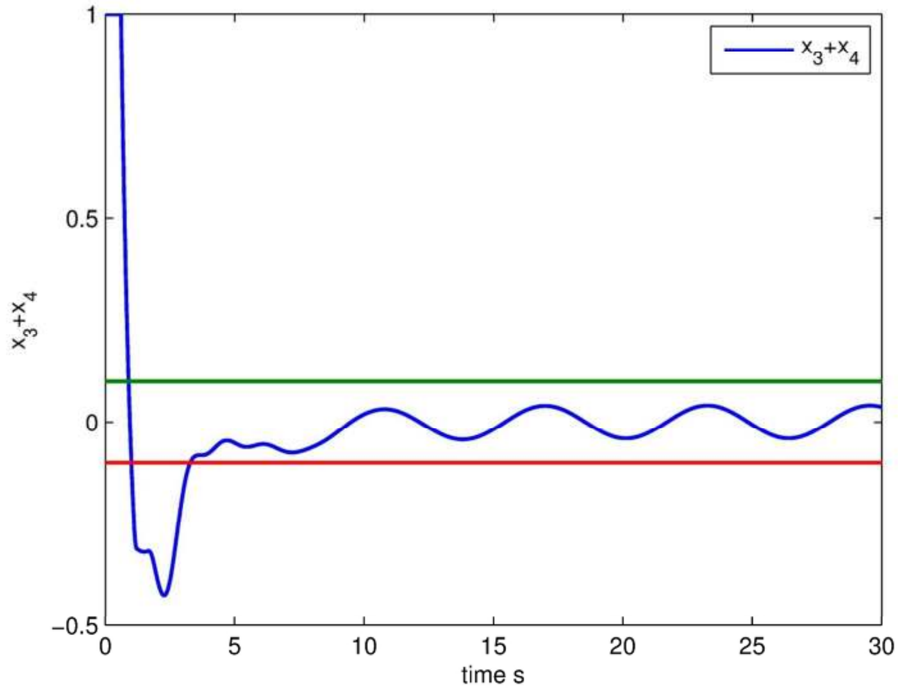

Figure 2: Trajectories of $x_1(t)+x_2(t)$ and $x_3(t)+x_4(t)$ of the closed-loop system.
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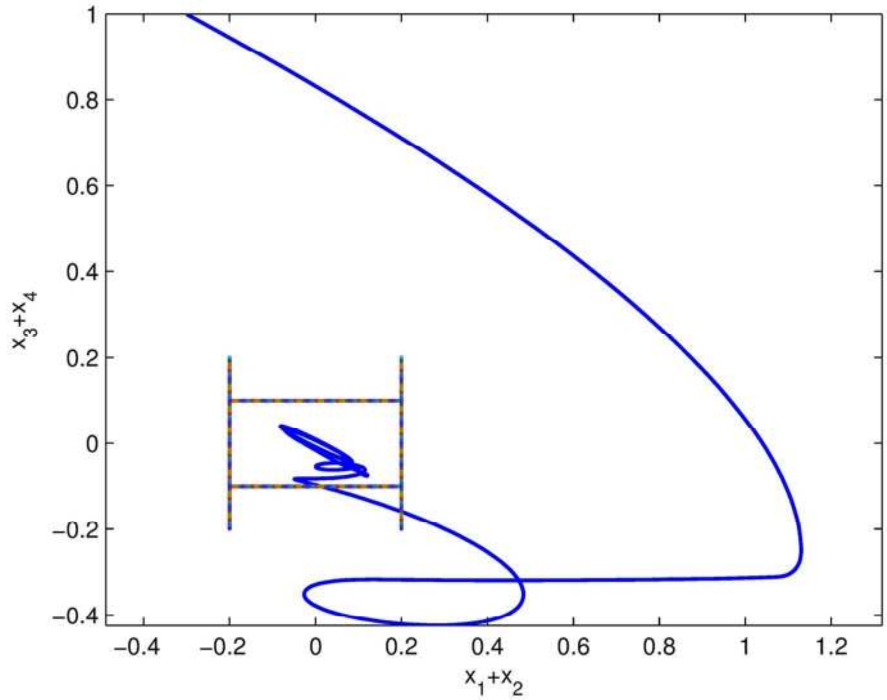

Figure 3: Trajectory of $(L_1x(t),L_2x(t))$ converges within a 0.4×0.2 rectangular.
111x83mm (300 x 300 DPI)

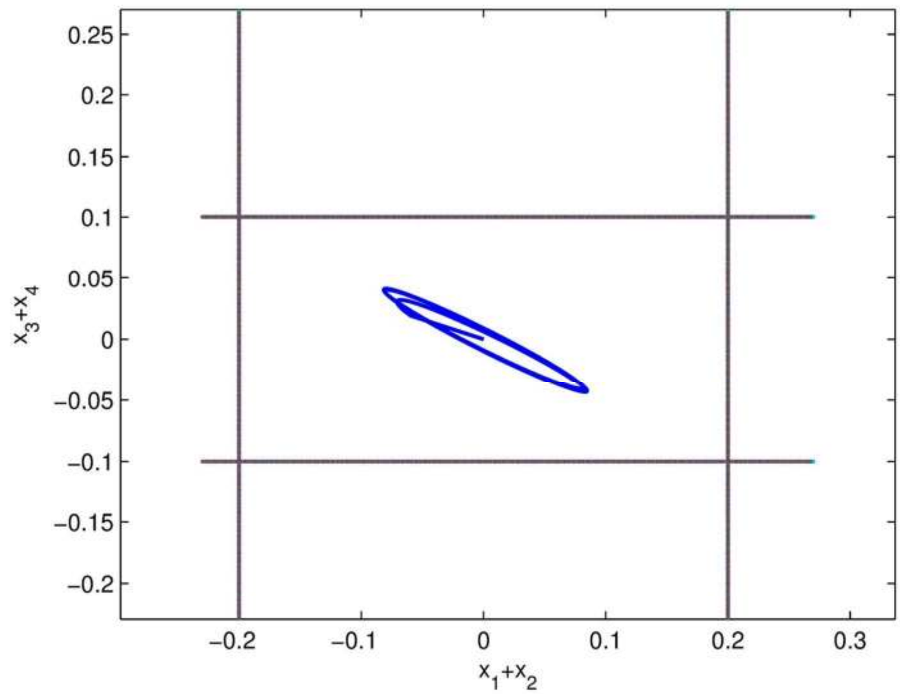

Figure 3: Trajectory of \$(L_1x(t),L_2x(t))\$ converges within a \$0.4\times 0.2\$ rectangular.
111x83mm (300 x 300 DPI)

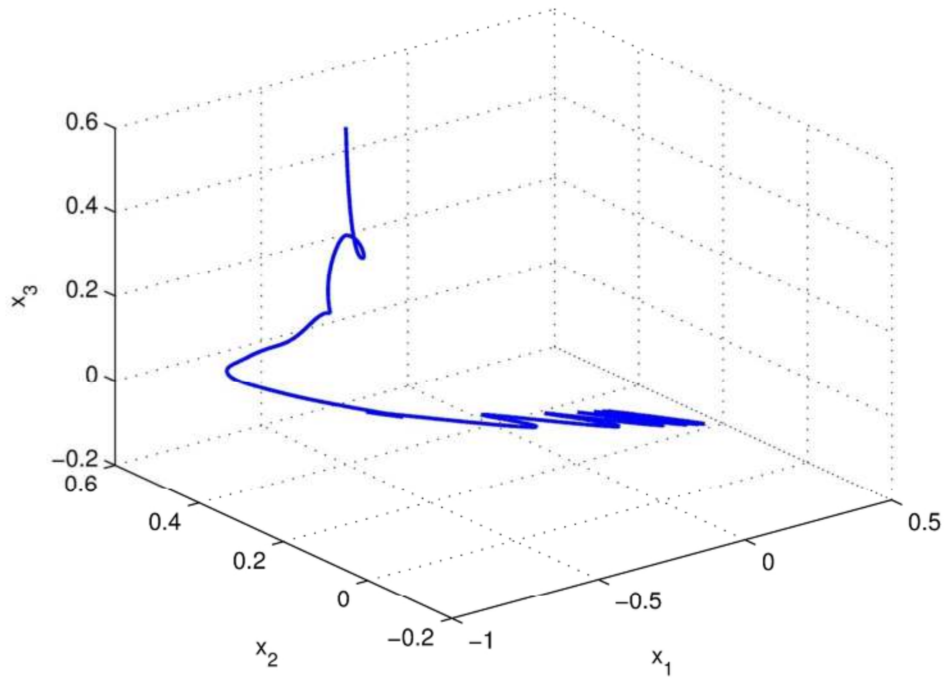

Figure 4: Trajectory of the closed-loop system, $x(t)$,
converges within the given box $\Omega=\{x\in \mathbb{R}^3: |x_j|\leq b_j, \ j=1,2,3\}$
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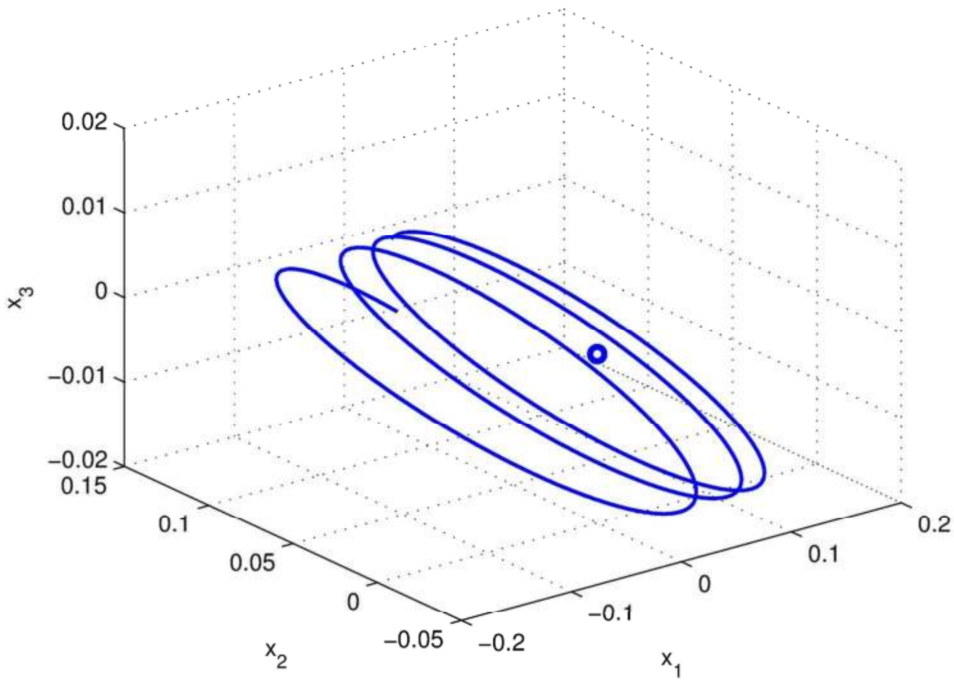

Figure 4: Trajectory of the closed-loop system, $x(t)$,
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111x83mm (300 x 300 DPI)

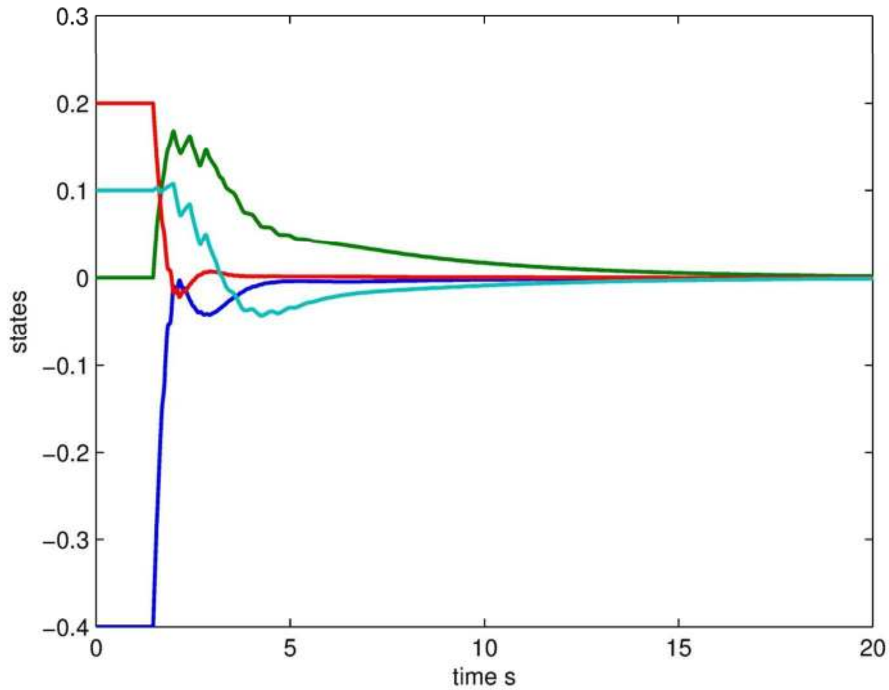

Figure 5: Trajectories of the closed-loop system in Example 3
111x83mm (300 x 300 DPI)