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Converse Lyapunov–Krasovskii theorems for systems described by neutral functional differential equations in Hale's form

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In this article, we show that the existence of a Lyapunov–Krasovskii functional is necessary and sufficient condition for the uniform global asymptotic stability and the global exponential stability (GES) of time-invariant systems described by neutral functional differential equations in Hale's form. It is assumed that the difference operator is linear and strongly stable, and that the map on the right-hand side of the equation is Lipschitz on bounded sets. A link between GES and input-to-state stability is also provided.

Keywords: converse Lyapunov theorem; Lyapunov–Krasovskii functionals; retarded functional differential equation; neutral functional differential equation

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1. Introduction

In Karafyllis (2006), Karafyllis, Pepe, and Jiang (2008a,b) and Karafyllis and Jiang (2010), many converse Lyapunov theorems have been presented, for many global stability notions, for systems described by retarded functional differential equations (RFDEs), in a very general setting. Time-varying delays, disturbances and time-varying equations are considered. Besides the global uniform asymptotic stability, the input-to-state stability (ISS) and the weighted input-to-output stability are also investigated. The reader can refer to the recent monograph Karafyllis and Jiang (2011) for an extensive presentation of this topic. As far as the systems described by neutral equations are concerned, converse Lyapunov–Krasovskii theorems are available for the (local, uniform) asymptotic stability (Cruz and Hale 1970; Bená and Godoy 2001). Methods for constructing the quadratic Lyapunov–Krasovskii functional are also provided, for the linear case, in Rodriguez, Kharitonov, Dion, and Dugard (2004), Kharitonov (2005, 2008) and Velazquez-Velazquez and Kharitonov (2009). Converse Lyapunov–Krasovskii methods are also used for establishing instability criteria for linear neutral systems in Mondié, Ochoa, and Ochoa (2011). As far as global asymptotic stability (GAS) notions are concerned, to our knowledge, converse Lyapunov–Krasovskii theorems for nonlinear neutral systems are not yet available in the literature.

In this article, we consider time-invariant systems described by neutral equations in Hale's form (Hale and Lunel 1993; Kolmanovskii and Myshkis 1999). The difference operator can involve an arbitrary number of arbitrary discrete time-delays. It is assumed to be linear and strongly stable (Hale and Lunel 1993). The map on the right-hand side of the equation is assumed to be Lipschitz on bounded sets. An arbitrary number of arbitrary discrete and distributed time delays can appear on the right-hand side of the equation. We prove here that the well-known conditions, extended to the whole state space, in the Lyapunov–Krasovskii theorem for the (uniform) local asymptotic stability of the origin (Kolmanovskii and Nosov 1982, 1986; Hale and Lunel 1993; Kolmanovskii and Myshkis 1999), are not only sufficient, but also necessary. Moreover, we show here converse Lyapunov–Krasovskii theorems for global exponential stability (GES), and a link between GES and ISS.

This article is organised as follows. In Section 2, converse Lyapunov–Krasovskii theorems for GAS and GES are provided. In Section 3, a link between GES and ISS is provided. In Section 4, conclusions are drawn. For the sake of readability,

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the proofs of the theorems are reported in the appendices.

Notations: R denotes the set of real numbers, $R^\star$ denotes the extended real line $[-\infty, +\infty]$, R^+ denotes the set of non-negative reals $[0, +\infty)$, Z^+ denotes the set of integers in R^+. For $s \in R^+$, $[s]$ denotes the largest number in Z^+, smaller or equal to s. The symbol $|\cdot|$ stands for the Euclidean norm of a real vector, or the induced Euclidean norm of a matrix. For a positive integer n, for a positive real Δ (maximum involved time-delay): $\mathcal{C}$ denotes the Banach space of the continuous functions mapping $[-\Delta, 0]$ into R^n, endowed with the supremum norm, indicated with the symbol $\|\cdot\|$; $W^{1,\infty}$ denotes the space of the absolutely continuous functions in $\mathcal{C}$, with essentially bounded derivative. With the symbol $\|\cdot\|_a$ is denoted any semi-norm in $\mathcal{C}$ (Pepe and Jiang 2006; Yeganefar, Pepe, and Dambrine 2008). For a function x: $[-\Delta, c) \to R^n$, with $0 < c \le +\infty$, for any real $t \in [0, c)$, x_t is the function in $\mathcal{C}$ defined as $x_t(\tau) = x(t+\tau)$, $\tau \in [-\Delta, 0]$. For positive real H, $\phi \in \mathcal{C}$, $\mathcal{C}_H(\phi)$ denotes the subset (of $\mathcal{C}$) $\{\psi \in \mathcal{C}: \|\psi - \phi\| \le H\}$. For $\mathcal{C}_H$, $\mathcal{C}_H(0)$ is meant. For positive integer m, positive real δ, B_δ denotes the subset (of R^m) $\{u \in R^m : |u| \le \delta\}$. For positive integer n, a map Q: $\mathcal{C} \to R^n$ is said to be: locally Lipschitz if, for any ϕ in $\mathcal{C}$, there exist positive reals H, L such that, for any $\phi_1, \phi_2 \in \mathcal{C}_H(\phi)$, the inequality $|Q(\phi_1) - Q(\phi_2)| \le L\|\phi_1 - \phi_2\|$ holds; Lipschitz on bounded sets if, for any positive real H, there exists a positive real L such that, for any $\phi_1, \phi_2 \in \mathcal{C}_H$, the inequality $|Q(\phi_1) - Q(\phi_2)| \le L\|\phi_1 - \phi_2\|$ holds; globally Lipschitz if there exists a positive real L such that, for any $\phi_1, \phi_2 \in \mathcal{C}$, the inequality $|Q(\phi_1) - Q(\phi_2)| \le L\|\phi_1 - \phi_2\|$ holds. For positive integers n, m, a map Q: $\mathcal{C} \times R^m \to R^n$ is said to be Lipschitz on bounded sets if, for any positive reals H, δ, there exists a positive real L such that, for any $\phi_1, \phi_2 \in \mathcal{C}_H$, for any $u_1, u_2 \in B_\delta$, the inequality $|Q(\phi_1, u_1) - Q(\phi_2, u_2)| \le L(\|\phi_1 - \phi_2\| + |u_1 - u_2|)$ holds. For positive integer m, a Lebesgue measurable function v: $R^+ \to R^m$ is said to be essentially bounded if $ess\,\sup_{t \ge 0}|v(t)| < \infty$. The essential supremum norm of a Lebesgue measurable and essentially bounded function is indicated again with the symbol $\|\cdot\|$. For given times $0 \le T_1 < T_2$, we indicate with $v_{[T_1,T_2)} : R^+ \to R^m$ the function given by $v_{[T_1,T_2)}(t) = v(t)$ for all $t \in [T_1, T_2)$ and $= 0$ elsewhere. An input v is said to be locally essentially bounded if, for any $T > 0, v_{[0,T)}$ is essentially bounded. Let us recall here that a function γ: $R^+ \to R^+$ is: of class $\mathcal{P}$ if it is continuous, zero at zero and positive for any positive real; of class $\mathcal{K}$ if it is of class $\mathcal{P}$ and strictly increasing; of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and it is unbounded; of class $\mathcal{L}$ if it is continuous and it monotonically decreases to zero as its argument tends to $+\infty$. A function β: $R^+ \times R^+ \to R^+$ is of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each $t \ge 0$ and $\beta(s, \cdot)$ is of class $\mathcal{L}$ for each $s \ge 0$. The symbol $\circ$ denotes composition of functions.

2. Neutral functional differential equations and converse Lyapunov–Krasovskii theorems for 0-GAS and 0-GES properties

Let us consider the following neutral functional differential equation (NFDE) in Hale's form (Hale and Lunel 1993; Kolmanovskii and Myshkis 1999):

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D}x_t = f(x_t), \quad t \ge 0, \\ &x(\tau) = \xi_0(\tau), \quad \tau \in [-\Delta, 0], \quad \xi_0 \in \mathcal{C}, \end{aligned} \tag{1}$$

where $x(t) \in R^n$, n is a positive integer; $\Delta > 0$ is the maximum involved time delay; the map f: $\mathcal{C} \to R^n$ is Lipschitz on bounded sets and satisfies $f(0) = 0$; the operator $\mathcal{D}$: $\mathcal{C} \to R^n$ is defined, for $\phi \in \mathcal{C}$, as

$$\mathcal{D}\phi = \phi(0) - \sum_{j=1}^{p} A_j \phi(-\Delta_j), \tag{2}$$

with p a positive integer, Δ_j positive reals satisfying $\Delta_j \le \Delta$, $j = 1, 2, \ldots, p$, A_j matrices in $R^{n \times n}$, $j = 1, 2, \ldots, p$. It is assumed that the operator $\mathcal{D}$ is strongly stable (see Definition 6.2, p. 284 of Hale and Lunel 1993). Let us recall here that: in the case $p = 1$, the operator $\mathcal{D}$ is strongly stable if and only if the eigenvalues of A_1 are located inside the open unitary disc; in the case of multiple delays, a necessary and sufficient condition for the strong stability of the operator $\mathcal{D}$ is provided in Theorem 6.1, pp. 284–287, of Hale and Lunel (1993). Sufficient conditions for the strong stability of the operator $\mathcal{D}$, in terms of matrix inequalities, are provided in Pepe and Verriest (2003), Pepe (2005), Gu and Liu (2009), Gu (2010), Li and Gu (2010) and Li (2012).

In the following well-known definition, reported here for reader's convenience, GAS is meant uniform with respect to $\mathcal{C}_H$, for any positive real H (see Definition 4.4, pp. 149–150, of Khalil 2000, for systems described by ODEs, see also Definition 1.1, pp. 130–131 of Hale and Lunel 1993, as far as the local, uniform asymptotic stability of systems described by RFDEs is concerned).

Definition 2.1: The system described by (1) is said to be 0-GAS if:

(i) For any $\epsilon > 0$ there exists $\delta > 0$ such that, for any initial condition $\xi_0 \in \mathcal{C}_\delta$, the solution exists for all $t \in R^+$ and, furthermore, satisfies

$$|x(t)| < \epsilon, \quad t \ge 0; \tag{3}$$

moreover, δ can be chosen arbitrarily large for sufficiently large ϵ.

(ii) For any positive real H, for any positive real ϵ, there exists a positive real T such that, for any initial condition $\xi_0 \in \mathcal{C}_H$, the corresponding solution exists for all $t \in R^+$ and, furthermore, satisfies

$$|x(t)| < \epsilon \quad \forall t \geq T. \tag{4}$$

Property (i) in Definition 2.1 is equivalent to Lyapunov and Lagrange stability for dynamical systems. The following well-known definition concerns the GES (see Definition 4.5, p. 150 of Khalil 2000, as far as systems described by ODEs are concerned, see Krasovskii 1963, as far as systems described by RFDEs are concerned).

Definition 2.2: The system described by (1) is said to be 0-GES if there exist positive reals M, λ such that, for any initial condition $\xi_0 \in \mathcal{C}$, the corresponding solution of (1) exists for all $t \in R^+$ and, furthermore, satisfies the inequality

$$|x(t)| \leq Me^{-\lambda t}\|\xi_0\|, \quad t \geq 0. \tag{5}$$

Notice that, in Definition 2.2, $M \geq 1$ is mandatory. For a locally Lipschitz functional V: $\mathcal{C} \to R^+$, the derivative of the functional V, D^+V: $\mathcal{C} \to R^\star$, is defined (in Driver 1962; Pepe 2007a), for $\phi \in \mathcal{C}$, as

$$D^+V(\phi) = \limsup_{h \to 0^+} \frac{1}{h}(V(\phi_h) - V(\phi)), \tag{6}$$

where for $0 < h < \Delta$, $\phi_h \in \mathcal{C}$ is given by

$$\phi_h(s) = \begin{cases} \phi(s+h), & s \in [-\Delta, -h], \\ \mathcal{D}\phi + f(\phi)(s+h) - \mathcal{D}\phi^\star_{s+h} + \phi(0), & s \in (-h, 0], \end{cases} \tag{7}$$

for $0 < \theta \leq h$, $\phi^\star_\theta \in \mathcal{C}$ is given by

$$\phi^\star_\theta(s) = \begin{cases} \phi(s+\theta), & s \in [-\Delta, -\theta], \\ \phi(0), & s \in (-\theta, 0]. \end{cases} \tag{8}$$

Let us recall here that for any locally Lipschitz functional V: $\mathcal{C} \to R^+$, the following result holds (Driver 1962; Pepe 2007a)

$$\limsup_{h \to 0^+} \frac{V(x_{t+h}) - V(x_t)}{h} = D^+V(x_t), \quad t \in [0, b), \tag{9}$$

where x_t is the solution of (1) in a maximal time interval $[0, b)$, $0 < b \leq +\infty$.

The main result of this article is given by the necessity part of the following theorem (see Theorem 8.1, p. 293, of Hale and Lunel 1993; and Theorem 7.2 of Cruz and Hale 1970; as far as local, uniform asymptotic stability is concerned).

Theorem 2.3: *The system described by* (1) *is* 0-*GAS if and only if there exist a locally Lipschitz functional* V: $\mathcal{C} \to R^+$, *functions* α_1, α_2 *of class* $\mathcal{K}_\infty$, *a function* α_3 *of class* $\mathcal{K}$, *such that the following conditions hold for all* $\phi \in \mathcal{C}$:

(i) $\alpha_1(|\mathcal{D}\phi|) \leq V(\phi) \leq \alpha_2(\|\phi\|)$;
(ii) $D^+V(\phi) \leq -\alpha_3(|\mathcal{D}\phi|)$.

The proof of Theorem 2.3, based on recently developed methodologies for converse Lyapunov–Krasovskii theorems concerning systems described by RFDEs (Karafyllis 2006; Karafyllis et al. 2008a,b), is reported in Appendix A. The following Theorems provide necessary and sufficient Lyapunov–Krasovskii conditions for the 0-GES property.

Theorem 2.4: *The system described by* (1) *is* 0-*GES if and only if there exist a locally Lipschitz functional* V: $\mathcal{C} \to R^+$ *and positive reals* a_1, a_2, a_3, *such that the following conditions hold for all* $\phi \in \mathcal{C}$:

(i) $a_1|\mathcal{D}\phi| \leq V(\phi) \leq a_2\|\phi\|$;
(ii) $D^+V(\phi) \leq -a_3V(\phi)$.

Theorem 2.5: *If the map* f *in* (1) *is globally Lipschitz, then the system described by* (1) *is* 0-*GES if and only if there exists a globally Lipschitz functional* V: $\mathcal{C} \to R^+$, *such that the following conditions hold for all* $\phi \in \mathcal{C}$:

(i) $a_1|\mathcal{D}\phi| \leq V(\phi) \leq a_2\|\phi\|$;
(ii) $D^+V(\phi) \leq -a_3V(\phi)$.

The proofs of Theorems 2.4 and 2.5, based on well-known methodologies for converse Lyapunov–Krasovskii theorems concerning systems described by RFDEs and exponential stability (Krasovskii 1963; Kim 1999), is reported in Appendices B and C. The following theorem will be used for establishing a link between 0-GES and ISS properties in the following section.

Theorem 2.6: *If the map* f *in* (1) *is globally Lipschitz, then the system described by* (1) *is* 0-*GES if and only if there exist a globally Lipschitz functional* V: $\mathcal{C} \to R^+$, *a semi-norm* $\|\cdot\|_a$ *and positive reals* a_1, a_2, a_3, a_4, *such that the following conditions hold for all* $\phi \in \mathcal{C}$:

(i) $a_1|\mathcal{D}\phi| \leq V(\phi) \leq a_2\|\phi\|_a$;
(ii) $D^+V(\phi) \leq -a_3\|\phi\|_a$;
(iii) $\|\phi\|_a \leq a_4\|\phi\|$.

The sufficiency part in Theorem 2.6 readily follows from Theorem 2.4. The necessity part in Theorem 2.6 can be proved almost identically as the one in Theorem 2.4 (just choose $\|\cdot\|_a = \|\cdot\|$ and make use of Lemma B.2). The proof of the global Lipschitz property of the functional V is identical to the one provided for Theorem 2.5. Therefore, the proof of Theorem 2.6 is omitted.

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3. A link between 0-GAS and ISS properties for systems described by NFDEs

A link between 0-GES and ISS properties is provided here (see Lemma 4.6, pp. 176–177, of Khalil 2000, as far as 0-GES systems described by ODEs are concerned, Yeganefar et al. 2008, as far as 0-GES systems described by RFDEs are concerned). Let us consider the following NFDE in Hale's form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D}x_t = f(x_t, u(t)), \quad t \geq 0, \\ x(\tau) = \xi_0(\tau), \quad \tau \in [-\Delta, 0], \quad \xi_0 \in \mathcal{C}, \tag{10}$$

where $x(t) \in R^n$, n is a positive integer; $\Delta > 0$ is the maximum involved time delay; the map f: $\mathcal{C} \times R^m \to R^n$ is Lipschitz on bounded sets and satisfies $f(0, 0) = 0$; m is a positive integer; $u(\cdot)$ is a Lebesgue measurable, locally essentially bounded input signal; the operator $\mathcal{D}$: $\mathcal{C} \to R^n$, defined as in (1), is assumed to be strongly stable. It is assumed that there exists a map $\bar{f}$: $R^n \times \mathcal{C} \times R^m \to R^n$, independent of the second argument at 0 (Pepe 2011, see Definition 5.1, p. 281, of Hale and Lunel 1993), such that, for any $\phi \in \mathcal{C}$, $u \in R^m$, the equality $f(\phi, u) = \bar{f}(\phi(0), \phi, u)$ holds.

Definition 3.1 (Sontag 1989)**:** The system described by (10) is said to be ISS if there exists a function β of class $\mathcal{KL}$ and a function γ of class $\mathcal{K}$ such that, for any initial condition $\xi_0 \in \mathcal{C}$, for any Lebesgue measurable, locally essentially bounded input signal u, the corresponding solution of (10) exists for all $t \geq 0$ and, furthermore, satisfies the inequality

$$|x(t)| \leq \beta(\|\xi_0\|, t) + \gamma(\|u_{[0,t)}\|), \quad t \in R^+. \tag{11}$$

For a locally Lipschitz functional V: $\mathcal{C} \to R^+$, the derivative of the functional V, D^+V: $\mathcal{C} \times R^m \to R^\star$, is defined (in Driver's form, see Driver 1962; Pepe 2007a), for $\phi \in \mathcal{C}$, $d \in R^m$, as

$$D^+V(\phi, d) = \limsup_{h \to 0^+} \frac{1}{h}\big(V(\phi_{h,d}) - V(\phi)\big), \tag{12}$$

where for $0 < h < \Delta$, $\phi_{h,d} \in \mathcal{C}$ is given by

$$\phi_{h,d}(s) = \begin{cases} \phi(s+h), & s \in [-\Delta, -h], \\ \mathcal{D}\phi + f(\phi, d)(s+h) - \mathcal{D}\phi^\star_{s+h} + \phi(0), & s \in (-h, 0], \end{cases} \tag{13}$$

for $0 < \theta \leq h$, $\phi^\star_\theta \in \mathcal{C}$ is given by (8).

Let us here recall that, for any locally Lipschitz functional V: $\mathcal{C} \to R^+$, the following result holds (Pepe 2007a)

$$\limsup_{h \to 0^+} \frac{V(x_{t+h}) - V(x_t)}{h} = D^+V(x_t, u(t)), \\ t \in [0, b), \ a.e., \tag{14}$$

where x_t is the solution of (10) in a maximal time interval $[0, b)$, $0 < b \leq +\infty$.

Theorem 3.2: *Let the system described by* (10), *with* $u(t) \equiv 0$, *be* 0-*GES. Let the map* f *in* (10) *satisfy the following hypotheses*:

(i) *There exists a positive real* L_0 *such that, for any* $\phi_i \in \mathcal{C}$, $i = 1, 2$, *the inequality holds*

$$|f(\phi_1, 0) - f(\phi_2, 0)| \leq L_0\|\phi_1 - \phi_2\|. \tag{15}$$

(ii) *there exists a function* L *of class* $\mathcal{K}$ *such that, for any* $\phi \in \mathcal{C}$, *for any* $u \in R^m$, *the inequality holds*

$$|f(\phi, u) - f(\phi, 0)| \leq L(|u|). \tag{16}$$

Then, the system described by (10) *is ISS.*

The proof of Theorem 3.2 is reported in Appendix D.

4. Conclusions

In this article we have dealt with converse Lyapunov–Krasovskii theorems for time-invariant systems described by NFDEs in Hale's form, with linear, strongly stable difference operator. We have proved that the well-known Lyapunov–Krasovskii conditions, sufficient for the 0-GAS property of these systems, are also necessary. Moreover, we have given necessary and sufficient Lyapunov–Krasovskii conditions for the 0-GES property. Finally, we have shown a link between the 0-GES property and the ISS property. Future investigation will concern the case of nonlinear difference operator, namely $\mathcal{D}\phi = \phi(0) - g(\phi)$, $\phi \in \mathcal{C}$, with g nonlinear, involving discrete as well as distributed time delays (Pepe, Karafyllis, and Jiang 2008b; Melchor-Aguilar 2012). Notions for the nonlinear difference operator such as, for instance, $g(\phi)$ independent of $\phi(0)$, $\phi \in \mathcal{C}$ (see Definition 5.1, p. 281, of Hale and Lunel 1993), ISS (Pepe, Jiang, and Fridman 2008a), incremental GAS, incremental ISS (see Angeli 2002, as far as systems described by ODEs are concerned), may be instrumental in order to further extend the results presented here to systems described by more general NFDEs. Another topic which will be considered concerns the converse Lyapunov–Krasovskii theorem for the ISS of systems described by NFDEs in Hale's form, for which sufficient Lyapunov–Krasovskii conditions are provided in Pepe (2007a) and Pepe et al. (2008b). We believe that a converse Lyapunov–Krasovskii theorem for the notion of robust 0-GAS (see Lin, Sontag, and Wang 1996; as far as systems described by ODEs are concerned) may be instrumental for deriving the converse Lyapunov–Krasovskii theorem for the ISS of these systems, as it happens for systems described by RFDEs (Karafyllis et al. 2008a,b).

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References

Angeli, D. (2002), 'A Lyapunov Approach to Incremental Stability Properties', *IEEE Transactions on Automatic Control*, 47, 410–421.

Bacciotti, A., and Rosier, L. (2005), *Lyapunov Functions and Stability in Control Theory*, Berlin, Heidelberg: Springer-Verlag.

Bená, M.A., and Godoy, S.M.S. (2001), 'On the Stability in Terms of Two Measures of Perturbed Neutral Functional Differential Equations', *Nonlinear Analysis: Theory, Methods, and Applications*, 47, 4567–4578.

Cruz, M.A., and Hale, J.K. (1970), 'Stability of Functional Differential Equations of Neutral Type', *Journal of Differential Equations*, 7, 334–355.

Driver, R.D. (1962), 'Existence and Stability of Solutions of a Delay-differential System', *Archive for Rational Mechanics and Analysis*, 10, 401–426.

Gu, K. (2010), 'Stability Problem of Systems with Multiple Delay Channels', *Automatica*, 46, 743–751.

Gu, K., and Liu, Y. (2009), 'Lyapunov–Krasovskii Functional for Uniform Stability of Coupled Differential-functional Equations', *Automatica*, 45, 798–804.

Hale, J.K., and Lunel, S.M.V. (1993), *Introduction to Functional Differential Equations*, New York: Springer Verlag.

Karafyllis, I. (2006), 'Lyapunov Theorems for Systems Described by RFDEs', *Nonlinear Analysis: Theory, Methods & Applications*, 64, 590–617.

Karafyllis, I., and Jiang, Z.-P. (2010), 'Necessary and Sufficient Lyapunov-like Conditions for Robust Nonlinear Stabilization', *ESAIM: Control, Optimisation and Calculus of Variations*, 16, 887–928.

Karafyllis, I., and Jiang, Z.P. (2011), *Stability and Stabilization of Nonlinear Systems*, London: Springer.

Karafyllis, I., Pepe, P., and Jiang, Z.-P. (2008a), 'Global Output Stability for Systems Described by Retarded Functional Differential Equations: Lyapunov Characterisations', *European Journal of Control*, 14, 516–536.

Karafyllis, I., Pepe, P., and Jiang, Z.-P. (2008b), 'Input-to-output Stability for Systems Described by Retarded Functional Differential Equations', *European Journal of Control*, 14, 539–555.

Khalil, H.K. (2000), *Nonlinear Systems*, (International Edition 3rd ed.), Upper Saddle River, NJ: Prentice Hall.

Kharitonov, V.L. (2005), 'Lyapunov Functionals and Lyapunov Matrices for Neutral Type Time Delay Systems: A Single Delay Case', *International Journal of Control*, 78, 783–800.

Kharitonov, V.L. (2008), 'Lyapunov Matrices for a Class of Neutral Type Time Delay Systems', *International Journal of Control*, 81, 883–893.

Kim, A.V. (1999), *Functional Differential Equations, Application of i-smooth Calculus*, Dordrecht: Kluwer Academic Publishers.

Kolmanovskii, V., and Myshkis, A. (1999), *Introduction to the Theory and Applications of Functional Differential Equations*, Dordrecht: Kluwer Academic.

Kolmanovskii, V.B., and Nosov, V.R. (1982), 'Stability of Neutral-type Functional Differential Equations', *Nonlinear Analysis: Theory, Methods and Applications*, 6, 873–910.

Kolmanovskii, V.B., and Nosov, V.R. (1986), *Stability of Functional Differential Equations*, London: Academic Press.

Krasovskii, N.N. (1963), *Stability of Motion*, Stanford: Stanford University Press.

Li, H. (2012), 'Discretized LKF Method for Stability of Coupled Differential-difference Equations with Multiple Discrete and Distributed Delays', *International Journal of Robust and Nonlinear Control*, 22, 875–891.

Li, H., and Gu, K. (2010), 'Discretized Lyapunov–Krasovskii Functional for Coupled Differential-difference Equations with Multiple Delay Channels', *Automatica*, 46, 902–909.

Lin, Y.D., Sontag, E.D., and Wang, Y. (1996), 'A Smooth Converse Lyapunov Theorem for Robust Stability', *SIAM Journal on Control and Optimization*, 34, 124–160.

Melchor-Aguilar, D. (2012), 'Exponential Stability of Some Linear Continuous Time Difference Systems', *Systems & Control Letters*, 61, 62–68.

Mondié, S., Ochoa, G., and Ochoa, B. (2011), 'Instability Conditions for Linear Time Delay Systems: A Lyapunov Matrix Function Approach', *International Journal of Control*, 84, 1601–1611.

Pepe, P. (2005), 'On the Asymptotic Stability of Coupled Delay Differential and Continuous Time Difference Equations', *Automatica*, 41, 107112.

Pepe, P. (2007a), 'On Lyapunov–Krasovskii Functionals under Carathé Odory Conditions', *Automatica*, 43, 701–706.

Pepe, P. (2007b), 'The Problem of the Absolute Continuity for Lyapunov–Krasovskii Functionals', *IEEE Transactions on Automatic Control*, 52, 953–957.

Pepe, P. (2011), 'On the Actuator Disturbance Attenuation for Systems Described by Neutral Equations', *IMA Journal of Mathematical Control and Information*, 28, 163–181.

Pepe, P., and Jiang, Z.-P. (2006), 'A Lyapunov–Krasovskii Methodology for ISS and ISS of Time-delay Systems', *Systems & Control Letters*, 55, 1006–1014.

Pepe, P., Jiang, Z.-P., and Fridman, E. (2008a), 'A New Lyapunov–Krasovskii Methodology for Coupled Delay Differential and Difference Equations', *International Journal of Control*, 81, 107–115.

Pepe, P., Karafyllis, I., and Jiang, Z.-P. (2008b), 'On the Lyapunov–Krasovskii Methodology for the ISS of Systems Described by Coupled Delay Differential and Difference Equations', *Automatica*, 44, 2266–2273.

Pepe, P., and Verriest, E.I. (2003), 'On the Stability of Coupled Delay Differential and Continuous Time Difference Equations', *IEEE Transactions on Automatic Control*, 48, 1422–1427.

Rodriguez, S.A., Kharitonov, V.L., Dion, J.M., and Dugard, L. (2004), 'Robust Stability of Neutral Systems: A Lyapunov–Krasovskii Constructive Approach',

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International Journal of Robust and Nonlinear Control, 14, 1345–1358.

Sontag, E.D. (1989), 'Smooth Stabilisation Implies Coprime Factorisation', *IEEE Transactions on Automatic Control*, 34, 435–443.

Sontag, E.D. (1998), 'Comments on Integral Variants of ISS', *Systems & Control Letters*, 34, 93–100.

Velazquez-Velazquez, J.E., and Kharitonov, V.L. (2009), 'Lyapunov–Krasovskii Functionals for Scalar Neutral Type Time Delay Equations', *Systems & Control Letters*, 58, 17–25.

Yeganefar, N., Pepe, P., and Dambrine, M. (2008), 'Input-to-state Stability of Time-delay Systems: A Link with Exponential Stability', *IEEE Transactions on Automatic Control*, 53, 1526–1531.

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Appendix A: Proof of Theorem 2.3

Lemma A.1: *The system described by* (1) *is* 0-*GAS if and only if there exist a function* β *of class* $\mathcal{KL}$ *such that, for any initial condition* $\xi_0 \in \mathcal{C}$, *the solution exists for all* $t \in R^+$ *and, furthermore, satisfies*

$$\|x_t\| \le \beta(\|\xi_0\|, t), \quad t \in R^+. \tag{A1}$$

Proof: See Theorem 2.2, p. 62 of Karafyllis and Jiang (2011). See also Lemma 4.5, p. 150 of Khalil (2000), as far as systems described by ODEs are concerned. □

Lemma A.2 (See Proposition 7 of Sontag 1998)**:** *For any function* β *of class* $\mathcal{KL}$, *there exist functions* $\overline{\alpha}_1, \overline{\alpha}_2$ *of class* $\mathcal{K}_\infty$, *such that, for all* $(s, t) \in R^+ \times R^+$, *the inequality holds*

$$\beta(s, t) \le \overline{\alpha}_1^{-1}\left(e^{-2t}\overline{\alpha}_2(s)\right). \tag{A2}$$

Lemma A.3 (Karafyllis 2006; Karafyllis et al. 2008a; Karafyllis and Jiang 2011)**:** *For any given function* α *of class* $\mathcal{K}_\infty$, *there exists a function* γ *of class* $\mathcal{K}_\infty$ *with the following properties*:

(i) $\gamma(s) \le \alpha(s)\ \forall s \in R^+$;
(ii) $|\gamma(s_1) - \gamma(s_2)| \le |s_1 - s_2|\ \forall s_1, s_2 \in R^+$.

Proof: In Karafyllis (2006), Karafyllis and Jiang (2011) and Karafyllis et al. (2008a), examples are provided for the function γ. For instance, the function γ: $R^+ \to R^+$ defined, for $s \ge 0$, as

$$\gamma(s) = \min_{0 \le y \le s}\{\alpha(y) + s - y\} \tag{A3}$$

has properties (i), (ii) and is a function of class $\mathcal{K}_\infty$. □

Lemma A.4 (Karafyllis 2006; Karafyllis et al. 2008a,b)**:** *There exists a continuous non-decreasing function* $\overline{L}$: $R^+ \to R^+$ *such that the following inequality holds, for all* $\phi_1, \phi_2 \in \mathcal{C}$,

$$|f(\phi_1) - f(\phi_2)| \le \overline{L}(\|\phi_1\| + \|\phi_2\|)\|\phi_1 - \phi_2\|. \tag{A4}$$

Proof: It is a consequence of the fact that the map f is assumed Lipschitz on bounded sets (see, e.g. Karafyllis 2006). □

For the system described by the continous-time difference equation (CTDE), for $t \ge 0$,

$$\mathcal{D}x_t = w(t), \quad x(\tau) = \xi_0(\tau), \ \tau \in [-\Delta, 0], \quad \xi_0 \in \mathcal{C}, \tag{A5}$$

with w: $R^+ \to R^n$ continuous, the following result holds.

Lemma A.5 (Hale and Lunel 1993)**:** *There exist positive constants* $\overline{M}$, ω, *such that, for any initial conditions* ϕ_1, ϕ_2 *in* $\mathcal{C}$, *for any continuous signals* w_1: $R^+ \to R$, w_2: $R^+ \to R$, *the corresponding solutions* $x(t, \phi_1, w_1)$ *and* $x(t, \phi_2, w_2)$ *of* (A5) *exist for all* $t \in R^+$ *and, furthermore, satisfy the following inequality*:

$$|x(t, \phi_1, w_1) - x(t, \phi_2, w_2)| \le \overline{M}e^{-\omega t}\|\phi_1 - \phi_2\| + \overline{M}\sup_{\theta \in [0,t]}|w_1(\theta) - w_2(\theta)|. \tag{A6}$$

Proof: Clearly the solutions exist in R^+. Let $y(t) = x(t, \phi_1, w_1) - x(t, \phi_2, w_2)$, $u(t) = w_1(t) - w_2(t)$ and $y_0 = \phi_1 - \phi_2$. Then y is solution of the CTDE

$$\mathcal{D}y_t = u(t), \quad y(\tau) = y_0(\tau), \quad \tau \in [-\Delta, 0]. \tag{A7}$$

Since the operator $\mathcal{D}$ is strongly stable, it follows from Theorem 3.5, p. 275 of Hale and Lunel (1993), that there exist positive constants $\overline{M}$, ω such that

$$|y(t)| \le \overline{M}e^{-\omega t}\|y_0\| + \overline{M}\sup_{\theta \in [0,t]}|u(\theta)|. \tag{A8}$$

Inequality (A6) follows directly from inequality (A8), taking into account of the previous definitions of $y(t)$, $u(t)$ and y_0. □

Notice that, in Lemma A.5, $\overline{M} \ge 1$ is mandatory.

Lemma A.6: *Let the system described by* (1) *be* 0-*GAS*, *with related function* β *of class* $\mathcal{KL}$ *as in Lemma* A.1. *Let* $\overline{M}$ *be a positive real as depicted in Lemma* A.5 *and let* $\overline{L}$ *be a continuous non-decreasing function as depicted in Lemma* A.4. *Define the positive real* M *and the continuous non-decreasing function* L: $R^+ \to R^+$ *as follows*:

$$M := \overline{M}\left(2 + \sum_{k=1}^{p}|A_k|\right), \quad L(s) := \overline{M}\,\overline{L}(2\beta(s, 0)), \ s \ge 0. \tag{A9}$$

Then, for any initial conditions $\phi_1, \phi_2 \in \mathcal{C}$, *the corresponding solutions* $x_t(\phi_1)$, $x_t(\phi_2)$ *of* (1) *satisfy the following inequality, for* $t \ge 0$,

$$\|x_t(\phi_1) - x_t(\phi_2)\| \le M\|\phi_1 - \phi_2\|e^{L(\|\phi_1\| + \|\phi_2\|)t}. \tag{A10}$$

Proof: The solutions $x(t, \phi_i)$, $i = 1, 2$, satisfy the following equations:

$$x(t, \phi_i) = \sum_{k=1}^{p}A_k x(t - \Delta_k, \phi_i) + w_i(t), \tag{A11}$$

where $w_i(t) = \mathcal{D}\phi_i + \int_0^t f(x_s(\phi_i))\mathrm{d}s$, $i = 1, 2$. Lemma A.5 implies that the following inequality holds:

$$|x(t, \phi_1) - x(t, \phi_2)| \le \overline{M}\|\phi_1 - \phi_2\| + \overline{M}\sup_{\theta \in [0,t]}|w_1(\theta) - w_2(\theta)|. \tag{A12}$$

Now, the following equality/inequalities hold

$$\begin{aligned}
|w_1(t) - w_2(t)| &= \left|\mathcal{D}\phi_1 + \int_0^t f(x_s(\phi_1))\mathrm{d}s - \mathcal{D}\phi_2 - \int_0^t f(x_s(\phi_2))\mathrm{d}s\right| \\
&\le |\mathcal{D}\phi_1 - \mathcal{D}\phi_2| + \left|\int_0^t (f(x_s(\phi_1)) - f(x_s(\phi_2)))\mathrm{d}s\right| \\
&\le \left(1 + \sum_{k=1}^{p}|A_k|\right)\|\phi_1 - \phi_2\| + \int_0^t \left|f(x_s(\phi_1)) - f(x_s(\phi_2))\right|\mathrm{d}s.
\end{aligned} \tag{A13}$$

From Lemma A.4, the inequality follows:

$$|w_1(t) - w_2(t)| \le \left(1 + \sum_{k=1}^{p} |A_k|\right)\|\phi_1 - \phi_2\| + \int_0^t \overline{L}(\|x_s(\phi_1)\| + \|x_s(\phi_2)\|)\|x_s(\phi_1) - x_s(\phi_2)\|\mathrm{d}s. \quad \text{(A14)}$$

From the fact that $\|x_s(\phi_i)\| \le \beta(\|\phi_i\|, s)$, the inequality follows:

$$|w_1(t) - w_2(t)| \le \left(1 + \sum_{k=1}^{p} |A_k|\right)\|\phi_1 - \phi_2\| + \int_0^t \overline{L}(\beta(\|\phi_1\|, 0) + \beta(\|\phi_2\|, 0))\|x_s(\phi_1) - x_s(\phi_2)\|\mathrm{d}s. \quad \text{(A15)}$$

From (A12) and (A15) the inequality follows

$$\|x_t(\phi_1) - x_t(\phi_2)\| \le \overline{M}\left(2 + \sum_{k=1}^{p} |A_k|\right)\|\phi_1 - \phi_2\| + \overline{M}\int_0^t \overline{L}(\beta(\|\phi_1\|, 0) + \beta(\|\phi_2\|, 0))\|x_s(\phi_1) - x_s(\phi_2\|\mathrm{d}s. \quad \text{(A16)}$$

Inequality (A10) is a direct consequence of inequality (A16) and the Bellman–Gronwall lemma. □

Lemma A.7: *Let the system described by* (1) *be* 0-*GAS, with* β *as related* $\mathcal{KL}$ *function, according to Lemma* A.1. *Let* (*taking into account of Lemma* A.3) $\overline{\alpha}_1$ *be a globally Lipschitz function, with Lipschitz constant equal to* 1, *of class* $\mathcal{K}_\infty$, $\overline{\alpha}_2$ *be a function of class* $\mathcal{K}_\infty$, *such that inequality* (A2) *in Lemma* A.2 *holds. Let M be the positive real and L be the continuous, non-decreasing function defined in* (A9), *in Lemma* A.6. *Let, for any positive integer q,* U_q: $\mathcal{C} \to R^+$ *be the functional defined, for all* $\phi \in \mathcal{C}$, *by*

$$U_q(\phi) = \sup_{t\ge 0}\left(\max\left\{0, \overline{\alpha}_1(\|x_t(\phi)\|) - \frac{1}{q}\right\}e^t\right), \quad \text{(A17)}$$

where $x_t(\phi)$ *is the solution of* (1) *with initial condition* ϕ. *Let G*: $R^+ \times Z^+ \to R^+$ *be the function defined as*

$$G(s, q) = M(1 + q\overline{\alpha}_2(s))^{(L(2s)+1)}, \quad s \in R^+,\ q \in Z^+. \quad \text{(A18)}$$

Then, for any positive integer q, the following results hold:

(1) $\max\left\{0, \overline{\alpha}_1(\|\phi\|) - \frac{1}{q}\right\} \le U_q(\phi) \le \overline{\alpha}_2(\|\phi\|)\ \forall \phi \in \mathcal{C}$;
(2) $U_q(x_t(\phi)) \le e^{-t}U_q(\phi)\ \forall t \in R^+, \forall \phi \in \mathcal{C}$;
(3) *for any positive real H, for any* $\phi_1, \phi_2 \in \mathcal{C}_H$, *the inequality holds*:

$$|U_q(\phi_1) - U_q(\phi_2)| \le G(H, q)\|\phi_2 - \phi_1\|. \quad \text{(A19)}$$

Proof: The proof follows the same lines of the one for the case of RFDEs studied in Karafyllis (2006) and Karafyllis et al. (2008a,b). For reader's convenience, the proof is reported here, with some slight modifications, mainly related to the definition of the function G (see also the definition of the functional V in (A28)). The first inequality in (1) follows by choosing $t = 0$ on the right-hand side of (A17). The second inequality in (1), taking into account of Lemma A.2, follows from the inequalities

$$\sup_{t\ge 0}\left(\max\left\{0, \overline{\alpha}_1(\|x_t(\phi)\|) - \frac{1}{q}\right\}e^t\right) \le \sup_{t\ge 0}\left(\max\left\{0, e^{-2t}\overline{\alpha}_2(\|\phi\|) - \frac{1}{q}\right\}e^t\right) \le \sup_{t\ge 0}\left(\max\left\{0, e^{-t}\overline{\alpha}_2(\|\phi\|) - \frac{e^t}{q}\right\}\right) \le \overline{\alpha}_2(\|\phi\|). \quad \text{(A20)}$$

Point (2) follows from the inequality

$$\sup_{\theta\ge 0}\left(\max\left\{0, \overline{\alpha}_1(\|x_\theta(x_t(\phi))\|) - \frac{1}{q}\right\}e^\theta\right) \le \sup_{\theta\ge 0}\left(\max\left\{0, \overline{\alpha}_1(\|x_\theta(\phi)\|) - \frac{1}{q}\right\}e^{\theta-t}\right). \quad \text{(A21)}$$

As far as Points (3) is concerned, the following equality holds (we consider $\log(0) = -\infty$):

$$\sup_{t\ge\max\left\{0, \frac{1}{2}\log(q\overline{\alpha}_2(\|\phi\|))\right\}}\left(\max\left\{0, \overline{\alpha}_1(\|x_t(\phi)\|) - \frac{1}{q}\right\}e^t\right) = 0. \quad \text{(A22)}$$

In order to prove this we can consider the two cases:

(i) $\max\left\{0, \frac{1}{2}\log(q\overline{\alpha}_2(\|\phi\|))\right\} = 0$;
(ii) $\max\left\{0, \frac{1}{2}\log(q\alpha_2(\|\phi\|))\right\} = \frac{1}{2}\log(q\alpha_2(\|\phi\|))$.

In case (i), we have $\overline{\alpha}_2(\|\phi\|) \le \frac{1}{q}$. It follows that $\overline{\alpha}_1(\|x_t(\phi)\|) \le \frac{1}{q}$, $t \ge 0$, and thus (A22) holds in this case. In case (ii), we have, for $t \ge \frac{1}{2}\log(q\overline{\alpha}_2(\|\phi\|))$,

$$\overline{\alpha}_1(\|x_t(\phi)\|) \le e^{-2\frac{1}{2}log(q\overline{\alpha}_2(\|\phi\|))}\overline{\alpha}_2(\|\phi\|) = \frac{1}{q}, \quad \text{(A23)}$$

and thus (A22) also holds in this case. Therefore, for any $\xi \ge \max\left\{0, \frac{1}{2}\log(q\overline{\alpha}_2(\|\phi\|))\right\}$, we have

$$U_q(\phi) = \sup_{0\le t\le\xi}\left(\max\left\{0, \overline{\alpha}_1(\|x_t(\phi)\|) - \frac{1}{q}\right\}e^t\right). \quad \text{(A24)}$$

Now, let H be a positive real. Let us choose $\xi = \max\left\{0, \frac{1}{2}\log(q\overline{\alpha}_2(H)\right\}$. Let us take any $\phi_1, \phi_2 \in \mathcal{C}_H$. Then, taking into account of the Lipschitz property of $\overline{\alpha}_1$, and of the fact that also the function $s \to \max\left\{0, s - \frac{1}{q}\right\}$ is globally Lipschitz with Lipschitz constant 1, we have

$$\begin{aligned}
|U_q(\phi_2) - U_q(\phi_1)| &= \left|\sup_{0\le t\le\xi}\left(\max\left\{0, \overline{\alpha}_1(\|x_t(\phi_2)\|) - \frac{1}{q}\right\}e^t\right) - \sup_{0\le t\le\xi}\left(\max\left\{0, \overline{\alpha}_1(\|x_t(\phi_1)\|) - \frac{1}{q}\right\}e^t\right)\right| \\
&\le \sup_{0\le t\le\xi}\left|\max\left\{0, \alpha_1(\|x_t(\phi_2)\|) - \frac{1}{q}\right\} - \max\left\{0, \alpha_1(\|x_t(\phi_1)\|) - \frac{1}{q}\right\}\right|e^t \\
&\le \sup_{0\le t\le\xi} |\overline{\alpha}_1(\|x_t(\phi_2)\|) - \overline{\alpha}_1(\|x_t(\phi_1)\|)|e^t \\
&\le \sup_{0\le t\le\xi} |\|x_t(\phi_2)\| - \|x_t(\phi_1)\||e^t \\
&\le \sup_{0\le t\le\xi} \|x_t(\phi_2) - x_t(\phi_1)\|e^t. \quad \text{(A25)}
\end{aligned}$$

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Thus, from Lemma A.6, we obtain

$$\begin{aligned} &|U_q(\phi_2) - U_q(\phi_1)| \\ &\quad \le e^{\xi} M e^{L(2H)\xi} \|\phi_2 - \phi_1\| \\ &\quad \le e^{\frac{1}{2}log(1+q\overline{\alpha}_2(H))} M e^{\frac{1}{2}L(2H)log(1+q\overline{\alpha}_2(H))} \|\phi_2 - \phi_1\| \\ &\quad \le G(H,q)\|\phi_2 - \phi_1\|. \end{aligned} \tag{A26}$$

The proof of the lemma is complete. □

Lemma A.8: *The system described by* (1) *is* 0*-GAS if and only if there exist a functional* V: $\mathcal{C} \to R^+$, *Lipschitz on bounded sets, and functions* α_1, α_2, α_3 *of class* $\mathcal{K}_\infty$, *such that,* $\forall \phi \in \mathcal{C}$, *the following inequalities hold*:

(i) $\alpha_1(\|\phi\|) \le V(\phi) \le \alpha_2(\|\phi\|)$;
(ii) $D^+V(\phi) \le -\alpha_3(\|\phi\|)$.

Proof: Let us prove first the necessity part. We will make use of Lemma A.7. Let the trivial solution of the system described by (1) be 0-GAS, with β as related $\mathcal{KL}$ function, according to Lemma A.1. Let $\overline{\alpha}_1$ be a globally Lipschitz (with Lipschitz constant equal to 1) function of class $\mathcal{K}_\infty$, $\overline{\alpha}_2$ be a function of class $\mathcal{K}_\infty$, such that, according to Lemma A.2,

$$\beta(s,t) \le \overline{\alpha}_1^{-1}\left(e^{-2t}\overline{\alpha}_2(s)\right), \quad s,t \in R^+. \tag{A27}$$

Let M be the positive real and L: $R^+ \to R^+$ be the continuous, non-decreasing function as depicted in Lemma A.6. Let, for any positive integer q, U_q: $\mathcal{C} \to R^+$ be the functional defined as in Lemma A.7. Let V: $\mathcal{C} \to R^+$ be the functional defined, for $\phi \in \mathcal{C}$, by (Karafyllis 2006; Karafyllis et al. 2008a,b)

$$V(\phi) = \sum_{q=1}^{+\infty} \frac{2^{-q}}{1+G(q,q)} U_q(\phi), \tag{A28}$$

where G: $R^+ \times Z^+ \to R^+$ is the function defined in (A18), in Lemma A.7. Notice that, from result (1) in Lemma A.7, and non-negativeness of G, it follows that, for any $\phi \in \mathcal{C}$, the sum in the right-hand side of (A28) is convergent. The functional V satisfies (i), (ii) in the lemma and is locally Lipschitz. Indeed, from result (1) in Lemma A.7, it follows that α_1, α_2 in (i) can be chosen, for $s \ge 0$, as

$$\begin{aligned} \alpha_1(s) &= \sum_{q=1}^{+\infty} \frac{2^{-q}}{1+G(q,q)} \max\left\{0, \overline{\alpha}_1(s) - \frac{1}{q}\right\}, \\ \alpha_2(s) &= 2\overline{\alpha}_2(s). \end{aligned} \tag{A29}$$

As far as (ii) is concerned, we have, for $\phi \in \mathcal{C}$ (see (6)–(8)),

$$\begin{aligned} D^+V(\phi) &= \limsup_{h\to 0^+} \frac{V(\phi_h) - V(\phi)}{h} \\ &\le \limsup_{h\to 0^+} \frac{V(\phi_h) - V(x_h(\phi))}{h} \\ &\quad + \limsup_{h\to 0^+} \frac{V(x_h(\phi)) - V(\phi)}{h}, \end{aligned} \tag{A30}$$

where $x_h(\phi)$ is the solution of (1) with initial condition ϕ. From item (2) on Lemma A.7, it follows that $V(x_h(\phi)) \le e^{-h}V(\phi)$, and therefore, for the second limit on the right-hand side of (A30), we have

$$\limsup_{h\to 0^+} \frac{V(x_h(\phi)) - V(\phi)}{h} \le -V(\phi). \tag{A31}$$

As far as the first limit on the right-hand side of the inequality in (A30) is concerned, since, from (A18), it follows that $G(s, q) \le G(q, q)$ for $0 \le s \le q$, recalling result (3) in Lemma A.7, taking into account of the continuity of the map $t \to x_t$ (see Lemma 2.1, p. 40 of Hale and Lunel 1993), we have, for $z = [\|\phi\|] + 1$, for sufficiently small positive h,

$$\begin{aligned} |V(\phi_h) - V(x_h(\phi))| &\le \sum_{q=1}^{+\infty} \frac{2^{-q}}{1+G(q,q)} |U_q(\phi_h) - U_q(x_h(\phi))| \\ &\le \left(2 + \sum_{q=1}^{z} \frac{2^{-q}G(z,q)}{1+G(q,q)}\right) \|\phi_h - x_h(\phi)\|. \end{aligned} \tag{A32}$$

Now, from (6)–(8), for sufficiently small h, we have

$$\begin{aligned} &\|\phi_h - x_h(\phi)\| \\ &\quad = \sup_{\theta\in(-h,0]} \left| \mathcal{D}\phi + (\theta+h)f(\phi) + \sum_{k=1}^{p} A_k\phi(-\Delta_k + \theta + h) \right. \\ &\qquad \left. -\mathcal{D}\phi - \int_0^{\theta+h} f(x_s(\phi))\mathrm{d}s - \sum_{k=1}^{p} A_k\phi(-\Delta_k + \theta + h) \right| \\ &\quad = \sup_{\theta\in(-h,0]} \left| (\theta+h)f(\phi) - \int_0^{\theta+h} f(x_s(\phi))\mathrm{d}s \right|. \end{aligned} \tag{A33}$$

Therefore,

$$\begin{aligned} &\limsup_{h\to 0^+} \frac{1}{h}\|\phi_h - x_h(\phi)\| \\ &\quad = \limsup_{h\to 0^+} \sup_{\theta\in(-h,0]} \left| \frac{1}{h}(\theta+h)f(\phi) - \frac{1}{h}\int_0^{\theta+h} f(x_s(\phi))\mathrm{d}s \right| \\ &\quad = \limsup_{h\to 0^+} \sup_{\theta\in(-h,0]} \frac{1}{h}|\theta+h| \left| f(\phi) - \frac{1}{\theta+h}\int_0^{\theta+h} f(x_s(\phi))\mathrm{d}s \right| \\ &\quad \le \limsup_{h\to 0^+} \sup_{\theta\in(-h,0]} \left| f(\phi) - \frac{1}{\theta+h}\int_0^{\theta+h} f(x_s(\phi))\mathrm{d}s \right| = 0. \end{aligned} \tag{A34}$$

So, the first limit on the right-hand side of (A30) is equal to zero. Taking into account of the result in (A31), taking into account of the already proved conditions (i), we can choose α_3 in (ii) equal to α_1. It remains to prove that the functional V is Lipschitz on bounded sets. Let H be a positive real. Let $z = [H] + 1$. Taking again into account of the definition of G in (A18), in Lemma A.7, taking into account of result (3) in the same lemma, we have, for any given $\phi_i \in \mathcal{C}_H$, $i = 1, 2$,

$$|V(\phi_1) - V(\phi_2)| \le \left(2 + \sum_{q=1}^{z} \frac{2^{-q}G(H,q)}{1+G(q,q)}\right) \|\phi_2 - \phi_1\|. \tag{A35}$$

The proof of the necessity part of the Lemma is complete. As far as the sufficiency part is concerned, by Lemma 6 in Pepe et al. (2008b), we can assume, without any loss of generality, that $\xi_0 \in W^{1,\infty}$ (see also Pepe, 2007b, as far as RFDEs are concerned). From Lemma 5 in Pepe et al. (2008b), it follows that the function $t \to w(t) = V(x_t(\xi_0))$ is locally absolutely continuous (thus its derivative exists almost everywhere). From conditions (i), (ii) it follows, for the function $t \to w(t) = V(x_t(\xi_0))$, taking into account of (9), that

$$\frac{\mathrm{d}w(t)}{\mathrm{d}t} = D^+V(x_t(\xi_0)) \le -\alpha_3 \circ \alpha_2^{-1}(w(t)), \quad t \in [0,b), \ a.e., \tag{A36}$$

where $[0, b)$, $0 < b \leq +\infty$, is the maximal interval of existence of the solution. From Lemma 4.4 in Lin et al. (1996), it follows that there exists a function $\overline{\beta}$ of class $\mathcal{KL}$ (depending only on α_2, α_3) such that $w(t) \leq \overline{\beta}(w(0), t)$, $t \in [0, b)$. From condition (i), it follows that

$$\|x_t\| \leq \alpha_1^{-1} \circ \overline{\beta}(\alpha_2(\|\xi_0\|), t), \quad t \in [0, b) \tag{A37}$$

Since the solution x_t is bounded in $[0, b)$ (and thus $\mathcal{D}x_t$ is bounded in $[0, b)$), it follows that $b = +\infty$ (see Hale and Lunel 1993; Lemma 3 in Pepe 2011). Since the function $(s, t) \to \beta(s, t) = \alpha_1^{-1} \circ \overline{\beta}(\alpha_2(s), t)$, $s,\ t \in R^+$, is of class $\mathcal{KL}$, the proof of the sufficiency part of the lemma is complete. The proof of the lemma is complete. □

We are now ready to prove Theorem 2.3. The necessity part follows from Lemma A.8, since, for all $\phi \in \mathcal{C}$, the inequality holds

$$|\mathcal{D}\phi| \leq \left(1 + \sum_{k=1}^{p} |A_k|\right)\|\phi\|. \tag{A38}$$

The sufficiency part is a standard result, as far as the local, uniform, asymptotic stability of the origin is concerned. The proof can be obtained by similar, though more involved, reasoning as the one used in the proof of Theorem 2.1, pp. 132–133 of Hale and Lunel (1993), for the local, uniform asymptotic stability of the origin of systems described by RFDEs, taking into account of Theorem 3.5, p. 275 of Hale and Lunel (1993). In the case of the uniform 0-GAS property, to our best knowledge, such proof is not available in the literature. For reader's convenience and for the sake of completeness, we report it here, using the derivative of the functional V in Driver's form, which does not involve the solution, not even formally (see (6)–(8)). Let ϵ be an arbitrary positive real. Let $\overline{M}$ be as in Lemma A.5. Let

$$\delta = \min\left\{\frac{\epsilon}{4\overline{M}}, \alpha_2^{-1} \circ \alpha_1\left(\frac{\epsilon}{4\overline{M}}\right)\right\}. \tag{A39}$$

Let $\|\xi_0\| < \delta$. Let $[0, b)$, $0 < b \leq +\infty$, be the maximal interval of existence of the solution $x(t)$. Then, $|x(t)| < \epsilon$, $t \in [0, b)$. In order to prove this, by Lemma 6.2 of Bacciotti and Rosier (2005), taking into account of (9) and ii), the following inequalities hold for $t \in [0, b)$,

$$\begin{aligned} |\mathcal{D}x_t| &\leq \alpha_1^{-1}(V(x_t)) \\ &\leq \alpha_1^{-1}(V(\xi_0)) \\ &\leq \alpha_1^{-1} \circ \alpha_2(\delta) \\ &\leq \alpha_1^{-1} \circ \alpha_2 \circ \alpha_2^{-1} \circ \alpha_1\left(\frac{\epsilon}{4\overline{M}}\right) = \frac{\epsilon}{4\overline{M}}. \end{aligned} \tag{A40}$$

Since $\mathcal{D}x_t$ is bounded in $[0, b)$, it follows that $b = +\infty$ (see Hale and Lunel 1993; Lemma 3 of Pepe 2011). From (A40), taking into account of Lemma A.5, the inequalities hold:

$$|x(t)| \leq \overline{M}\|\xi_0\| + \overline{M} \sup_{\tau \in [0,t]} |\mathcal{D}x_\tau| \leq \overline{M}\frac{\epsilon}{4\overline{M}} + \overline{M}\frac{\epsilon}{4\overline{M}} = \frac{\epsilon}{2}, \quad t \in R^+. \tag{A41}$$

Therefore, we have proved Lagrange and Lyapunov stability. Now, let H, ϵ be arbitrary positive reals. By Lemma 6 of Pepe et al. (2008b), we can assume, without any loss of generality, that $\xi_0 \in W^{1,\infty}$. From Lemma 5 of Pepe et al. (2008b), it follows that the function $t \to w(t) = V(x_t(\xi_0))$ is locally absolutely continuous (thus its derivative exists almost everywhere). We wish to show that: for any $\xi_0 \in C_H \cap W^{1,\infty}$, the solution exists for all $t \in R^+$; there exists a positive real T such that, for all $t \geq T$, $|x(t, \xi_0)| < \epsilon$ $\forall \xi_0 \in C_H \cap W^{1,\infty}$. First, let us notice that, for $\xi_0 \in \mathcal{C}_H \cap W^{1,\infty}$, and $[0, b)$ the related maximal interval of existence of the solution, we have that

$$|\mathcal{D}x_t| \leq \alpha_1^{-1} \circ \alpha_2(H), \quad t \in [0, b), \tag{A42}$$

and thus, from Lemma A.5,

$$|x(t)| \leq \overline{M}H + \overline{M}\alpha_1^{-1} \circ \alpha_2(H), \quad t \in [-\Delta, b). \tag{A43}$$

Again, since $\mathcal{D}x_t$ is bounded in $[0, b)$, it follows $b = +\infty$. Let us choose δ as in (A39). Let $\overline{\delta} = \frac{\delta}{4\overline{M}}$. Let j be the smallest positive integer such that

$$e^{-\omega j\Delta}\left(\overline{M}H + \overline{M}\alpha_1^{-1} \circ \alpha_2(H)\right) < \overline{\delta}, \tag{A44}$$

where ω is the positive real in inequality (A6), in Lemma A.5 (see Theorem 3.5, p. 275 of Hale and Lunel 1993). Let $\overline{\Delta} = (j+1)\Delta$. Let $\overline{k}$ be the smallest integer satisfying

$$\overline{k} - 1 \geq \frac{L\alpha_2(H)}{\overline{\delta}\alpha_3\left(\frac{\overline{\delta}}{2}\right)}, \tag{A45}$$

where

$$L = \max\left\{\sup_{\phi \in C, \|\phi\| \leq \overline{M}H + \overline{M}\alpha_1^{-1} \circ \alpha_2(H)} |f(\phi)|, \ \frac{2\overline{\delta}}{\overline{\Delta}}\right\}. \tag{A46}$$

Let $T = 2\overline{k}\,\overline{\Delta}$. Let $\xi_0 \in \mathcal{C}_H \cap W^{1,\infty}$. From the time-invariant character of the system described by (1) and the conditions on the functional V, it follows that, if, at a time $\overline{t}$, $\|x_{\overline{t}}\| < \delta$, then $|x(t)| < \epsilon$ $\forall t \geq \overline{t}$. Now, for some $\tau \in \left[\overline{\Delta}, T\right]$, the inequality holds:

$$\sup_{\theta \in [\tau - \overline{\Delta}, \tau]} |\mathcal{D}x_\theta| < \overline{\delta}. \tag{A47}$$

In order to prove this, by contradiction, if such τ does not exist, in each interval $[(2k-1)\overline{\Delta}, 2k\overline{\Delta}]$, $k = 1, 2, , \overline{k}$, there exists a time t_k such that $|\mathcal{D}x_{t_k}| \geq \overline{\delta}$. Since $|f(x_t)| \leq L$ $\forall t \in R^+$, it follows that $|\mathcal{D}x_t| \geq \frac{\overline{\delta}}{2}$ $\forall t \in I_k = [t_k - \frac{\overline{\delta}}{2L}, t_k + \frac{\overline{\delta}}{2L}]$. In order to prove this, take into account that, from the equality

$$\mathcal{D}x_{\max(t,t_k)} = \mathcal{D}x_{\min(t,t_k)} + \int_{\min(t,t_k)}^{\max(t,t_k)} f(x_\theta)\mathrm{d}\theta, \tag{A48}$$

the inequality follows

$$|\mathcal{D}x_t| \geq |\mathcal{D}x_{t_k}| - \int_{\min(t,t_k)}^{\max(t,t_k)} |f(x_\theta)|\mathrm{d}\theta. \tag{A49}$$

Notice that the intervals I_k, $k = 1, 2, , \overline{k}$, do not overlap, because of the choice of L. Now, taking into account of the absolute continuity property of the functional V, and of (9),

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we have

$$\begin{aligned}V(x_{t_{\bar{k}}}) &= V(\xi_0) + \int_0^{t_{\bar{k}}} \frac{\mathrm{d}V(x_\tau)}{\mathrm{d}\tau}\mathrm{d}\tau \\ &= V(\xi_0) + \int_0^{t_{\bar{k}}} D^+V(x_\tau)\mathrm{d}\tau \\ &\le V(\xi_0) - \int_0^{t_{\bar{k}}} \alpha_3(|\mathcal{D}x_\theta|)\mathrm{d}\theta \le \alpha_2(H) \\ &\quad - \sum_{k=1}^{\bar{k}-1} \int_{t_k-\frac{\bar{\delta}}{2L}}^{t_k+\frac{\bar{\delta}}{2L}} \alpha_3\left(\frac{\delta}{2}\right)\mathrm{d}\theta \\ &= \alpha_2(H) - (\bar{k}-1)\alpha_3\left(\frac{\delta}{2}\right)\frac{\bar{\delta}}{L} \\ &\le \alpha_2(H) - \frac{L\alpha_2(H)}{\bar{\delta}\alpha_3\left(\frac{\delta}{2}\right)}\alpha_3\left(\frac{\delta}{2}\right)\frac{\bar{\delta}}{L} = 0, \qquad \text{(A50)}\end{aligned}$$

which, by (i), is a contradiction, since we hypothesised $|\mathcal{D}x_{t_{\bar{k}}}| \ge \bar{\delta}$. Now, for $t \in [\tau - \Delta, \tau]$, taking into account of Theorem 3.5, p. 275 of Hale and Lunel (1993) (see Lemma A.5) and of (A44), we have

$$\begin{aligned}|x(t)| &\le \overline{M}e^{-\omega(t-(\tau-\bar{\Delta}))}\left(\overline{M}H + \overline{M}\alpha_1^{-1} \circ \alpha_2(H)\right) + \overline{M}\,\bar{\delta} \\ &\le \overline{M}e^{-\omega(j\Delta)}\left(\overline{M}H + \overline{M}\alpha_1^{-1} \circ \alpha_2(H)\right) + \overline{M}\,\bar{\delta} \\ &\le 2\overline{M}\,\bar{\delta} = \frac{\delta}{2}. \qquad \text{(A51)}\end{aligned}$$

Therefore, $\|x_\tau\| < \delta$ and, for any $t \ge T$, $|x(t)| < \epsilon$. Since this result holds for any $\xi_0 \in \mathcal{C}_H \cap W^{1,\infty}$, the proof of the theorem is complete.

Appendix B: proof of Theorem 2.4

Lemma B.1: *Let the system described by* (1) *be* 0-*GES*, *with related positive reals* M, λ. *Let* H *be a positive real. Let* $\overline{M}$ *be a positive real as depicted in Lemma* A.5 (*see* (A6)). *Let* $\overline{L}$ *be a positive real such that, for all* $\phi_1, \phi_2 \in \mathcal{C}_{(H+MH)}$, *the inequality follows*:

$$|f(\phi_1) - f(\phi_2)| \le \overline{L}\|\phi_1 - \phi_2\|. \qquad \text{(B1)}$$

Let L, P *be the positive reals defined as follows*:

$$P = \overline{M}\left(2 + \sum_{k=1}^{p}|A_k|\right), \quad L = \overline{M}\,\overline{L}. \qquad \text{(B2)}$$

Then, for any given initial conditions $\phi_1, \phi_2 \in \mathcal{C}_H$, *the corresponding solutions* $x_t(\phi_1)$ *and* $x_t(\phi_2)$ *satisfy the inequality*

$$\|x_t(\phi_1) - x_t(\phi_2)\| \le Pe^{Lt}\|\phi_1 - \phi_2\|, \quad t \ge 0. \qquad \text{(B3)}$$

Proof: The solutions $x(t, \phi_i)$, $i = 1, 2$, satisfy the following equation, for $t \ge 0$,

$$x(t, \phi_i) = \sum_{k=1}^{p} A_k x(t - \Delta_k, \phi_i) + w_i(t), \qquad \text{(B4)}$$

where $w_i(t) = \mathcal{D}\phi_i + \int_0^t f(x_s(\phi_i))\mathrm{d}s$, $i = 1, 2$. From Lemma A.5, the inequality follows:

$$|x(t, \phi_1) - x(t, \phi_2)| \le \overline{M}\|\phi_1 - \phi_2\| + \overline{M}\sup_{\theta \in [0,t]}|w_1(\theta) - w_2(\theta)|. \qquad \text{(B5)}$$

Now, the following equality/inequalities holds:

$$\begin{aligned}&|w_1(t) - w_2(t)| \\ &= \left|\mathcal{D}\phi_1 + \int_0^t f(x_s(\phi_1))\mathrm{d}s - \mathcal{D}\phi_2 - \int_0^t f(x_s(\phi_2))\mathrm{d}s\right| \\ &\le |\mathcal{D}\phi_1 - \mathcal{D}\phi_2| + \left|\int_0^t (f(x_s(\phi_1)) - f(x_s(\phi_2)))\mathrm{d}s\right| \\ &\le \left(1 + \sum_{k=1}^{p}|A_k|\right)\|\phi_1 - \phi_2\| + \int_0^t \left|f(x_s(\phi_1)) - f(x_s(\phi_2))\right|\mathrm{d}s. \qquad \text{(B6)}\end{aligned}$$

Taking into account that $\|x_t(\phi_i)\| \le H + MH$, $t \ge 0$, from the Lipschitz property of the map f it follows that the inequality holds:

$$\begin{aligned}&|w_1(t) - w_2(t)| \\ &\le \left(1 + \sum_{k=1}^{p}|A_k|\right)\|\phi_1 - \phi_2\| + \int_0^t \overline{L}\|x_s(\phi_1) - x_s(\phi_2)\|\mathrm{d}s. \qquad \text{(B7)}\end{aligned}$$

The inequality follows

$$\begin{aligned}\|x_t(\phi_1) - x_t(\phi_2)\| &\le \overline{M}\left(2 + \sum_{k=1}^{p}|A_k|\right)\|\phi_1 - \phi_2\| \\ &\quad + \overline{M}\int_0^t \overline{L}\|x_s(\phi_1) - x_s(\phi_2)\|\mathrm{d}s. \qquad \text{(B8)}\end{aligned}$$

Inequality (B3) follows from inequality (B8) by the Bellman–Gronwall lemma. □

Lemma B.2: *The system described by* (1) *is* 0-*GES if and only if there exist a functional* V: $\mathcal{C} \to R^+$, *Lipschitz on bounded sets, and positive reals* a_1, a_2, a_3, *such that the following conditions hold for all* $\phi \in \mathcal{C}$:

(i) $a_1|\mathcal{D}\phi| \le V(\phi) \le a_2\|\phi\|$;
(ii) $D^+V(\phi) \le -a_3\|\phi\|$

Proof: Let us prove the sufficiency part. Let $x(t)$, $t \in [0, b)$, $0 < b \le +\infty$, be the solution of (1) corresponding to an initial condition ξ_0. By Lemma 6 of Pepe et al. (2008b), we can assume, without any loss of generality, that $\xi_0 \in W^{1,\infty}$. From Lemma 5 of Pepe et al. (2008b), it follows that the function $t \to w(t) = V(x_t(\xi_0))$ is locally absolutely continuous (thus its derivative exists almost everywhere). From condition (ii), taking into account of (9), it follows, for the function $t \to w(t) = V(x_t(\xi_0))$, that

$$\frac{\mathrm{d}w(t)}{\mathrm{d}t} = D^+V(x_t(\xi_0)) \le -a_3\|x_t(\xi_0)\|, \quad t \in [0, b), \quad a.e., \qquad \text{(B9)}$$

and, from (i),

$$\frac{\mathrm{d}w(t)}{\mathrm{d}t} \le -\frac{a_3}{a_2}w(t), \quad t \in [0, b), \ a.e. \qquad \text{(B10)}$$

From (B10), taking into account of the absolute continuity property of the function $t \to w(t)$, by the Bellman–Gronwall lemma, the inequality follows

$$w(t) \le e^{-\bar{\lambda}t}w(0), \quad t \in [0, b), \qquad \text{(B11)}$$

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with $\overline{\lambda}=\frac{a_3}{a_2}$. Finally, from (i), the inequality follows:

$$|\mathcal{D}x_t| \le \frac{a_2}{a_1}e^{-\overline{\lambda}t}\|\xi_0\|, \quad t\in[0,b). \tag{B12}$$

Moreover, $b=+\infty$, otherwise the function $t\to\mathcal{D}x_t$, in $[0,b)$, was unbounded (see Hale and Lunel 1993; see Lemma 3 of Pepe 2011). From (B12), from Theorem 4.5, p. 275 of Hale and Lunel (1993), it follows that there exists a positive real q such that, for any $t\ge 0$, for any $\xi_0\in W^{1,\infty}$, the inequality holds:

$$\|x_t\| \le q\|\xi_0\|. \tag{B13}$$

Now, taking into account of Theorem 4.5, p. 275 of Hale and Lunel (1993), and of the time-invariant character of the system described by (1), there exist positive reals q_1, q_2, such that, for any $t\ge t_0\ge 0$, the inequality holds:

$$|x(t)| \le q_1\left(e^{-q_2(t-t_0)}q\|\xi_0\| + \frac{a_2}{a_1}e^{-\overline{\lambda}t_0}\|\xi_0\|\right). \tag{B14}$$

Then the 0-GES property follows by choosing $t_0=t/2$, $M=q_1\left(q+\frac{a_2}{a_1}\right)$, $\lambda=\frac{1}{2}\min\{q_2,\overline{\lambda}\}$. As far as the necessity part is concerned, let the solution satisfy the inequality

$$|x(t)| \le Me^{-\lambda t}\|\xi_0\|_\infty, \quad t\in[0,+\infty), \tag{B15}$$

for suitable positive reals M, λ. It follows that

$$\|x_t\| \le Me^{\lambda\Delta}e^{-\lambda t}\|\xi_0\|, \quad t\in[0,+\infty). \tag{B16}$$

Let T be a positive real such that

$$Me^{\lambda\Delta}e^{-\lambda T} = \frac{1}{2}. \tag{B17}$$

Let $V\colon\mathcal{C}\to R^+$ be defined (Krasovskii 1963), for $\phi\in\mathcal{C}$, as

$$V(\phi) = \int_0^T \|x_t\|\mathrm{d}t + \sup_{t\in[0,T]}\|x_t\|, \tag{B18}$$

where $x(t)$ is the solution of (1), corresponding to an initial condition $\xi_0=\phi$. From the inequality

$$|\mathcal{D}\phi| \le \left(1+\sum_{k=1}^{p}|A_k|\right)\|\phi\|, \quad \phi\in\mathcal{C}, \tag{B19}$$

it follows that the functional V satisfies the first inequality in (i). As far as the second inequality in (i) is concerned, the following inequality helps:

$$V(\phi) \le \left(\int_0^T Me^{\lambda\Delta}e^{-\lambda t}dt + Me^{\lambda\Delta}\right)\|\phi\|. \tag{B20}$$

As far as inequality (ii) is concerned, taking into account of (9), the following equality/inequalities holds (recall that $x(t)$ is the solution corresponding to ϕ),

$$\begin{aligned} D^+V(\phi) &= \limsup_{h\to 0^+}\frac{V(x_h)-V(\phi)}{h} \\ &\le \limsup_{h\to 0^+}\frac{\int_h^{T+h}\|x_t\|\mathrm{d}t - \int_0^T\|x_t\|\mathrm{d}t}{h} \\ &\quad + \limsup_{h\to 0^+}\frac{\sup_{t\in[h,T+h]}\|x_t\| - \sup_{t\in[0,T]}\|x_t\|}{h}. \end{aligned} \tag{B21}$$

Taking into account of (B16) and (B17), it follows that

$$\limsup_{h\to 0^+}\frac{\sup_{t\in[h,T+h]}\|x_t\| - \sup_{t\in[0,T]}\|x_t\|}{h} \le 0. \tag{B22}$$

Therefore,

$$D^+V(\phi) \le \|x_T\| - \|\phi\|. \tag{B23}$$

Taking account of (B16) and (B17), the inequality follows:

$$D^+V(\phi) \le -\frac{1}{2}\|\phi\|. \tag{B24}$$

It remains to prove that the functional V is Lipschitz on bounded sets. Let H be a positive real. Let $\phi_1,\ \phi_2\in\mathcal{C}_H$. The inequality follows

$$\begin{aligned} &|V(\phi_1)-V(\phi_2)| \\ &\quad\le \int_0^T \big|\|x_t(\phi_1)\|-\|x_t(\phi_2)\|\big|\mathrm{d}t \\ &\qquad + \left|\sup_{t\in[0,T]}\|x_t(\phi_1)\| - \sup_{t\in[0,T]}\|x_t(\phi_2)\|\right|. \end{aligned} \tag{B25}$$

Taking into account that

$$\begin{aligned} &\left|\sup_{t\in[0,T]}\|x_t(\phi_1)\| - \sup_{t\in[0,T]}\|x_t(\phi_2)\|\right| \\ &\quad\le \sup_{t\in[0,T]}\big|\|x_t(\phi_1)\|-\|x_t(\phi_2)\|\big| \\ &\quad\le \sup_{t\in[0,T]}\|x_t(\phi_1)-x_t(\phi_2)\|, \end{aligned} \tag{B26}$$

by Lemma B.1 the inequality follows, for suitable positive reals P, L,

$$|V(\phi_1)-V(\phi_2)| \le \int_0^T Pe^{Lt}\|\phi_1-\phi_2\|\mathrm{d}t + \sup_{t\in[0,T]}Pe^{Lt}\|\phi_1-\phi_2\|. \tag{B27}$$

Therefore

$$|V(\phi_1)-V(\phi_2)| \le L_V\|\phi_1-\phi_2\| \quad \forall\phi_i\in\mathcal{C}_H,\ i=1,2, \tag{B28}$$

where $0<L_V\le P(T+1)e^{\mathrm{LT}}$. □

The necessity part of Theorem 2.4 follows straightforwardly by Lemma B.2. As far as the sufficiency part is concerned, let $x(t)$, $t\in[0,b)$, $0<b\le+\infty$, be the solution of (1) corresponding to an initial condition ξ_0. By Lemma 6 of Pepe et al. (2008b), we can assume, without any loss of generality, that $\xi_0\in W^{1,\infty}$. From Lemma 5 of Pepe et al. (2008b), it follows that the function $t\to w(t)=V(x_t(\xi_0))$ is locally absolutely continuous (thus its derivative exists almost everywhere). From condition (ii), taking into account of (9), it follows, for the function $t\to w(t)=V(x_t(\xi_0))$, that

$$\frac{\mathrm{d}w(t)}{\mathrm{d}t} = D^+V(x_t(\xi_0)) \le -a_3w(t), \quad t\in[0,b), \quad a.e. \tag{B29}$$

From (B29), taking into account of the absolute continuity property of the function $t\to w(t)$, by the Bellman–Gronwall lemma, the inequality follows:

$$w(t) \le e^{-a_3t}w(0), \quad t\in[0,b), \tag{B30}$$

Finally, from (i), the inequality follows:

$$|\mathcal{D}x_t| \le \frac{a_2}{a_1} e^{-a_3 t}\|\xi_0\| \quad t \in [0, b). \tag{B31}$$

From here on, the same lines of the proof of the sufficiency part of Lemma B.2 can be followed (just use (B13) and (B14) with $\overline{\lambda} = a_3$).

Appendix C: proof of Theorem 2.5

The sufficiency part follows from Theorem 2.4. As far as the necessity part is concerned, we have the following lemma.

Lemma C.1: *Let the system described by* (1) *be* 0-*GES, with related positive reals* M, λ. *Let* $\overline{M}$ *be a positive real as depicted in* (A6), *in Lemma A.5. Let* $\overline{L}$ *be a positive real such that, for all* $\phi_1, \phi_2 \in \mathcal{C}$, *the inequality follows*:

$$|f(\phi_1) - f(\phi_2)| \le \overline{L}\|\phi_1 - \phi_2\|. \tag{C1}$$

Let L, P *be the positive reals defined as follows*

$$P = \overline{M}\left(2 + \sum_{k=1}^{p} |A_k|\right), \quad L = \overline{M}\,\overline{L}. \tag{C2}$$

Then, for any given initial conditions $\phi_1, \phi_2 \in \mathcal{C}$, *the corresponding solutions* $x_t(\phi_1)$ *and* $x_t(\phi_2)$ *satisfy the inequality*

$$\|x_t(\phi_1) - x_t(\phi_2)\| \le Pe^{Lt}\|\phi_1 - \phi_2\|, \quad t \ge 0. \tag{C3}$$

Proof: The same steps of the proof of Lemma B.1 can be used here. Just take into account that, from (C1), the inequalities (B7), (B8) hold for any $\phi_i \in \mathcal{C}$, $i = 1, 2$. □

Now, take the same functional V defined in (B18). Such functional satisfies the conditions (i), (ii) of Lemma B.2 and thus satisfies (i), (ii) of Theorem 2.5. It remains to prove that such functional V is globally Lipschitz. Let $\phi_1, \phi_2 \in \mathcal{C}$. The inequality follows

$$\begin{aligned} &|V(\phi_1) - V(\phi_2)| \\ &\quad \le \int_0^T \big|\|x_t(\phi_1)\| - \|x_t(\phi_2)\|\big|\mathrm{d}t \\ &\qquad + \left|\sup_{t\in[0,T]} \|x_t(\phi_1)\| - \sup_{t\in[0,T]} \|x_t(\phi_2)\|\right|. \end{aligned} \tag{C4}$$

Taking into account of (B26), by Lemma C.1 the inequality follows, for suitable positive reals P, L,

$$|V(\phi_1) - V(\phi_2)| \le \int_0^T Pe^{Lt}\|\phi_1 - \phi_2\|\mathrm{d}t + \sup_{t\in[0,T]} Pe^{Lt}\|\phi_1 - \phi_2\|. \tag{C5}$$

Therefore

$$|V(\phi_1) - V(\phi_2)| \le L_V\|\phi_1 - \phi_2\|, \quad \forall \phi_i \in \mathcal{C},\ i = 1, 2, \tag{C6}$$

where $0 < L_V \le P(T+1)e^{LT}$.

Appendix D: proof of Theorem 3.2

We will make use of Theorem 2.6 and of Theorem 4 of Pepe (2007a). From the 0-GES property and condition (i), it follows that there exists a globally Lipschitz functional V and a semi-norm $\|\cdot\|_a$ as depicted in Theorem 2.6 (see conditions (i), (ii), (iii) in Theorem 2.6), for the system described by (10) with $u(t) \equiv 0$. Now, for the same functional V and semi-norm $\|\cdot\|_a$, taking into account of Theorem 2.6, the following equality/inequalities holds, for any $\phi \in \mathcal{C}$, $u \in R^m$, by (8), (12), (13),

$$\begin{aligned} D^+V(\phi, u) &= \limsup_{h\to 0^+} \frac{1}{h}\big(V(\phi_{h,u}) - V(\phi)\big) \\ &\le \limsup_{h\to 0^+} \frac{1}{h}\big(V(\phi_{h,u}) - V(\phi_{h,0})\big) \\ &\quad + \limsup_{h\to 0^+} \frac{1}{h}\big(V(\phi_{h,0}) - V(\phi)\big) \\ &\le \limsup_{h\to 0^+} \frac{1}{h}\big(V(\phi_{h,u}) - V(\phi_{h,0})\big) - a_3\|\phi\|_a, \end{aligned} \tag{D1}$$

where a_3 is the positive real given in Theorem 2.6. Taking into account of the global Lipschitz property of the functional V and of condition (ii), we obtain, for suitable positive real l,

$$\begin{aligned} &\limsup_{h\to 0^+} \frac{1}{h}\big(V(\phi_{h,u}) - V(\phi_{h,0})\big) \\ &\quad \le \limsup_{h\to 0^+} \frac{1}{h}\big(l\,\|\phi_{h,u} - \phi_{h,0}\|\big) \\ &\quad = \limsup_{h\to 0^+} \frac{1}{h} \sup_{s\in[-h,0]} l\,|\mathcal{D}\phi + f(\phi, u)(s+h) - \mathcal{D}\phi^\star_{s+h} + \phi(0) \\ &\qquad - \mathcal{D}\phi - f(\phi, 0)(s+h) + \mathcal{D}\phi^\star_{s+h} - \phi(0)| \\ &\quad \le \limsup_{h\to 0^+} \frac{1}{h} \sup_{s\in[-h,0]} (s+h)l\,|f(\phi, u) - f(\phi, 0)| \le lL(|u|). \end{aligned} \tag{D2}$$

From (D1), (D2), we obtain

$$D^+V(\phi, u) \le -a_3\|\phi\|_a + lL(|u|). \tag{D3}$$

From Theorem 4 of Pepe (2007a), taking into account of condition (i) in Theorem 2.6, the ISS of the system described by (10) follows. The proof of the theorem is complete.

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