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Convergences of a stage-structured predator-prey model with modified Leslie-Gower and Holling-type II schemes

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Abstract

A stage-structured predator-prey model (stage structure for both predator and prey) with modified Leslie-Gower and Holling-II schemes is studied in this paper. Using the iterative technique method and the fluctuation lemma, sufficient conditions which guarantee the global stability of the positive equilibrium and boundary equilibrium are obtained. Our results indicate that for a stage-structured predator-prey community, both the stage structure and the death rate of the mature species are the important factors that lead to the permanence or extinction of the system.

MSC: 34D23; 92B05; 34D40

Keywords: global stability; extinction; stage-structure; Leslie-Gower; Holling-type II; predator-prey

1 Introduction

Since the pioneer work of Aiello and Freedman [1], the stage-structured population models have been investigated extensively and many excellent results have been obtained (see [1–35]). Recently, Huo *et al.* [5] considered a stage-structured predator-prey model with modified Leslie-Gower and Holling-type II schemes as follows:

$$\begin{aligned} x_1'(t) &= r_1 x_2(t) - d_{11} x_1(t) - r_1 e^{-d_{11}\tau_1} x_2(t-\tau_1), \\ x_2'(t) &= r_1 e^{-d_{11}\tau_1} x_2(t-\tau_1) - b x_2^2(t) - \frac{a_1 y(t) x_2(t)}{x_2(t)+k_1}, \\ y'(t) &= y(t)\left(r_2 - \frac{a_2 y(t)}{x_2(t)+k_2}\right), \end{aligned} \tag{1.1}$$

where x_1, x_2, and y represent the population densities of immature prey, mature prey and predator at time t, respectively; r_1 is the birth rate of immature prey x_1; d_{11} denotes the death rate of the immature prey x_1; r_2 is the intrinsic growth rate of predator y; b represents the strength of intra-specific competition in the mature prey; a_1 represents the maximum value that mature x_2 can be captured by predator y, and the meaning of a_2 is similar to a_1; k_1 and k_2 measure the protection degree that the environment could afford for prey x_2 and predator y, respectively; τ_1 is the time to maturity for prey; $r_1 e^{-d_{11}\tau_1} x_2(t-\tau_1)$ represents the prey who were born at time $t-\tau_1$ and survive and become mature at time t.

In [5], the authors analyzed the dynamics of system (1.1), specially, by using the iterative technique, the authors obtained a set of sufficient conditions which guarantee the existence of a unique globally attractive positive equilibrium. Li *et al.* [6] found that some of the conditions in [5] are redundant, and they obtained the following result.

Theorem A *Suppose that*

$$\text{(H)}\quad \lambda_0 = r_1 e^{-d_{11}\tau_1} a_2 k_1 b - a_1 k_2 r_2 b - a_1 r_2 r_1 e^{-d_{11}\tau_1} > 0$$

holds, then the system (1.1) *has a unique globally attractive positive equilibria E.*

As we can see, [5] and [6] only considered the stage structure of immature and mature of the prey species, yet they ignored that of the predator ones. Already, several scholars had proposed and investigated the dynamic behaviors of the predator-prey system with stage structure for predator species [8–11, 17, 24–26]. Indeed, Wang and Chen [17] considered the following predator-prey system with stage structure for the predator population:

$$\begin{aligned}
&\dot{x}(t) = x(t)\bigl(r - ax(t-\tau_1) - by_2(t)\bigr),\\
&\dot{y}_1(t) = kbx(t-\tau_2)y_2(t-\tau_2) - (D+\nu_1)y_1(t),\\
&\dot{y}_2(t) = Dy_1(t) - \nu_1 y_2(t).
\end{aligned}$$

In [17], the authors studied the asymptotic behavior of the above system. When a time delay due to gestation of the predators and a time delay from a crowding effect of the prey are incorporated, they establish conditions for the permanence of the populations and sufficient conditions under which a positive equilibrium of the above system is globally stable; Zhang and Luo [8] argued that above system is not a realistic model because it is an autonomous system, and they incorporated a type IV functional response into above system; by using the continuation theorem of coincidence degree theory, the existence of multiple positive periodic solutions for the system is established. Recently, Chen *et al.* [10, 11] studied the persistence and extinction property of the following stage-structured predator-prey system (stage structure for both predator and prey, respectively):

$$\begin{aligned}
&\dot{x}_1(t) = r_1(t)x_2(t) - d_{11}x_1(t) - r_1(t-\tau_1)e^{-d_{11}\tau_1}x_2(t-\tau_1),\\
&\dot{x}_2(t) = r_1(t-\tau_1)e^{-d_{11}\tau_1}x_2(t-\tau_1) - d_{12}x_2(t) - b_1(t)x_2^2(t) - c_1(t)x_2(t)y_2(t),\\
&\dot{y}_1(t) = r_2(t)y_2(t) - d_{22}y_1(t) - r_2(t-\tau_2)e^{-d_{22}\tau_2}y_2(t-\tau_2),\\
&\dot{y}_2(t) = r_2(t-\tau_2)e^{-d_{22}\tau_2}y_2(t-\tau_2) - d_{21}y_2(t) - b_2(t)y_2^2(t) + c_2(t)y_2(t)x_2(t),
\end{aligned}$$

where $x_1(t)$ and $x_2(t)$ denote the densities of the immature and mature prey species at time t, respectively; $y_1(t)$ and $y_2(t)$ represent the immature and mature population densities of predator species at time t, respectively; $r_i(t)$, $b_i(t)$, $c_i(t)$ $(i = 1, 2)$ are all continuous functions bounded above and below by positive constants for all $t \geq 0$. d_{ij}, τ_i, $i, j = 1, 2$ are all positive constants. There are many interesting properties of this system, for example, due to the influence of the stage structure, the extinction of the predator species could not directly imply the permanence of the prey species. The extinction of the prey species could not lead to the extinction of the predator species. Under certain assumptions, the system would be broken, which means that both predators and prey species would be driven to extinction. However, all the works of [9–11] did not take the functional response of the predator species into consideration.

Now, stimulated by the work of [5, 6, 9–11], we consider the following stage-structured predator-prey model (stage structure for both predator and prey, respectively) with modified Leslie-Gower and Holling-type II schemes:

$$\begin{aligned} x_1'(t) &= r_1x_2(t) - d_{11}x_1(t) - r_1e^{-d_{11}\tau_1}x_2(t-\tau_1), \\ x_2'(t) &= r_1e^{-d_{11}\tau_1}x_2(t-\tau_1) - d_{12}x_2(t) - bx_2^2(t) - \frac{a_1y_2(t)x_2(t)}{x_2(t)+k_1}, \\ y_1'(t) &= r_2y_2(t) - d_{22}y_1(t) - r_2e^{-d_{22}\tau_2}y_2(t-\tau_2), \\ y_2'(t) &= r_2e^{-d_{22}\tau_2}y_2(t-\tau_2) - d_{21}y_2(t) - \frac{a_2y_2^2(t)}{x_2(t)+k_2}, \end{aligned} \tag{1.2}$$

where d_{12} and d_{21} represent the death rate of mature prey x_2 and mature predator y_2, respectively; τ_1 is the time length of the prey species from immature ones to mature ones, τ_2 is the time length of the predator from immature ones to mature ones. Other parameters have the same biological meaning as that of system (1.1). All parameters are positive constants in system (1.2).

The initial conditions for system (1.2) take the form of

$$x_i(\theta) = \varphi_i(\theta) > 0, \qquad y_i(\theta) = \psi_i(\theta) > 0, \quad \theta \in [-\tau, 0], i = 1, 2, \tag{1.3}$$

where $\tau = \max\{\tau_1, \tau_2\}$. For continuity of the initial conditions, we assume that

$$\begin{aligned} x_1(0) &= \varphi_1(0) = \int_{-\tau_1}^0 r_1e^{d_{11}u}\varphi_2(u)\,du, \\ y_1(0) &= \psi_1(0) = \int_{-\tau_2}^0 r_2e^{d_{22}u}\psi_2(u)\,du. \end{aligned} \tag{1.4}$$

Integrating both sides of the first and third equation of system (1.2) (see [12]) over the interval $(0, t)$ leads to

$$\begin{aligned} x_1(t) &= \int_{t-\tau_1}^t r_1e^{-d_{11}(t-u)}x_2(u)\,du, \\ y_1(t) &= \int_{t-\tau_2}^t r_2e^{-d_{22}(t-u)}y_2(u)\,du. \end{aligned} \tag{1.5}$$

This suggests that the dynamics of model (1.1) is completely determined by its second and fourth equations. Therefore, in the rest of this paper, we investigate the asymptotic behavior for the subsystem of system (1.1) as follows:

$$\begin{aligned} x_2'(t) &= r_1e^{-d_{11}\tau_1}x_2(t-\tau_1) - d_{12}x_2(t) - bx_2^2(t) - \frac{a_1y_2(t)x_2(t)}{x_2(t)+k_1}, \\ y_2'(t) &= r_2e^{-d_{22}\tau_2}y_2(t-\tau_2) - d_{21}y_2(t) - \frac{a_2y_2^2(t)}{x_2(t)+k_2}. \end{aligned} \tag{1.6}$$

The organization of this paper is as follows: The main results are stated and proved in Sections 2 and 3, respectively. In Section 4, several examples together with their numerical simulations are presented to illustrate the feasibility of our main results. We end this paper by a brief discussion. For more work on the Leslie-Gower predator-prey system, one may refer to [36–39] and the references cited therein.

2 Main results

For convenience, we denote

$$\lambda_1 \stackrel{\text{def}}{=} r_1 e^{-d_{11}\tau_1} - d_{12}, \qquad \lambda_2 \stackrel{\text{def}}{=} r_2 e^{-d_{22}\tau_2} - d_{21}.$$

Let $x_2'(t) = 0$, $y_2'(t) = 0$ in system (1.6), we can get four equilibria as follows:

$$\begin{aligned} &E_0 = (0,0), \qquad E_1 = \left(\frac{\lambda_1}{b}, 0\right) \stackrel{\text{def}}{=} (x_{1*}, 0), \\ &E_2 = \left(0, \frac{\lambda_2 k_2}{a_2}\right) \stackrel{\text{def}}{=} (0, y_{2*}), E\left(x_2^*, y_2^*\right), \end{aligned}$$

where E is an interior equilibrium point in system (1.6). The components of E are given by

$$y_2^* = \frac{\lambda_2(k_2 + x_2^*)}{a_2},$$

where x_2^* is a positive solution of the second order equation as follows:

$$a_2 b x^2 + (a_2 k_1 b + a_1 \lambda_2 - a_2 \lambda_1)x + C = 0,$$

where $C = a_1 k_2 \lambda_2 - a_2 k_1 \lambda_1$, we can see that there exists a unique $x_2^* > 0$ if $C < 0$, *i.e.*,

$$0 < a_1 k_2 \lambda_2 < a_2 k_1 \lambda_1. \tag{2.1}$$

E_1, E_2 are two of the boundary equilibria of the system (1.6) if $\lambda_1 > 0$, $\lambda_2 > 0$.

Consequently, we have the following theorem.

Theorem 2.1 *Assume that inequality* (2.1) *holds, then system* (1.6) *admits a unique positive equilibrium point E.*

Theorem 2.2 *Suppose that*

$$(\mathrm{H}_1) \quad \lambda_1 > 0, \qquad \lambda_2 > 0,$$

$$(\mathrm{H}_2) \quad \lambda_3 = b(a_2 k_1 \lambda_1 - a_1 k_2 \lambda_2) - a_1 \lambda_1 \lambda_2 > 0$$

hold, then the unique positive equilibrium E is globally attractive.

Remark 2.1 If $d_{12} = d_{21} = 0$, $\tau_2 \neq 0$, that is, we only consider the stage structure of the predator species and ignore the death rate of the mature predator and prey species, in this case, (H_1) holds naturally, and λ_3 in condition (H_2) would reduce to

$$(H_2') \quad \lambda_3 = r_1 e^{-d_{11}\tau_1} a_2 k_1 b - a_1 k_2 b r_2 e^{-d_{22}\tau_2} - a_1 r_1 e^{-d_{11}\tau_1} r_2 e^{-d_{22}\tau_2} > 0.$$

Noting that $r_2 e^{-d_{22}\tau_2} < r_2$, ($H_2'$) is weaker than (H) in Theorem A, this means that the stage structure of predator species has benefit for the coexistence of the system.

Remark 2.2 Suppose that $\tau_2 = 0$, *i.e.*, we did not consider the stage structure of the predator species, then λ_2 and λ_3 in Theorem 2.2 become $\lambda_2 = r_2 - d_{21} > 0$, $\lambda_3 = \lambda_0 - (a_2 k_1 b - a_1 r_2 + a_1 d_{21}) d_{12} + (a_1 k_2 b + a_1 r_1 e^{-d_{11}\tau_1}) d_{21} > 0$, respectively.

(i) If $d_{12} = 0$, $d_{21} \neq 0$, in this case, $\lambda_3 > \lambda_0$, that is, introducting the mortality item of the predator species improving the coexistence rate of the two species.

(ii) If $d_{12} \neq 0$, $d_{21} = 0$, in this case, $\lambda_0 > \lambda_3$, that is, introducting the mortality item of the prey species decreasing the chance of coexistence of both species.

Theorem 2.3 *Suppose that*

$$(H_3) \quad \lambda_1 < 0, \qquad \lambda_2 < 0$$

holds, then both of the predator and prey species will be driven to extinction, that is, E_0 *is globally attractive.*

Theorem 2.4 *Suppose that*

$$(H_4) \quad \lambda_1 > 0, \qquad \lambda_2 < 0$$

holds, then E_1 *is globally attractive.*

Remark 2.3 From [5], we know that $E_0(0,0,0)$ of the system (1.1) is unstable, which implies the extinction of both predator and prey species is impossible. However, if the death rates of the mature prey and predator species are large enough, (H_3) in Theorem 2.3 would hold, and consequently both the prey and the predator species will be driven to extinction. By constructing a suitable Lyapunov function, Korobeinikov [27] showed that the unique positive equilibrium of the traditional Leslie-Gower predator-prey model is globally attractive, which means that it is impossible for the predator species to become extinct. However, Theorem 2.4 shows that if the death rate of the mature predator species is large enough, (H_4) would hold and the predator species will be driven to extinction. Theorems 2.3 and 2.4 show that the death rates of the mature predator and prey species are two of the essential factors to determine the persistent property of the system.

Theorem 2.5 *Suppose that*

$$(H_5) \quad \lambda_2 > 0, \qquad \lambda_1 < \min\left\{ k_1 b, \frac{a_1 k_2 \lambda_2}{a_2 k_1} \right\}$$

holds, then E_2 *is globally attractive.*

Corollary 2.1 *If the parameters of system* (1.2) *satisfy the condition* (2.1), *then system* (1.2) *has a unique positive equilibrium point* $E'(x_1^*, x_2^*, y_1^*, y_2^*)$, *where* $x_1^* = \frac{r_1 x_2^*(1-e^{-d_{11}\tau_1})}{d_{11}}$, $y_1^* = \frac{r_2 y_2^*(1-e^{-d_{22}\tau_2})}{d_{22}}$.

Corollary 2.2 *If the parameters of system* (1.2) *satisfy the conditions* (H_1) *and* (H_2), *then* E' *is globally attractive.*

Corollary 2.3 *If the parameters of system* (1.2) *satisfy the condition* (H_3), *then* $E_0' = (0, 0, 0, 0)$ *is globally attractive.*

Corollary 2.4 *If the parameters of system* (1.2) *satisfy the condition* (H_4), *then* $E_1' = (x_{1*}, x_{2*}, 0, 0)$ *is globally attractive, where* $x_{1*} = \frac{r_1 x_{2*}(1-e^{-d_{11}\tau_1})}{d_{11}}$.

Corollary 2.5 *If the parameters of system* (1.2) *satisfy the condition* (H_5), *then* $E_2' = (0, 0, y_{1*}, y_{2*})$ *is globally attractive, where* $y_{1*} = \frac{r_2 y_{2*}(1-e^{-d_{22}\tau_2})}{d_{22}}$.

3 Proof of the main results

Now let us state several lemmas which will be useful in proving the main results.

Lemma 3.1 *Assume that* $x_2(\theta) \geq 0$, $y_2(\theta) \geq 0$ *are continuous on* $\theta \in [-\tau, 0]$, *and* $x_2(0) > 0$, $y_2(0) > 0$. *Let* $(x_2(t), y_2(t))^T$ *be a any solution of system* (1.6), *then* $x_2(t) > 0$, $y_2(t) > 0$ *for all* $t > 0$.

The proof of Lemma 3.1 is similar to the proof of Theorem 1 in [1], so we omit its proof.

Lemma 3.2 [2] *Consider the following equation*:

$$x'(t) = bx(t-\delta) - a_1 x(t) - a_2 x^2(t),$$
$$x(t) = \phi(t) > 0, \quad -\delta \leq t \leq 0,$$

and assume that $b, a_2 > 0$, $a_1 \geq 0$, *and* $\delta \geq 0$ *is a constant. Then*

(i) *if* $b \geq a_1$, *then* $\lim_{t\to+\infty} x(t) = \frac{b-a_1}{a_2}$;

(ii) *if* $b \leq a_1$, *then* $\lim_{t\to+\infty} x(t) = 0$.

Lemma 3.3 (Fluctuation lemma [23]) *Let* $x(t)$ *be a bounded differentiable function on* (α, ∞), *Then there exist sequences* $\gamma_n \to \infty$, $\sigma_n \to \infty$ *such that*

(i) $x'(\gamma_n) \to 0$ *and* $x(\gamma_n) \to \limsup_{t\to+\infty} x(t) = \overline{x}$ *as* $n \to \infty$,

(ii) $x'(\sigma_n) \to 0$ *and* $x(\sigma_n) \to \liminf_{t\to+\infty} x(t) = \underline{x}$ *as* $n \to \infty$.

Lemma 3.4 *Assume that* $x_2(\theta), y_2(\theta) \geq 0$ *are continuous on* $\theta \in [-\tau, 0]$, *and* $x_2(0) > 0$, $y_2(0) > 0$. *Let* $(x_2(t), y_2(t))^T$ *be a any solution of system* (1.6). *If* $\lambda_2 > 0$, *then*

$$\liminf_{t\to+\infty} y_2(t) \geq \frac{k_2\lambda_2}{a_2}.$$

Proof From the second equation of system (1.6), we have

$$y_2'(t) \geq r_2 e^{-d_{22}\tau_2} y_2(t-\tau_2) - d_{21} y_2(t) - \frac{a_2 y_2^2(t)}{k_2}.$$

Since $\lambda_2 > 0$, and by applying Lemma 3.2(i), and the standard comparison theorem, we have

$$\liminf_{t\to+\infty} y_2(t) \geq \frac{k_2(r_2 e^{-d_{22}\tau_2} - d_{21})}{a_2} = \frac{k_2\lambda_2}{a_2} > 0.$$

This completes the proof of Lemma 3.4. □

Now we start to prove the above results.

Proof of Theorem 2.2 From the first equation of system (1.6), we have

$$x_2'(t) < r_1 e^{-d_{11}\tau_1} x_2(t-\tau_1) - d_{12}x_2(t) - bx_2^2(t).$$

According to condition (H_1), we know $r_1 e^{-d_{11}\tau_1} - d_{12} > 0$. By applying Lemma 3.2(i), and the standard comparison theorem, we have

$$\limsup_{t\to+\infty} x_2(t) \leq \frac{\lambda_1}{b}.$$

So, for any small constant $\varepsilon > 0$, there exists a $T_1 > 0$ such that

$$x_2(t) \leq \frac{\lambda_1}{b} + \varepsilon \overset{\text{def}}{=} M_1^{(1)}, \quad t > T_1. \tag{3.1}$$

For $t > T_1 + \tau_2$, substituting (3.1) into the second equation of system (1.6), we have

$$y_2'(t) < r_2 e^{-d_{22}\tau_2} y_2(t-\tau_2) - d_{21}y_2(t) - \frac{a_2 y_2^2(t)}{M_1^{(1)} + k_2}.$$

According to condition (H_1), we have $r_2 e^{-d_{22}\tau_2} - d_{21} > 0$. By applying Lemma 3.2(i), and the standard comparison theorem, we have

$$\limsup_{t\to+\infty} y_2(t) \leq \frac{\lambda_2(M_1^{(1)} + k_2)}{a_2}.$$

Then, for the above ε, there exists a $T_2 > T_1 + \tau_2$, such that

$$y_2(t) < \frac{\lambda_2(M_1^{(1)} + k_2)}{a_2} + \varepsilon \overset{\text{def}}{=} M_2^{(1)}, \quad t > T_2. \tag{3.2}$$

For $t > T_2 + \tau_1$, substituting (3.2) into the first equation of system (1.6), we have

$$\begin{aligned} x_2'(t) &> r_1 e^{-d_{11}\tau_1} x_2(t-\tau_1) - d_{12}x_2(t) - bx_2^2(t) - \frac{a_1 M_2^{(1)} x_2(t)}{k_1} \\ &= r_1 e^{-d_{11}\tau_1} x_2(t-\tau_1) - \left(d_{12} + \frac{a_1 M_2^{(1)}}{k_1}\right) x_2(t) - bx_2^2(t). \end{aligned}$$

Let

$$r_1 e^{-d_{11}\tau_1} - \left(d_{12} + \frac{a_1 M_2^{(1)}}{k_1}\right) \overset{\text{def}}{=} \Delta.$$

Then substituting (3.1) and (3.2) into Δ, we have

$$\Delta = \frac{b(a_2k_1\lambda_1 - a_1k_2\lambda_2) - a_1\lambda_1\lambda_2}{a_2k_1b} - \frac{a_1}{k_1}\left(\frac{\lambda_2}{a_2} + 1\right)\varepsilon.$$

Then, for small enough $\varepsilon > 0$ and condition (H_2), we have

$$\Delta = r_1e^{-d_{11}\tau_1} - \left(d_{12} + \frac{a_1M_2^{(1)}}{k_1}\right) > 0. \tag{3.3}$$

By applying Lemma 3.2(i), and the standard comparison theorem, we have

$$\liminf_{t\to+\infty} x_2(t) \geq \frac{\lambda_1 - \frac{a_1M_2^{(1)}}{k_1}}{b}.$$

Then, for the above $\varepsilon > 0$, there exists a $T_3 > T_2 + \tau_1$, such that

$$x_2(t) > \frac{\lambda_1 - \frac{a_1M_2^{(1)}}{k_1}}{b} - \varepsilon \stackrel{\text{def}}{=} m_1^{(1)}, \quad t > T_3. \tag{3.4}$$

For $t > T_3 + \tau_2$, substituting (3.4) into the second equation of system (1.6), we have

$$y_2'(t) > r_2e^{-d_{22}\tau_2}y_2(t-\tau_2) - d_{21}y_2(t) - \frac{a_2y_2^2(t)}{m_1^{(1)} + k_2}.$$

By applying Lemma 3.2(i), and the standard comparison theorem, we have

$$\liminf_{t\to+\infty} y_2(t) \geq \frac{\lambda_2(m_1^{(1)} + k_2)}{a_2}.$$

Then for the above $\varepsilon > 0$, there exists a $T_4 > T_3 + \tau_2$, such that

$$y_2(t) > \frac{\lambda_2(m_1^{(1)} + k_2)}{a_2} - \varepsilon \stackrel{\text{def}}{=} m_2^{(1)}, \quad t > T_4. \tag{3.5}$$

According to (3.1), (3.2), (3.4), and (3.5), we obtain

$$0 < m_1^{(1)} < x_2(t) < M_1^{(1)}, \qquad 0 < m_2^{(1)} < y_2(t) < M_2^{(1)}, \quad t > T_4. \tag{3.6}$$

Then for $t > T_4 + \tau_1$, substituting (3.1) and (3.5) into the first equation of system (1.6), we have

$$\begin{aligned} x_2'(t) &< r_1e^{-d_{11}\tau_1}x_2(t-\tau_1) - d_{12}x_2(t) - bx_2^2(t) - \frac{a_1m_2^{(1)}x_2(t)}{k_1 + M_1^{(1)}} \\ &= r_1e^{-d_{11}\tau_1}x_2(t-\tau_1) - \left(d_{12} + \frac{a_1m_2^{(1)}}{k_1 + M_1^{(1)}}\right)x_2(t) - bx_2^2(t). \end{aligned}$$

According to the inequalities (3.3) and (3.6), we have $r_1e^{-d_{11}\tau_1} - (d_{12} + \frac{a_1m_2^{(1)}}{k_1+M_1^{(1)}}) > 0$. By applying Lemma 3.2(i), and the standard comparison theorem, we have

$$\limsup_{t\to+\infty} x_2(t) \leq \frac{\lambda_1 - \frac{a_1m_2^{(1)}}{k_1+M_1^{(1)}}}{b}.$$

Then, for the above $\varepsilon > 0$, there exists a $T_5 > T_4 + \tau_1$, such that

$$x_2(t) < \frac{\lambda_1 - \frac{a_1 m_2^{(1)}}{k_1 + M_1^{(1)}}}{b} + \frac{\varepsilon}{2} \stackrel{\text{def}}{=} M_1^{(2)}, \quad t > T_5. \tag{3.7}$$

From inequalities (3.1) and (3.7), we obtain

$$x_2(t) < M_1^{(2)} < M_1^{(1)}, \quad t > T_5. \tag{3.8}$$

For $t > T_5 + \tau_2$, substituting (3.7) into the second equation of system (1.6), we have

$$y_2'(t) < r_2 e^{-d_{22}\tau_2} y_2(t - \tau_2) - d_{21} y_2(t) - \frac{a_2 y_2^2(t)}{M_1^{(2)} + k_2}.$$

By applying Lemma 3.2(i), and the standard comparison theorem, we have

$$\limsup_{t\to+\infty} y_2(t) \le \frac{\lambda_2(M_1^{(2)} + k_2)}{a_2}.$$

Then for above $\varepsilon > 0$, there exists a $T_6 > T_5 + \tau_2$, such that

$$y_2(t) < \frac{\lambda_2(M_1^{(2)} + k_2)}{a_2} + \frac{\varepsilon}{2} \stackrel{\text{def}}{=} M_2^{(2)}, \quad t > T_6. \tag{3.9}$$

From inequalities (3.2), (3.8), and (3.9), we have

$$y_2(t) < M_2^{(2)} < M_2^{(1)}, \quad t > T_6. \tag{3.10}$$

For $t > T_6 + \tau_1$, substituting inequalities (3.4) and (3.9) into the first equation of system (1.6), we have

$$\begin{aligned} x_2'(t) &> r_1 e^{-d_{11}\tau_1} x_2(t - \tau_1) - d_{12} x_2(t) - b x_2^2(t) - \frac{a_1 M_2^{(2)} x_2(t)}{k_1 + m_1^{(1)}} \\ &= r_1 e^{-d_{11}\tau_1} x_2(t - \tau_1) - \left(d_{12} + \frac{a_1 M_2^{(2)}}{k_1 + m_1^{(1)}}\right) x_2(t) - b x_2^2(t). \end{aligned}$$

According to inequalities (3.3) and (3.10), we can obtain $r_1 e^{-d_{11}\tau_1} - d_{12} - \frac{a_1 M_2^{(2)}}{k_1 + m_1^{(1)}} > 0$. By applying Lemma 3.2(i), and the standard comparison theorem, we have

$$\liminf_{t\to+\infty} x_2(t) \ge \frac{\lambda_1 - \frac{a_1 M_2^{(2)}}{k_1 + m_1^{(1)}}}{b}.$$

Then, for the above $\varepsilon > 0$, there exists a $T_7 > T_6 + \tau_1$, such that

$$x_2(t) > \frac{\lambda_1 - \frac{a_1 M_2^{(2)}}{k_1 + m_1^{(1)}}}{b} - \frac{\varepsilon}{2} \stackrel{\text{def}}{=} m_1^{(2)}, \quad t > T_7. \tag{3.11}$$

According to the inequalities (3.4), (3.10), and (3.11), we can obtain

$$x_2(t) > m_1^{(2)} > m_1^{(1)}, \quad t > T_7. \tag{3.12}$$

Substituting inequality (3.11) into the second equation of system (1.6), we have

$$y_2'(t) > r_2 e^{-d_{22}\tau_2} y_2(t-\tau_2) - d_{21} y_2(t) - \frac{a_2 y_2^2(t)}{m_1^{(2)} + k_2}, \quad t > T_7 + \tau_2.$$

By applying Lemma 3.2(i), and the standard comparison theorem, we have

$$\liminf_{t\to+\infty} y_2(t) \ge \frac{\lambda_2(m_1^{(2)} + k_2)}{a_2}.$$

Then, for the above $\varepsilon > 0$, there exists a $T_8 > T_7 + \tau_2$, such that

$$y_2(t) > \frac{\lambda_2(m_1^{(2)} + k_2)}{a_2} - \frac{\varepsilon}{2} \stackrel{\text{def}}{=} m_2^{(2)}, \quad t > T_8. \tag{3.13}$$

According to the inequalities (3.5), (3.12), and (3.13), we can obtain

$$y_2(t) > m_2^{(2)} > m_2^{(1)}, \quad t > T_8. \tag{3.14}$$

For $t > T_8$, according to (3.8), (3.10), (3.12), and (3.14), we have

$$\begin{aligned} &m_1^{(1)} < m_1^{(2)} < x_2(t) < M_1^{(2)} < M_1^{(1)}, \\ &m_2^{(1)} < m_2^{(2)} < y_2(t) < M_2^{(2)} < M_2^{(1)}. \end{aligned} \tag{3.15}$$

Repeating the above process, we get four sequences

$$\begin{aligned} &M_1^{(n)} = \frac{\lambda_1 - \frac{a_1 m_2^{(n-1)}}{k_1 + M_1^{(n-1)}}}{b} + \frac{\varepsilon}{n}, \qquad M_2^{(n)} = \frac{\lambda_2(M_1^{(n)} + k_2)}{a_2} + \frac{\varepsilon}{n}, \\ &m_1^{(n)} = \frac{\lambda_1 - \frac{a_1 M_2^{(n)}}{k_1 + m_1^{(n-1)}}}{b} - \frac{\varepsilon}{n}, \qquad m_2^{(n)} = \frac{\lambda_2(m_1^{(n)} + k_2)}{a_2} - \frac{\varepsilon}{n}. \end{aligned} \tag{3.16}$$

For $i = 1, 2$, we claim that $M_i^{(n)}$ are monotonic decreasing sequences, and $m_i^{(n)}$ are monotone increasing sequences. In the following we will prove this claim by induction. First of all, according to inequalities (3.15), we have

$$m_i^{(1)} < m_i^{(2)}, \qquad M_i^{(2)} < M_i^{(1)}, \quad i = 1, 2.$$

Second, we suppose that our claim is true for n, that is,

$$m_i^{(n-1)} < m_i^{(n)}, \qquad M_i^{(n)} < M_i^{(n-1)}, \quad i = 1, 2. \tag{3.17}$$

Noting that

$$\begin{aligned} M_1^{(n+1)} &= \frac{\lambda_1 - \frac{a_1 m_2^{(n)}}{k_1 + M_1^{(n)}}}{b} + \frac{\varepsilon}{n+1}, \qquad M_2^{(n+1)} = \frac{\lambda_2 (M_1^{(n+1)} + k_2)}{a_2} + \frac{\varepsilon}{n+1}, \\ m_1^{(n+1)} &= \frac{\lambda_1 - \frac{a_1 M_2^{(n+1)}}{k_1 + m_1^{(n)}}}{b} - \frac{\varepsilon}{n+1}, \qquad m_2^{(n+1)} = \frac{\lambda_2 (m_1^{(n+1)} + k_2)}{a_2} - \frac{\varepsilon}{n+1}. \end{aligned} \tag{3.18}$$

According to inequalities (3.16), (3.17), and (3.18), one could easily see that

$$M_i^{(n+1)} < M_i^{(n)}, \qquad m_i^{(n)} < m_i^{(n+1)}, \quad i = 1, 2.$$

Then for $t > T_{4n}$, we have

$$\begin{aligned} &0 < m_1^{(1)} < m_1^{(2)} < \cdots < x_2(t) < M_1^{(n)} < \cdots < M_1^{(2)} < M_1^{(1)}, \\ &0 < m_2^{(1)} < m_2^{(2)} < \cdots < y_2(t) < M_2^{(n)} < \cdots < M_2^{(2)} < M_2^{(1)}. \end{aligned}$$

Therefore the limits of $M_i^{(n)}$, $m_i^{(n)}$ ($i = 1, 2$, $n = 1, 2, \ldots$) exist. Denote that

$$\begin{aligned} &\lim_{t \to +\infty} M_1^{(n)} = \overline{x}_2, \qquad \lim_{t \to +\infty} m_1^{(n)} = \underline{x}_2, \\ &\lim_{t \to +\infty} M_2^{(n)} = \overline{y}_2, \qquad \lim_{t \to +\infty} m_2^{(n)} = \underline{y}_2. \end{aligned}$$

Consequently, $\overline{x}_2 \geq \underline{x}_2$, $\overline{y}_2 \geq \underline{y}_2$. In order to complete the proof, we just need to show that $\overline{x}_2 = \underline{x}_2$, $\overline{y}_2 = \underline{y}_2$. Letting $n \to +\infty$ in (3.16), we have

$$\begin{aligned} b\overline{x}_2 &= \lambda_1 - \frac{a_1 \underline{y}_2}{\overline{x}_2 + k_1}, \qquad a_2 \overline{y}_2 = \lambda_2 (\overline{x}_2 + k_2), \\ b\underline{x}_2 &= \lambda_1 - \frac{a_1 \overline{y}_2}{\underline{x}_2 + k_1}, \qquad a_2 \underline{y}_2 = \lambda_2 (\underline{x}_2 + k_2). \end{aligned}$$

It follows from the above four equations that

$$\begin{aligned} a_1 \lambda_2 (\underline{x}_2 + k_2) &= a_2 (\lambda_1 - b\overline{x}_2)(\overline{x}_2 + k_1), \\ a_1 \lambda_2 (\overline{x}_2 + k_2) &= a_2 (\lambda_1 - b\underline{x}_2)(\underline{x}_2 + k_1). \end{aligned} \tag{3.19}$$

Subtracting the first equation of (3.19) from the second equation, we get

$$\big(a_2 \lambda_1 + a_1 \lambda_2 - a_2 b(\overline{x}_2 + \underline{x}_2) - k_1 a_2 b\big)(\overline{x}_2 - \underline{x}_2) = 0.$$

Suppose that $\overline{x}_2 \neq \underline{x}_2$, it follows from the above equation that

$$a_2 \lambda_1 + a_1 \lambda_2 - k_1 a_2 b = a_2 b(\overline{x}_2 + \underline{x}_2). \tag{3.20}$$

Substituting (3.20) into (3.19), we find $\overline{x}_2$ and $\underline{x}_2$ both satisfy the following equation:

$$a_2^2 b(\lambda_1 - bx)(x + k_1) = a_1 \lambda_2 (a_1 \lambda_2 + a_2 \lambda_1 - a_2 k_1 b - a_2 bx + a_2 k_2 b). \tag{3.21}$$

Simplifying equality (3.21), we get

$$a_2^2b^2x^2 + a_2b(a_2k_1b - a_2\lambda_1 - a_1\lambda_2)x + D = 0, \tag{3.22}$$

where

$$D = a_2(a_1\lambda_1\lambda_2 + ba_1k_2\lambda_2 - ba_2k_1\lambda_1) + a_1\lambda_2(a_1\lambda_2 - a_2k_1b).$$

According to condition (H_2), we can immediately obtain

$$-\lambda_3 = a_1\lambda_1\lambda_2 + ba_1k_2\lambda_2 - ba_2k_1\lambda_1 < 0.$$

It implies that

$$a_1\lambda_2 - ba_2k_1 < 0.$$

Therefore, we have $D < 0$, that is, equation (3.22) has only one positive root. Then $\overline{x}_2 = \underline{x}_2$, and consequently, $\overline{y}_2 = \underline{y}_2$. Obviously, conditions (H_1), (H_2) imply inequality (2.1), so system (1.6) has a unique positive equilibrium $E(x_2^*, y_2^*)$. That is,

$$\lim_{t\to+\infty} x_2(t) = x_2^*, \qquad \lim_{t\to+\infty} y_2(t) = y_2^*.$$

This completes the proof of Theorem 2.2. □

Proof of Theorem 2.3 It follows from the first equation of system (1.6) that

$$x_2'(t) < r_1e^{-d_{11}\tau_1}x_2(t-\tau_1) - d_{12}x_2(t) - bx_2^2(t).$$

According to first inequality of condition (H_3), we have $r_1e^{-d_{11}\tau_1} - d_{12} < 0$. By applying Lemma 3.2(ii) and the standard comparison theorem, we have $\limsup_{t\to+\infty} x_2(t) \le 0$. That is,

$$\lim_{t\to+\infty} x_2(t) = 0.$$

Then, for any $\varepsilon > 0$, there exists a $T > 0$ such that

$$0 < x_2(t) < \varepsilon.$$

Therefore, it follows from the second equation of system (1.6) that

$$y_2'(t) < r_2e^{-d_{22}\tau_2}y_2(t-\tau_2) - d_{21}y_2(t) - \frac{a_2y_2^2(t)}{\varepsilon + k_2}, \quad t > T + \tau_2.$$

Similar to the above analysis, we also have

$$\lim_{t\to+\infty} y_2(t) = 0.$$

Therefore, $E_0 = (0,0)$ is globally attractive. This completes the proof of Theorem 2.3. □

Proof of Theorem 2.4 According to the first inequality of condition (H_4), we have $r_1e^{-d_{11}\tau_1} - d_{12} > 0$. Therefore from the proof of Theorem 2.2, we know that

$$\limsup_{t\to+\infty} x_2(t) \le \frac{\lambda_1}{b}. \tag{3.23}$$

And for any small positive constant $\varepsilon > 0$, there exists a $T_1 > 0$ such that

$$y_2'(t) < r_2e^{-d_{22}\tau_2}y_2(t-\tau_2) - d_{21}y_2(t) - \frac{a_2y_2^2(t)}{M_1^{(1)} + k_2}, \quad t > T_1 + \tau_2.$$

According the second inequality of condition (H_4), we have $r_2e^{-d_{22}\tau_2} - d_{21} < 0$. By applying Lemma 3.2(ii) and the standard comparison theorem, we have $\limsup_{t\to+\infty} y_2(t) \le 0$. That is,

$$\lim_{t\to+\infty} y_2(t) = 0.$$

Then, for any small $\varepsilon > 0$, there exists $T_2 > T_1 + \tau_2$, such that

$$0 < y_2(t) < \varepsilon, \quad t > T_2. \tag{3.24}$$

Substituting inequality (3.24) into the first equation of system (1.6), we have

$$\begin{aligned} x_2'(t) &> r_1e^{-d_{11}\tau_1}x_2(t-\tau_1) - d_{12}x_2(t) - bx_2^2(t) - \frac{a_1\varepsilon x_2(t)}{k_1} \\ &= r_1e^{-d_{11}\tau_1}x_2(t-\tau_1) - \left(d_{12} + \frac{a_1\varepsilon}{k_1}\right)x_2(t) - bx_2^2(t), \quad t > T_2 + \tau_1. \end{aligned}$$

Since $r_1e^{-d_{11}\tau_1} - d_{12} > 0$, we can choose sufficiently small $\varepsilon > 0$ such that $r_1e^{-d_{11}\tau_1} - d_{12} - \frac{a_1\varepsilon}{k_1} > 0$. By applying Lemma 3.2(i), and the standard comparison theorem, we have

$$\liminf_{t\to+\infty} x_2(t) \ge \frac{r_1e^{-d_{11}\tau_1} - d_{12} - \frac{a_1\varepsilon}{k_1}}{b} = \frac{\lambda_1 - \frac{a_1\varepsilon}{k_1}}{b}.$$

For the above formula, letting $\varepsilon \to 0$, we have

$$\liminf_{t\to+\infty} x_2(t) \ge \frac{\lambda_1}{b}. \tag{3.25}$$

From inequalities (3.23) and (3.25), we get

$$\frac{\lambda_1}{b} \le \liminf_{t\to+\infty} x_2(t) \le \limsup_{t\to+\infty} x_2(t) \le \frac{\lambda_1}{b}.$$

Then we have

$$\lim_{t\to+\infty} x_2(t) = \frac{\lambda_1}{b} = x_{2*}.$$

Therefore, $E_1(x_{2*}, 0)$ is globally attractive. This completes the proof of Theorem 2.4. □

Proof of Theorem 2.5 According to the fluctuation lemma, there exist two sequences $\gamma_n \to \infty$, $\sigma_n \to \infty$ such that $x_2'(\gamma_n) \to 0$, $x_2(\gamma_n) \to \limsup_{t\to+\infty} x(t) = \overline{x}_2$, and $y_2'(\sigma_n) \to 0$, $y_2(\sigma_n) \to \liminf_{t\to+\infty} y_2(t) = \underline{y}_2$ as $n \to \infty$. Since from Lemma 3.1, we know $\overline{x}_2 \geq 0$. In order to prove $\lim_{t\to\infty} x_2(t) = 0$, we only need to prove $\overline{x}_2 = 0$, so for getting a contradiction, we suppose that $\overline{x}_2 > 0$. Since for $\lambda_2 > 0$, and according to Lemma 3.4, we know $\underline{y}_2 > 0$, it follows from the first equation of the system (1.6) that

$$\begin{aligned} x_2'(\gamma_n) &= r_1 e^{-d_{11}\tau_1} x_2(\gamma_n - \tau_1) - d_{12} x_2(\gamma_n) - b x_2^2(\gamma_n) - \frac{a_1 y_2(\gamma_n) x_2(\gamma_n)}{x_2(\gamma_n) + k_1} \\ &\leq r_1 e^{-d_{11}\tau_1} \sup_{t\geq\gamma_n-\tau_1} x_2(t) - d_{12} x_2(\gamma_n) - b x_2^2(\gamma_n) - \frac{a_1 x_2(\gamma_n)}{x_2(\gamma_n) + k_1} \inf_{t\geq\gamma_n} y_2(\gamma_n). \end{aligned}$$

Letting $n \to \infty$ in the above inequality, we obtain

$$0 \leq \lambda_1 - b\overline{x}_2 - \frac{a_1 \underline{y}_2}{\overline{x}_2 + k_1},$$

that is,

$$0 \leq \lambda_1(\overline{x}_2 + k_1) - b\overline{x}_2(\overline{x}_2 + k_1) - a_1 \underline{y}_2. \tag{3.26}$$

From the second equation of system (1.6), by a similar argument, we have

$$0 \geq k_2\lambda_2 - a_2\underline{y}_2. \tag{3.27}$$

It follows from inequalities (3.26) and (3.27) that

$$0 \leq a_2\lambda_1(k_1 + \overline{x}_2) - a_2 b\overline{x}_2(k_1 + \overline{x}_2) - a_1 k_2 \lambda_2.$$

Simplifying the above inequality, we have

$$a_2 b\overline{x}_2^2 + (a_2 k_1 b - a_2\lambda_1)\overline{x}_2 + a_1 k_2\lambda_2 - a_2 k_1\lambda_1 \leq 0. \tag{3.28}$$

According to the second inequality of condition (H_5), we know $a_1k_2\lambda_2 - a_2k_1\lambda_1 > 0$, $a_2k_1b - a_2\lambda_1 > 0$, then only $\overline{x}_2 < 0$ can ensure (3.28) holds. And $\overline{x}_2 < 0$ contradicts the hypothesis $\overline{x}_2 > 0$, then we get

$$\lim_{t\to\infty} x_2(t) = 0.$$

Then, for any small enough $\varepsilon > 0$, there exists a $T > 0$, such that

$$0 < x_2(t) < \varepsilon, \quad t > T. \tag{3.29}$$

Substituting inequality (3.29) into the second equation of system (1.6), we have

$$y_2'(t) < r_2 e^{-d_{22}\tau_2} y_2(t - \tau_2) - d_{21} y_2(t) - \frac{a_2 y_2^2(t)}{\varepsilon + k_2}, \quad t > T + \tau_2.$$

According to second inequality of condition (H_5), we know $r_2e^{-d_{22}\tau_2} - d_{21} > 0$. By applying Lemma 3.2(i), and the standard comparison theorem, we have

$$\limsup_{t\to+\infty} y_2(t) \le \frac{\lambda_2(\varepsilon + k_2)}{a_2}.$$

Letting $\varepsilon \to 0$ in the above inequality, we have

$$\limsup_{t\to+\infty} y_2(t) \le \frac{\lambda_2 k_2}{a_2}. \tag{3.30}$$

According to Lemma 3.4 and (3.30), we obtain

$$\frac{\lambda_2 k_2}{a_2} \le \liminf_{t\to+\infty} y_2(t) \le \limsup_{t\to+\infty} y_2(t) \le \frac{\lambda_2 k_2}{a_2}.$$

Then we have

$$\lim_{t\to+\infty} y_2(t) = \frac{\lambda_2 k_2}{a_2} = y_{2*}.$$

This completes the proof of Theorem 2.5. □

4 Numerical simulations

The following examples show the feasibility of our main results.

Example 4.1

$$\begin{aligned} x_2'(t) &= 2e^{-0.4}x_2(t-1) - 0.4x_2(t) - 0.8x_2^2(t) - \frac{y_2(t)x_2(t)}{x_2(t)+1}, \\ y_2'(t) &= 1.2e^{-0.5}y_2(t-1) - 0.6y_2(t) - \frac{2y_2^2(t)}{x_2(t)+2}, \end{aligned} \tag{4.1}$$

where $r_1 = 2$; $r_2 = 1.2$; $d_{11} = 0.4$; $d_{22} = 0.5$; $\tau_1 = 1$; $\tau_2 = 1$; $d_{12} = 0.4$; $d_{21} = 0.6$; $b = 0.8$; $a_1 = 1$; $a_2 = 2$; $k_1 = 1$; $k_2 = 2$. One could easily verify that $\lambda_1 \approx r_1e^{-d_{11}\tau_1} - d_{12} = 0.9406 > 0$, $\lambda_2 \approx r_2e^{-d_{22}\tau_2} - d_{21} = 0.1278 > 0$, $\lambda_3 \approx b(a_2k_1\lambda_1 - a_1k_2\lambda_2) - a_1\lambda_1\lambda_2 = 1.1802 > 0$, which shows conditions (H_1) and (H_2) hold. According to Theorem 2.2, system (1.6) has a unique and globally attractive positive equilibrium $E(1.0571, 0.1954)$. Figure 1 indicates the dynamical behavior of system (4.1).

Example 4.2

$$\begin{aligned} x_2'(t) &= e^{-0.5}x_2(t-1) - 0.7x_2(t) - 0.8x_2^2(t) - \frac{y_2(t)x_2(t)}{x_2(t)+1}, \\ y_2'(t) &= 1.2e^{-0.7}y_2(t-1) - 0.8y_2(t) - \frac{2y_2^2(t)}{x_2(t)+2}, \end{aligned} \tag{4.2}$$

where $r_1 = 1$; $r_2 = 1.2$; $d_{11} = 0.5$; $d_{22} = 0.7$; $\tau_1 = 1$; $\tau_2 = 1$; $d_{12} = 0.7$; $d_{21} = 0.8$; $b = 0.8$; $a_1 = 1$; $a_2 = 2$; $k_1 = 1$; $k_2 = 2$. By simple computation, one could see that $\lambda_1 \approx -0.0935 < 0$, $\lambda_2 \approx -0.2041 < 0$, which shows condition (H_3) holds. It follows from Theorem 2.3 that the

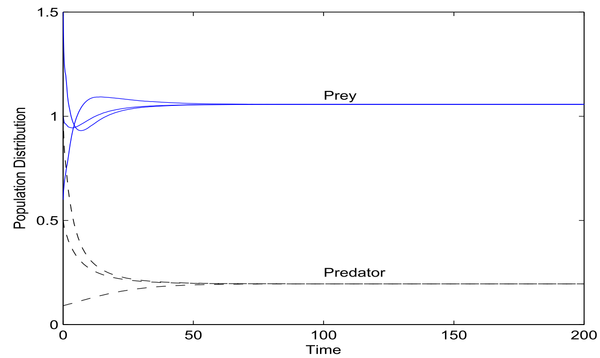

Figure 1 **Dynamics behaviors of system (4.1) with the initial values $(\varphi(\theta), \psi(\theta))^T = (0.6, 0.09)^T$, $(1, 0.5)^T$, and $(1.5, 1)^T$, $\theta \in [-1, 0]$.**

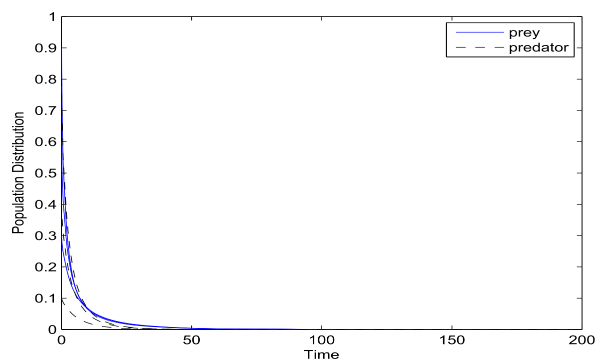

Figure 2 **Dynamics behaviors of system (4.2) with initial values $(\varphi(\theta), \psi(\theta))^T = (0.3, 0.1)^T$, $(0.6, 0.4)^T$, and $(0.9, 0.8)^T$, $\theta \in [-1, 0]$.**

solution $E_0(0, 0)$ of system (4.2) is globally attractive. Figure 2 shows the feasibility of this case.

Example 4.3

$$\begin{aligned} x_2'(t) &= 2.2e^{-1}x_2(t-2) - 0.7x_2(t) - 0.8x_2^2(t) - \frac{y_2(t)x_2(t)}{x_2(t)+1}, \\ y_2'(t) &= 1.5e^{-0.8}y_2(t-2) - 0.8y_2(t) - \frac{2y_2^2(t)}{x_2(t)+2}, \end{aligned} \tag{4.3}$$

where $r_1 = 2.2$; $r_2 = 1.5$; $d_{11} = 0.5$; $d_{22} = 0.4$; $\tau_1 = 2$; $\tau_2 = 2$; $d_{12} = 0.7$; $d_{21} = 0.8$; $b = 0.8$; $a_1 = 1$; $a_2 = 2$; $k_1 = 1$; $k_2 = 2$. By computation, we have $\lambda_1 \approx 0.1093 > 0$, $\lambda_2 \approx -0.1260 < 0$, which shows condition (H_4) holds. It follows from Theorem 2.4 that the boundary equilibrium $E_1(0.1367, 0)$ of system (4.3) is globally attractive. Figure 3 supports this assertion.

Example 4.4

$$\begin{aligned} x_2'(t) &= 1.5e^{-0.4}x_2(t-1) - 0.8x_2(t) - 0.5x_2^2(t) - \frac{y_2(t)x_2(t)}{x_2(t)+0.6}, \\ y_2'(t) &= 1.5e^{-0.5}y_2(t-1) - 0.6y_2(t) - \frac{1.5y_2^2(t)}{x_2(t)+2}, \end{aligned} \tag{4.4}$$

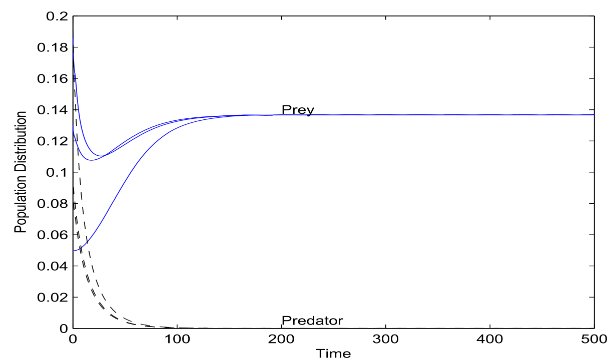

Figure 3 Dynamics behaviors of system (4.3) with initial values $(\varphi(\theta),\psi(\theta))^T = (0.05, 0.09)^T$, $(0.13, 0.1)^T$, and $(0.19, 0.2)^T$, $\theta \in [-2, 0]$.

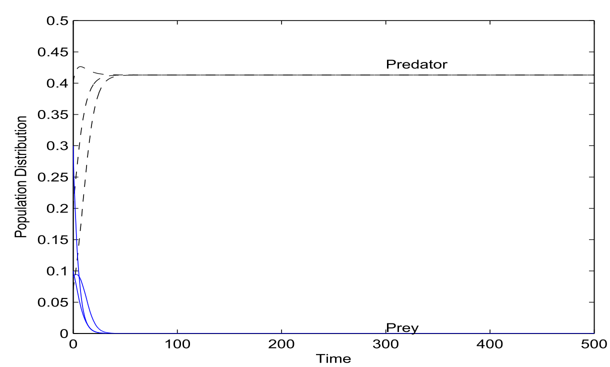

Figure 4 Dynamics behaviors of system (4.4) with initial values $(\varphi(\theta),\psi(\theta))^T = (0.09, 0.07)^T$, $(0.1, 0.2)^T$ and $(0.3, 0.4)^T$, $\theta \in [-1, 0]$.

where $r_1 = 1.5$; $r_2 = 1.5$; $d_{11} = 0.4$; $d_{22} = 0.5$; $\tau_1 = 1$; $\tau_2 = 1$; $d_{12} = 0.8$; $d_{21} = 0.6$; $b = 0.5$; $a_1 = 1$; $a_2 = 1.5$; $k_1 = 0.6$; $k_2 = 2$. By computation, we have $\lambda_1 \approx 0.2055$, $\lambda_2 \approx 0.3098 > 0$, $\lambda_1 - \frac{a_1k_2\lambda_2}{a_2k_1} \approx -0.9945 < 0$, $\lambda_1 - k_1 b \approx -0.0945 < 0$, which shows condition (H_5) holds. It follows from Theorem 2.5 that the solution $E_2(0, 0.4131)$ of system (4.4) is globally attractive. Figure 4 shows the feasibility of this case.

Example 4.5

$$\begin{aligned} x_2'(t) &= 2e^{-0.5}x_2(t-1) - 0.7x_2(t) - 0.8x_2^2(t) - \frac{y_2(t)x_2(t)}{x_2(t)+1}, \\ y_2'(t) &= 1.5e^{-0.7}y_2(t-1) - 0.6y_2(t) - \frac{2y_2^2(t)}{x_2(t)+2}, \end{aligned} \tag{4.5}$$

where $r_1 = 2$; $r_2 = 1.5$; $d_{11} = 0.5$; $d_{22} = 0.7$; $\tau_1 = 1$; $\tau_2 = 1$; $d_{12} = 0.7$; $d_{21} = 0.6$; $b = 0.8$; $a_1 = 1$; $a_2 = 2$; $k_1 = 1$; $k_2 = 2$. And so $a_2k_1\lambda_1 \approx 1.0261 > 0.2898 \approx a_1k_2\lambda_2 > 0$, which shows condition (2.1) holds. However, $\lambda_3 \approx -0.3145 < 0$, and so, condition (H_2) does not hold. A numeric simulation (Figure 5) shows that the system still admits a unique globally attractive positive equilibrium $E(0.4900, 0.1804)$.

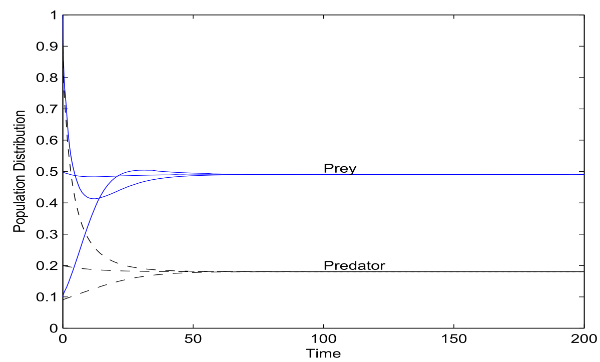

Figure 5 Dynamics behaviors of system (4.5) with initial values $(\varphi(\theta), \psi(\theta))^T = (0.1, 0.09)^T$, $(0.5, 0.2)^T$, and $(1, 0.9)^T$, $\theta \in [-1, 0]$.

5 Conclusion

Huo *et al.* [5] and Li *et al.* [6] studied the stability property of the positive equilibrium of a stage-structured predator-prey model with modified Leslie-Gower and Holling-type II schemes. In those two papers, the authors only consider the stage structure of prey species and ignore that of predator species. Stimulated by [9–11], we consider a model with stage structure for both predator and prey species. By applying an iterative technique and the fluctuation lemma, sufficient conditions which guarantee the global attractivity of all the nonnegative equilibria are obtained. Our study indicates that both the stage structure of the species and the death rate of the mature predator and prey species are the important factors on the dynamic behaviors of the system. If the death rates of the mature prey and predator species are too large or the degree of the stage structure of the species is large enough, then at least one of the species will be driven to extinction. We would like to mention here that Example 4.5 shows that our result in Theorem 2.2 has room for improvement. We conjecture that condition (2.1) is enough to ensure the global attractivity of the positive equilibrium. We leave this for future work.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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