CONVERGENT HIGHER DERIVATIONS ON LOCAL RINGS

BY
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I. **Introduction.** In this paper we define a quasi-local ring R, or (R, M), to be a commutative ring with unity having a unique maximal ideal M such that $\bigcap_{n=1}^{\infty} M^n=\{0\}$. Thus a Noetherian quasi-local ring is a local ring. A higher derivation $D=\{D_i\}_{i=1}^{\infty}$ on a quasi-local ring R is said to be convergent if, for all a in R, $\sum_{i=0}^{\infty} D_i(a)$ is a convergent series in the M-adic topology. D_0 always denotes the identity mapping. If R is complete the mapping α_D: $a \to \sum_{i=0}^{\infty} D_i(a)$ is an endomorphism of R which induces the identity mapping on the residue field of R (Lemma 1). With suitable restrictions on D, α_D is an automorphism and hence an inertial automorphism. A seemingly "natural" additional condition sufficient to insure that α_D is an automorphism is the condition

$$(1) \qquad D_i(M) \subset M^2, \qquad i \geqq 1.$$

A convergent higher derivation which satisfies (1) is said to be M-convergent.

In a number of recent papers [4], [5], [7], Neggers, Wishart, and the author have used convergent higher derivations to study the inertial automorphisms of particular kinds of complete local rings. In particular Neggers [5] used higher derivations to relate properties of the higher ramification groups of a ramified v-ring to its derivation structure. The author has shown [4, Theorem 3.1] that if R is an unramified n-dimensional complete regular local ring then every inertial automorphism of R is of the form α_D where $D=\{D_{i_1,\dots,i_n}\}$ is a convergent higher derivation on n-indices. By defining H_m to be $\sum_{i_1+\dots+i_n=m} D_{i_1,\dots,i_n}$ one obtains a higher derivation H on one index such that $\alpha_H=\alpha_D$, and H is, in fact, what is called "strongly convergent" in this paper (Definition 3). The representation of inertial automorphisms by higher derivations provides a convenient means for determining the factor groups of the higher ramification groups of R in this case [4, Theorems 2.1, 2.2, 2.3].

This paper is primarily concerned with convergent higher derivations as such. A bit of calculation with the possibility of defining a composition of higher derivations so that the condition $\alpha_{D\circ D'}=\alpha_D\alpha_{D'}$ obtains leads to Definition 2. Theorem 1 asserts that the set of all higher derivations $H(R, R)$ on any (noncommutative) ring R is a group with respect to this composition. §II is concerned with closure properties of various convergent subsets of $H(R, R)$ with respect to both the group

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operation and taking inverses, all in the case in which R is quasi-local. Theorem 2 states that the convergent higher derivations form a subsemigroup $H_c(R, R)$ of $H(R, R)$ and Theorem 3 states that the subsets $H_c^M(R, R)$, $H_u^M(R, R)$, and H_t^M, (R, R) of M-convergent, uniformly M-convergent and strongly M-convergent higher derivations (see Definition 3) form subgroups of $H(R, R)$. An example following the proof of Theorem 2 illustrates the fact that D may be convergent and α_D may be an automorphism whereas D^{-1} is not convergent.

It is readily seen that if D is M-convergent then α_D is in H_1, the subgroup of those inertial automorphisms α satisfying the condition $\alpha(a)-a \in M^2$ for all a in M. Conversely, if $\alpha=\alpha_D$ and α is in H_1 then D is M-convergent. If R is a v-ring (unramified) every inertial automorphism is in H_1. If R is an unramified complete regular local ring then the mapping Δ: $D \to \alpha_D$ is a homomorphism of $H_c^M(R, R)$ onto H_1. As a matter of fact Δ restricted to $H_t^M(R, R)$ still maps onto H_1. It follows from work of Wishart [7, pp. 50, 51] that a ramified v-ring may have inertial automorphisms represented by D in $H_c^M(R, R)$ but not by D in $H_t^M(R, R)$.

§III deals with the question of necessary and sufficient conditions on a complete local ring R that every convergent higher derivation be uniformly convergent. Theorem 5 asserts that if the residue field k has characteristic p, the condition that k have a finite p-basis is sufficient and if R is regular this condition is necessary. If R is regular and k has characteristic zero (R is a power series ring over k) then every convergent higher derivation is uniformly convergent if and only if k has finite transcendency degree over its prime field.

II. **Closure properties.** Initially we assume S to be an arbitrary associative ring and R an over ring of S.

Definition 1. A higher derivation D of S into R is a set $\{D_i\}_{i=1}^{\infty}$ of mappings of S into R such that for all $i \geqq 1$ and all a, b in S,

(i) $D_i(a+b)=D_i(a)+D_i(b)$,

(ii) $D_i(ab)=\sum_{j=0}^{i} D_j(a)D_{i-j}(b)$,

where D_0 denotes the identity mapping. The symbol $H(S, R)$ will designate the set of all higher derivations of S into R, and Q will represent the higher derivation $\{Q_i\}$ such that Q_i is the zero mapping for all $i \geqq 1$.

Definition 2. If H and D are in $H(S, S)$ then $K=H \circ D$ is the set of mappings $\{K_i\}_{i=1}^{\infty}$ where

$$K_i = \sum_{j=0}^{i} H_j D_{i-j}. \tag{2}$$

Proposition 1. *The set of mappings K as defined by* (2) *is a higher derivation.*

Proof. Proposition 1 and Theorem 1, below, follow immediately from the following fact first observed by Schmidt [6]. Let G represent the group of all automorphisms α on the power series ring $R[[X]]$ satisfying the conditions (i) $\alpha(X)=X$ and (ii) $\eta\alpha(a)=a$ for a in R where $\eta(\sum a_i X^i)=a_0$. Given $\alpha \in G$,

$D^{\alpha}=\{D^{\alpha}_i\}$ is in $H(R, R)$ where $D^{\alpha}_i(a)$ is the coefficient of X^i in $\alpha(a)$. The mapping $\alpha \to D_{\alpha}$ is a one-to-one correspondence between G and $H(R, R)$ which then induces a group structure on $H(R, R)$, the induced operation being (2). Thus, we have

THEOREM 1. *Given any ring R, $H(R, R)$ is a group with respect to the composition* (2).

For later use we exhibit below an explicit description of D^{-1} in terms of D. Let (r, n) be a partition of the integer n into r nonnegative summands. If $(r, n)=i_1, \ldots, i_r$ we let $[D]_{(r,n)}$ be the sum of the formally distinct products of the r maps $D_{i_1}, \ldots, D_{i_r}$. Thus, if $(3, 5)=\{1, 2, 2\}$ then $[D]_{(3,5)}$ is $D_1D_2^2+D_2D_1D_2+D_2^2D_1$. Given D in $H(R, R)$ we define $\bar{D}$ by

$$\bar{D}_n = \sum_{(r,n)} (-1)^r[D]_{(r,n)}, \qquad n \geqq 1, \tag{3}$$

and contend that $\bar{D}=D^{-1}$.

The expression $\sum_{i=0}^{n} D_i\bar{D}_{n-i}$ is a sum of terms of the form $D_{j_1} \cdots D_{j_{r+1}}$ each such terms occuring twice in $D_{j_1}\bar{D}_{n-j_1}$ with coefficient $(-1)^r$ and in $D_0\bar{D}_n$ with coefficient $(-1)^{r+1}$. Hence $D \circ \bar{D}=Q$. But this equality uniquely determines the set of maps $\bar{D}$ and thus $\bar{D}=D^{-1}$.

LEMMA 1. *Let (R, M) be a quasi-local ring and let S be a subring of R with the property that every nonunit of S is in M. If D in $H(S, R)$ converges then $D^{(i)}(S)\subset M$ for $i>0$.*

Proof. Let u be a unit in S such that $D_i(u)$ is a unit for some $i>0$ and let n be the least such integer. Since

$$0 = D_n(1) = D_n(uu^{-1}) = uD_n(u^{-1})+u^{-1}D_n(u)+\sum_{i=1}^{n-1} D_i(u)D_{n-i}(u^{-1}),$$

it follows that $D_n(u^{-1})$ is also a unit. Since D converges there is a largest integer, say s, such that $D_s(u)$ is a unit, and a largest integer t such that $D_t(u^{-1})$ is a unit. Now $0=D_{s+t}(1)=D_{s+t}(uu^{-1})$ and $D_{s+t}(uu^{-1})=D_s(u)D_t(u^{-1})$, mod M, which yields a contradiction. Thus $D_i(u)$ is in M for all units u. Next, let a be in $S \cap M$. Then $D_i(1+a)=D_i(1)+D_i(a)$ is in M and thus $D_i(a)$ is in M. This proves Lemma 1.

THEOREM 2. *If R is a quasi-local ring the set $H_c(R, R)$ of convergent higher derivations on R is a subsemigroup of $H(R, R)$.*

Proof. Let D and H be in $H_c(R, R)$. Given a in R and a positive integer n, there is an integer m such that if $i\geqq m$ then $H_i(a)$ is in M^n and there exists an integer t such that if $i\geqq t$ then $D_iH_j(a)$ is in M^n for $j=0, 1, \ldots, m-1$. It is readily seen from (ii) of Definition 1 and from Lemma 1 that $D_i(M^t)\subset M^t$ for all $i>0$, and $t>0$. Thus, if s is the maximum of $2m$ and $2t$ and if $j>s$ then $\sum_{i=0}^{j} D_jH_{i-j}(a)$ is in M^t. Thus $D \circ H$ is in $H_c(R, R)$.

A simple example illustrates the fact that a convergent higher derivation need not have a convergent inverse. Let k be any field and let $k[[X]]$ be the power series ring in the indeterminate X over k. We define $D \in H(k[[X]], k[[X]])$ by the conditions

(i) $D_j(a) = 0$ for $a \in k$ and all $j > 0$;

(ii) $D_1(X) = X$, $D_i(X) = 0$ for $i \geqq 2$.

These conditions determine an obviously unique higher derivation by [2, Theorem 2] and Proposition 2 which appears later in this paper. We note that:

$$D_n^{-1}(X) = \sum_{(r,n)} (-1)^r [D]_{(r,n)}(X) = (-1)^n D_1^n(X) = (-1)^n X.$$

Since this is true for any $n > 0$ it follows that D^{-1} does not converge. Note, however, that α_D is an automorphism. As this example suggests a sufficient condition for $D \in H_c(R, R)$ to have a convergent inverse is that $D(M) \subset M^2$, by which is meant $D_i(M) \subset M^2$ for all $i > 0$. We shall see (Lemma 5) that this condition is fullfilled if R is a v-ring, a one-dimensional complete regular local ring having characteristic zero with residue field having characteristic $p \neq 0$.

Definition 3. Let (R, M) be a quasi-local ring and let S be a subring. D in $H_c(S, R)$ is said to be

(a) M-convergent if $D(S \cap M) \subset M^2$;

(b) uniformly M convergent if D is M-convergent and converges uniformly;

(c) strongly convergent if $D_i(S) \subset M^i$ for $i = 1, 2, \ldots$ Strong M-convergence is defined as in (b).

The symbols $H_u(S, R)$ and $H_t(S, R)$ will represent the subsets of $H(S, R)$ consisting of the uniformly convergent D and the strongly convergent D respectively. A superscript M indicates M convergence i.e. $H_u^M(S, R)$ is the set of all uniformly M convergent D in $H(S, R)$.

Theorem 3. *Let R be a quasi-local ring. $H_c^M(R, R)$, $H_u^M(R, R)$ and $H_t^M(R, R)$ are all subgroups of $H_c(R, R)$. $H_u(R, R)$ is a subsemigroup of $H_c(R, R)$.*

Proof. Obviously the product of M-convergent higher derivations is M-convergent. We note that if D and H of the proof of Theorem 2 are in $H_u(R, R)$ then the proof is independent of the choice of a and hence $D \circ H$ is in $H_u(R, R)$. If D is in $H_t^M(R, R)$ then

$$D_i(M^j) \subset M^{i+j}, \qquad i \geqq 1, \quad u \geqq 0. \tag{4}$$

Relation (4) implies closure in $H_t^M(R, R)$ and also leads immediately to the conclusion that if (r, i) is any partition of i and $D \in H_t^M(R, R)$ then $[D]_{(r,i)}(M^j) \subset M^{i+j}$. Thus if D is in $H_t^M(R, R)$ so is D^{-1}. The example following Theorem 2 is a strongly convergent higher derivation. If D represents the higher derivation in question and $H = D \circ D$ then $H_2(X) = X$, illustrating the fact that $H_t(R, R)$ is neither closed with respect to product nor with respect to taking inverse.

In order to verify that the inverse of D in $H_c^M(R, R)$ is in $H_c^M(R, R)$ it is sufficient to show that, given a in R and $m \geqq 0$, there is an integer n such that if $i_1, \ldots, i_r$ is any partition into positive integers of $t > n$ then,

$$D_{i_1} \cdots D_{i_r}(a) \in M^m. \tag{5}$$

Since for D in $H_c^M(R, R)$

$$D_i(M^j) \subset M^{j+1}, \qquad i > 0, \;\; j \geqq 0$$

it follows that (5) holds if $r \geqq m$. There is an integer n_1 such that if $i > n_1$ then $D_i(a) \in M^m$ and an integer n_2 such that if $i_2 > n_2$ then $D_{i_2} D_{i_1}(a) \in M^m$ for $i_1 = 1, 2, \ldots, n_1$.

Iteratively, we define integers $n_1, n_2, \ldots, n_{m-1}$ such that, if $0 < j < m$ and $i_j > n_j$, then $D_{i_j} D_{i_{j-1}}, \ldots, D_{i_1}(a) \in M^m$ if $0 < i_t \leqq n_t$ for $t = 1, \ldots, j-1$. Let n' be the maximum of $n_1, n_2, \ldots, n_{m-1}$ and let $n = m(n'+1)$. If $j_1, \ldots, j_r$ are positive integers such that $j_1 + \cdots + j_r > n$ then either $r \geqq m$ or $j_t > n'$ for some t. In either case $D_{i_1}, \ldots, D_{i_r}(a) \in M^m$. It follows then from (3) that if D is M convergent so is D^{-1}. If D is in $H_u^M(R, R)$ the above argument again applies independently of the choice of a. We conclude that D^{-1} is in $H_u^M(R, R)$ if D is in $H_u^M(R, R)$.

III. **Uniformly convergent higher derivations.** We begin with some basic facts about extensions of higher derivations and their convergence properties. Let T be a commutative overring of a ring S and let $a \in S$ be invertible in S. Then if $D \in H(S, T)$

$$D_n(a^{-1}) = \sum_{(r,n)} (-1)^{r+1} a^{-(r+1)} C(r, n)[D(a)]_{(r,n)} \tag{6, n}$$

where $C(r, n) = r!/(n_1! \cdots n_t!)$ and $n_1, \ldots, n_t$ represent the number of times the distinct integers of (r, n) occur in (r, n). Also if $(r, n) = j_1, \ldots, j_r$ then $[D(a)]_{(r,n)}$ is the sum of all the formally distinct products of the r quantities $D_{j_1}(a), \ldots, D_{j_r}(a)$. For $n=1$ we have $D_1(a^{-1}) = -a^{-2} D_1(a)$. Proceeding by induction, $0 = D_n(aa^{-1}) = \sum D_i(a) D_{n-i}(a^{-1})$ or $D_n(a^{-1}) = -a^{-1} \sum_{i=0}^{n-1} D_{n-i}(a) D_i(a^{-1})$. Substitution of (6, i) in the right hand side of this equality for $i = 1, \ldots, n-1$ yields (6, n) without difficulty. Let T and S be as above and let D be in $H(S, T)$. The mapping $\tau_D \colon S \to T[[X]]$ given by (7) is an isomorphism with the property $\eta\tau_D$ is the

$$\tau_D(a) = \sum_{i=0}^{\infty} D_i(a) x^i \tag{7}$$

identity on S where again $\eta(\sum a_i X^i) = a_0$. Conversely, if $\tau \colon S \to T[[X]]$ is a homomorphism such that $\eta\tau$ is the identity on S then $\tau(a) = a + \sum X^i D_i^\tau(a)$ and $D^\tau = \{D_i^\tau\}$ is in $H(S, T)$. As in the proof of Theorem 1, $D \to \tau_D$ is a one-to-one correspondence between $H(S, T)$ and the set of isomorphisms τ of S into $T[[X]]$ such that $\eta\tau$ is the identity map on S.

Let M be a multiplicatively closed subset of S each element of which has an inverse in T. Thus S_M the ring of quotients with respect to M is a subring of T.

LEMMA 2. *Each D in $H(S, T)$ has a unique extension to $H(S_M, T)$.*

Proof. The lemma follows from the existence and uniqueness of the extension of τ_D to S_M.

LEMMA 3. *Let S be a subring of the quasi-local ring (R, M) and let B be a subset of R. Let D be in $H(S[B], R)$.*

(i) *If D converges on S and on B then $D \in H_c(S[B], R)$.*

(ii) *If D is uniformly convergent on S and on B and $D(S[B]) \subset M$ then $D \in H_u(S[B], R)$.*

(iii) *If D is strongly convergent on S and on B then $D \in H_t(S[B], R)$.*

(iv) *If $D(S \cap M) \subset M^2$ and $D(B \cap M) \subset M^2$ then $D(S[B] \cap M) \subset M^2$.*

Proof. Each element in $S[B]$ is a sum of terms of the form $sb_1 \cdots b_t$ where $s \in S$; $b_1, \ldots, b_t \in B$ *and* $t \geqq 0$. Now

$$D_n(s, b_1, \ldots, b_t) = \sum_{i_0 + \cdots + i_t = n} D_{i_0}(s) D_{i_1}(b_1) \cdots D_{i_t}(b_t). \tag{8}$$

Clearly, if D converges at $s, b_1, \ldots, b_t$ then D converges at $sb_1 \cdots b_t$.

Statement (ii) is a consequence of the following lemma which will be useful elsewhere.

LEMMA 4. *Let S be a subring of a quasi-local ring (R, M) and let B be a subset of S. If $D \in H(S, R)$ converges uniformly on B and $D(B) \subset M$ then given $n > 0$, there is an $m > 0$ such that given any product $b_1 \cdots b_t$ of $t \geqq 1$ elements in B, $D_{i_1}(b_1) \cdots D_{i_t}(b_t) \in M^n$ whenever $i_1 + \cdots + i_t > m$.*

Proof. There is an integer r such that if $i > r$, then $D_i(B) \subset M^n$. Let $m = nr$. Then, if $i_1 + \cdots + i_t > m$ either n of the i's are different from zero or one of them is greater than r. In either case $D_{i_1}(b_1) \cdots D_{i_t}(b_t)$ is in M^n.

To prove (iii) of Lemma 3 we simply observe that if $D_i(a) \in M^i$ for a in S or in B then (8) is in M^n. Statement (iv) is immediate.

COROLLARY 3.1. *If $D \in H_c(S[B], R)$ converges uniformly on S, where B is a finite set and $D(S[B]) \subset M$, then $D \in H_u(S[B], R)$.*

COROLLARY 3.2. *Let M be a multiplicatively closed subset of S each element of which has an inverse in R and let $\bar{D} \in H(S_M, R)$ be the extension of $D \in H(S, R)$. If $D(S) \subset M$, it follows that*

(i) *if $D \in H_c(S, R)$ then $\bar{D} \in H_c(S_M, R)$;*

(ii) *if $D \in H_u(S, R)$ then $\bar{D} \in H_u(S_M, R)$;*

(iii) *if $D \in H_t(S, R)$ then $\bar{D} \in H_t(S_M, R)$.*

Proof. Let M^{-1} denote the set of inverses of the elements of M. Then $\bar{D}(M^{-1}) \subset M$ in view of (6) and the assumption that $D(S) \subset M$. Also, it follows from Lemma 4 and (6) that if $D \in H_c(S, R)$ then $\bar{D}$ converges on M^{-1}, and if $\bar{D} \in H_u(S, R)$ then $\bar{D}$ converges uniformly on M^{-1}. If $D \in H_t(S, R)$ it is apparent from (6) that D is strongly convergent on M^{-1}. The observation that $S_M = S[M^{-1}]$ and an appeal to Lemma 3 completes the proof.

The symbol V will represent a valuation ring having characteristic zero with residue field k of characteristic $p\neq 0$. Let π be a prime element of V and let e be the ramification of V, that is $pV=\pi^e V$, and we write $e=p^s r$ where $(p, r)=1$. Let (R, M) be a regular local ring containing V such that $\pi V=V\cap M$.

LEMMA 5. *Each D in $H_c(V, R)$ has the property $D(\pi V)\subset M^2$ and thus $H_c(V, R) =H_c^M(V, R)$.*

Proof. For some positive integer t, $\pi V\subset M^t\backslash M^{t+1}$, i.e. $\pi V\subset M^t$ but $\pi V\not\subset M^{t+1}$. Thus $\pi\in M^t\backslash M^{t+1}$. Let i be the least integer such that $D_i(\pi)\notin M^2$. We assume $t>1$. Now

$$D_{ir}(\pi^r) = [D_i(\pi)]^r + \sum_{i_1+\cdots+i_r=ir;\ \text{some } i_j<i} D_{i_1}(\pi),\ldots, D_{i_r}(\pi).$$

Since $[D_i(\pi)]^r\in M^r\backslash M^{r+1}$ and the second term is seen to be in M^{r+1} we have $D_{ir}(\pi^r)\in M^r\backslash M^{r+1}$. Similarly,

$$(9)\quad D_{p^s ir}(\pi^{p^s r}) = [D_{ir}(\pi^r)]^{p^s} + \sum_{i_1+\cdots+i_{ps}=p^s ir;\ i_j\neq i_k \text{ for some } j,k.} D_{i_1}(\pi^r)\cdots D_{i_{ps}}(\pi^r).$$

Again $[D_{ir}(\pi^r)]^{p^s}\in M^{rp^s}\backslash M^{rp^s+1}$ and the remaining term on the right of (9) is seen to be in M^{rp^s+1} since each summand occurs a multiple of p times. We conclude from (9) that

$$D_{p^s ir}(\pi^{p^s r})\in M^{p^s r}\backslash M^{p^s r+1}.$$

For some unit u in $Vp=u\pi^{p^s r}$ and

$$(10)\qquad 0 = D_{p^s ir}(p) = uD_{p^s ir}(\pi^{p^s r}) + \sum_{j=1}^{p^s ir-j} D_j(u)D_{p^s ir-j}(\pi^{p^s r}).$$

By an argument like that above applied to the right side of (10) we conclude that $D_{p^s ir}(p)\in M^{p^s r}\backslash M^{p^s r+1}$ which is the desired contradiction.

If $t=1$ then we observe as above that $D_i(\pi^r)\in M^r\backslash M^{r+1}$ and hence that $D_{p^s i}(\pi^{p^s r})\in M^{p^s r}\backslash M^{p^s r+1}$. It follows that $D_{p^s i}(p)=D_{p^s i}(u\pi^{p^s r})\notin M^{p^s r+1}$; a contradiction. This proves Lemma 5.

LEMMA 6. *If D is in $H(V, R)$ and a is in V then $D_i(a^{p^n})\subset M^j$ for $i<p^{n-j}$.*

Proof. We note that

$$(11)\quad \begin{aligned} D_i(a^{p^n}) &= \sum_{i_1+\cdots+i_{pn}=i} D_{i_1}(a)\cdots D_{i_{pn}}(a) \\ &= C[p^n; q_1,\ldots,q_t][D_{j_1}(a)]^{q_1},\ldots,[D_{j_t}(a)]^{q_t} \end{aligned}$$

where the set $i_1,\ldots, i_{p_n}$ consists of q_r integers j_r for $r=1,\ldots, t$ and $C[p^n; q_1,\ldots, q_t]$ is the indicated multinomial coefficient. Since $i<p^{n-j}$, and hence $q_r<p^{n-j}$ for at least one q_r, it follows that the maximum integer t such that $p^t|q_r$, for all q_r is less than $n-j$. Thus $p^j|C[p^n; q_1,\ldots, q_t]$. (Here we are using the fact that if s is the largest integer such that $p^s|q_r$ for all r then $p^{n-s}|C[p^n; q_1,\ldots, q_t]$.) It follows from (11) that $D_i(a^{p^n})\subset M^j$.

We now make an additional assumption on V and R, namely that R is complete in the M-adic topology and V is a complete subring with $e=1$.

THEOREM 4. *Let $\bar{S}$ be a p-basis for k the residue field of V and let $S \subset V$ be a set of representatives of the elements in S. If f is a mapping of $S \times I$ into R where I denotes the positive integers then*

(a) *There is one and only one $D \in H(V, R)$ with the property $D_i(s)=f(s, i)$ for (s, i) in $S \times I$.*

(b) *D is in* (i) $H_c^M(V, R)$, (ii) $H_u^M(V, R)$, (iii) $H_t^M(V, R)$ *if and only if $D(S) \subset M$ and* (i) *D converges on S,* (ii) *D converges uniformly on S,* (iii) *$D_i(S) \subset M^i$ for $i=1, 2, \ldots$.*

Proof. In order to prove part (a) we consider V_0 the complete subring of V having residue field k_0, the maximal perfect subfield of k. Since $\bar{S}$ is an algebraically independent set over k_0, S is algebraically independent over V_0. Thus by a standard Zorn's Lemma argument using [2, Theorem 2] we can define $H \in H(V_0(S), R)$ by the conditions (i) H restricted to V_0 is the zero higher derivation and (ii) $H_i(s)=f(s, i)$ for $s \in S$ and $i \in I$.

Let $\bar{U}$ be a basis for k as a linear space over $k_0(\bar{S})$ and let U be a set of representatives in V of the elements in $\bar{U}$. We assume that 1 is in U.

The set $\bar{U}^{p^n}$ of p^nth powers of elements of $\bar{U}$ is also a basis for k over $k_0(\bar{S})$ [3, p. 347]. If $V_0(S)$ is the ring of rational functions over V_0 in the elements of S then $V_1=V_0(S) \cap V$ is a valuation ring with residue field $k_0(S)$. Thus, given $a \in V$, there are elements $a_1, \ldots, a_n$ in V_1 and $u_1, \ldots, u_n$ in U such that

$$a = \sum a_i u_i^{p^n}, \qquad \bmod p^n. \tag{12}$$

Moreover, the a_i are uniquely determined, mod p^n, by the condition (12).

For $i=1, \ldots, m$ and $a \in V$, let

$$D_i^{(m)}(a+p^m V) = \sum H_i(a_j)u_j^{p^{3m}} + M^m, \tag{13}$$

where $a=\sum a_j u_j^{p^{3m}}$, mod p^m, according to (12). The fact that the a_j are determined, mod p^n, assures that $D_i^{(m)}$ is a well defined map of $V/p^m V$ into R/M^m. We define the desired $D \in H(V, R)$ by the coset intersection

$$D_i(a) = \bigcap_{m>i} D_i^{(m)}(a+p^m V). \tag{14}$$

The following equalities which will be verified in turn, permit us to conclude that D is a higher derivation. For A and B in $V/p^m V$

$$D_i^{(m)}(A+B) = D_i^{(m)}(A)+D_i^{(m)}(B) \qquad \text{for } i = 1, \ldots, m, \tag{15}$$

$$D_i^{(m)}(AB) = \sum_{j=0}^{i} D_j^{(m)}(A)D_{i-j}^{(m)}(B) \tag{16}$$

and for $a \in V$ the following coset inclusion holds.

$$D_i^{(m)}(a+p^m V) \supset D_i^{(m+1)}(a+p^{m+1} V). \tag{17}$$

Statement (15) is clear from the definition. In order to establish (16) we let $A=\sum a_k u_k^{p^{3m}}+p^m V$ and $B=\sum b_j u_j^{p^{3m}}+p^m V$, using (12). Thus

$$AB = \sum a_k b_j u_k^{p^{3m}} u_j^{p^{3m}} + p^m V. \tag{18}$$

Using (12) we have $u_k u_j = \sum d_r u_r$, mod pV. Thus [2, Lemma 1],

$$u_k^{p^{3m}} u_j^{p^{3m}} = \sum_{t=0}^{3m-1} p^t \sum_l s_{k,j,t,l} c_{k,j,t,l}^{p^{3m-t}} u_l^{p^{3m}}, \qquad \bmod p^{3m} V, \tag{19}$$

where $s_{k,j,t,l}$ is a rational integer and $c \in V_1$. Substituting (19) into (18) we have

$$D_i^{(m)}(AB) = \sum H_i\left(\sum a_k b_j p^t s_{k,j,t,l} c_{k,j,t,l}^{p^{3m-t}}\right) u_l^{p^{3m}} + M^m. \tag{20}$$

Since p and $s_{k,j,t,l}$ are rational integers $H_i(p^t)=H_i(s_{k,j,t,l})=0$, for all i. Also, by Lemma 6, $H_i(c_{k,j,t,l}^{p^{3m-t}})$ is in M^m if $t<m$, since $i \leqq m$. Thus, mod M^m, we have

$$\begin{aligned} H_i\left(\sum a_k b_j p^t s_{k,j,t,l} c_{k,j,t,l}^{p^{3m-t}}\right) &= \sum H_i(a_k b_j) p^t s_{k,j,t,l} c_{k,j,t,l}^{p^{3m-t}} \\ &= \sum \sum_{r=0}^{i} H_r(a_k) H_{i-r}(b_j) p^t s_{k,j,t,l} c_{k,j,t,l}^{p^{3m-t}}. \end{aligned}$$

Thus, substituting this last expression into (20) and then using (19) we find that (20) reduces to $\sum H_r(a_k)H_{i-r}(b_j) u_k^{p^{3m}} u_j^{p^{3m}} + M^m$ from which (16) follows.

Relation (17) can be verified as follows. Using (12) for $n=1$ we have $u_k^{p^3}=\sum a_i u_i$, mod p, the a_i being in V_1. Thus [2, Lemma 1]

$$u_k^{p^{3(m+1)}} = \left[\sum a_i u_i\right]^{p^{3m}} = \sum_{t=0}^{3m-1} p^t \sum s_{k,t,n} c_{k,t,n}^{p^{3m-t}} u_n^{p^{3m}}, \qquad \bmod p^{3m} V. \tag{21}$$

Again, $s_{k,t,n}$ is a rational integer and $c_{k,t,n} \in V_1$. Thus if $a+p^{m+1}V=\sum a_k u_k^{p^{3(m+1)}} + p^{m+1}V$ then

$$\begin{aligned} D_i^{(m+1)}(a+p^{m+1}V) &= \sum H_i(a_k) u_k^{p^{3(m+1)}} + M^{m+1} \\ &= \sum H_i(a_k) \sum_{t=0}^{3m-1} p^t \sum s_{k,t,n} c_{k,t,n}^{p^{3m-t}} u_n^{p^{3m}} + M^{m+1} \\ &= \sum H_i\left(a_k \sum p^t \sum s_{k,t,n} c_{k,t,n}^{p^{3m-t}}\right) u_n^{p^{3m}} + M^{m+1}. \end{aligned} \tag{22}$$

But, $a+p^m V = \sum (a_k \sum p^t \sum s_{k,t,n} c_{k,t,n}^{p^{3m-t}}) u_n^{p^{3m}} + p^m V$ and

$$D_i^{(m)}(a+p^m V) = \sum H_i\left(a_k \sum p^t s_{k,t,n} c_{k,t,n}^{p^{3m-t}}\right) u_n^{p^{3m}} + M^m.$$

Relation (17) then follows in view of (22).

Since $\bigcap_{n=1}^{\infty} M^n = 0$, D_i as defined by (14) is a uniquely determined element of R. Properties (15) and (16) assure that conditions (i) and (ii) of Definition 1 hold mod M^m for all m. Thus D is a higher derivation.

In order to show that D is an extension of H we note that if $a \in V_1$ then $D_i^{(m)}(a) = H_i(a) + M^m$ since $1 \in U$. Thus $D_i(a) = \bigcap_m D_i^{(m)}(a+p^m V) = H_i(a)$.

It remains to show that D is determined by $W=\{D_i(s)\}_{i=1,s\in S}^{\infty}$. Certainly, the restriction of D to $V_1 \subset V_0(S)$ is completely determined by W since $D_i(a)=0$ for $i>0$ and a in V_0 by Lemma 6 and the fact that V_0 is for each $n>0$ the completion of the subring generated by the p^nth powers of elements in V_0. Let a be any element in V. By (12) $a=\sum a_i u_i^{p^{3m}}$, mod p^{3m+1}, where the a_i are in V_1. If $j<m$,

$$D_j\left(\sum a_i u_i^{p^{3m}}\right) = \sum D_j(a_i) u_i^{p^{3m}}, \qquad \text{mod } M^m,$$

by Lemma 6. Hence $D_j(a)=\sum D_j(a_i)u_i^{p^{3m}}$, mod M^m. Thus D is determined, mod M^m by its restriction to S. But m is arbitrary. It follows that D is uniquely determined by its action on S. This proves (a) of Theorem 4.

If D in $H(V, R)$ converges then $D(V) \subset M$. Hence the condition $D(S) \subset M$ is necessary for D to be in $H_c^M(V, R)$, $H_u^M(V, R)$ or $H_t^M(V, R)$. The remaining condition is clearly necessary in each case.

Let D in $H(V, R)$ be such that $D(S) \subset M$ and $\sum D_i(s)$ converges for all $s \in S$. To show that D is in $H_c^M(V, R)$ it is only necessary to show that D converges in view of Lemma 5. Given $n>0$. By Lemma 6 $D_j(V^{p^{n+1}}) \subset M$ for $j \leq n$. But $V=V^{p^{n+1}}[S]+pV$ and hence $D_j(V) \subset M$ or

$$D(V) \subset M. \tag{23}$$

Lemma 7. *Let (T, M) be a quasi-local ring with residue field having characteristic $p \neq 0$. Let S be a subring of T. If $D \in H(S, T)$ maps S into M then*

$$D(S^{p^n}) \subset M^{n+1}, \qquad \text{for } n = 1, 2, \ldots. \tag{24}$$

Proof. We argue by induction on n using

$$D_i(a^p) = pa^{p-1}D_i(a) + \sum_{i_1+\cdots+i_p=i;\, i_j<i} D_{i_1}(a)\cdots D_{i_p}(a). \tag{25}$$

Since at least two of the integers $i_1, \ldots, i_p$ are different from zero $D_i(a^p)$ is in M^2. If in (25) $a=b^{p^n}$ then, by induction, $D(b^{p^n}) \in M^{n+1}$ and hence $D_i(b^{p^{n+1}}) \in M^{n+2}$.

By relation (23) and Lemma 7 then $D(V^{p^n}) \subset M^{n+1}$. Given a in V and $t>0$, $a=f(s_1, \ldots, s_q)$, mod $p^t V$, where $f \in V^{p^t}[X_1, \ldots, X_q]$ has degree $<p^t$ in each X_i, and $\{s_1, \ldots, s_q\} \subset S$. We choose n so that if $i>n/qp^t$ then $D_i(s_j) \in M^t$ for $j=1, \ldots, q$.

$$D_i(bs_1^{n_1}\cdots s_q^{n_q}) = \sum_{i_0,\ldots,i_{n_1+\ldots+n_q}=i} D_{i_0}(b)D_{i_1}(s_1)\cdots D_{i_{n_1+\cdots+n_q}}(s_q). \tag{26}$$

If $i>n$ in (26) either $i_0>0$ or $i_j>n/qp^t$ for some $j>0$. Thus, since $b \in V^{p^t}$, $D_{i_0}(b)D_{i_1}(s_1)\cdots D_{i_{n_1+\cdots+n_q}}(s_q)$ is in M^t. Since every term in $V^{p^t}[s_1, \ldots, s_q]$ is of the type treated in (26) it follows that, if $i>n$, $D_i(a) \in M^t$. Thus D converges.

If D converges uniformly on S then the n of the previous paragraph can be chosen so that if $i>n/qp^t$ then $D_i(S) \subset M^t$, from which it follows that $D_i(V) = D_i(V^{p^t}[S]+p^tV) \subset M^t$.

Thus $D \in H_u^M(V, R)$. Similarly, if $D_i(S) \subset M^i$ a like argument leads to the conclusion that $D_i(V) \subset M^i$.

THEOREM 5. *If (R, M) is a complete local ring with residue field k having characteristic $p\neq 0$ then $H_c(R, R)=H_u(R, R)$ if k has a finite p-basis. If R is regular $H_c(R, R)=H_u(R, R)$ only if k has a finite p-basis.*

Proof. As in Theorem 4 we let S be a set of units in R which map biuniquely onto a p-basis $\bar{S}$ for k under the canonical map of R onto k. It is assumed that $\bar{S}$ is finite. Let $\mathscr{M}$ be the set of multiplicative representatives of the element in k_0, the maximal perfect subfield of k. We choose an arbitrary D in $H_c(R, R)$ and observe first that $D(\mathscr{M})=\{0\}$, by Lemma 7 since each a in $\mathscr{M}$ is a p^mth power for all m. Thus if T is the subring of R generated by $\mathscr{M}$ then $D|_T$, the restriction of D to T is the zero higher derivation. By Corollary 3.1 $D|_{T[S]}$ is uniformly convergent.

Let U be a subset of R which maps biuniquely onto $\bar{U}$ a basis for k as a linear space over $k_0(S)$. As we have observed before the set U_n of p^nth powers of the elements in U maps onto a basis for k over $k_0(\bar{S})$.

Let $t>0$ be fixed. If $M=\sum_{i=1}^{s} w_iR$, then $a\in R\Rightarrow$:

$$a = \sum_i f_i u_i^{p^t}+\mu, \qquad \mu\in M^t,\ \ f_i\in T[S][w_1,\dots, w_s].$$

Hence applying Corollary 3.1 to obtain $D|T[S][w_1,\dots, w_s]$ uniformly convergent, we pick an n such that $j>n$ implies

$$D_j(T[S][w_1,\dots, w_s]) \subset M^t.$$

Thus since $D(M^t)\subset M^t$,

$$D_j(a) = D_j\Big(\sum_i f_i u_i^{p^t}\Big)+D_j(\mu)\in M^t.$$

Since the choice of n depends only on t, S, and $\{w_1,\dots, w_s\}$ it follows that D converges uniformly on R. Inclusion the other way is obvious so the first part of Theorem 5 is proved.

In proving the rest of Theorem 5 we will have use for the following proposition whose proof is standard and will be omitted.

PROPOSITION 2. *Let S be a subring of a complete local ring (R, M) and let D be in $H(S, R)$. If D is continuous in the induced topology then D extends and in only one way to a higher derivation D^* on S^* the completion of S in R. If D is convergent so is D^*. If D is uniformly convergent so is D^*. If $D(S)\subset M$ then $D^*(S^*)\subset M$.*

Assuming R to be regular we consider the converse. If R has characteristic p then R is a power series ring $k[[X_1,\dots, X_n]]$ in a finite number of indeterminates $X_1,\dots, X_n$ over its residue field k. We assume that k possesses a p-basis S with infinite cardinal. Let $\{s_i\}_{i=1}^{\infty}$ be a countable sequence of elements in S. A higher derivation $D^{(i)}$ in $H(k, k[[X_1,\dots, X_n]])$ is uniquely determined by the conditions (i) $D_j^{(i)}(s_i)=\delta_{i,j}$ (ii) $D_j^{(i)}(s)=0$ for $j\geqq 1$ and $s\in S$, $s\neq s_i$ [2, Theorem 1]. The theorem referred to here applies to $D\in H(k, k)$ but the proof applies to the case in which the range of D is a ring containing k. Let $H^{(i)}$ be defined by $H^{(i)}_{ni}=X_1^n D^{(i)}_n$, $n\geqq 1$, and $H^{(i)}_m=\theta$, for m not a multiple of i, θ being the zero map. $H^{(i)}$ so defined is a convergent higher derivation. $H^{(i)}$ is extended to $H^{(i)}$ on $k[X_1,\dots,X_n]$ by the condition $H^{(i)}_j(X_t)=0$ for $j\geqq 1$, and $t=1,\dots,n$. $H^{(i)}$ extended is again, by Lemma 3, a convergent higher derivation. Finally, let $E=H^{(1)}\circ H^{(2)}\circ\cdots\circ H^{(n)}\circ\cdots$. Thus $E_n=(H^{(1)}\circ\cdots\circ H^{(n)})_n$ since $H^{(i)}_m=\theta$ for $m<i$. It follows readily that E is a well-defined higher derivation, and is clearly convergent. Let E^* represent the extension of E to $k[[X_1,\dots,X_n]]$. Again by Proposition 2, E^* is a convergent higher derivation. It follows immediately from the definition of E^* that $E^*_i(s_i)$ is in M and not in M^2. Hence, E^* is not uniformly convergent.

Assume now that R has characteristic zero. Then $R=R_1[\pi]$ where

$$R_1 = V[[X_1,\dots,X_n]]$$

is a power series ring in n indeterminates over an unramified v-ring V and π is a root of an Eisenstein polynomial f over R [1, Theorem 1].

The following facts will be useful. Let K be the quotient field of R_1.

(A) A given higher derivation on R_1 has a unique extension to a higher derivation on K. This follows from Lemma 2.

(B) A higher derivation D on K has a unique extension $\bar{D}$ to $K[\pi]$ [2, Theorem 3]. If D is convergent on K, $\bar{D}$ will be convergent if and only if $\sum \bar{D}_i(\pi)$ converges. If $D(R_1)\subset R$ then $\bar{D}(R)\subset R$ if and only if $\bar{D}(\pi)\in R$.

Let the minimal polynomial f of π over R, be $f=X^e+f_{e-1}X^{e-1}+\cdots+f_0$ and let f' denote the ordinary derivative of f.

LEMMA 8. *If $f'(\pi)\in M^t\backslash M^{t+1}$ and $D\in H_c(R_1,R_1)$ is such that $D(f_j)\in M^{3t-j}$ for $j=0,\dots,e-1$ then the extension of D to R will be convergent and will map R into R.*

Proof. We choose the same symbol D for the extension of the given higher derivation. Application of the defining properties of a higher derivation to $D_i(f(\pi))=0$ yields

$$\begin{aligned} f'(\pi)D_i(\pi) = {} & -f^{D_i}(\pi)-\sum_{j_1+\cdots+j_e=i;\ 0\leqq j_q<1} D_{j_1}(\pi)\cdots D_{j_e}(\pi) \\ & -\sum_{t=0}^{n-1}\ \sum_{j_0+\cdots+j_t=i;\ 0\leqq j_q<i} D_{j_0}(f_t)D_{j_1}(\pi)\cdots D_{j_t}(\pi) \end{aligned} \tag{27}$$

where $f^{D_i}=D_i(f_{e-1})X^{e-1}+\cdots+D_i(f_0)$. For $i=1$ we have the familiar formula $D_1(\pi)=f^{D_1}(\pi)/f'(\pi)$ and hence, since $D_1(f_j)\in[f'(\pi)]^2M^{t-j}$ for $j=0,\dots,e-1$ we have $D_1(\pi)\in f'(\pi)M^t$. If, for $i<r$, $D_i(\pi)\in f'(\pi)M^t$ then by (27) $D_r(\pi)\in f'(\pi)M^t$. Thus $D(R)\subset R$. In order to show that $\sum D_i(\pi)$ converges we assume that for any integer s, $1<s<r$, there is an integer $N_s>eN_{s-1}$ such that if $i>N_s$ then $D_i(f_j)\in M^{st}$ for $j=0,\dots,n$ and $D_i(\pi)\in M^{(s-1)t}$. Then since D converges on R_1 there is an N such that if $i>N$ then $D_i(f_j)\in M^{rt}$ for all j. Let N_r be the larger of eN and eN_{r-1}. It follows then from (27) that for $i>N_r$, $D_i(\pi)\in M^{(r-1)t}$ and the lemma is proved.

If S is a set of representatives in V of a p-basis for its residue field k then $V=V^{p^m}[S]+p^m V$ for any $m>0$. Thus there is a finite subset S_1 of S such that $f_j \in V^{p^{3t}}[S_1]+p^{3t}V$. Assuming S to be an infinite set we enumerate a countable subset $\{s_i\}_{i=1}^{\infty}$ of $S-S_1$ and we define a higher derivation $D \in H_c^M(V, R)$ by $D_i(s_j)=\delta_{ij}[f'(\pi)]^2$, for $i, j>0$, and $D(s)=0$ for $s \in S-\{s_i\}_{i=1}^{\infty}$. By Theorem 4, D is in $H_c^M(V, R)$ and is not in $H_u(V, R)$ since D does not converge uniformly on S. We extend D to $V[X_1, \ldots, X_m]$ and hence, by Proposition 2, to R_1 by the requirement $D(X_i)=0$ for $i=1, \ldots, n$, using the same symbol for the extended map. Since $\sum D_i(X_j)$ converges for $j=1, \ldots, n$, $D \in H_c^M(R_1, R)$, $D \notin H_u(R_1, R)$. By construction of D the conditions of Lemma 8 are met and hence D extends to a higher derivation in $H_c(R, R)$ which is not in $H_u(R, R)$.

The following lemma is needed in order to obtain an analogue to Theorem 5 in case the residue field R has characteristic zero.

LEMMA 9. *Let k_0, k_1, and k be fields such that $k_0 \subset k_1 \subset k$. Let*

$$D \in H_c(k_1, k[[X_1, \ldots, X_n]])$$

and assume k_1 separable algebraic over k_0. If D restricted to k_0 is uniformly convergent then D is also uniformly convergent. If $D \in H(k_1, k[[X_1, \ldots, X_n]])$ is convergent (M convergent) on k then

$$D \in H_c(k_1, k[[X_1, \ldots, X_n]]) \qquad (D \in H_c^M(k_1, k[[X_1, \ldots, X_n]])).$$

Proof. Let u be in k_1 and let f be its minimal polynomial over k_0. If

$$f = X^n + \sum_{i=0}^{n-1} f_i X^i$$

then, as in the proof of Lemma 8,

$$\begin{aligned} f'(u)D_i(u) = -f^{D_i}(u) - &\sum_{j_1+\cdots+j_n=i;\, 0 \leq j_t < i} D_{j_1}(u) \cdots D_{j_n}(u) \\ &- \sum_{t=0}^{n-1} \sum_{j_0+\cdots+j_t=i;\, 0 \leq j_q < i} D_{j_0}(f_t) D_{j_1}(u) \cdots D_{j_t}(u) \end{aligned} \tag{28, i}$$

for $i=1, 2, \ldots$.

Using (28) and induction on i we observe below that $D_i(u)$ is a sum of terms of the form

$$bD_{i_1}(a_1) \cdots D_{i_r}(a_r), \qquad i_1+\cdots+i_r = i. \tag{29, i}$$

Relation (28, i) exhibits a representation of $D_{(1)}(u)$ as a sum of terms of the form (29, 1). Assuming that, for $i<j$, $D_i(u)$ is a sum of the form (29, i) we substitute these sums in (28, j) and conclude that $D_j(u)$ is of the same form. The first assertion of Lemma 9 now follows from Lemma 4.

Let $D \in H(k_1, k[[X_1, \ldots, X_n]])$ be convergent on k_0 and let u be as above. Now $f'(u)D_i(u)$ was observed to be a sum of terms of the form (29, i) from which fact one concludes that $\sum D_i(u)$ converges if D converges on k_0. The remaining statement is obvious.

THEOREM 6. *If (R, M) is a complete regular local ring having residue field k with characteristic zero then $H_c(R, R) = H_u(R, R)$ if and only if k has finite transcendency degree over its prime field.*

Proof. In this case R is a power series ring $k[[X_1, \ldots, X_n]]$ in n-indeterminates over k. Let k_0 be the prime field of k and let B be a transcendency basis of k. Then, by Proposition 2 and Lemma 9, it is sufficient to show that $H_c(k_0(B), R) = H_u(k_0(B), R)$ if and only if B is finite. Since the first nonzero mapping of a higher derivation is a derivation and there are no nonzero derivations with domain k_0 it follows that every higher derivation on k is trivial on k_0. Hence if $D \in H_c(k_0(B), R)$ then D is uniformly convergent on k_0 and, if B is finite, D is uniformly convergent on $k_0[B]$ by Lemma 1 and Corollary 3.1, and hence is uniformly convergent on $k_0(B)$ by Corollary 3.2.

If B is infinite we choose a countable subset $\{b_i\}_{i=1}^{\infty} = B'$ in B and define a $D \in H_M(k_0(B), R)$ by the conditions $D_i(b_j) = \delta_{ij} X_1$, for $i, j \geqq 1$, and $D_i(b) = 0$ for $i \geqq 1$ and b in B, b not in B'. D is M-convergent on k_0 and on B and hence D is in $H_c^M(k_0(B), R)$ by Lemma 3. Since $D_j(b_j) \notin M^2$ for all j, D is not uniformly convergent.

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