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CONVERSE THEOREMS FOR GL_n

by J. W. COGDELL* *and* I. I. PIATETSKI-SHAPIRO**

The objective of this article is to prove a criterion for a given irreducible representation Π of $GL_n(\mathbf{A})$ to be automorphic. This criterion traditionally is called a Converse Theorem, after Hecke's celebrated Converse Theorem [17, 18]. The converse theorems of this paper have an application to the problem of Langlands' lifting of automorphic representations from classical groups to GL_n. This application will be considered in a future joint publication with S. Gelbart, D. Ginzburg, S. Rallis, and D. Soudry.

The first converse theorem was actually proved by Hamburger in 1921 [5]. This theorem states that any Dirichlet series satisfying the functional equation of the Riemann zeta function $\zeta(s)$ and suitable regularity conditions must be a multiple of $\zeta(s)$. The generalization to L-functions corresponding to holomorphic modular forms was done by Hecke in 1936 [17]. The leading idea of Hecke was the connection of L-functions which satisfy a certain functional equation with modular forms. However Hecke was able to prove this connection only for holomorphic modular forms with respect to the full modular group. In 1944 Maass extended Hecke's method to his non-holomorphic forms, but still only for the full modular group [35]. The next very important step was made by Weil in 1967 [42]. Weil showed how to work with Dirichlet series corresponding to holomorphic modular forms with respect to congruence subgroups of the full modular group. Weil proved that if a Dirichlet series together with a sufficient number of twists satisfy nice functional equations with suitable regularity then it comes from a holomorphic modular form with respect to a congruence subgroup. The work of Weil marks the beginning of the modern era in the study of the connection between L-functions and automorphic forms.

In 1970 a remarkable new book came out: " Automorphic Forms on GL(2) " by Jacquet and Langlands [21]. In this book, instead of automorphic forms, a new object came into this scheme: automorphic representations. The basic result of Jacquet and Langlands was the following. They attached to each automorphic representation of GL(2) an L-function and proved that the nice properties of this L-function, i.e., holo-

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morphic continuation and nice functional equations, are equivalent to the representation being automorphic.

In the following we will use the language of automorphic representations rather than the classical language of automorphic forms. However, for the applications to the problem of lifting of automorphic representations, we need results which are more similar to Weil's theorem rather than that of Jacquet and Langlands. In order to get this result we have to use Weil's idea, but disguised in the language of automorphic representations. A preliminary version of these results over $\mathbf{Q}$ was given in [39].

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1. Some basic definitions and notation

Throughout we will take k to be a global field. Let $\mathfrak{o}$ denote its ring of integers. For each place v of k we will let k_v denote the completion of k at v. At the non-archimedean places we will let $\mathfrak{o}_v$ denote the ring of integers of k_v, $\mathfrak{p}_v$ the unique prime ideal of $\mathfrak{o}_v$, ϖ_v a choice of generator of $\mathfrak{p}_v$ and we will normalize the absolute value so that $|\varpi_v|_v = q_v^{-1}$ where $q_v = |\mathfrak{o}_v/\mathfrak{p}_v|$. We will use either $\mathfrak{o}_v^\times$ or $\mathfrak{u}_v$ for the group of local units. The symbol $\mathbf{A}$ will denote the ring of adeles of k and $\mathbf{A}^\times$ its group of ideles. Thus, $\mathbf{A}$ is the restricted product $\prod'_v k_v$ of the completions of k with respect to the compact subrings $\mathfrak{o}_v$. If S is a finite set of places of k we will let $k_S = \prod_{v\in S} k_v$ and $\mathbf{A}^S = \prod'_{v\notin S} k_v$ so that $\mathbf{A} = k_S\, \mathbf{A}^S$. We will use a similar notation for ideles.

For each finite set of places S of k containing all archimedean places, the ring of S-integers is $\mathfrak{o}_S = k \cap k_S \prod_{v\notin S} \mathfrak{o}_v$. We may view $\mathfrak{o}_S$ as a discrete subgroup of k_S through the embedding of k into k_S. Let $\mathfrak{u}^S = \prod_{v\notin S} \mathfrak{u}_v \subset (\mathbf{A}^\times)^S$. The class number h_S of $\mathfrak{o}_S$, called the S class number of k, is the cardinality of the S-class group

$$\mathscr{C}_S = k^\times\backslash\mathbf{A}^\times/k_S^\times\, \mathfrak{u}^S = k^\times\backslash(\mathbf{A}^\times)^S/\mathfrak{u}^S.$$

We fix a non-trivial normalized additive character ψ of $\mathbf{A}$ which is trivial on k.

Fix a basis $\{ e_i \}$ of k^n with respect to which the matrix structure of GL_n is defined. Let B_n denote the Borel subgroup of upper triangular matrices, A_n its Levi subgroup consisting of all diagonal matrices, and N_n its maximal unipotent subgroup. Let P'_n denote the standard parabolic subgroup of GL_n associated to the partition $(n-1, 1)$ of n. Let $P_n \subset P'_n$ be the mirabolic subgroup consisting of those matrices in P'_n whose last row is $(0, \ldots, 0, 1)$. Let $\bar{P}_n \subset \bar{P}'_n$ denote the opposite mirabolic and parabolic. So $\bar{P}'_n = {}^t(P'_n)^{-1}$. By Z_n, we denote the center of GL_n.

For each non-archimedean place v we will let $K_v = GL_n(\mathfrak{o}_v)$ be a maximal compact subgroup. We will always consider admissible representations Π_v of $GL_n(k_v)$ on a complex vector space V_{Π_v} in the usual sense [6, 8, 9]. As is common, we will not distinguish between admissible representations of $GL_n(k_v)$ and of its Hecke algebra [6]. We will call an admissible representation *unramified* if the space of vectors fixed by K_v is one-dimensional.

At an archimedean place v, we select as maximal compact subgroup K_v either $O(n)$ or $U(n)$ defined with respect to the basis above. At an archimedean place v of k by an *admissible* representation Π_v of $GL_n(k_v)$ we will mean a smooth representation of $GL_n(k_v)$ on a complete Frechet space V_{Π_v} whose subspace of K_v-finite vectors is an admissible representation of its Hecke algebra [6] and such that (Π_v, V_{Π_v}) is a canonical smooth model of moderate growth (in the sense of Casselman and Wallach) of the underlying representation of its Hecke algebra [10, 27].

Let v be any place of k and let ψ_v be any non-trivial additive character of k_v. Then ψ_v defines a character of $N_n(k_v)$, which by abuse of notation we again denote by ψ_v, by $\psi_v(n) = \psi_v(n_{1,2} + n_{2,3} + \ldots + n_{n-1,n})$ where $n = (n_{i,j}) \in N_n(k_v)$ relative to the basis above. Let (Π_v, V_{Π_v}) be a finitely generated admissible representation of $GL_n(k_v)$. We let $V^*_{\psi_v}$ denote the space of ψ_v-Whittaker functionals on V_{Π_v}, i.e., the space of continuous linear functionals λ_v on V_{Π_v} such that $\lambda_v(\Pi_v(n)\,\xi_v) = \psi_v(n)\,\lambda_v(\xi_v)$ for all $n \in N_n(k_v)$ and all $\xi_v \in V_{\Pi_v}$. A representation Π_v of $GL_n(k_v)$ is of *Whittaker type* if Π_v is finitely generated, admissible, and $\dim(V^*_{\psi_v}) = 1$. In this case we have a non-zero intertwining map from V_{Π_v} to the Whittaker space $\mathrm{ind}_{N_n(k_v)}^{GL_n(k_v)}(\psi_v)$ given by

$$\xi_v \mapsto W_{\xi_v}(g) = \lambda_v(\Pi_v(g)\,\xi_v)$$

where $\lambda_v \in V^*_{\psi_v}$ is a non-zero Whittaker functional. We will call the space of functions $\mathscr{W}(\Pi_v, \psi_v) = \{ W_{\xi_v}(g) \mid \xi_v \in \Pi_v \}$ the *Whittaker model* of Π_v (even though it is a model for the Whittaker quotient of Π_v unless the Whittaker map above is injective) and it is unique. An irreducible admissible representation of Whittaker type is called *generic*.

For our purposes, we will only need consider representations of Whittaker type of a certain nature. Let Q be the parabolic subgroup of GL_n associated to the partition $(r_1, \ldots, r_m)$ of n. For each i let π_i be a quasi-square integrable representation of $GL_{r_i}(k_v)$ (i.e., an irreducible admissible representation whose matrix coefficients become square-integrable modulo the center after twisting by a suitable character of $GL_{r_i}(k_v)$). Then the (unitarily) induced representations

$$\Xi_v = \mathrm{Ind}_{Q(k_v)}^{GL_n(k_v)}[\pi_1 \otimes \ldots \otimes \pi_m]$$

are of Whittaker type [4, 27]. Throughout this paper, by an *induced representation of Whittaker type* (or, more succinctly, an induced of Whittaker type) we will always mean one of these induced representations. From this it is clear that induced representations of Whittaker type have well-defined central characters. Also, the subspace of K_v-fixed vectors is at most one-dimensional.

In particular, let Q be the parabolic subgroup of GL_n associated to the partition $(r_1, \ldots, r_m)$ of n. For each i, let $\rho_{i,0}$ be a tempered representation of $GL_{r_i}(k_v)$. Let $u_1 > u_2 > \ldots > u_m$ be a sequence of real numbers. Set

$$\Xi_v = \mathrm{Ind}_{Q(k_v)}^{GL_n(k_v)}[(\rho_{1,0} \otimes |\ |^{u_1}) \otimes \ldots \otimes (\rho_{m,0} \otimes |\ |^{u_m})]$$

(unitary induction). We call such a representation an induced representation of *Langlands type*. Then Ξ_v is an induced representation of Whittaker type [19, 26, 27]. If Π_v is an irreducible generic representation of $GL_n(k_v)$ then Π_v is necessarily an irreducible induced representation of Langlands type [26, 27]. The *Langlands classification* for $GL_n(k_v)$ says that every such Ξ_v has a unique irreducible quotient Π_v and every irreducible admissible Π_v occurs uniquely as the quotient of some Ξ_v [7].

Consider a representation Π of $GL_n(\mathbf{A})$ on a space $V = V_\Pi$. This representation is called *factorizable* if there are local representations Π_v of $GL_n(k_v)$ on spaces V_v such that Π_v is unramified for almost all places v and (Π, V) is the restricted tensor product of the (Π_v, V_v) as in Flath [12]. We will denote this simply by $\Pi = \bigotimes \Pi_v$. We will always consider admissible representations of $GL_n(\mathbf{A})$ in the sense of [6] or [12]. If $\Pi = \bigotimes \Pi_v$ is factorizable and admissible then each Π_v is admissible, and conversely. If S is a finite set of places of k we will let $G_S = GL_n(k_S) = \prod_{v \in S} GL_n(k_v)$ and $G^S = GL_n(\mathbf{A}^S) = \prod'_{v \notin S} GL_n(k_v)$. Similarly, for $\Pi = \bigotimes \Pi_v$ factorizable we shall let $\Pi_S = \bigotimes_{v \in S} \Pi_v$ be the associated representation of G_S and $\Pi^S = \bigotimes_{v \notin S} \Pi_v$ be the associated representation of G^S, so that $\Pi = \Pi_S \otimes \Pi^S$.

If $\Pi = \bigotimes \Pi_v$ is an admissible factorizable representation of $GL_n(\mathbf{A})$ we will say that Π is of Whittaker type, induced of Whittaker type, or generic if each Π_v is and, in addition, at the places v where Π_v is unramified the space of K_v-fixed vectors is not in the kernel of the map to the Whittaker quotient. (This last condition is automatic if Π_v is generic or induced of Langlands type since in these cases the map to the Whittaker model is an isomorphism [26].) In these cases there is a unique global Whittaker functional λ (up to scalars) given by the product of the local Whittaker functionals λ_v suitably normalized. At the places v where Π_v is unramified, there is a distinguished unramified vector ξ_v^0 with respect to which the restricted tensor product is taken. At these places we always normalize the Whittaker functionals λ_v so that $\lambda_v(\xi_v^0) = 1$. In terms of the local Whittaker models, this implies that $W_{\xi_v^0}(I_n) = 1$. If $\mathscr{W}(\Pi_v, \psi_v)$ are the local Whittaker models, then $\mathscr{W}(\Pi, \psi) = \bigotimes \mathscr{W}(\Pi_v, \psi_v)$. It is again clear that global induced representations of Whittaker type have central characters.

If $\Pi = \bigotimes \Pi_v$ and $\Pi' = \bigotimes \Pi'_v$ are two factorizable admissible representations of $GL_n(\mathbf{A})$ then we will say that they are *quasi-isomorphic* if $\Pi_v \simeq \Pi'_v$ for all non-archimedean v for which both Π_v and Π'_v are unramified.

By an *automorphic* representation of $GL_n(\mathbf{A})$ we will mean an admissible subquotient representation of the space of automorphic forms $\mathscr{A}(GL_n(k)\backslash GL_n(\mathbf{A}))$ [6]. By a *proper automorphic representation* we will mean an admissible subrepresentation of the space of automorphic forms $\mathscr{A}(GL_n(k)\backslash GL_n(\mathbf{A}))$. By a cuspidal automorphic representation we will always mean an *irreducible* cuspidal automorphic representation. These are of course always proper.

2. Basic converse theorems

Let $\Pi = \bigotimes \Pi_v$ be an admissible factorizable representation of $GL_n(\mathbf{A})$ such that each Π_v is either irreducible or induced of Whittaker type. Let τ denote a factorizable automorphic representation of $GL_m(\mathbf{A})$ for some m with $1 \leqslant m \leqslant n-1$ such that each τ_v is irreducible or induced of Whittaker type. Then from the local theory of L-functions for $GL_n(k_v)$ [24, 27] for each place v we have a local L-function $L(\Pi_v \times \tau_v, s)$ and local ε-factor $\varepsilon(\Pi_v \times \tau_v, s, \psi_v)$ attached to Π and τ. We may then formally define a global L-function

$$L(\Pi \times \tau, s) = \prod_v L(\Pi_v \times \tau_v, s)$$

and a global ε-factor

$$\varepsilon(\Pi \times \tau, s, \psi) = \prod_v \varepsilon(\Pi_v \times \tau_v, s, \psi_v).$$

To see that these are actually well-defined we need the following elementary lemmas.

Lemma **2.1.** — *The ε-factor $\varepsilon(\Pi \times \tau, s, \psi)$ is absolutely convergent and if the central character ω_Π of Π is invariant under $k^\times$ then*

$$\varepsilon(\Pi \times \tau, s) = \prod_v \varepsilon(\Pi_v \times \tau_v, s, \psi_v)$$

is independent of ψ.

Proof. — For almost all v, Π_v, τ_v and ψ_v will be unramified and so

$$\varepsilon(\Pi_v \times \tau_v, s, \psi_v) \equiv 1$$

for these places. Thus $\varepsilon(\Pi \times \tau, s, \psi)$ is convergent.

To prove that the product is independent of the choice of additive character we must consider how the local ε-factor changes when we change our additive character $\psi(x)$ to $\psi^\lambda(x) = \psi(\lambda x)$ with $\lambda \in k^\times$. Recall that the local ε-factor is defined by the local functional equation [24, 27]

$$\frac{\Psi(W, W'; s)}{L(\Pi_v \times \tau_v, s)} \varepsilon(\Pi_v \times \tau_v, s, \psi_v)\, \omega_{\tau_v}(-1)^{n-1} = \frac{\widetilde{\Psi}(\rho(w_{n,m})\, \widetilde{W}, \widetilde{W}'; 1-s)}{L(\widetilde{\Pi}_v \times \widetilde{\tau}_v, 1-s)}$$

where $W \in \mathcal{W}(\Pi_v, \psi_v)$, $W' \in \mathcal{W}(\tau_v, \psi_v^{-1})$, $\widetilde{W}(g) = W(w_n\, {}^t g^{-1}) \in \mathcal{W}(\widetilde{\Pi}_v, \psi_v^{-1})$, and $\widetilde{W}'(g) = W(w_m\, {}^t g^{-1}) \in \mathcal{W}(\widetilde{\tau}_v, \psi_v)$. The Weyl elements involved are

$$w_r = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix},$$

the longest Weyl element of GL_r, and

$$w_{n,m} = \begin{pmatrix} I_m & 0 \\ 0 & w_{n-m} \end{pmatrix},$$

whereas ρ denotes right translation in the Whittaker model. The integrals themselves are given by

$$\Psi(W, W'; s) = \int_{N_m(k_v)\backslash GL_m(k_v)} W\begin{pmatrix} g & 0 \\ 0 & I_{n-m} \end{pmatrix} W'(g) \,|\det(g)|^{s-(n-m)/2}\, dg$$

and, setting $k = n - m - 1$,

$$\widetilde{\Psi}(W, W'; s) = \int_{N_m(k_v)\backslash GL_m(k_v)} \int_{M_{k,m}(k_v)} W\begin{pmatrix} g & 0 & 0 \\ x & I_k & 0 \\ 0 & 0 & 1 \end{pmatrix} W'(g) \,|\det(g)|^{s-(n-m)/2}\, dx\, dg.$$

To change the Whittaker model from those with respect to ψ_v to those with respect to ψ_v^λ, note that if we set

$$a_t(\lambda) = \begin{pmatrix} \lambda^{t-1} & & & & \\ & \lambda^{t-2} & & & \\ & & \ddots & & \\ & & & \lambda & \\ & & & & 1 \end{pmatrix} \in B_t(k),$$

then for $W \in \mathscr{W}(\Pi_v, \psi_v)$ we have $\ell(a_n(\lambda))\, W(g) = W(a_n(\lambda)\, g) \in \mathscr{W}(\Pi_v, \psi_n^\lambda)$. Note that the local L-functions are independent of the choice of ψ_v. Hence $\varepsilon(\Pi_v \times \tau_v, s, \psi_v^\lambda)$ is defined by the local functional equation

$$\frac{\Psi(\ell(a_n(\lambda))\, W, \ell(a_m(\lambda))\, W'; s)}{L(\Pi_v \times \tau_v, s)}\, \varepsilon(\Pi_v \times \tau_v, s, \psi_v^\lambda)\, \omega_{\tau_v}(-1)^{n-1} = \frac{\widetilde{\Psi}(\rho(w_{n,m})\, (\ell(a_n(\lambda))\, W)^\sim, (\ell(a_m(\lambda))\, W')^\sim; 1-s)}{L(\widetilde{\Pi}_v \times \widetilde{\tau}_v, 1-s)}.$$

Now a straightforward computation gives

$$\Psi(\ell(a_n(\lambda))\, W, \ell(a_m(\lambda))\, W'; s) = |\lambda|_v^{As+B}\, \omega_{\tau_v}(\lambda)^{m-n}\, \Psi\left(\rho\begin{pmatrix} I_m & \\ & a_{n-m}(\lambda) \end{pmatrix} W, W'; s\right)$$

with

$$A = m(m-n) - \frac{1}{2} m(m-1)$$

$$B = \frac{1}{2}(m-n)\left(m(m-n) - \frac{1}{2} m(m-1)\right) + \frac{1}{6}(m-1)\, m(m+1).$$

An equally straightforward calculation gives

$$\widetilde{\Psi}(\rho(w_{n,m})\,(\ell(a_n(\lambda))\,W)^{\sim}, (\ell(a_m(\lambda))\,W')^{\sim}; 1-s)$$

$$= |\lambda|_v^{C(1-s)+D}\, \omega_{\Pi_v}(\lambda)^m\, \omega_{\tau_v}(\lambda)^m\, \widetilde{\Psi}\left(\rho(\omega_{n,m})\left(\rho\begin{pmatrix} I_m & \\ & a_{n-m}(\lambda)\end{pmatrix} W\right)^{\sim}, \widetilde{W}'; 1-s\right)$$

with

$$C = -m^2 + \frac{1}{2} m(m-1)$$

$$D = \frac{1}{2} m(n-m-1)(n-m-2) + \frac{1}{2}(n-m)\left(m^2 - \frac{1}{2} m(m-1)\right) + \frac{1}{6}(m-1)\, m(m+1).$$

Then using the definition of the respective local ε-factors, we find

$$\varepsilon(\Pi_v \times \tau_v, s, \psi_v^\lambda) = \omega_{\Pi_v}(\lambda)^m\, \omega_{\tau_v}(\lambda)^n\, |\lambda|_v^{nms-d}\, \varepsilon(\Pi_v \times \tau_v, s, \psi_v)$$

with $d = nm - \frac{1}{2} m(m+1)$.

Taking the product over all places of k and using the product formula we find

$$\varepsilon(\Pi \times \tau, s, \psi^\lambda) = \omega_\Pi(\lambda)^m\, \omega_\tau(\lambda)^n\, \varepsilon(\Pi \times \tau, s, \psi).$$

Since τ is automorphic its central character is invariant under $k^\times$. Hence if ω_Π is invariant under $k^\times$ we see that the product $\varepsilon(\Pi \times \tau, s, \psi) = \varepsilon(\Pi \times \tau, s)$ will be independent of the choice of ψ. □

Lemma **2.2.** — *Suppose* $L(\Pi, s) = \prod_v L(\Pi_v, s)$ *is absolutely convergent in some half-plane. Then for any automorphic representation* $\tau = \bigotimes \tau_v$ *of* $GL_m(\mathbf{A})$ *which is either irreducible or induced of Whittaker type the Euler product for* $L(\Pi \times \tau, s)$ *is also absolutely convergent in some half-plane.*

Proof. — Let T be a finite set of places of k containing all archimedean places such that Π_v is unramified for $v \notin T$. Then the local L-factor for the places $v \notin T$ will be of the form

$$L(\Pi_v, s) = \prod_{i=1}^{n} (1 - a_{i,v}\, q_v^{-s})^{-1}.$$

Globally, let us write

$$L(\Pi, s) = L_T(\Pi, s)\, L^T(\Pi, s),$$

where

$$L_T(\Pi, s) = \prod_{v \in T} L(\Pi_v, s)$$

is a finite product, and hence always absolutely convergent, and

$$L^T(\Pi, s) = \prod_{v \notin T} L(\Pi_v, s) = \prod_{v \notin T} \prod_{i=1}^{n} (1 - a_{i,v}\, q_v^{-s})^{-1}.$$

Then if the Euler product for $L(\Pi, s)$ is absolutely convergent for $\operatorname{Re}(s) > c_0$, we have the estimate $| a_{i,v} | \ll q_v^{c_0}$ for all $v \notin T$, with the implied constant independent of i and v.

Let $\tau = \otimes \tau_v$ be an automorphic representation of $GL_m(\mathbf{A})$ which is irreducible or induced of Whittaker type. Then we know that the Euler product for $L(\tau, s)$ converges absolutely in some half-plane, say $\operatorname{Re}(s) > c_1$. Enlarging T if necessary, we may assume that τ_v is also unramified for $v \notin T$. Then, as above, we have

$$L(\tau_v, s) = \prod_{j=1}^{m} (1 - b_{j,v}\, q_v^{-s})^{-1}$$

with the estimate $| b_{j,v} | \ll q_v^{c_1}$.

The Euler factor for $L(\Pi_v \times \tau_v, s)$ for $v \notin T$ is given by

$$L(\Pi_v \times \tau_v, s) = \prod_{i=1}^{n} \prod_{j=1}^{m} (1 - a_{i,v}\, b_{j,v}\, q_v^{-s})^{-1} = \prod_{k=1}^{nm} (1 - c_{k,v}\, q_v^{-s})^{-1}.$$

Since we have the estimate $| c_{k,v} | = | a_{i,v} | \, | b_{j,v} | \ll q_v^{c_0 + c_1}$ for $v \notin T$, we see that the Euler product for $L(\Pi \times \tau, s)$ is absolutely convergent for $\operatorname{Re}(s) > c_0 + c_1 + 1$. □

Let g^ι denote the outer automorphism $g \mapsto g^\iota = {}^t g^{-1}$ of GL_n. For any representation π of GL_n over a local or global field, let $\pi^\iota(g) = \pi(g^\iota)$. If Π_v is an induced of Whittaker type, then so is Π_v^ι. If Π_v is irreducible, then so is Π_v^ι and in fact $\Pi_v^\iota \simeq \widetilde{\Pi}_v$, the contragredient representation.

Lemma **2.3.** — *Suppose* $L(\Pi, s)$ *converges in some half-plane and that the central character of* $\Pi = \otimes \Pi_v$ *is invariant under* $k^\times$. *Then the Euler product for* $L(\Pi^\iota, s)$ *also converges absolutely in some half-plane, as do the* $L(\Pi^\iota \times \tau^\iota, s)$ *for any automorphic representation* τ *of* $GL_m(\mathbf{A})$ *which is irreducible or induced of Whittaker type.*

Proof. — We may assume that the central character ω_Π of Π is unitary. For if it is not, we have $| \omega_\Pi(a) | = | a |^d$ for some $d \neq 0$. If we let $\omega_{-d/n}(a) = | a |^{-d/n}$ and set $\Pi' = \Pi \otimes \omega_{-d/n}$ then Π' has a unitary central character. Since $L(\Pi', s) = L(\Pi, s - d/n)$ we see that $L(\Pi, s)$ is absolutely convergent in some half-plane if and only if $L(\Pi', s)$ is.

For $v \notin T$, with T as in Lemma 2.2, Π_v will be unramified and we have unramified characters $\mu_{1,v}, \ldots, \mu_{n,v}$ of $GL_1(k_v)$ such that Π_v is the unramified constituent of $\mathrm{Ind}_{B_n(k_v)}^{GL_n(k_v)}(\mu_{1,v} \otimes \ldots \otimes \mu_{n,v})$. The local factor $L(\Pi_v, s)$ is then $\Pi(1 - a_{i,v} q_v^{-s})^{-1}$ with $a_{i,v} = \mu_{i,v}(\varpi_v)$. The central character of Π_v is $\omega_{\Pi_v} = \Pi \mu_{i,v}$. Since this is unitary, we have

$$1 = | \omega_{\Pi_v}(\varpi_v) | = \prod_{i=1}^{n} | \mu_{i,v}(\varpi_v) | = \prod_{i=1}^{n} | a_{i,v} |.$$

Still for $v \notin T$, if Π_v is as above, then Π_v^ι will be an unramified constituent of $\mathrm{Ind}_{B_n(k_v)}^{GL_n(k_v)}(\mu_{1,v}^{-1} \otimes \ldots \otimes \mu_{n,v}^{-1})$. Its local factor will then be $L(\Pi_v^\iota, s) = \Pi(1 - b_{i,v} q_v^{-1})^{-1}$ with $b_{i,v} = \mu_{i,v}(\varpi_v)^{-1} = a_{i,v}^{-1}$.

Now assume that $L(\Pi, s)$ converges absolutely for $\mathrm{Re}(s) > c$, so that we have the estimate $| a_{i,v} | \ll q_v^c$. Then for $| b_{i,v} |$ we have

$$| b_{i,v} | = \prod_{j \neq i} | b_{j,v} |^{-1} = \prod_{j \neq i} | a_{i,v} | \ll q_v^{(n-1)c}.$$

Hence the Euler product for $L(\Pi^\iota, s)$ converges absolutely for $\mathrm{Re}(s) > (n-1)c + 1$.

The rest of the lemma now follows form Lemma 2.2 applied to Π^ι. □

Definition. — Let $\Pi = \otimes \Pi_v$ *be a factorizable admissible representation of* $GL_n(\mathbf{A})$ *such that each local component* Π_v *is either irreducible or induced of Whittaker type and such that its central character* ω_Π *is invariant under* $k^\times$ *and its L-function* $L(\Pi, s)$ *is absolutely convergent in some half-plane. Let* τ *be an automorphic representation of* $GL_m(\mathbf{A})$ *which is either irreducible or induced of Whittaker type. We will say that* $L(\Pi \times \tau, s)$ *is* nice *if* $L(\Pi \times \tau, s)$ *and* $L(\Pi^\iota \times \tau^\iota, s)$ *have an analytic continuation to entire functions of s which are bounded in vertical strips and satisfy the functional equation*

$$L(\Pi \times \tau, s) = \varepsilon(\Pi \times \tau, s)\, L(\Pi^\iota \times \tau^\iota, 1 - s).$$

A converse theorem for GL_n is a criterion in terms of the $L(\Pi \times \tau, s)$ for determining when Π is actually an automorphic representation. Our first converse theorem, modeled on that of Jacquet and Langlands, is one of the end products of years of collaboration of the second author with H. Jacquet and J. Shalika (for example [22-24]). In the function field case, this theorem was proven in the 1970's by the second author [38]. The same method of proof works in the number field case now that the local archimedean theory has been completed by Jacquet and Shalika [27].

Theorem 1. — Let Π *be an irreducible admissible representation of* $GL_n(\mathbf{A})$ *whose central character* ω_Π *is invariant under* $k^\times$ *and whose L-function* $L(\Pi, s)$ *is absolutely convergent in some half-plane. Suppose that* $L(\Pi \times \tau, s)$ *is nice for every cuspidal automorphic representation* τ *of* $GL_m(\mathbf{A})$ *for all m with* $1 \leqslant m \leqslant n - 1$. *Then* Π *is a cuspidal automorphic representation of* $GL_n(\mathbf{A})$.

This theorem yields maximal information about Π, namely that it is actually cuspidal automorphic, but it requires nice behavior of the L-functions under twists by all cuspidal automorphic representations on all smaller GL_m's. We will give variants of this theorem where we require the L-functions to be nice under a smaller set of twists. For this we must use the ideas of Weil. The most typical converse theorem of this type is the following.

Fix a finite set of places S of k containing all archimedean places. For each integer m, let

$$\Omega_S(m) = \{ \pi : \pi \text{ is an irreducible generic automorphic representation of } GL_m(\mathbf{A}), \text{ unramified at all } v \notin S \}.$$

Similarly, let $\Omega_S^0(m)$ be the set of cuspidal elements of $\Omega_S(m)$.

Theorem 3. — Let $n \geqslant 3$. *Let* Π *be an irreducible admissible representation of* $GL_n(\mathbf{A})$ *whose central character* ω_Π *is invariant under* $k^\times$ *and whose* L*-function* $L(\Pi, s)$ *is absolutely convergent in some half-plane. Let* S *be a non-empty finite set of places of* k *containing all archimedean places such that the ring* $\mathfrak{o}_S$ *of* S*-integers has class number one. Suppose that for every* m *with* $1 \leqslant m \leqslant n-1$ *and every* $\tau \in \Omega_S^0(m)$ *the* L*-function* $L(\Pi \times \tau, s)$ *is nice. Then there exists an irreducible automorphic representation* Π' *of* $GL_n(\mathbf{A})$ *such that* $\Pi'_v \simeq \Pi_v$ *for all non-archimedean places* v *where* Π_v *is unramified.*

This will be proved in Section 11.

We will also give a version of this theorem where we put the extra hypothesis that Π be generic. In this case we can draw slightly stronger conclusions. These are stated as Theorem 2 and its corollaries, which can be found in Section 7.

We believe that it is not necessary to have control of so many twists to be able to draw conclusions about the automorphic nature of Π. Twists by characters of GL_1 might be enough. We state this in the following conjecture.

Conjecture. — Let $\Pi = \otimes \Pi_v$ *be an irreducible admissible representation of* $GL_n(\mathbf{A})$ *whose central character* ω_Π *is invariant under* $k^\times$ *and whose* L*-function* $L(\Pi, s)$ *is absolutely convergent in some half-plane. Assume that* $L(\Pi \otimes \omega, s)$ *is nice for all characters* ω *of* $k^\times \backslash \mathbf{A}^\times$. *Then there exists an automorphic representation* Π' *of* $GL_n(\mathbf{A})$ *which is quasi-isomorphic to* Π *and such that* $L(\Pi \otimes \omega, s) = L(\Pi' \otimes \omega, s)$ *and* $\varepsilon(\Pi \otimes \omega, s) = \varepsilon(\Pi' \otimes \omega, s)$.

The validity of this conjecture would have very fundamental applications to the problem of Langlands lifting.

This conjecture is known to be true for $n = 2$ [21] and $n = 3$ [22] and we actually have $\Pi = \Pi'$. The first example where $\Pi \neq \Pi'$ was constructed in [38] for $n = 4$ and the construction provides examples for all $n \geqslant 4$.

3. Outline of the proof of Theorem 1

Let us first outline the proof Theorem 1 under the more restrictive hypothesis that $\Pi = \otimes \Pi_v$ is generic, i.e., each Π_v is generic.

Let us begin with an arbitrary $\xi \in V_\Pi$. Our goal is to embed V_Π in

$\mathscr{A}(GL_n(k)\backslash GL_n(\mathbf{A}))$ such that the actions of $GL_n(\mathbf{A})$ are intertwined. Since V_Π is linearly spanned by decomposable vectors we may assume that ξ is decomposable, i.e., $\xi = \otimes \xi_v$ with $\xi_v \in V_{\Pi_v}$. As a first step let us associate to ξ some function on $GL_n(\mathbf{A})$. This is where the assumption that Π is generic comes into play. Each Π_v has a unique Whittaker model $\mathscr{W}(\Pi_v, \psi_v)$ and to each ξ_v is associated a function $W_{\xi_v}(g_v) \in \mathscr{W}(\Pi_v, \psi_v)$. For almost all v, Π_v will be unramified and there is a distinguished unramified vector ξ_v^0 with respect to which the restricted tensor product is taken. At these places we normalize the Whittaker model so that $W_{\xi_v^0}(I_n) = 1$. Now to $\xi \in V_\Pi$ associate the global function $W_\xi(g) = \prod_v W_{\xi_v}(g_v)$. Since for almost all v, ξ_v is the distinguished unramified vector ξ_v^0 in V_{Π_v} and $g_v \in GL_n(\mathfrak{o}_v)$, this product converges absolutely to a continuous function on $GL_n(\mathbf{A})$.

We first attempt to make an automorphic function from ξ by averaging as much as possible over $GL_n(k)$. First note that $W_\xi(g)$ is left invariant under both $N_n(k)$ and $Z_n(k)$. To get further invariance, consider the sum

$$U_\xi(g) = \sum_{\gamma \in N_n(k)\backslash P_n(k)} W_\xi(\gamma g) = \sum_{\gamma' \in N_{n-1}(k)\backslash GL_{n-1}(k)} W_\xi\left(\begin{pmatrix} \gamma' & 0 \\ 0 & 1 \end{pmatrix} g\right).$$

This sum converges absolutely and uniformly on compact subsets to a continuous function on $GL_n(\mathbf{A})$ which is cuspidal along the unipotent radical of any maximal parabolic subgroup of GL_n containing B_n. As a function on $GL_n(\mathbf{A})$, $U_\xi(g)$ is left invariant with respect to $P_n(k)$ and $Z_n(k)$ and hence with respect to the full parabolic subgroup $P'_n(k)$ associated to the partition $(n-1, 1)$ of n.

We next construct a second function $V_\xi(g)$ associated to ξ which will be related to $U_\xi(g)$ via the functional equation of the L-function. Put

$$\alpha_n = \begin{pmatrix} & 1 \\ I_{n-1} & \end{pmatrix} = w_n \begin{pmatrix} w_{n-1} & \\ & 1 \end{pmatrix},$$

where

$$w_r = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$$

is the longest Weyl element of GL_r. Then, if we consider $W_\xi(\alpha_n g)$, this is left invariant under

$$\alpha_n^{-1} N_n(k)\, \alpha_n = \left\{ \begin{pmatrix} 1 & * & * & 0 \\ 0 & \ddots & * & \vdots \\ 0 & 0 & 1 & 0 \\ * & \dots & * & 1 \end{pmatrix} \right\}$$

which we will denote by $N'_n(k)$. Note that $N'_n(k) \subset \overline{P}_n(k)$ where $\overline{P}_n$ is the mirabolic opposite to P_n.

Then let us set

$$V_\xi(g) = \sum_{\gamma \in N'_n(k)\backslash \bar{P}_n(k)} W_\xi(\alpha_n \gamma g) = \sum_{\gamma' \in N_{n-1}(k)\backslash GL_{n-1}(k)} W_\xi\left(\alpha_n \begin{pmatrix} \gamma' & 0 \\ 0 & 1 \end{pmatrix} g\right).$$

To deduce the properties of V_ξ from those above for U_ξ, let us recall that if we set

$$\widetilde{W}_\xi(g) = W_\xi(w_n\, {}^t g^{-1}),$$

then $\widetilde{W}_\xi$ is in the Whittaker model for $\widetilde{\Pi}$, the contragredient representation [24, 25, 27]. Then we have

$$V_\xi(g) = \sum_{\gamma' \in N_{n-1}(k)\backslash GL_{n-1}(k)} \widetilde{W}_{\xi'}\left(\begin{pmatrix} \gamma' & 0 \\ 0 & 1 \end{pmatrix} g'\right)$$

where

$$g' = \begin{pmatrix} w_{n-1} & \\ & 1 \end{pmatrix} . {}^t g^{-1} . \begin{pmatrix} w_{n-1} & \\ & 1 \end{pmatrix}$$

and

$$\xi' = \widetilde{\Pi}\begin{pmatrix} w_{n-1} & 0 \\ 0 & 1 \end{pmatrix} \xi.$$

We may conclude that $V_\xi(g)$ converges absolutely and uniformly on compact subsets to a continuous function an $GL_n(\mathbf{A})$. It is left invariant with respect to $\bar{P}'_n(k) = {}^t P'_n(k)^{-1}$.

To any ξ we have attached two functions on $GL_n(\mathbf{A})$, one invariant under $P'_n(k)$ and the other under $\bar{P}'_n(k)$. Note that together $P'_n(k)$ and $\bar{P}'_n(k)$ generate all of $GL_n(k)$. Our strategy will be to use the global functional equation to prove that $U_\xi(g) = V_\xi(g)$, which will show that this function is in fact invariant under $GL_n(k)$ and hence automorphic.

To relate U_ξ and V_ξ to the L-function we consider the following integrals. If we restrict $U_\xi(g)$ or $V_\xi(g)$ to $GL_{n-1}(\mathbf{A}) \subset GL_n(\mathbf{A})$ embedded in the standard way, then $U_\xi\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$ and $V_\xi\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$ are rapidly decreasing automorphic forms on $GL_{n-1}(\mathbf{A})$. Let τ be an irreducible proper automorphic representation of $GL_{n-1}(\mathbf{A})$ and let φ be an automorphic form in the space of τ. Set

$$I(\xi, \varphi; s) = \int_{GL_{n-1}(k)\backslash GL_{n-1}(\mathbf{A})} U_\xi\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \varphi(h)\, |\det(h)|^{s-(1/2)}\, dh.$$

The integral $I(\xi, \varphi; s)$ converges absolutely for $\mathrm{Re}(s) \gg 0$. If we unfold the series defining $U_\xi\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$, we find

$$I(\xi, \varphi; s) = \int_{N_{n-1}(\mathbf{A})\backslash GL_{n-1}(\mathbf{A})} W_\xi\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} W_\varphi(h)\, |\det(h)|^{s-(1/2)}\, dh = \Psi(W_\xi, W_\varphi; s)$$

where

$$W_\varphi(h) = \int_{N_{n-1}(k)\backslash N_{n-1}(\mathbf{A})} \varphi(nh)\, \psi(n)\, dn$$

i.e., $W_\varphi(h) \in \mathscr{W}(\tau, \psi^{-1})$.

Similarly, for V_ξ we may define the integral

$$\tilde{I}(\xi, \varphi; s) = \int_{GL_{n-1}(k)\backslash GL_{n-1}(\mathbf{A})} V_\xi \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \varphi(h) \mid \det(h) \mid^{s-(1/2)} dh.$$

This will converge for $\mathrm{Re}(s) \ll 0$. If we unfold it, we find

$$\tilde{I}(\xi, \varphi; s) = \int_{N_{n-1}(\mathbf{A})\backslash GL_{n-1}(\mathbf{A})} \tilde{W}_\xi \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \tilde{W}_\varphi(h) \mid \det(h) \mid^{(1-s)-(1/2)} dh$$
$$= \Psi(\tilde{W}_\xi, \tilde{W}_\varphi; 1-s),$$

where, as before, we set

$$\tilde{W}_\xi(g) = W_\xi(w_n\, {}^t g^{-1}) \qquad \tilde{W}_\varphi(h) = W_\varphi(w_{n-1}\, {}^t h^{-1}).$$

Both of these families will have an analytic continuation to entire functions of s, bounded in vertical strips. To see this we must relate these global integrals to the global L-function.

Up to this point, nothing is used other than general properties of Whittaker functions. To prove the continuation of these integrals and relate them, we must use our assumptions on the L-functions. The integrals are related to the global L-functions through their expressions as Whittaker integrals. In fact, we have

$$I(\xi, \varphi; s) = \Psi(W_\xi, W_\varphi; s) = L(\Pi \times \tau, s)\, E(s)$$
$$\tilde{I}(\xi, \varphi; s) = \Psi(\tilde{W}_\xi, \tilde{W}_\varphi; 1-s) = L(\tilde{\Pi} \times \tilde{\tau}, 1-s)\, \tilde{E}(s)$$

where $E(s)$ and $\tilde{E}(s)$ are entire functions of s. The analytic continuation of the global L-functions then implies that $I(\xi, \varphi; s)$ and $\tilde{I}(\xi, \varphi; s)$ both have continuation to entire functions of s which are bounded in vertical strips.

The global functional equation for $L(\Pi \times \tau, s)$ will allow us to relate $I(\xi, \varphi; s)$ and $\tilde{I}(\xi, \varphi; s)$ and hence U_ξ and V_ξ. From the local functional equation we have

$$\frac{\Psi(W_{\xi_v}, W_{\varphi_v}; s)}{L(\Pi_v \times \tau_v, s)}\, \varepsilon(\Pi_v \times \tau_v, s, \psi_v)\, \omega_{\tau_v}(-1)^{n-1} = \frac{\Psi(\tilde{W}_{\xi_v}, \tilde{W}_{\varphi_v}; 1-s)}{L(\tilde{\Pi}_v \times \tilde{\tau}_v, 1-s)}.$$

Using this, the global functional equation will imply that upon taking products we have

$$\Psi(W_\xi, W_\varphi; s) = \Psi(\tilde{W}_\xi, \tilde{W}_\varphi; 1-s)$$

or $$I(\xi, \varphi; s) = \tilde{I}(\xi, \varphi; s)$$

for all s.

If we set $F_\xi = U_\xi - V_\xi$ then $F_\xi\begin{pmatrix} h & 0 \\ 0 & 1\end{pmatrix}$ is rapidly decreasing on $GL_{n-1}(k)\backslash GL_{n-1}(\mathbf{A})$. If we restrict to $SL_{n-1}(\mathbf{A})$ then $F_\xi\begin{pmatrix} h & 0 \\ 0 & 1\end{pmatrix}$ will be in $L^2(SL_{n-1}(k)\backslash SL_{n-1}(\mathbf{A}))$ and if we interpret the above equality in terms of F_ξ, we see that

$$\int_{SL_{n-1}(k)\backslash SL_{n-1}(\mathbf{A})} F_\xi\begin{pmatrix} h & 0 \\ 0 & 1\end{pmatrix} \varphi(h)\, dh = 0$$

for all φ occurring in irreducible automorphic subrepresentations of $SL_{n-1}(\mathbf{A})$. If we then apply the weak form of Langlands' spectral theory we may conclude that $F_\xi\begin{pmatrix} h & 0 \\ 0 & 1\end{pmatrix} \equiv 0$.

Since $F_\xi\begin{pmatrix} h & 0 \\ 0 & 1\end{pmatrix} \equiv 0$, we have that $U_\xi\begin{pmatrix} h & 0 \\ 0 & 1\end{pmatrix} = V_\xi\begin{pmatrix} h & 0 \\ 0 & 1\end{pmatrix}$ for all $h \in SL_{n-1}(\mathbf{A})$ and in particular $U_\xi(1) = V_\xi(1)$. Since this is true for all ξ, then

$$U_\xi(g) = U_{\Pi(g)\,\xi}(1) = V_{\Pi(g)\,\xi}(1) = V_\xi(g)$$

for all $g \in GL_n(\mathbf{A})$. We now have that $U_\xi(g)$ is invariant under $P_n(k)$, $\overline{P}_n(k)$, and $Z_n(k)$. Since these generate $GL_n(k)$ we see that $U_\xi \in \mathscr{A}(GL_n(k)\backslash GL_n(\mathbf{A}))$. Thus the map $\xi \mapsto U_\xi(g)$ embeds Π into $\mathscr{A}(GL_n(k)\backslash GL_n(\mathbf{A}))$. Hence Π is an automorphic subrepresentation.

To see that Π is cuspidal, since $U_\xi(g)$ is given by the convergent "Fourier expansion"

$$U_\xi(g) = \sum_{\gamma \in N_{n-1}(k)\backslash GL_{n-1}(k)} W_\xi\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1\end{pmatrix} g\right)$$

without constant term, we observe that for any parabolic Q, the constant term of U_ξ along Q is 0. Hence $U_\xi \in \mathscr{A}^0(GL_n(k)\backslash GL_n(\mathbf{A}))$, i.e., U_ξ is cuspidal and hence Π is cuspidal. This is the conclusion of Theorem 1.

4. Preliminary considerations on Whittaker models

Before we turn to the rigorous proof of Theorem 1, we would like to gather together some known results which we will need. We begin with the following local and global estimates for Whittaker functions.

If v is a place of k then by a *gauge* on $GL_n(k_v)$ is meant a function β_v which is left invariant under $N_n(k_v)$, invariant on the right under $K_{n,v}$ and which on $A_n(k_v)$ has the form

$$\beta_v(a) = |a_1 a_2 \dots a_{n-1}|^{-t}\, \Phi(a_1, a_2, \dots, a_{n-1}),$$

where

$$a = \begin{pmatrix} a_1 a_2 \ldots a_n & & & \\ & a_2 \ldots a_n & & \\ & & \ddots & \\ & & & a_n \end{pmatrix} \in A_n(k_v),$$

t is real and non-negative, and Φ is a Schwartz-Bruhat function on k_v^{n-1} [22, 27]. A global gauge on $GL_n(\mathbf{A})$ is defined analogously [22]. Then the standard estimates on Whittaker functions are the following.

Lemma **4.1.** — *a) Let* Π_v *be a generic representation or an induced representation of Whittaker type of* $GL_n(k_v)$ *with* ω_{Π_v} *its central character. Let* $|\omega_{\Pi_v}(x)| = |x|_v^d$. *Then for any* $W_v \in \mathcal{W}(\Pi_v, \psi_v)$ *there is a gauge* β_v *such that* $|W_v(g_v)| \leqslant \beta_v(g_v) |\det(g_v)|_v^{d/n}$.

b) Let Π *be a generic representation or an induced representation of Whittaker type of* $GL_n(\mathbf{A})$ *whose central character* ω_Π *is invariant under* $k^\times$ *and whose* L*-function* $L(\Pi, s)$ *is absolutely convergent in some half-plane. Let* $|\omega_\Pi(x)| = |x|^d$. *Then for any* $\xi \in \Pi$ *there exist a global gauge* β *such that* $|W_\xi(g)| \leqslant \beta(g) |\det(g)|^{d/n}$.

Proof. — *a)* When v is a non-archimedean place of k and Π_v is generic this is just Proposition 2.3.6 of [22]. As noted in Remark 2.2.5 of [22], the proof is still valid if Π_v is induced of Whittaker type. For v an archimedean place this follows from Proposition 2.1 of [27] and the comments following it.

b) Let $\Pi' = \Pi \otimes \omega_{-d/n}$ as in the proof of Lemma 2.3, so that Π' has a unitary central character. The product $L(\Pi', s)$ also converges absolutely in some half-plane. Let T be a finite set of places of k such that Π_v, and hence Π'_v, is unramified outside T. As in the proof of Lemma 2.3, for $v \notin T$ we have that Π'_v is the unramified constituent of $\mathrm{Ind}_{B_n(k_v)}^{GL_n(k_v)}(\mu_{1,v} \otimes \ldots \otimes \mu_{n,v})$. If $L(\Pi', s)$ converges absolutely for $\mathrm{Re}(s) > c_0$ then we have seen that we have the uniform estimate $|\mu_{i,v}(\varpi_v)| \ll q_v^{c_0}$. Since Π' has a unitary central character, we also have a uniform lower estimate of $q_v^{-(n-1)c_0} \ll |\mu_{i,v}(\varpi_v)|$. Hence there exists a uniform d_0, independent of v, such that $q_v^{-d_0} < |\mu_{i,v}(\varpi_v)| < q_v^{d_0}$ for all $v \notin T$.

Now applying Proposition 2.4.1 of [22] we see that we can choose a compatible family of local gauges $\{\beta_v\}$ for the Π_v as in *a)* such that $\beta = \prod \beta_v$ is a global gauge and gives the estimate in *b)*. □

5. Preliminary considerations on Langlands' spectral theory

For the proof of Theorem 1 we will need the weak form of Langlands' spectral theory for SL_{n-1}. We recall here what we will need, specialized to SL_{n-1}. For details, see [33], [14] or [36].

The standard parabolic subgroups of SL_{n-1} are parameterized by partitions $(n_1, \ldots, n_r)$ of $n-1$. To such a partition we associate the parabolic subgroup P containing the upper triangular matrices having Levi component

$$M = \left\{ m = \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_r \end{pmatrix} \mid m_i \in GL_{n_i} \text{ and } \prod_{i=1}^{r} \det(m_i) = 1 \right\}.$$

We will denote this group by $M = S(GL_{n_1} \times \ldots \times Gl_{n_r})$. The group R_M of rational characters of M is isomorphic to $\mathbf{Z}^{r-1}$. If $a = (a_1, \ldots, a_{r-1}) \in \mathbf{Z}^{r-1}$ then we associate to it the character

$$\chi_a(m) = \prod_{i=1}^{r-1} \det(m_i)^{a_i}.$$

We will also denote this by $\chi_a(m) = m^a$.

Every rational character defines a homomorphism from $M(\mathbf{A})$ to the idele group $\mathbf{A}^\times$. Hence for each $\chi \in R_M$ we have $|\chi| : M(\mathbf{A}) \to \mathbf{R}_+^\times$. Let $M^1(\mathbf{A}) \subset M(\mathbf{A})$ be the subgroup defined by

$$M^1(\mathbf{A}) = \bigcap_{\chi \in R_M} \ker |\chi|.$$

Then if we let $R_M(\mathbf{C}) = R_M \otimes_{\mathbf{Z}} \mathbf{C} \simeq \mathbf{C}^{r-1}$ and let $s = (s_1, \ldots, s_{r-1}) \in \mathbf{C}^{r-1}$, every s defines a character of $M(\mathbf{A})$ trivial on $M^1(\mathbf{A})$ by

$$\chi_s(m) = \prod_{i=1}^{r-1} |\det(m_i)|^{s_i},$$

which we will also denote by $|m|^s$.

Consider the homomorphism $\nu : M(\mathbf{A}) \to (\mathbf{R}_+^\times)^{r-1}$ given by

$$\nu(m) = (|\det(m_1)|, \ldots, |\det(m_{r-1})|).$$

The kernel of this map is precisely $M^1(\mathbf{A})$. Its image is $(\mathbf{R}_+^\times)^{r-1}$ if k is a number field and is $(q^{\mathbf{Z}})^{r-1} \subset (\mathbf{R}_+^\times)^{r-1}$ if k is a function field and q is the order of its field of constants. Let V_M denote this image.

If we let X_M denote the group of characters of $M(\mathbf{A})$ which are trivial on $M^1(\mathbf{A})$, then X_M is the character group of V_M. Hence if k is a number field we have

$$X_M = R_M(\mathbf{C}) \simeq \mathbf{C}^{r-1}$$

whereas if k is a function field then

$$X_M = R_M(\mathbf{C}) \Big/ \frac{2\pi i}{\log q} R_M \simeq \mathbf{C}^{r-1} \Big/ \frac{2\pi i}{\log q} \mathbf{Z}^{r-1}.$$

In either case, X_M has the structure of a complex manifold.

Now let σ be an irreducible admissible cuspidal representation of $M(\mathbf{A})$. Then we may form the induced representation $\mathrm{Ind}_{P(\mathbf{A})}^{SL_{n-1}(\mathbf{A})}(\sigma)$ which we view as the space of functions $\varphi : SL_{n-1}(\mathbf{A}) \to \mathbf{C}$ such that for all $g \in SL_{n-1}(\mathbf{A})$, the function $m \mapsto \varphi(mg)$ is in $\sigma \otimes \delta_P^{1/2}$. Let $I(\sigma)$ denote the subspace of admissible vectors of this induced representation. So $\varphi \in I(\sigma)$ if it is smooth, K-finite, and satisfies the previous condition.

If $g \in SL_{n-1}(\mathbf{A})$, then g will have an Iwasawa decomposition $g = umk$ relative to P, where $u \in U(\mathbf{A})$ the unipotent radical of $P(\mathbf{A})$, $m \in M(\mathbf{A})$, and $k \in K = \prod_v K_v$. Write $m = m(g)$. This is not unique, but its image $\nu(m(g))$ in $M(\mathbf{A})/M^1(\mathbf{A})$ is uniquely defined. If $\varphi \in I(\sigma)$ then the function

$$\varphi\chi_s : g \mapsto \varphi(g) \mid m(g) \mid^s$$

is in $I(\sigma \otimes \chi_s)$.

We are now ready to define the Eisenstein series we will use. If M is a Levi subgroup of a parabolic subgroup P of SL_{n-1}, σ a unitary cuspidal representation of M, $\varphi \in I(\sigma)$, and $s \in \mathbf{C}^{r-1}$, set

$$E_\varphi(g; s) = \sum_{\gamma \in P(k)\backslash SL_{n-1}(k)} \varphi(\gamma g) \mid m(\gamma g) \mid^s$$

whenever this converges. The facts we will need about the Eisenstein series are contained in the following theorem.

Theorem S1. — The series defining $E_\varphi(g; s)$ converges absolutely and uniformly on compact subsets for all s in the positive cone

$$X_M^+ = \{ s \in X_M \mid \mathrm{Re}(s_i) - \mathrm{Re}(s_{i+1}) > 1 \}$$

(set $s_r = 0$). In this region, $E_\varphi(g; s)$ is a holomorphic function of s and is of moderate growth on $SL_{n-1}(k)\backslash SL_{n-1}(\mathbf{A})$. Moreover for $s \in X_M^+$

$$E_\varphi(g; s) \in \mathscr{A}(SL_{n-1}(k)\backslash SL_{n-1}(\mathbf{A})).$$

For generic $s \in X_M^+$, $I(\sigma \otimes \chi_s)$ is irreducible and the map $\varphi \mapsto E_\varphi(g; s)$ defines an embedding of $I(\sigma \otimes \chi_s)$ as an automorphic subrepresentation of $\mathscr{A}(SL_{n-1}(k)\backslash SL_{n-1}(\mathbf{A}))$.

Besides the Eisenstein series we need another family of functions which seem to go by many names (incomplete theta series, pseudo Eisenstein series, etc.). Let us introduce them through the Paley-Wiener functions on X_M. If k is a number field, so $X_M \simeq \mathbf{C}^{r-1}$, then $P(X_M)$, the space of Paley-Wiener functions on X_M, is the space of holomorphic functions $f : X_M \to \mathbf{C}$ which satisfy an estimate of the following type. For each $f \in P(X_M)$ there exists a real number r and for each $n \in \mathbf{N}$ there exists a constant C_n such that

$$|f(s)| \leqslant C_n e^{r\|\mathrm{Re}(s)\|}(1 + \| s \|)^{-n}.$$

If k is a function field, then $X_M \simeq \mathbf{C}^{r-1} \Big/ \frac{2\pi i}{\log q} \mathbf{Z}^{r-1}$ and $P(X_M)$ is the set of functions which are given by polynomials in $q^{s_1}, \ldots, q^{s_{r-1}}$ and their inverses.

If we define the Fourier transform on functions on X_M by

$$\hat{f}(m) = \int_{\mathrm{Re}(s) = \mathrm{Re}(s_0)} f(s) \mid m \mid^s ds,$$

then $\hat{f}(m)$ is a function on $V_M = M(\mathbf{A})/M^1(\mathbf{A})$ and the space of Paley-Wiener functions on X_M has the equivalent characterization by $f \in P(X_M)$ if and only if $\hat{f} \in C_c^\infty(V_M)$.

For $\varphi \in I(\sigma)$ and $f \in P(X_M)$ we define

$$\theta_{\varphi, f}(g) = \sum_{\gamma \in P(k) \backslash SL_{n-1}(k)} \varphi(\gamma g) \hat{f}(m(\gamma g)).$$

Then first basic result on these functions is the following.

Theorem S2. — The sum $\theta_{\varphi, f}(g)$ converges absolutely and uniformly on compact sets to a rapidly decreasing function on $SL_{n-1}(k) \backslash SL_{n-1}(\mathbf{A})$. It has an expansion in terms of Eisenstein series by

$$\theta_{\varphi, f}(g) = \int_{\mathrm{Re}(s) = \mathrm{Re}(s_0)} E_\varphi(g; s) f(s)\, ds$$

for any $s_0 \in X_M^+$.

To state what we have called the weak spectral theorem, let us recall the convention that for $M = SL_{n-1}$ itself, both the Eisenstein series $E_\varphi(g; s)$ and the series $\theta_{\varphi, f}(g)$ reduce to just the cusp forms φ in the cuspidal representation σ of $SL_{n-1}(\mathbf{A})$. Then by *weak spectral theory* we mean the following result [36, Theorem II.1.12].

Theorem S3. — The collection of all functions of the form $\theta_{\varphi, f}(g)$ obtained as M runs over all Levi subgroups of SL_{n-1}, σ all unitary cuspidal representations of $M(\mathbf{A})$, $\varphi \in I(\sigma)$, and $f \in P(X_M)$ are dense in $L^2(SL_{n-1}(k) \backslash SL_{n-1}(\mathbf{A}))$.

We will use this in the form of the following standard corollary. We repeat the proof for the convenience of the reader.

Corollary. — Let $F(g)$ be a smooth function of rapid decay on $SL_{n-1}(k) \backslash SL_{n-1}(\mathbf{A})$. Suppose that

$$\int_{SL_{n-1}(k) \backslash SL_{n-1}(\mathbf{A})} F(g)\, E_\varphi(g; s)\, dg = 0$$

for all Einstein series $E_\varphi(g; s)$ as M runs over all Levi subgroups of SL_{n-1}, σ all unitary cuspidal representations of $M(\mathbf{A})$, $\varphi \in I(\sigma)$, and all s in a Zariski open subset of X_M^+. Then $F(g) \equiv 0$.

Proof. — Since $F(g)$ is smooth and of rapid decay it lies in $L^2(SL_{n-1}(k) \backslash SL_{n-1}(\mathbf{A}))$, and hence by Theorem S3 it suffices to show that

$$I(\varphi, f) = \int_{SL_{n-1}(k) \backslash SL_{n-1}(\mathbf{A})} F(g)\, \theta_{\varphi, f}(g)\, dg = 0$$

for all $\theta_{\varphi,f}(g)$ as in the statement of that theorem. If we replace $\theta_{\varphi,f}(g)$ by its expansion in terms of Eisenstein series from Theorem S2, we have

$$I(\varphi,f) = \int_{SL_{n-1}(k)\backslash SL_{n-1}(\mathbf{A})} \int_{Re(s)=Re(s_0)} F(g)\, E_\varphi(g;s) f(s)\, ds\, dg.$$

Since $F(g)$ is of rapid decay, $|E_\varphi(g;s)|$ satisfies a moderate growth estimate depending only on $Re(s)$, and f is Paley-Wiener, we may interchange the order of integration to obtain

$$I(\varphi,f) = \int_{Re(s)=Re(s_0)} \left(\int_{SL_{n-1}(k)\backslash SL_{n-1}(\mathbf{A})} F(g)\, E_\varphi(g;s)\, dg\right) f(s)\, ds.$$

By our assumption,

$$\int_{SL_{n-1}(k)\backslash SL_{n-1}(\mathbf{A})} F(g)\, E_\varphi(g;s)\, dg = 0$$

except possibly on a set of measure zero in the set $Re(s) = Re(s_0)$. Hence

$$I(\varphi,f) = \int_{SL_{n-1}(k)\backslash SL_{n-1}(\mathbf{A})} F(g)\, \theta_{\varphi,f}(g)\, dg = 0$$

and we are done. □

6. Proof of Theorem 1

Let $\Pi = \otimes \Pi_v$ be an irreducible, admissible, not necessarily generic representation of $GL_n(\mathbf{A})$ whose central character ω_Π is invariant under $k^\times$ and whose L-function $L(\Pi, s)$ is absolutely convergent in some half-plane.

By the Langlands classification for $GL_n(k_v)$ at each place v there is an admissible induced representation Ξ_v of Langlands type such that Π_v is the unique irreducible quotient of Ξ_v [7]. The representation Ξ_v is induced of Whittaker type and $\Xi_v = \Pi_v$ only if Π_v is generic [26]. The induced representation Ξ_v has a well-defined central character ω_{Ξ_v} and this will be the central character of any constituent of Ξ_v. In particular Ξ_v and Π_v will have the same central character. The point of introducing the Ξ_v is that for non-generic representations like Π_v their local L-function is defined through the L-functions of the Ξ_v where an integral representation via Whittaker models can be used [22]. More specifically, from the definition of the local L-functions [24, 27], for every irreducible admissible representation τ_v of $GL_n(k_v)$ we have

$$L(\Pi_v \times \tau_v, s) = L(\Xi_v \times \tau_v, s)$$

with a similar equality for the ε-factors. Therefore if we consider the admissible representation $\Xi = \otimes \Xi_v$, this representation will have the same central character as Π and its L-function will be nice for all twists by every cuspidal automorphic representation τ of $GL_m(\mathbf{A})$ for all m with $1 \leqslant m \leqslant n-1$. It has the extra advantage that it is induced of Whittaker type. If Π was generic to begin with, then $\Pi = \Xi$.

For the convenience of the reader, let us recall how the local L-functions $L(\Xi_v \times \tau_v, s)$ are defined through their Whittaker models. The representations Ξ_v and τ_v are both induced representations of Whittaker type and thus have Whittaker models. For each pair of smooth functions $W_v(g) \in \mathscr{W}(\Xi_v, \psi_v)$ and $W'_v(g) \in \mathscr{W}(\tau_v, \psi_v^{-1})$ there is associated an integral

$$\Psi(W_v, W'_v; s) = \int_{N_m(k_v)\backslash GL_m(k_v)} W_v\begin{pmatrix} g & 0 \\ 0 & I_{n-m}\end{pmatrix} W'_v(g) \mid \det(g) \mid^{s-(n-m)/2} dg$$

which is absolutely convergent for $\mathrm{Re}(s) \gg 0$ by the estimates in Section 4.

If k_v is non-archimedean then by Theorem 2.7 of [24] we know the following. The integrals $\Psi(W_v, W'_v; s)$ define rational functions of q_v^{-s}. As the functions W_v and W'_v run over their respective Whittaker spaces, this family of integrals form a $\mathbf{C}[q_v^s, q_v^{-s}]$-fractional ideal in $\mathbf{C}(q_v^{-s})$. The local L-factor $L(\Xi_v \times \tau_v, s)$ is the generator of this ideal of the form $L(\Xi_v \times \tau_v, s) = P(q_v^{-s})^{-1}$ with $P(X) \in \mathbf{C}[X]$ a polynomial having $P(0) = 1$. Moreover, these integrals satisfy a local functional equation of the form

$$\frac{\Psi(W_v, W'_v; s)}{L(\Pi_v \times \tau_v, s)} \varepsilon(\Pi_v \times \tau_v, s, \psi_v)\, \omega_{\tau_v}(-1)^m = \frac{\widetilde{\Psi}(\rho(w_{n,m})\, \widetilde{W}_v, \widetilde{W}'_v; 1-s)}{L(\widetilde{\Pi}_v \times \widetilde{\tau}_v, 1-s)}.$$

In this functional equation, the function $\widetilde{\Psi}(W_v, W'_v; s)$ is defined by the integral

$$\widetilde{\Psi}(W_v, W'_v; s) = \int_{N_m(k_v)\backslash GL_m(k_v)} \int_{M_{k,m}(k_v)} W_v\begin{pmatrix} g & 0 & 0 \\ x & I_k & 0 \\ 0 & 0 & 1\end{pmatrix} W'_v(g) \mid \det(g) \mid^{s-(n-m)/2} dx\, dg,$$

where $k = n - m - 1$. (Note that if $m = n - 1$ then $\widetilde{\Psi}(W_v, W'_v; s) = \Psi(W_v, W'_v; s)$.) The Whittaker functions involved are $\widetilde{W}_v(g) = W_v(w_n\, {}^t g^{-1}) \in \mathscr{W}(\Xi_v^\iota, \psi_v^{-1})$, and $\widetilde{W}'_v(g) = W'_v(w_m\, {}^t g^{-1}) \in \mathscr{W}(\tau_v^\iota, \psi_v)$, where Ξ_v^ι is the representation of $GL_n(k_v)$ on the same space as Ξ_v but with action $\Xi_v^\iota(g) = \Xi_v({}^t g^{-1})$ and similarly for τ_v^ι. The Weyl elements involved are

$$w_r = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix},$$

the longest Weyl element of GL_r and

$$w_{n,m} = \begin{pmatrix} I_m & 0 \\ 0 & w_{n-m}\end{pmatrix}.$$

As before, ρ denotes right translation in the Whittaker model. These integrals have the same analytic properties as the $\Psi(W_v, W'_v; s)$. The ε-factor is of the form

$\varepsilon(\Xi_v \times \tau_v, s, \psi_v) = A q_v^{-Bs}$ for appropriate constants A and B. The local functional equation is also written as

$$\Psi(W_v, W'_v; s)\, \gamma(\Xi_v \times \tau_v, s, \psi_v) = \widetilde{\Psi}(\widetilde{W}_v, \widetilde{W}'_v; 1 - s)$$

where

$$\gamma(\Xi_v \times \tau_v, s, \psi_v) = \frac{\omega_{\tau_v}(-1)^m\, \varepsilon(\Xi_v \times \tau_v, s, \psi_v)\, L(\Xi_v^{\iota} \times \tau_v^{\iota}, 1 - s)}{L(\Xi_v \times \tau_v, s)}.$$

If the local field k_v is archimedean, then the integrals $\Psi(W_v, W'_v; s)$ extend to meromorphic functions of s. For the L-function $L(\Xi_v \times \tau_v, s)$ and the ε-factor $\varepsilon(\Xi_v \times \tau_v, s, \psi_v)$ we may take the L-function and ε-factor of the nm-dimensional representation of the local Weil group associated to the pair (Ξ_v, τ_v) by the archimedean local Langlands correspondence as in [5, 27, 29]. The ratio $\Psi(W_v, W'_v; s)/L(\Xi_v \times \tau_v, s)$ is again entire and satisfies the same functional equation as in the non-archimedean case. These results are all due to Jacquet and Shalika and the details can be found in [27].

To prove Theorem 1, let us begin with an arbitrary $\xi \in V_\Xi$. Since V_Ξ is linearly spanned by decomposable vectors we may assume that ξ is decomposable, i.e., $\xi = \otimes \xi_v$ with $\xi_v \in V_{\Xi_v}$. Each Ξ_v has a unique Whittaker model $\mathcal{W}(\Xi_v, \psi_v)$ and to each ξ_v is associated a function $W_{\xi_v}(g_v) \in \mathcal{W}(\Xi_v, \psi_v)$. Now to $\xi \in V_\Xi$ associate the global function $W_\xi(g) = \prod_v W_{\xi_v}(g_v)$. Since ξ_v is the distinguished unramified vector ξ_v^0 in V_{Ξ_v} for almost all v and $g_v \in GL_n(\mathfrak{o}_v)$ for almost all v, this product converges absolutely to a continuous function on $GL_n(\mathbf{A})$. The function $W_\xi(g)$ is left invariant under both $N_n(k)$ and $Z_n(k)$.

Consider the sum

$$U_\xi(g) = \sum_{\gamma \in N_n(k)\backslash P_n(k)} W_\xi(\gamma g) = \sum_{\gamma' \in N_{n-1}(k)\backslash GL_{n-1}(k)} W_\xi\left(\begin{pmatrix} \gamma' & 0 \\ 0 & 1 \end{pmatrix} g\right).$$

From the global gauge estimate of Lemma 4.1 we may estimate $U_\xi(g)$ and find the following.

Lemma **6.1.** — *The sum* $U_\xi(g)$ *converges absolutely and uniformly on compact subsets to a continuous function on* $GL_n(\mathbf{A})$. *Moreover it is cuspidal along the unipotent radical of any maximal parabolic subgroup of* GL_n *containing* B_n. *If k is a number field, Ω a compact subset of* $GL_n(\mathbf{A})$ *and $c > 0$ there exists t_0 such that if $t \geqslant t_0$ then there is a constant c' with property that*

$$| U_\xi(a\omega) | \leqslant c' \prod_{i=1}^{n-1} | a_i/a_{i+1} |^{-ti + i(n-1-i)} \, | \det(a) |^{d/n}$$

for $\omega \in \Omega$ *and*

$$a = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$$

satisfying $| a_i/a_{i+1} | \geqslant c$ *for* $1 \leqslant i \leqslant n-2$, *where d is such that* $| \omega_\Xi(a) | = | a |^d$.

Proof. — This is just Propositions 12.2 and 12.3 of [22]. □

Lemma **6.2.** — *Let $h \in \mathrm{GL}_{n-1}(\mathbf{A})$ and consider the function* $\mathrm{U}_\xi\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$. *Then* $\mathrm{U}_\xi\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$ *is a rapidly decreasing automorphic function on $\mathrm{GL}_{n-1}(\mathbf{A})$ and furthermore it satisfies the estimate*

$$\left| \mathrm{U}_\xi\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \right| \leqslant c_t \mid \det(h) \mid^{-t}$$

for sufficiently large $t > 0$.

Proof. — That $\mathrm{U}_\xi\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$ is automorphic follows from the formula

$$\mathrm{U}_\xi\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} = \sum_{\gamma' \in \mathrm{N}_{n-1}(k)\backslash \mathrm{GL}_{n-1}(k)} \mathrm{W}_\xi\begin{pmatrix} \gamma' h & 0 \\ 0 & 1 \end{pmatrix}.$$

First, assume that k is a number field. Then, by reduction theory for $\mathrm{GL}_{n-1}(\mathbf{A})$, we may write $h = \gamma a \omega$ where $\omega \in \Omega$, a compact subset of $\mathrm{GL}_{n-1}(\mathbf{A})$, $\gamma \in \mathrm{GL}_{n-1}(k)$ and $a = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_{n-1} \end{pmatrix}$ with $\mid a_i/a_{i+1} \mid > c$ for some c and $i = 1, 2, \ldots, n-2$. Then from Lemma 6.1 we have (setting $a_n = 1$) the estimate

$$\left| \mathrm{U}_\xi\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \right| \leqslant c'_t \prod_{i=1}^{n-1} \mid a_i/a_{i+1} \mid^{-ti+i(n-1-i)} \mid \det(a) \mid^{d/n}.$$

Since the ratios a_i/a_{i+1} for $1 \leqslant i \leqslant n-2$ are the simple roots of GL_{n-1}, this shows that $\mathrm{U}_\xi\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$ is rapidly decreasing on $\mathrm{GL}_{n-1}(\mathbf{A})$.

On the other hand, since $\mid \det(a) \mid = \prod_{i=1}^{n-1} \mid a_i/a_{i+1} \mid^i$ we see that

$$\prod_{i=1}^{n-1} \mid a_i/a_{i+1} \mid^{-ti} = \mid \det(a) \mid^{-t}$$

$$\prod_{i=1}^{n-1} \mid a_i/a_{i+1} \mid^{i(n-1-i)} \leqslant c \mid \det(a) \mid^{(n-1)}$$

and therefore

$$\left| \mathrm{U}_\xi\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \right| \leqslant c'_t \mid \det(a) \mid^{-t+(n-1)+(d/n)}$$

$$\leqslant c'_{t'} \mid \det(a) \mid^{-t'}$$

for $t' > 0$. Since $h = \gamma a \omega$ with $\mid \det(\gamma) \mid = 1$ and $\mid \det(\omega) \mid$ bounded for $\omega \in \Omega$, this gives the estimate when k is a number field.

Now assume that k is a function field having a finite field of q elements as its field of constants. It is easy to see from the transformation property defining the Whittaker function that there is a sequence of constants $c = \{ c_v \}$ with $c_v = 0$ for almost all v such that if $W_\xi(g) \neq 0$ with $g = nak$, where $n \in N_n(\mathbf{A})$, $k \in K = \prod GL_n(\mathfrak{o}_v)$, and

$$a = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix},$$

then $| a_i/a_{i+1} |_v \leqslant q^{c_v}$ for $1 \leqslant i \leqslant n-1$. Taking $a_n = 1$ we see that $W_\xi \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$ vanishes identically for $| \det(h) |$ sufficiently large and so the same will be true for $U_\xi \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$. This establishes the estimate for $| \det(h) |$ large.

On a set $\{ h \in GL_{n-1}(\mathbf{A}) \mid | \det(h) | = q^r \}$ of matrices with fixed determinant, the function $U_\xi \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$ is compactly supported mod $GL_{n-1}(k)$, and hence is rapidly decreasing as an automorphic form on $GL_{n-1}(\mathbf{A})$. To see this, recall that by the reduction theory for $GL_{n-1}(\mathbf{A})$ [16] there exists a set of constants $\lambda = \{ \lambda_v \}$ with $\lambda_v = 0$ for almost all v and a compact subset $\Omega \subset N_{n-1}(\mathbf{A})$ such that, if we set

$$\mathfrak{S}(\lambda, \Omega) = \{ h = nak \in GL_{n-1}(\mathbf{A}) \mid a = \operatorname{diag}(a_1, \ldots, a_{n-1}) \in A_{n-1}(\mathbf{A}), \\ n \in \Omega,\ k \in \prod GL_{n-1}(\mathfrak{o}_v) \quad \text{with } | a_i/a_{i+1} |_v \geqslant q^{\lambda_v} \quad \text{for } 1 \leqslant i \leqslant n-2 \},$$

then

$$GL_{n-1}(\mathbf{A}) = GL_{n-1}(k)\, \mathfrak{S}(\lambda, \Omega).$$

Hence it suffices to prove that $U_\xi \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$ has compact support in

$$\mathfrak{S}_r(\lambda, \Omega) = \{ h \in \mathfrak{S}(\lambda, \Omega) \mid | \det(h) | = q^r \}.$$

On such a set if suffices to prove that $| a_1 |$ is bounded if $U_\xi \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \neq 0$. If $U_\xi \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \neq 0$ then there must be a $\gamma \in GL_{n-1}(k)$ such that $W_\xi \begin{pmatrix} \gamma h & 0 \\ 0 & 1 \end{pmatrix} \neq 0$. Write $\gamma = (\gamma_{i,j})$ as a matrix. First we assume that $\gamma_{n-1,1} \neq 0$. Write $\gamma h = \gamma nak$. It is easy to see that the $(n-1, 1)$ entry of γna is $\gamma_{n-1,1}\, a_1$. Write $\gamma na = bk'$ with $k' \in K$ and $b \in B_{n-1}(\mathbf{A})$. Then we have $| \gamma_{n-1,1}\, a_1 |_v \leqslant | b_{n-1,n-1} |_v \leqslant q^{c_v}$ for all places v. Hence

$$| a_1 | = | \gamma_{n-1,1}\, a_1 | = \prod_v | \gamma_{n-1,1}\, a_1 |_v \leqslant \prod_v q^{c_v} = q^{|c|}$$

where $| c | = \Sigma c_v$. The general case proceeds in the same way using the first $\gamma_{\nu,1}$ such that $\gamma_{\nu,1} \neq 0$ and $\gamma_{k,1} = 0$ for all $k > \nu$, giving $| a_1 | \leqslant q^{|c'|}$ where $c'_v = c_v^{n-\nu}$.

Finally, the polynomial estimate for small determinant is now a consequence of the reduction theory and the gauge estimate of Lemma 4.1 as in the number field case. □

Before we proceed, let us note the following.

Lemma **6.3.** — *The function* $U_\xi(g)$ *is not identically* 0.

Proof. — If we compute the ψ-Fourier coefficient of $U_\xi(g)$ we find

$$\int_{N_n(k)\backslash N_n(\mathbf{A})} U_\xi(ng)\,\psi^{-1}(n)\,dn = \int_{N_n(k)\backslash N_n(\mathbf{A})} \sum_{N_n(k)\backslash P_n(k)} W_\xi(\gamma ng)\,\psi^{-1}(n)\,dn$$

$$= \int_{N_n(k)\backslash N_n(\mathbf{A})} \sum_{N_{n-1}(k)\backslash GL_{n-1}(k)} W_\xi\left(\begin{pmatrix}\gamma' & 0\\ 0 & 1\end{pmatrix} ng\right)\psi^{-1}(n)\,dn.$$

We now proceed by induction. Let $N'' \subset N_n$ be the unipotent radical of P_n, i.e.,

$$N'' = \left\{\begin{pmatrix} & & & * \\ & I_{n-1} & & \vdots \\ & & & * \\ 0 & \dots & 0 & 1\end{pmatrix}\right\}.$$

Then since N'' is normal in N_n we may integrate over it first, and the formula for an individual term in the ψ-Fourier coefficient becomes

$$\int_{N_{n-1}(k)\backslash N_{n-1}(\mathbf{A})} \int_{N''(k)\backslash N''(\mathbf{A})} W_\xi\left(\begin{pmatrix}\gamma' & 0\\ 0 & 1\end{pmatrix} n''\, ng\right)\psi^{-1}(n'')\,dn''\,\psi^{-1}(n)\,dn.$$

Now, GL_{n-1} normalizes N'' so

$$W_\xi\left(\begin{pmatrix}\gamma' & 0\\ 0 & 1\end{pmatrix} n''\, ng\right) = \psi\left(\begin{pmatrix}\gamma' & 0\\ 0 & 1\end{pmatrix} n'' \begin{pmatrix}\gamma' & 0\\ 0 & 1\end{pmatrix}^{-1}\right) W_\xi\left(\begin{pmatrix}\gamma' & 0\\ 0 & 1\end{pmatrix} ng\right).$$

But

$$\int_{N''(k)\backslash N''(\mathbf{A})} \psi\left(\begin{pmatrix}\gamma' & 0\\ 0 & 1\end{pmatrix} n'' \begin{pmatrix}\gamma' & 0\\ 0 & 1\end{pmatrix}^{-1}\right)\psi^{-1}(n'')\,dn'' = \begin{cases} 0 & \gamma' \notin P_{n-1}(k) \\ 1 & \gamma' \in P_{n-1}(k)\end{cases}.$$

Hence this term vanishes unless $\gamma' \in P_{n-1}(k)$. We now proceed by induction in this way and finally conclude

$$\int U_\xi(ng)\,\psi^{-1}(n)\,dn = W_\xi(g).$$

Since $W_\xi(g)$ is not identically 0, because $\xi \mapsto W_\xi(g)$ is injective, this shows that $U_\xi(g)$ cannot be identically 0. □

As a function on $GL_n(\mathbf{A})$, $U_\xi(g)$ is left invariant with respect to $P_n(k)$ and $Z_n(k)$ and hence with respect to the full parabolic subgroup $P'_n(k)$ associated to the partition $(n-1, 1)$ of n.

Consider now a second function $V_\xi(g)$ associated to ξ. Let us set

$$V_\xi(g) = \sum_{\gamma \in N'_n(k)\backslash \overline{P}_n(k)} W_\xi(\alpha_n \gamma g) = \sum_{\gamma' \in N_{n-1}(k)\backslash GL_{n-1}(k)} W_\xi\left(\alpha_n \begin{pmatrix}\gamma' & 0\\ 0 & 1\end{pmatrix} g\right)$$

where the notation is as in Section 3. To deduce the analytic properties of V_ξ from those above for U_ξ, let us recall that if we set

$$\widetilde{W}_\xi(g) = W_\xi(w_n\, {}^t g^{-1})$$

then $\widetilde{W}_\xi$ is in the Whittaker model for Ξ^ι. Then we have

$$V_\xi(g) = \sum_{\gamma' \in N_{n-1}(k)\backslash GL_{n-1}(k)} \widetilde{W}_{\xi'}\left(\begin{pmatrix}\gamma' & 0\\ 0 & 1\end{pmatrix} g'\right),$$

where

$$g' = \begin{pmatrix}w_{n-1} & \\ & 1\end{pmatrix} . {}^t g^{-1} . \begin{pmatrix}w_{n-1} & \\ & 1\end{pmatrix}$$

and

$$\xi' = \Xi^\iota \begin{pmatrix}w_{n-1} & 0\\ 0 & 1\end{pmatrix} \xi.$$

We may conclude that $V_\xi(g)$ converges absolutely and uniformly on compact subsets to a continuous function an $GL_n(\mathbf{A})$. It is left invariant with respect to $\overline{P}'_n(k) = {}^t P'_n(k)^{-1}$. The function $V_\xi(g)$ does not vanish identically. Furthermore, if we consider the function $V_\xi\begin{pmatrix}h & 0\\ 0 & 1\end{pmatrix}$ on $GL_{n-1}(\mathbf{A})$ it is a rapidly decreasing automorphic function on $GL_{n-1}(\mathbf{A})$. The only difference is that our determinant estimate becomes

$$\left| V_\xi \begin{pmatrix}h & 0\\ 0 & 1\end{pmatrix}\right| \leqslant c_t \, |\det(h)|^t \quad \text{for } t > 0.$$

These facts follow from Lemma 6.2.

From ξ we have produced two functions on $GL_n(\mathbf{A})$, one invariant under $P'_n(k)$ and the other under $\overline{P}'_n(k)$. Note that together $P'_n(k)$ and $\overline{P}'_n(k)$ generate all of $GL_n(k)$.

To relate U_ξ and V_ξ to the L-function consider the following integrals. Let τ be an irreducible proper automorphic subrepresentation of $GL_{n-1}(\mathbf{A})$ and let $\varphi \in V_\tau$. Set

$$I(\xi, \varphi; s) = \int_{GL_{n-1}(k)\backslash GL_{n-1}(\mathbf{A})} U_\xi \begin{pmatrix}h & 0\\ 0 & 1\end{pmatrix} \varphi(h)\, |\det(h)|^{s-(1/2)}\, dh.$$

As a function on $\mathrm{GL}_{n-1}(\mathbf{A})$, $\varphi(h)$ is of moderate growth and transforms via a central character ω_τ. On the other hand, $U_\xi\begin{pmatrix} h & 0\\ 0 & 1\end{pmatrix}$ is rapidly decreasing on $\mathrm{GL}_{n-1}(k)\backslash\mathrm{GL}_{n-1}(\mathbf{A})$ and, in terms of the determinant, satisfies $\left| U_\xi\begin{pmatrix} h & 0\\ 0 & 1\end{pmatrix}\right| \leqslant c_t \mid \det(h)\mid^{-t}$ for every $t > 0$. Hence $\mathrm{I}(\xi, \varphi; s)$ converges absolutely for $\mathrm{Re}(s) \gg 0$.

On the other hand, if we unfold the series defining $U_\xi\begin{pmatrix} h & 0\\ 0 & 1\end{pmatrix}$ we find

$$\begin{aligned}
\mathrm{I}(\xi, \varphi; s) &= \int_{\mathrm{GL}_{n-1}(k)\backslash\mathrm{GL}_{n-1}(\mathbf{A})} U_\xi\begin{pmatrix} h & 0\\ 0 & 1\end{pmatrix} \varphi(h) \mid \det(h)\mid^{s-(1/2)}\, dh\\
&= \int_{\mathrm{GL}_{n-1}(k)\backslash\mathrm{GL}_{n-1}(\mathbf{A})} \sum_{\gamma' \in \mathrm{N}_{n-1}(k)\backslash\mathrm{GL}_{n-1}(k)} W_\xi\begin{pmatrix} \gamma' h & 0\\ 0 & 1\end{pmatrix} \varphi(h) \mid \det(h)\mid^{s-(1/2)}\, dh\\
&= \int_{\mathrm{N}_{n-1}(k)\backslash\mathrm{GL}_{n-1}(\mathbf{A})} W_\xi\begin{pmatrix} h & 0\\ 0 & 1\end{pmatrix} \varphi(h) \mid \det(h)\mid^{s-(1/2)}\, dh\\
&= \int_{\mathrm{N}_{n-1}(\mathbf{A})\backslash\mathrm{GL}_{n-1}(\mathbf{A})} W_\xi\begin{pmatrix} h & 0\\ 0 & 1\end{pmatrix} W_\varphi(h) \mid \det(h)\mid^{s-(1/2)}\, dh\\
&= \Psi(W_\xi, W_\varphi; s),
\end{aligned}$$

where

$$W_\varphi(h) = \int_{\mathrm{N}_{n-1}(k)\backslash\mathrm{N}_{n-1}(\mathbf{A})} \varphi(nh)\, \psi(n)\, dn$$

i.e., $W_\varphi(h) \in \mathscr{W}(\tau, \psi^{-1})$. Hence we have:

Lemma **6.4.** — *For any* τ, *irreducible automorphic subrepresentation of* $\mathrm{GL}_{n-1}(\mathbf{A})$, *and* $\varphi \in V_\tau$ *the integral* $\mathrm{I}(\xi, \varphi; s)$ *converges for* $\mathrm{Re}(s) \gg 0$. *Moreover, in this range,* $\mathrm{I}(\xi, \varphi; s) = \Psi(W_\xi, W_\varphi; s)$. *Hence* $\mathrm{I}(\xi, \varphi; s) \equiv 0$ *if* τ *is not generic.*

Similarly, for V_ξ we may define the integral

$$\widetilde{\mathrm{I}}(\xi, \varphi; s) = \int_{\mathrm{GL}_{n-1}(k)\backslash\mathrm{GL}_{n-1}(\mathbf{A})} V_\xi\begin{pmatrix} h & 0\\ 0 & 1\end{pmatrix} \varphi(h) \mid \det(h)\mid^{s-(1/2)}\, dh$$

which will converge for $\mathrm{Re}(s) \ll 0$. If we unfold this, then we find

$$\begin{aligned}
\widetilde{\mathrm{I}}(\xi, \varphi; s) = \int_{\mathrm{N}_{n-1}(\mathbf{A})\backslash\mathrm{GL}_{n-1}(\mathbf{A})} \widetilde{W}_\xi\begin{pmatrix} h & 0\\ 0 & 1\end{pmatrix} \widetilde{W}_\varphi(h) \mid \det(h)\mid^{(1-s)-(1/2)}\, dh&\\
= \Psi(\widetilde{W}_\xi, \widetilde{W}_\varphi; 1-s)&
\end{aligned}$$

where, as before, we set

$$\widetilde{W}_\xi(g) = W_\xi(w_n\, {}^t g^{-1}), \qquad \widetilde{W}_\varphi(h) = W_\varphi(w_{n-1}\, {}^t h^{-1}).$$

Hence we have proven:

Lemma **6.5.** — *For* τ *any irreducible automorphic representation of* $GL_{n-1}(\mathbf{A})$ *the integral* $\widetilde{I}(\xi, \varphi; s)$ *converges for* $\mathrm{Re}(s) \ll 0$. *Moreover, in this range,* $\widetilde{I}(\xi, \varphi; s) = \Psi(\widetilde{W}_\xi, \widetilde{W}_\varphi; 1 - s)$. *Hence* $\widetilde{I}(\xi, \varphi; s) \equiv 0$ *if* τ *is not generic.*

Both of these families will have an analytic continuation to entire functions of s, bounded in vertical strips. To see this we must relate these global integrals to the global L-function. We will work with $I(\xi, \varphi; s)$ in detail, then $\widetilde{I}(\xi, \varphi; s)$ proceeds in the same way.

Proposition **6.1.** — *The integral* $I(\xi, \varphi; s)$ *has an analytic continuation to an entire function of* s.

Proof. — We will consider two cases, although this is not really necessary.

First, assume that τ is cuspidal. We take $I(\xi, \varphi; s) = \Psi(W_\xi, W_\varphi; s)$. Assume that ξ and φ are decomposable. (This is possible since the decomposable vectors span V_Ξ and V_τ.) Then we have

$$I(\xi, \varphi; s) = \prod_v \Psi(W_{\xi_v}, W_{\varphi_v}; s).$$

Now, from the local theory of L-functions

$$\frac{\Psi(W_{\xi_v}, W_{\varphi_v}; s)}{L(\Xi_v \times \tau_v, s)} = E_v(s)$$

is an entire function of s. If v is non-archimedean, $E_v(s) \in \mathbf{C}[q_v^s, q_v^{-s}]$ and if both ξ_v and φ_v are the distinguished unramified vectors, which is true for almost all v, $E_v(s) \equiv 1$ [24]. If v is archimedean, then $E_v(s)$ is an entire function of s [27]. Hence, setting $E(s) = \prod_v E_v(s)$, we find

$$I(\xi, \varphi; s) = \prod_v L(\Xi_v \times \tau_v, s)\, E_v(s) = L(\Xi \times \tau, s)\, E(s).$$

So if τ is cuspidal, then by our hypothesis on $L(\Xi \times \tau, s)$ we have that $L(\Xi \times \tau, s)$ is an entire function. The same holds true of $E(s)$ and hence for $I(\xi, \varphi; s)$.

Now suppose τ is not necessarily cuspidal. Since $I(\xi, \varphi; s) \equiv 0$ unless τ is generic, we may assume that τ is an irreducible generic automorphic subrepresentation of $GL_{n-1}(\mathbf{A})$. Then by the work of Langlands [34] there exists a partition $(r_1, \dots, r_m)$ of $n - 1$ and irreducible cuspidal representations σ_i of $GL_{r_i}(\mathbf{A})$ such that τ is a subrepresentation of $\Upsilon = \mathrm{Ind}_{Q(\mathbf{A})}^{GL_{n-1}(\mathbf{A})}(\sigma_1 \otimes \dots \otimes \sigma_m)$, where Q is the standard parabolic associated to this partition. The theorem as stated in [34] only gives τ as a subquotient of Υ. But if one begins with τ an automorphic subrepresentation, the proof presents τ as a subrepresentation of the induced Υ. Let us sketch Langlands' proof, referring to [34] for more details. Langlands begins with the realization of the automorphic representation τ on V/U where V is a space of automorphic forms generated by a single form φ and U is a subspace of V. Since we are assuming that τ is an irreducible subrepresentation of the space of automorphic forms, we may take V irreducible and $U = \{0\}$. Langlands then realizes τ in the space of constant terms $\varphi_P \in V_P$ of the forms $\varphi \in V$ along a suitably chosen parabolic P with Levi M. Since our V is irreducible, we get a realization of τ on a subspace V_P of these constant terms and $U_P = \{0\}$. Again, using the fact that $U_P = \{0\}$ for this realization of τ, the argument of Lemma 6 in [34] produces a generator φ_P of V_P such that $\varphi_P(ag) = \chi(a)\,\varphi_P(g)$ with χ a character of the center of $M(\mathbf{A})$ for all $g \in G(\mathbf{A})$ and a in the center of $M(\mathbf{A})$. Langlands then projects each $\varphi_P \in V_P$ to a function φ'_P in the space of constant terms transforming by an irreducible cuspidal representation σ of $M(\mathbf{A})$ having the central character χ. Since V_P is still irreducible as a $G(\mathbf{A})$ representation, this mapping is an injection and realizes V_P as a subspace V'_P of these functions, i.e., a subspace of the induced representation from this cuspidal representation σ of $M(\mathbf{A})$ to $G(\mathbf{A})$. Taking $G = GL_{n-1}$ and $P = Q$ we obtain the conclusion stated above.

Now, locally for each place v of k, τ_v will be a generic irreducible subrepresentation of $\Upsilon_v = \mathrm{Ind}_{Q(k_v)}^{GL_{n-1}(k_v)}(\sigma_{1,v} \otimes \ldots \otimes \sigma_{m,v})$. Since each σ_i is cuspidal and hence generic, the local components $\sigma_{i,v}$ must also be generic. Then the results of Rodier [40] and Jacquet [19] imply that each Υ_v is of Whittaker type, that is, has a one-dimensional space of Whittaker functionals. Hence it has at least one generic constituent. If k_v is non-archimedean, the results of Bernstein and Zelevinsky [4] imply that there is a unique generic constituent and so τ_v must be it. If k_v is archimedean, then using the Casselman subrepresentation theorem for each $\sigma_{i,v}$ and the transitivity of induction we can embed Υ_v into a representation Υ'_v which is induced off the Borel subgroup. Now the results of Kostant [31] imply that Υ'_v has a unique generic constituent. Since Υ_v is a subrepresentation of Υ'_v we have that Υ_v can have at most one generic constituent. Since we already have seen that it has one generic constituent, namely τ_v, we have that τ_v must be the unique generic constituent of Υ_v in the archimedean case as well. τ_v cannot lie in the kernel of the map from Υ_v to its Whittaker model, since if it did this would imply that Υ_v would have at least two generic constituents. Hence, the Whittaker model of τ_v will be a subspace of the Whittaker model of Υ_v. In particular, the family of integrals defining $L(\Xi_v \times \tau_v, s)$ will be a subspace of those defining $L(\Xi_v \times \Upsilon_v, s)$. At those places where τ_v is unramified, these families agree. Hence from the computation of the local L-functions in [24] we see that at all non-archimedean places we have

$$L(\Xi_v \times \tau_v, s) \in \prod_{i=1}^{m} L(\Xi_v \times \sigma_{i,v}, s)\, \mathbf{C}[q_v^s, q_v^{-s}]$$

and if Ξ_v and τ_v are unramified then

$$L(\Xi_v \times \tau_v, s) = \prod_{i=1}^{m} L(\Xi_v \times \sigma_{i,v}, s).$$

Hence for v non-archimedean we have

$$L(\Xi_v \times \tau_v, s) = \prod_{i=1}^{m} L(\Xi_v \times \sigma_{i,v}, s)\, E'_v(s),$$

where $E'_v(s)$ is entire, bounded in strips, and identically one for almost all v. If v is archimedean then from [27] we have

$$L(\Xi_v \times \tau_v, s) = \prod_{i=1}^{m} L(\Xi_v \times \sigma_{i,v}, s).$$

Hence, globally we have

$$L(\Xi \times \tau, s) = E'(s) \prod_{i=1}^{m} L(\Xi \times \sigma_i, s).$$

By our hypothesis, each $L(\Xi \times \sigma_i, s)$ is entire and bounded in strips since each σ_i is cuspidal. The same is true of $E'(s) = \prod_v E'_v(s)$. Hence it is true of $L(\Xi \times \tau, s)$. (We will use the boundedness in strips in the proof of Proposition 6.3.)

If we now write

$$I(\xi, \varphi; s) = L(\Xi \times \tau, s)\, E(s)$$

as above we see that $I(\xi, \varphi; s)$ is entire as desired. □

Proposition **6.2.** — *The integral* $\widetilde{I}(\xi, \varphi; s)$ *has an analytic continuation to an entire function of* s.

Proof. — In this case we write

$$\widetilde{I}(\xi, \varphi; s) = \Psi(\widetilde{W}_\xi, \widetilde{W}_\varphi; 1 - s).$$

By the local theory of L-functions, we may relate this integral to the global L-function $L(\Xi^\iota \times \tau^\iota; 1 - s)$ and proceed as before. □

We next relate these integrals, again using the properties of the global L-function —this time the functional equation.

Proposition **6.3.** — *As entire functions of* s, $I(\xi, \varphi; s) = \widetilde{I}(\xi, \varphi; s)$. *Moreover, this function is bounded in vertical strips.*

Proof. — If we write these integrals as Euler products

$$I(\xi, \varphi; s) = \prod_v \Psi(W_{\xi_v}, W_{\varphi_v}; s)$$

$$\widetilde{I}(\xi, \varphi; s) = \prod_v \Psi(\widetilde{W}_{\xi_v}, \widetilde{W}_{\varphi_v}; 1 - s)$$

then the local factors are related by the local functional equation [24, 27]

$$\frac{\Psi(W_{\xi_v}, W_{\varphi_v}; s)}{L(\Xi_v \times \tau_v, s)} \varepsilon(\Xi_v \times \tau_v, s, \psi_v)\, \omega_{\tau_v}(-1)^{n-1} = \frac{\Psi(\widetilde{W}_{\xi_v}, \widetilde{W}_{\varphi_v}; 1-s)}{L(\Xi_v^{\iota} \times \tau_v^{\iota}, 1-s)}$$

or

$$\Psi(W_{\xi_v}, W_{\varphi_v}; s)\, \gamma(\Xi_v \times \tau_v, s, \psi_v) = \Psi(\widetilde{W}_{\xi_v}, \widetilde{W}_{\varphi_v}; 1-s).$$

Therefore, taking the product over v, we have

$$\Psi(W_{\xi}, W_{\varphi}; s)\, \gamma(\Xi \times \tau, s) = \Psi(\widetilde{W}_{\xi}, \widetilde{W}_{\varphi}; 1-s).$$

Since $L(\Xi \times \tau, s)$ is assumed to satisfy the global functional equation, $\gamma(\Xi \times \tau, s) \equiv 1$ if τ is cuspidal and hence $I(\xi, \varphi; s) = \widetilde{I}(\xi, \varphi; s)$ in the cuspidal case. If τ is not cuspidal, then we may assume it is generic (since otherwise both sides are identically zero) and hence, as in the proof of Proposition 6.1, is a subrepresentation of an induced representation

$$\tau \subset \mathrm{Ind}_{Q(\mathbf{A})}^{GL_{n-1}(\mathbf{A})}(\sigma_1 \otimes \ldots \otimes \sigma_m)$$

where each σ_i is cuspidal and generic on $GL_{r_i}(\mathbf{A})$. Now, from the local theory,

$$\gamma(\Xi_v \times \tau_v, s, \psi_v) = \prod_{i=1}^{m} \gamma(\Xi_v \times \sigma_{i,v}, s, \psi_v)$$

and hence globally

$$\gamma(\Xi \times \tau, s) = \prod_{i=1}^{m} \gamma(\Xi \times \sigma_i, s).$$

Now, by assumption, since each σ_i is cuspidal the global $\gamma(\Xi \times \sigma_i, s) \equiv 1$. Hence in the case of non-cuspidal τ we still have $I(\xi, \varphi; s) = \widetilde{I}(\xi, \varphi; s)$.

We need to show that this function is bounded in vertical strips. Note that from the integral representations, $I(\xi, \varphi; s)$ is bounded in vertical strips in its half-plane of absolute convergence $\mathrm{Re}(s) \gg 0$ and $\widetilde{I}(\xi, \varphi; s)$ is bounded in strips in its half-plane of absolute convergence $\mathrm{Re}(s) \ll 0$. To verify that it is bounded in any vertical strip we just need to see that it grows sufficiently slowly that the Phrágmen-Lindelöff principle applies. From the proof of Proposition 6.1, we have

$$I(\xi, \varphi; s) = L(\Xi \times \tau, s) \prod_v E_v(s).$$

The factor $L(\Xi \times \tau, s)$ is bounded in any vertical strip as in the proof of Proposition 6.1. The factor $E_v(s)$ is identically 1 for almost all places. At the remaining non-archimedean places $E_v(s)$ belongs to $\mathbf{C}[q^s, q^{-s}]$ and is thus bounded in any vertical strip. If v is archimedean, then

$$E_v(s) = \frac{\Psi(W_{\xi_v}, W_{\varphi_v}; s)}{L(\Xi_v \times \tau_v, s)}.$$

From the local archimedean theory [27] the numerator decreases like 1 over a polynomial in s at infinity in vertical strips while the denominator is a linear exponential factor times a product of Γ-functions. Then Stirling's formula applied to this product of Γ-functions gives a bound on $|L(\Xi_v \times \tau_v, s)|^{-1}$ of the form Ce^{At} at infinity in any vertical strip, where we have written $s = \sigma + it$ as usual. Hence Phrágmen-Lindelöff applies to $I(\xi, \varphi; s)$ and we may conclude that it is indeed bounded in any vertical strip. $\square$

This concludes our use of the L-function. We now maneuver ourselves into a position where we can apply the weak form of Langlands' spectral theory for automorphic representations.

For each idele a let us set

$$I_1(\xi, \varphi; a) = \int_{SL_{n-1}(k)\backslash SL_{n-1}(\mathbf{A})} U_\xi\left(\begin{pmatrix} h & \\ & 1 \end{pmatrix}\begin{pmatrix} a & \\ & I_{n-1} \end{pmatrix}\right) \varphi\left(h \begin{pmatrix} a & \\ & I_{n-2} \end{pmatrix}\right) dh$$

and similarly for $\widetilde{I}_1(\xi, \varphi; a)$. The integrals $I_1(\xi, \varphi; a)$ and $\widetilde{I}_1(\xi, \varphi; a)$ are continuous functions on $k^\times\backslash\mathbf{A}^\times$. Note that if we replace τ by $\tau \otimes \omega$ for a (unitary) character ω then

$$I_1(\xi, \varphi . \omega; a) = \omega(a)\, I_1(\xi, \varphi; a)$$

and similarly for $\widetilde{I}_1(\xi, \varphi; a)$. Hence we may write

$$I(\xi, \varphi . \omega; s) = \int_{k^\times\backslash\mathbf{A}^\times} I_1(\xi, \varphi; a)\, \omega(a)\, |a|^{s-(1/2)}\, d^\times a \quad \text{for } \mathrm{Re}(s) \gg 0$$

$$\widetilde{I}(\xi, \varphi . \omega; s) = \int_{k^\times\backslash\mathbf{A}^\times} I_1(\xi, \varphi; a)\, \omega(a)\, |a|^{s-(1/2)}\, d^\times a \quad \text{for } \mathrm{Re}(s) \ll 0.$$

We may now apply the following elementary lemma of Jacquet-Langlands [21, Lemma 11.3.1].

Lemma. — Let f_1 and f_2 be two continuous functions on $k^\times\backslash\mathbf{A}^\times$. Assume there is a constant c so that for all (unitary) characters ω of $k^\times\backslash\mathbf{A}^\times$ the integral

$$\int_{k^\times\backslash\mathbf{A}^\times} f_1(a)\, \omega(a)\, |a|^s\, d^\times a$$

is absolutely convergent for $\mathrm{Re}(s) > c$ *and the integral*

$$\int_{k^\times\backslash\mathbf{A}^\times} f_2(a)\, \omega(a)\, |a|^s\, d^\times a$$

is absolutely convergent for $\mathrm{Re}(s) < -c$. *Assume that the functions represented by these integrals can be analytically continued to the same entire function and that this entire function is bounded in strips. Then f_1 and f_2 are equal.*

Therefore we may conclude that $I_1(\xi, \varphi; a) = \tilde{I}_1(\xi, \varphi; a)$ for all $a \in \mathbf{A}^\times$ and all φ. In particular, for $a = 1$, we have

$$\int_{SL_{n-1}(k)\backslash SL_{n-1}(\mathbf{A})} U_\xi\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \varphi(h)\, dh = \int_{SL_{n-1}(k)\backslash SL_{n-1}(\mathbf{A})} V_\xi\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \varphi(h)\, dh.$$

Note that since U_ξ and V_ξ are rapidly decreasing on $GL_{n-1}(k)\backslash GL_{n-1}(\mathbf{A})$ they are also on $SL_{n-1}(k)\backslash SL_{n-1}(\mathbf{A})$ and hence $U_\xi, V_\xi \in L^2(SL_{n-1}(k)\backslash SL_{n-1}(\mathbf{A}))$. Let $F_\xi(g) = U_\xi(g) - V_\xi(g)$. Then $F_\xi\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$ is rapidly decreasing and

$$\int_{SL_{n-1}(k)\backslash SL_{n-1}(\mathbf{A})} F_\xi\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \varphi(h)\, dh = 0$$

for all φ occurring in irreducible automorphic subrepresentations of $GL_{n-1}(\mathbf{A})$.

Proposition **6.4.** — *We have* $F_\xi\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \equiv 0.$

Proof. — We wish to apply the weak form of Langlands spectral theory for SL_{n-1} as formulated in Section 5. Before doing so we must relate automorphic representations of $SL_{n-1}(\mathbf{A})$ to the restrictions of automorphic representations of $GL_{n-1}(\mathbf{A})$. By a result of Labesse-Schwermer [32] (see also Lemme 5.6 of Clozel [11]) given any irreducible cuspidal representation τ_1 of $SL_{n-1}(\mathbf{A})$ there exists an irreducible unitary cuspidal representation τ of $GL_{n-1}(\mathbf{A})$ whose restriction to $SL_{n-1}(\mathbf{A})$ contains τ_1. The argument in [32] extends to yield that any irreducible cuspidal representation τ_1 of $M_1 = S(GL_{n_1}(\mathbf{A}) \times \ldots \times GL_{n_r}(\mathbf{A}))$ extends in this way to an irreducible unitary cuspidal representation τ of $M = GL_{n_1}(\mathbf{A}) \times \ldots \times GL_{n_r}(\mathbf{A})$.

If we apply this fact in the construction of Eisenstein series, we find that for any partition $(n_1, \ldots, n_r)$ of $n - 1$ and any irreducible cuspidal representation σ_1 of the Levi subgroup $M_1 = S(GL_{n_1}(\mathbf{A}) \times \ldots \times GL_{n_r}(\mathbf{A}))$ the space of Eisenstein series $E_\varphi(h; s)$ for $\varphi \in I(\sigma_1)$ and $s \in X^+_{M_1}$ is obtained by the restriction of Eisenstein series on $GL_{n-1}(\mathbf{A})$ formed with the extention σ of σ_1 to $M = GL_{n_1}(\mathbf{A}) \times \ldots \times GL_{n_r}(\mathbf{A})$. In the realm of absolute convergence for these $GL_{n-1}(\mathbf{A})$ Eisenstein series, the induced representations are irreducible for all parameters in a Zariski open subset, and hence for these values of the parameter the Eisenstein series generate irreducible automorphic subrepresentations. Hence for all s in a Zariski dense subset of $X^+_{M_1}$ the Eisenstein series $E_\varphi(h; s)$ are obtained from restriction of irreducible automorphic subrepresentations of $GL_{n-1}(\mathbf{A})$.

As a consequence, we see that

$$\int_{SL_{n-1}(k)\backslash SL_{n-1}(\mathbf{A})} F_\xi\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} E_\varphi(h; s)\, dh = 0$$

for all Eisenstein series $E_\varphi(h; s)$ as M_1 runs over all Levi subgroups of SL_{n-1}, σ_1 all unitary cuspidal representations of $M_1(\mathbf{A})$, $\varphi \in I(\sigma_1)$, and all s in a Zariski open subset of $X^+_{M_1}$. Hence by the Corollary to Theorem S3, $F_\xi\begin{pmatrix} h & 0 \\ 0 & 1\end{pmatrix} \equiv 0$. □

Since $F_\xi\begin{pmatrix} h & 0 \\ 0 & 1\end{pmatrix} \equiv 0$, we have that $U_\xi\begin{pmatrix} h & 0 \\ 0 & 1\end{pmatrix} = V_\xi\begin{pmatrix} h & 0 \\ 0 & 1\end{pmatrix}$ for all $h \in SL_{n-1}(\mathbf{A})$ and in particular $U_\xi(1) = V_\xi(1)$. Since this is true for all ξ,

$$U_\xi(g) = U_{\Pi(g)\xi}(1) = V_{\Pi(g)\xi}(1) = V_\xi(g)$$

for all $g \in GL_n(\mathbf{A})$. We now have that $U_\xi(g)$ is invariant under $P_n(k)$, $\overline{P}_n(k)$, and $Z_n(k)$. Since these generate $GL_n(k)$ we see that $U_\xi \in \mathscr{A}(GL_n(k)\backslash GL_n(\mathbf{A}))$. Thus the map $\xi \mapsto U_\xi(g)$ embeds Ξ into $\mathscr{A}(GL_n(k)\backslash GL_n(\mathbf{A}))$. Hence Ξ is an automorphic subrepresentation.

In fact the map $\xi \mapsto U_\xi(g)$ embeds Ξ in the space of cusp forms. To see this, we must show that for any parabolic Q, with unipotent radical N_Q, the constant term of U_ξ along Q is 0, i.e.,

$$\int_{N_Q(k)\backslash N_Q(\mathbf{A})} U_\xi(ng)\, dn \equiv 0.$$

Since $U_\xi(g)$ is left invariant under $GL_n(k)$ and all k-rational Borel subgroups of GL_n are conjugate under $GL_n(k)$ it suffices to compute the constant term along the unipotent radicals of standard parabolic subgroups $Q \supset B_n$, so that $N_Q \subset N_n$. If Q' is a maximal parabolic subgroup such that $Q' \supset Q \supset B_n$, then $N_{Q'}$ is a normal subgroup of N_Q and in computing the constant term along N_Q we can integrate along $N_{Q'}$ first. Hence to show that $U_\xi(g)$ is cuspidal it suffices to show that it is cuspidal along the unipotent radical of any standard maximal parabolic subgroup. But this is guaranteed by Lemma 6.1. Hence $U_\xi \in \mathscr{A}^0(GL_n(k)\backslash GL_n(\mathbf{A}))$, i.e., U_ξ is cuspidal for every ξ and hence Ξ is cuspidal. As a constituent of Ξ, Π will then be cuspidal automorphic as well.

However, we can say a little more. Since Π is cuspidal, it is generic. Thus each local component Π_v is generic. But as we have pointed out, when Π_v is generic, $\Pi_v = \Xi_v$. Hence $\Pi = \Xi$. □

7. A second converse theorem

Theorem 1 is a generalization of results of Jacquet and Langlands for GL(2). It gives the most information about Π, namely that it is not only automorphic but also cuspidal. However Theorem 1 requires information about $L(\Pi \times \tau, s)$ which is usually not available. More precisely, in Theorem 1 we assume that $L(\Pi \times \tau, s)$ is entire for twists by all cuspidal automorphic forms on all $GL_m(\mathbf{A})$ with $m < n$. It is very difficult to obtain such information. André Weil, even before Jacquet-Langlands, suggested a different method of proving this type of theorem, which will allow us to obtain a result suitable for applications. In the method of Weil the first step is the construction of some periodic holomorphic function which is supposed to be an automorphic form. From given information about the functional equations satisfied by the associated Dirichlet series and their twists, Weil derived the conclusion that this function was an automorphic form with respect to some congruence subgroup. In the following Theorems 2 and 3 we will follow the method of Weil disguised in the language of automorphic representations.

For each finite set of places S of k containing all archimedean places and for each integer m, let

$$\Omega_S(m) = \{ \pi : \pi \text{ is an irreducible generic automorphic representation of } GL_m(\mathbf{A}), \text{ unramified at all } v \notin S \}.$$

Similarly, let $\Omega_S^0(m)$ be the set of cuspidal elements of $\Omega_S(m)$.

Theorem 2. — Let $n \geqslant 3$. Let Π be an irreducible admissible generic representation of $GL_n(\mathbf{A})$ whose central character ω_Π is invariant under $k^\times$ and whose L-function $L(\Pi, s)$ is absolutely convergent in some half-plane. Fix a non-empty finite set of places S of k containing all archimedean places such that the ring $\mathfrak{o}_S$ of S-integers of k has class number one. Suppose that for every m with $1 \leqslant m \leqslant n-1$ and every $\tau \in \Omega_S^0(m)$ the L-function $L(\Pi \times \tau, s)$ is nice. Then there exists an irreducible automorphic representation Π' of $GL_n(\mathbf{A})$ such that $\Pi_v \simeq \Pi'_v$ for all $v \in S$ and for all non-archimedean v such that Π_v is unramified.

In order to prove Theorem 2, we will first use the framework of Theorem 1 to construct an embedding of Π_S in the space of smooth functions on $\Gamma_S \backslash G_S$ for a congruence subgroup Γ_S of G_S with respect to an appropriate Hecke algebra. Let us recall that according to the general Duality Theorem [13], it is known that " classical " automorphic forms with respect to a group Γ are in duality with embeddings of given irreducible representations of $GL_2(\mathbf{R})$ into the space $L^2(\Gamma \backslash GL_2(\mathbf{R}))$. In the case $n \geqslant 3$ there is a simplification compared with Weil's theory, which in fact says that the set of assumptions (i.e. necessary twists) does not depend on the conductor of the representation Π. The reason for this simplification is that the congruence subgroup theorem is true for SL_n for $n \geqslant 3$.

There are two extensions of this which follow after some extra arguments. Currently they are separate statements, but we hope that they will eventually coalesce.

Corollary 1. — With the hypotheses of Theorem 2, there exists a proper automorphic representation Π'' with $\Pi''_v \simeq \Pi_v$ for all non-archimedean v for which Π_v is unramified.

The next Corollary is the one which is most useful for the application to Langlands' lifting.

Corollary 2. — With the hypotheses of Theorem 2, there is a unique irreducible generic automorphic representation Π'' such that $\Pi''_v \simeq \Pi_v$ for all $v \in S$ and all non-archimedean v for which Π_v is unramified.

8. The conductor of a representation

Let $\Pi = \otimes \Pi_v$ be an irreducible admissible generic representation of $GL_n(\mathbf{A})$. Let S be a finite set of places of k containing all archimedean places. For almost all places $v \notin S$, the representation Π_v is unramified, that is, Π_v contains a vector which is fixed by the maximal compact subgroup $K_v = GL_n(\mathfrak{o}_v)$. This vector is unique up to scalar multiples.

Let T denote the smallest finite set of places containing S such that Π_v is unramified for $v \notin T$ and let $T' = T \backslash S$. So T' is the set of places not in S for which Π_v is ramified. For those places $v \in T'$, it is known from [23] that there is a unique integer $m_v > 0$ such that if we set

$$K_{1,v}(\mathfrak{p}_v^{m_v}) = \left\{ g \in GL_n(\mathfrak{o}_v) : g \equiv \begin{pmatrix} & & * \\ & * & \vdots \\ & & * \\ 0 & \dots \; 0 & 1 \end{pmatrix} \pmod{\mathfrak{p}_v^{m_v}} \right\}$$

then the dimension of the space of $K_{1,v}(\mathfrak{p}_v^{m_v})$-fixed vectors in Π_v is one.

Set $m_v = 0$ for $v \notin T$. We will call the compact subring $\mathfrak{n} = \prod_{v \notin S} \mathfrak{p}_v^{m_v} \subset \mathbf{A}^S$ the S-*conductor of* Π. If S is precisely the set of archimedean places, hence is empty in the function field case, we will call $\mathfrak{n}$ the conductor of Π. It determines (and is determined by) an ideal of $\mathfrak{o}_S$ by $\mathfrak{n}_S = k \cap k_S \mathfrak{n} \subset \mathfrak{o}_S$. To simplify notation we will denote $\mathfrak{n}_S$ simply by $\mathfrak{n}$, since they can be distinguished by context. Note that $\mathfrak{o}_S/\mathfrak{n} \simeq \prod_{v \notin S} \mathfrak{o}_v/\mathfrak{p}_v^{m_v}$.

If we set

$$K_1(\mathfrak{n}) = \left\{ g \in \prod_{v \notin S} GL_n(\mathfrak{o}_v) : g \equiv \begin{pmatrix} & & * \\ & * & \vdots \\ & & * \\ 0 & \dots \; 0 & 1 \end{pmatrix} \pmod{\mathfrak{n}} \right\}$$

$$= \prod_{v \notin S} K_{1,v}(\mathfrak{p}_v^{m_v}) \subset G^S,$$

then the dimension of the space of $K_1(\mathfrak{n})$-fixed vectors in Π^S is exactly one.

We may similarly define

$$K_{0,v}(\mathfrak{p}_v^{m_v}) = \left\{ g \in GL_n(\mathfrak{o}_v) : g \equiv \begin{pmatrix} & & * \\ & * & \vdots \\ & & * \\ 0 & \dots \; 0 & * \end{pmatrix} \pmod{\mathfrak{p}_v^{m_v}} \right\}$$

and
$$K_0(\mathfrak{n}) = \left\{ g \in \prod_{v \notin S} GL_n(\mathfrak{o}_v) : g \equiv \begin{pmatrix} & & * \\ & * & \vdots \\ & & * \\ 0 & \dots\ 0 & * \end{pmatrix} \pmod{\mathfrak{n}} \right\}$$
$$= \prod_{v \notin S} K_{0,v}(\mathfrak{p}_v^{m_v}) \subset G^S.$$

The group $K_{1,v}(\mathfrak{p}_v^{m_v})$ will then be a normal subgroup of $K_{0,v}(\mathfrak{p}_v^{m_v})$ with abelian quotient given by $K_{0,v}(\mathfrak{p}_v^{m_v})/K_{1,v}(\mathfrak{p}_v^{m_v}) \simeq (\mathfrak{o}_v/\mathfrak{p}_v^{m_v})^\times$ and $K_1(\mathfrak{n})$ is a normal subgroup of $K_0(\mathfrak{n})$ with quotient $\prod_{v \notin S}(\mathfrak{o}_v/\mathfrak{p}_v^{m_v})^\times \simeq (\mathfrak{o}_S/\mathfrak{n})^\times$. Then the action of $K_0(\mathfrak{n})$ will preserve the one-dimensional space of $K_1(\mathfrak{n})$-fixed vectors and act on it by a character of $K_0(\mathfrak{n})$ trivial on $K_1(\mathfrak{n})$.

It is easy to compute the action of $K_0(\mathfrak{n})$ on the space of $K_1(\mathfrak{n})$-fixed vectors. Let ξ_v^0 be a non-trivial $K_{1,v}(\mathfrak{p}_v^{m_v})$-fixed vector in Π_v for $v \notin S$. Then the tensor product $\xi^0 = \bigotimes \xi_v^0$ is a non-trivial $K_1(\mathfrak{n})$-vector in Π^S. If $v \notin T$ then
$$K_{1,v}(\mathfrak{p}_n^{m_v}) = K_{0,v}(\mathfrak{p}_v^{m_v}) = GL_n(\mathfrak{o}_v)$$
and so for $g_v \in K_{0,v}(\mathfrak{p}_v^{m_v})$ we have $\Pi_v(g_v)\,\xi_v^0 = \xi_v^0$. If $v \in T'$ and $g_v = (g_{i,j}) \in K_{0,v}(\mathfrak{p}_v^{m_v})$ then from the congruence condition we have $|\,g_{n,j}\,|_v < 1$ for $1 \leqslant j < n$. Since $g_v \in GL_n(\mathfrak{o}_v)$ we must have $\max_j\{|\,g_{n,j}\,|_v\} = 1$ and hence $|\,g_{n,n}\,|_v = 1$ and so $g_{n,n} \in \mathfrak{o}_v^\times$. Then we may write $g_v = (g_{n,n}\,I_n)\,g_{1,v}$ with $g_{1,v} \in K_{1,v}(\mathfrak{p}_v^{m_v})$. Then
$$\Pi_v(g_v)\,\xi_v^0 = \Pi_v(g_{n,n}\,I_n)\,\xi_v^0 = \omega_v(g_{n,n})\,\xi_n^0$$
where ω_v is the central character of Π_v. So we may define a character $\chi = \bigotimes \chi_v$ of $K_0(\mathfrak{n})$ by $\chi_v(g_v) = 1$ if $v \notin T$ and $\chi_v(g_v) = \omega_v(g_{n,n})$ if $v \in T'$. This is guaranteed to be a character by construction. If we wish to emphasize the dependence on the central character ω of Π we will write $\chi = \chi_\omega$. We have $\Pi^S(g)\,\xi^0 = \chi_\omega(g)\,\xi^0$ for $g \in K_0(\mathfrak{n})$.

There is another useful construction of χ_ω. Consider the central character ω of Π. If $v \notin T$ then for any local unit $u_v \in \mathfrak{o}_v^\times$ we have $u_v\,I_n \in GL_n(\mathfrak{o}_v) = K_{1,v}(\mathfrak{p}_v^{m_v})$ and so $\omega_v(u_v)\,\xi_v^0 = \Pi_v(u_v\,I_n)\,\xi_v^0 = \xi_v^0$ so that $\omega_v(u_v) = 1$. Similarly, if $v \in T'$ and u_v is a local unit of the form $1 + \mathfrak{p}_v^{m_v}$ then $\omega_v(u_v) = 1$. So ω_v is unramified at $v \notin T$ and has conductor at least $\mathfrak{p}_v^{m_v}$ at the places $v \in T'$. Since $(\mathfrak{o}_S/\mathfrak{n})^\times \simeq \prod_v(\mathfrak{o}_v/\mathfrak{p}_n^{m_v})^\times$, the character ω defines a character χ_ω of $(\mathfrak{o}_S/\mathfrak{n})^\times$ via this isomorphism by $\chi_\omega = \prod_{v \notin S}\omega_v$. Then, through the isomorphism $K_0(\mathfrak{n})/K_1(\mathfrak{n}) \simeq (\mathfrak{o}_S/\mathfrak{n})^\times$, this character χ_ω defines a character of $K_0(\mathfrak{n})$ trivial on $K_1(\mathfrak{n})$ which is easily seen to be the same character as defined above. Hence we could write $\chi_\omega(g) = \chi_\omega(g_{n,n})$ for $g = (g_{i,j}) \in K_0(\mathfrak{n})$.

9. Generation of congruence subgroups

Let $n \geqslant 3$. Let S denote a non-empty finite set of places of k containing all archimedean places. Let $\mathfrak{o}_S$ denote the S-integers of k. Since

$$GL_n(\mathfrak{o}_S) = GL_n(k) \cap G_S \prod_{v \notin S} GL_n(\mathfrak{o}_v),$$

we may view $GL_n(\mathfrak{o}_S)$ as a subgroup of $GL_n(k)$ embedded in G_S. Then $GL_n(\mathfrak{o}_S)$ is a discrete subgroup of G_S.

For the proof of Theorem 2 we will need a preliminary result on the generation of certain congruence subgroups of $GL_n(\mathfrak{o}_S)$. The heart of this proof is Lemma 9.1 which is extracted from the proof of Theorem 4.2 of Bass [1]. This result from the stable algebra of GL_n plays a role in the solution of the congruence subgroup problem for SL_n [2]. This is the place where the restriction $n \geqslant 3$ comes from, as in the congruence subgroup theorem.

Let T' be a finite set of places disjoint from S and let $T = S \cup T'$. For each $v \in T'$ let m_v be a positive integer and for $v \notin T$ set $m_v = 0$. Let $\mathfrak{n} = \prod_{v \notin S} \mathfrak{p}_v^{m_v} \subset k^S$. As in Section 8, $\mathfrak{n}$ defines an ideal, again denoted $\mathfrak{n}$, in $\mathfrak{o}_S$. The congruence subgroups of $GL_n(\mathfrak{o}_S)$ we are interested in are

$$\Gamma_1(\mathfrak{n}) = \left\{ \gamma \in GL_n(\mathfrak{o}_S) : \gamma \equiv \begin{pmatrix} & & & * \\ & * & & \vdots \\ & & & * \\ 0 & \dots & 0 & 1 \end{pmatrix} \pmod{\mathfrak{n}} \right\}$$

and

$$\Gamma_0(\mathfrak{n}) = \left\{ \gamma \in GL_n(\mathfrak{o}_S) : \gamma \equiv \begin{pmatrix} & & & * \\ & * & & \vdots \\ & & & * \\ 0 & \dots & 0 & * \end{pmatrix} \pmod{\mathfrak{n}} \right\}.$$

If we define $K_i(\mathfrak{n}) \subset G^S$ for $i = 0, 1$ as in Section 8 then we can also characterize $\Gamma_i(\mathfrak{n})$ by $\Gamma_i(\mathfrak{n}) = GL_n(k) \cap G_S . K_i(\mathfrak{n})$.

Consider the following subgroups of $\Gamma_i(\mathfrak{n})$. Set

$$P'_i(\mathfrak{o}_S) = P'_n(k) \cap G_S . K_i(\mathfrak{n}) = P'_n(k) \cap G_S . K^S,$$

where as usual we have set $K^S = \prod_{v \notin S} K_v$. This is the set of all matrices in $GL_n(\mathfrak{o}_S)$ whose last row is of the form $(0, \dots, 0, *)$ if $i = 0$ or $(0, \dots, 0, 1)$ if $i = 1$. It is independent of $\mathfrak{n}$. Set $\overline{P}'_i(\mathfrak{n}) = \overline{P}'_n(k) \cap G_S . K_i(\mathfrak{n})$. This is the subgroup of $\Gamma_i(\mathfrak{n})$ consisting of those matrices whose last column is ${}^t(0, \dots, 0, *)$. There is a congruence condition on the last row of these matrices.

Proposition **9.1.** — *The groups* $P_i'(\mathfrak{o}_S)$ *and* $\overline{P}_i'(\mathfrak{n})$ *together generate the congruence subgroup* $\Gamma_i(\mathfrak{n})$ *for* $i = 0, 1$.

For now, let $\Delta_i(\mathfrak{n})$ denote the subgroup of $GL_n(\mathfrak{o}_S)$ generated by $P_i'(\mathfrak{o}_S)$ and $\overline{P}_i'(\mathfrak{n})$. Note that $\Delta_1(\mathfrak{n}) \subset \Delta_0(\mathfrak{n})$.

Lemma **9.1.** — *Let* $(a_1, \ldots, a_n) \in \mathfrak{o}_S^n$ *be a unimodular sequence such that* $(a_1, \ldots, a_n) \equiv (0, \ldots, 0, d) \bmod \mathfrak{n}$. *Then there exists an element* $\gamma \in \Delta_1(\mathfrak{n})$ *such that* $(a_1, \ldots, a_n)\gamma = (0, \ldots, 0, d)$.

Proof. — The sequence $(a_1, \ldots, a_n)$ is unimodular in the sense that there exist $c_1, \ldots, c_n$ in $\mathfrak{o}_S$ such that $1 = \Sigma c_i a_i$. Therefore $a_1 = \Sigma a_1 c_i a_i = a_1 c_1 a_1 + \Sigma_{i=2}^n a_1 c_i a_i$. If we substitute this expression for a_1 into $1 = \Sigma c_i a_i$, and let $q = a_1 c_1$, we find

$$1 = c_1 q a_1 + \sum_{n=2}^{n} c_i(q+1) a_i.$$

Since $a_1 \in \mathfrak{n}$, we have $a_1 c_1 = q \in \mathfrak{n}$, and we see that the sequence $(qa_1, a_2, \ldots, a_n)$ is again unimodular. Since $\mathfrak{o}_S$ is a Dedekind domain, $n = 2$ defines a stable range for $\mathfrak{o}_S$ in the sense of [1]. (Note that there is a shift of one in the definition of stable range between [1] and [2].) This implies that there exist $a_i' = a_i + b_i q a_1$ with $b_i \in \mathfrak{o}_S$ such that the sequence $(a_2', \ldots, a_n')$ is unimodular. Let

$$\tau_1 = \begin{pmatrix} 1 & b_2 q & \ldots & b_n q \\ & 1 & & 0 \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix} \in P_1'(\mathfrak{o}_S).$$

Then $(a_1, \ldots, a_n)\tau_1 = (a_1, a_2', \ldots, a_n')$. Note that we still have

$$(a_1, a_2', \ldots, a_n') \equiv (0, \ldots, 0, d) \pmod{\mathfrak{n}}.$$

Since $(a_2', \ldots, a_n')$ is unimodular, we may write $1 = \Sigma_{i=2}^n c_i' a_i'$. Write $a_n' = d + q_n'$ with $q_n' \in \mathfrak{n}$. Then we have $q_n' - a_1 \in \mathfrak{n}$ and by the unimodularity of $(a_2', \ldots, a_n')$ we may write this element as $q_n' - a_1 = \Sigma_{n=2}^n d_i a_i'$ with $d_i \in \mathfrak{n}$. Now let

$$\tau_2 = \begin{pmatrix} 1 & & & \\ d_2 & 1 & & \\ \vdots & & \ddots & \\ d_n & 0 & \ldots & 1 \end{pmatrix}.$$

Then $\tau_2 \in \overline{P}_1'(\mathfrak{n})$ since $d_n \in \mathfrak{n}$. So $(a_1, a_2', \ldots, a_n')\tau_2 = (q_n', a_2', \ldots, a_n')$. Now set

$$\sigma = \begin{pmatrix} 1 & & -1 \\ & \ddots & \\ & & 1 \end{pmatrix} \in P_1'(\mathfrak{o}_S)$$

so that $(q'_n, a'_2, \ldots, a'_n)\,\sigma = (q'_n, a'_2, \ldots, a'_{n-1}, d)$. Note that we still have

$$q'_n, a'_2, \ldots, a'_{n-1} \in \mathfrak{n}.$$

So if we set

$$\tau_3 = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \ddots & & \\ & & & \ddots & \\ -q'_n & -a'_2 & \ldots & -a'_{n-1} & 1 \end{pmatrix},$$

then $\tau_3 \in \overline{P}'_1(\mathfrak{n})$ and $(q'_n, a'_2, \ldots, a'_{n-1}, 1)\,\tau_3 = (0, \ldots, 0, d)$. Therefore

$$(a_1, \ldots, a_n)\,\tau_1\,\tau_2\,\sigma\tau_3 = (0, \ldots, 0, d).$$

Since $\tau_1, \sigma \in P'_1(\mathfrak{o}_S)$ and $\tau_2, \tau_3 \in \overline{P}'_1(\mathfrak{n})$ we see that $\tau_1\,\tau_2\,\sigma\tau_3 \in \Delta_1(\mathfrak{n})$. □

Proof of the proposition. — Since $P'_i(\mathfrak{o}_S)$, $\overline{P}'_i(\mathfrak{n}) \subset \Gamma_i(\mathfrak{n})$, it is clear that $\Delta_i(\mathfrak{n}) \subset \Gamma_i(\mathfrak{n})$.

Now let $\gamma \in \Gamma_i(\mathfrak{n})$, so

$$\gamma \equiv \begin{pmatrix} & & & * \\ & * & & \vdots \\ & & & * \\ 0 & \ldots & 0 & d \end{pmatrix} \pmod{\mathfrak{n}}$$

with $d = 1$ if $i = 1$. Let $u = \det(\gamma)$. This is a unit in $\mathfrak{o}_S$ and the diagonal matrix $\mathrm{diag}(u, 1, \ldots, 1)$ is in $P'_i(\mathfrak{o}_S)$. Then $\mathrm{diag}(u^{-1}, 1, \ldots, 1)\,\gamma$ has determinant 1. Hence its last row is unimodular in the sense of the lemma and we still have

$$\begin{pmatrix} u^{-1} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \gamma \equiv \begin{pmatrix} & & & * \\ & * & & \vdots \\ & & & * \\ 0 & \ldots & 0 & d \end{pmatrix} \pmod{\mathfrak{n}}.$$

Now, by our lemma, there exists $\gamma_1 \in \Delta_1(\mathfrak{n})$ such that

$$\begin{pmatrix} u^{-1} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \gamma\gamma_1 = \begin{pmatrix} & & & * \\ & * & & \vdots \\ & & & * \\ 0 & \ldots & 0 & d \end{pmatrix} = p.$$

But then $p \in P'_i(\mathfrak{o}_S)$. Hence $\gamma = \mathrm{diag}(u, 1, \ldots, 1)\,p\gamma_1^{-1} \in P'_i(\mathfrak{o}_S)\,\Delta_1(\mathfrak{n}) \subset \Delta_i(\mathfrak{n})$. □

10. Proofs of Theorem 2 and its corollaries

Let $\mathfrak{n}$ denote the S-conductor of Π and $\omega = \omega_\Pi$ its central character. Let $\xi^0 = \bigotimes_{v \notin S} \xi_v^0$ be a non-zero $K_1(\mathfrak{n})$-fixed vector in Π^S as in Section 8. So ξ^0 transforms by the character χ_ω of $K_0(\mathfrak{n})$ as in Section 8.

Proof of Theorem 2. — For each $\xi_S \in \Pi_S$ consider the functions $U_{\xi_S \otimes \xi^0}(g)$ and $V_{\xi_S \otimes \xi^0}(g)$ associated to the vector $\xi = \xi_S \otimes \xi^0 \in \Pi$. The function $U_{\xi_S \otimes \xi^0}$ is left invariant under $P_n(k)$ and $V_{\xi_S \otimes \xi^0}$ is left invariant under $\overline{P}_n(k)$. Both are invariant under $Z_n(k)$. In addition, $U_{\xi_S \otimes \xi^0}$ and $V_{\xi_S \otimes \xi^0}$ are right invariant under $K_1(\mathfrak{n})$ for all $\xi_S \in \Pi_S$. Now, if we restrict these functions to $GL_{n-1}(\mathbf{A})$ we find:

Lemma **10.1.** — *In addition to the properties from Lemma* 6.1 *and Lemma* 6.2, *the functions* $U_{\xi_S \otimes \xi^0}\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$ *and* $V_{\xi_S \otimes \xi^0}\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$ *are unramified at all* $v \notin S$, *i.e., they are right invariant under* $K_{n-1}^S = \prod_{v \notin S} GL_{n-1}(\mathfrak{o}_v)$.

Proof. — For v such that Π_v is unramified this is clear since ξ_v^0 is fixed by $GL_n(\mathfrak{o}_v) \supset GL_{n-1}(\mathfrak{o}_v)$. At the remaining places ξ_v^0 is fixed by $K_{1,v}(\mathfrak{p}_v^{m_v}) \supset GL_{n-1}(\mathfrak{o}_v)$. □

Now consider the integrals $I(\xi_S \otimes \xi^0, \varphi; s)$ and $\widetilde{I}(\xi_S \otimes \xi^0, \varphi; s)$ for φ lying in a proper automorphic representation τ of $GL_{n-1}(\mathbf{A})$ as defined in Section 6. Since $U_{\xi_S \otimes \xi^0}\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$ is unramified, we find that

$$\begin{aligned} I(\xi_S \otimes \xi^0, \varphi; s) &= \int_{GL_{n-1}(k)\backslash GL_{n-1}(\mathbf{A})} U_{\xi_S \otimes \xi^0}\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \varphi(h) \,|\det(h)|^{s-(1/2)}\, dh \\ &= \int_{N_{n-1}(\mathbf{A})\backslash GL_{n-1}(\mathbf{A})} W_{\xi_S \otimes \xi^0}\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} W_\varphi(h) \,|\det(h)|^{s-(1/2)}\, dh \\ &= \int_{N_{n-1}(\mathbf{A})\backslash GL_{n-1}(\mathbf{A})/K_{n-1}^S} W_{\xi_S \otimes \xi^0}\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \int_{K_{n-1}^S} W_\varphi(hk)\, dk \,|\det(h)|^{s-(1/2)}\, dh. \end{aligned}$$

Hence $I(\xi_S \otimes \xi^0, \varphi; s) \equiv 0$ unless τ contains a vector fixed by K_{n-1}^S. The same is true for $\widetilde{I}(\xi_S \otimes \xi^0, \varphi; s)$. Hence, if $\tau \notin \Omega_S(n-1)$, then $I(\xi_S \otimes \xi^0, \varphi; s) = 0 = \widetilde{I}(\xi_S \otimes \xi^0, \varphi; s)$.

On the other hand if $\tau \in \Omega_S^0(n-1)$, then by our assumption on the L-function we have as in the proof of Theorem 1 that $I(\xi_S \otimes \xi^0, \varphi; s) = \widetilde{I}(\xi_S \otimes \xi^0, \varphi; s)$. If $\tau \in \Omega_S(n-1)$ but is non-cuspidal, then τ must still be generic for the integrals to be non-zero and as before we have that τ is a subrepresentation of a representation induced from cuspidals. Since τ is unramified for $v \notin S$ and generic, these cuspidal representations must also be unramified for $v \notin S$ and generic. Then arguing as before, we find that, in this case as well, $I(\xi_S \otimes \xi^0, \varphi; s) = \widetilde{I}(\xi_S \otimes \xi^0, \varphi; s)$. Hence we have the following result.

Proposition **10.1.** — *For all proper automorphic representations* τ *of* $GL_{n-1}(\mathbf{A})$ *we have* $I(\xi_S \otimes \xi^0, \varphi; s) = \widetilde{I}(\xi_S \otimes \xi^0, \varphi; s)$ *for all* $\xi_S \in \Pi_S$.

From here, applying the weak form of Langlands' spectral theory as in the proof of Theorem 1, we find

Proposition **10.2.** — *The equality* $U_{\xi_S \otimes \xi^0}(1) = V_{\xi_S \otimes \xi^0}(1)$ *holds for all* $\xi_S \in \Pi_S$.

Since ξ_S was arbitrary in Π_S and ξ_S^0 transforms by the character χ_ω of $K_0(\mathfrak{n})$ as in Section 8, we find that in fact

$$U_{\xi_S \otimes \xi^0}(g) = V_{\xi_S \otimes \xi^0}(g)$$

for all $g \in G_S . K_0(\mathfrak{n}) \subset GL_n(\mathbf{A})$ and all $\xi_S \in \Pi_S$. However, since we have fixed the vector ξ^0 at places $v \notin S$, we cannot conclude equality for all $g \in GL_n(\mathbf{A})$. For this reason we are unable to embed Π as a subrepresentation of $\mathscr{A}(GL_n(k)\backslash GL_n(\mathbf{A}))$. We will only be able to embed Π_S as a subrepresentation of a space of classical modular forms on G_S transforming by the Nebentypus character χ_ω^{-1} of $\Gamma_0(\mathfrak{n})$.

To simplify notation, let us introduce the functions

$$\Phi_{\xi_S}(g_S) = U_{\xi_S \otimes \xi^0}((g_S, 1^S)) = V_{\xi_S \otimes \xi^0}((g_S, 1^S)),$$

where $1^S = \Pi_{v \notin S} 1_v \in G^S$ and $(g_S, 1^S) \in G_S G^S = GL_n(\mathbf{A})$. This associates to each $\xi_S \in \Pi_S$ a function on G_S. Let $P_0'(\mathfrak{o}_S)$ and $\overline{P}_0'(\mathfrak{n})$ be the discrete subgroups of G_S defined in Section 9. These are both subgroups of $\Gamma_0(\mathfrak{n})$. Since $\Gamma_1(\mathfrak{n})$ is a normal subgroup of $\Gamma_0(\mathfrak{n})$ with abelian quotient $(\mathfrak{o}_S/\mathfrak{n})^\times$, the central character ω of Π induces a character χ_ω of $\Gamma_0(\mathfrak{n})$ through the character χ_ω of $(\mathfrak{o}_S/\mathfrak{n})^\times$ defined in Section 8.

Lemma **10.2.** — *The function* Φ_{ξ_S} *is left invariant under* $P_1(\mathfrak{o}_S)$ *and* $\overline{P}_1(\mathfrak{n})$ *and transforms by the character* χ_ω^{-1} *under* $P_0'(\mathfrak{o}_S)$ *and* $\overline{P}_0'(\mathfrak{n})$.

Proof. — This is the standard argument. Write an element $g \in GL_n(\mathbf{A})$ as $g = (g_S, g^S)$ with $g_S \in G_S$, $g^S \in G^S$. Then for $\gamma \in P_0'(\mathfrak{o}_S)$ we have

$$\Phi_{\xi_S}(\gamma g_S) = U_{\xi_S \otimes \xi^0}((\gamma g_S, 1^S)).$$

Since $U_{\xi_S \otimes \xi^0}$ is left invariant under $P_n'(k)$ this is

$$\Phi_{\xi_S}(\gamma g_S) = U_{\xi_S \otimes \xi^0}((g_S, \gamma^{-1})) = U_{\xi_S \otimes \Pi^S(\gamma^{-1})\xi^0}((g_S, 1^S)).$$

But now $\gamma^{-1} \in K_0(\mathfrak{n})$. Since $U_{\xi_S \otimes \xi^0}$ transforms by χ_ω under $K_0(\mathfrak{n})$ we have $\Phi_{\xi_S}(\gamma g_S) = \chi_\omega^{-1}(\gamma)\, \Phi_{\xi_S}(g_S)$. The argument for $\overline{P}_0'(\mathfrak{n})$ is the same, but using $V_{\xi_S \otimes \xi^0}$.

Since χ_ω is trivial on the subgroups $P_1'(\mathfrak{o}_S)$ and $\overline{P}_1'(\mathfrak{n})$ we obtain the invariance of Φ_{ξ_S} under these groups. □

By Proposition 9.1, the groups $P'_i(\mathfrak{o}_S)$ and $\overline{P}'_i(\mathfrak{n})$ generate the congruence subgroup $\Gamma_i(\mathfrak{n}) \subset GL_n(\mathfrak{o}_S)$ for $i = 0, 1$. Hence we may conclude that for every $\xi_S \in \Pi_S$ the function $\Phi_{\xi_S}(g_S)$ is left invariant under $\Gamma_1(\mathfrak{n})$ and transforms by the character χ_ω^{-1} under $\Gamma_0(\mathfrak{n})$. Let

$$\mathscr{A}(\Gamma_0(\mathfrak{n})\backslash G_S;\, \omega_S,\, \chi_\omega^{-1})$$

be the set of automorphic forms Φ on G_S in the sense of [6] which also satisfy

$$\Phi(\gamma g_S) = \chi_\omega^{-1}(\gamma)\, \Phi(g_S) \quad \text{for } \gamma \in \Gamma_0(\mathfrak{n}) \tag{1}$$

$$\Phi(z_S\, g_S) = \omega_S(z_S)\, \Phi(g_S) \quad \text{for } z_S \in Z_n(k_S) \simeq k_S^\times, \tag{2}$$

where ω_S is the central character of Π_S. The character χ_ω^{-1} is referred to as the Nebentypus character [18]. We then have the following.

Proposition **10.3.** — *The map* $\xi_S \mapsto \Phi_{\xi_S}(g_S)$ *embeds* Π_S *as an irreducible subrepresentation of* $\mathscr{A}(\Gamma_0(\mathfrak{n})\backslash G_S;\, \omega_S,\, \chi_\omega^{-1})$.

From Section 1 of the appendix, we know that $\mathscr{A}(\Gamma_0(\mathfrak{n})\backslash G_S;\, \omega_S,\, \chi_\omega^{-1})$ is naturally isomorphic to the space $\mathscr{A}(GL_n(k)\backslash GL_n(\mathbf{A});\, \omega)^{K_1(\mathfrak{n})}$ of $K_1(\mathfrak{n})$-invariant functions in the space of automorphic forms transforming by the character ω under the center. To relate irreducible subrepresentations of $\mathscr{A}(\Gamma_0(\mathfrak{n})\backslash G_S;\, \omega_S,\, \chi_\omega^{-1})$ to automorphic representations of $GL_n(\mathbf{A})$ occurring in $\mathscr{A}(GL_n(k)\backslash GL_n(\mathbf{A});\, \omega)$ we need to know that the representation consists of Hecke eigenforms for an appropriate Hecke algebra. In the appendix we explain this relationship and the Hecke algebras involved when the S-class number is equal to one. We refer the reader to the appendix for the notation to be used.

Let T be the smallest finite set of places containing S and such that Π_v is unramified at all $v \notin T$. Let $T' = T\backslash S$. So T' consists of those places dividing the S-conductor $\mathfrak{n}$. Then Π^T is an irreducible unramified representation of G^T and hence corresponds to a character Λ of the Hecke algebra $\mathscr{H}(G^T, K^T)$ of compactly supported K^T-bi-invariant functions on G^T. Since ξ^0 is the unique $K_1(\mathfrak{n}) \supset K^T$-fixed vector in Π^S we see that for all $\widetilde{\Phi} \in \mathscr{H}(G^T, K^T)$

$$\Pi^T(\widetilde{\Phi})\, \xi^0 = \Lambda(\widetilde{\Phi})\, \xi^0.$$

There is a natural Hecke algebra, which we will denote by $\mathscr{H}_c(\mathfrak{n})$, acting on the space $\mathscr{A}(\Gamma_0(\mathfrak{n})\backslash G_S;\, \omega_S,\, \chi_\omega^{-1})$. To describe $\mathscr{H}_c(\mathfrak{n})$, let $M = GL_n(k) \cap (\prod_{v \in T'} K_{0,v}(\mathfrak{p}_v^{m_v}))\, G^T$. This M consists of those rational matrices $\gamma \in GL_n(k)$ such that for all $v \in T'$ the v-component γ_v lies in $K_{0,v}(\mathfrak{p}_v^{m_v})$. Then $\Gamma_1(\mathfrak{n}) \subset M$. Let $\mathscr{H}_c(\mathfrak{n})$ depote the $\mathbf{C}$-span of the double cosets $\Gamma_1(\mathfrak{n})\backslash M/\Gamma_1(\mathfrak{n})$.

The algebra $\mathscr{H}_c(\mathfrak{n})$ is related to the following adelic Hecke algebra. Let $G^S(\mathfrak{n}) = (\prod_{v \in T'} K_{0,v}(\mathfrak{p}_v^{m_v}))\, G^T$. Then $G^S(\mathfrak{n}) \supset K_1(\mathfrak{n})$ and we may from the associated Hecke algebra $\mathscr{H}(G^S(\mathfrak{n}), K_1(\mathfrak{n}))$ of compactly supported $K_1(\mathfrak{n})$-bi-invariant functions on $G^S(\mathfrak{n})$. From the appendix we know that this algebra is isomorphic to

$\mathbf{C}[(\mathfrak{o}_S/\mathfrak{n})^\times] \otimes \mathscr{H}(G^T, K^T)$ and so contains $\mathscr{H}(G^T, K^T)$ as a subalgebra. Then there is a natural isomorphism $\alpha : \mathscr{H}_c(\mathfrak{n}) \to \mathscr{H}(\mathfrak{n})$ which takes the double coset $\Gamma_1(\mathfrak{n})\, t\Gamma_1(\mathfrak{n})$ to the normalized characteristic function $\widetilde{\Phi}_t$ of the double coset $K_1(\mathfrak{n})\, tK_1(\mathfrak{n})$. The algebra structure on $\mathscr{H}_c(\mathfrak{n})$ is the pull back of that of $\mathscr{H}(\mathfrak{n})$ via α. In particular, $\mathscr{H}_c(\mathfrak{n})$ has a subalgebra $\mathscr{H}_c^T$ corresponding to $\mathscr{H}(G^T, K^T)$ via α. If $\Gamma_1(\mathfrak{n})\, t\Gamma_1(\mathfrak{n}) \in \mathscr{H}_c(\mathfrak{n})$ then the associated Hecke operator $\mathscr{E}_t$ acting on $\mathscr{A}(\Gamma_0(\mathfrak{n})\backslash G_S; \omega_S, \chi_\omega^{-1})$ is defined as follows. For $f \in \mathscr{A}(\Gamma_0(\mathfrak{n})\backslash G_S; \omega_S, \chi_\omega^{-1})$ and $\Gamma_1(\mathfrak{n})\, t\Gamma_1(\mathfrak{n}) = \coprod a_j\, \Gamma_1(\mathfrak{n})$ the action is

$$(\mathscr{E}_t f)\,(g_S) = \sum_j f(a_j^{-1} g_S).$$

The algebra $\mathscr{H}(\mathfrak{n})$ acts on $\mathscr{A}(GL_n(k)\backslash GL_n(\mathbf{A}); \omega)^{K_1(\mathfrak{n})}$ by convolution

$$(\widetilde{\Phi} * \varphi)\,(g) = \int_{G^S(\mathfrak{n})} \widetilde{\Phi}(h)\, \varphi(gh)\, dh$$

for $\widetilde{\Phi} \in \mathscr{H}(\mathfrak{n})$ and $\varphi \in \mathscr{A}(GL_n(k)\backslash GL_n(\mathbf{A}); \omega)^{K_1(\mathfrak{n})}$. These facts can be found in Section 3 of the appendix.

Proposition **10.4.** — *For each* $\xi_S \in \Pi_S$ *the function* Φ_{ξ_S} *is a Hecke eigenform for* $\mathscr{H}_c^T$ *with eigencharacter* Λ, *i.e.,* $\mathscr{E}_t\, \Phi_{\xi_S} = \Lambda(\widetilde{\Phi}_t)\, \Phi_{\xi_S}$ *for each* $\Gamma_1(\mathfrak{n})\, t\Gamma_1(\mathfrak{n}) \in \mathscr{H}_c^T$.

Proof. — Let $\Gamma_1(\mathfrak{n})\, t\Gamma_1(\mathfrak{n})$ be a double coset in $\mathscr{H}_c^T$ and $\mathscr{E}_t$ the associated Hecke operator. Write $\Gamma_1(\mathfrak{n})\, t\Gamma_1(\mathfrak{n}) = \coprod a_j\, \Gamma_1(\mathfrak{n})$ with $a_j = p_j\, \gamma_j \in P'_n(k)\, \Gamma_0(\mathfrak{n})$. This choice of coset representatives is possible by Lemma A.2 of the appendix. Then since Φ_{ξ_S} transforms by the Nebentypus character χ_ω^{-1} we have

$$\begin{aligned}(\mathscr{E}_t\, \Phi_{\xi_S})\,(g_S) &= \sum_j \Phi_{\xi_S}(a_j^{-1} g_S)\\ &= \sum_j \Phi_{\xi_S}(\gamma_j^{-1} p_j^{-1} g_S)\\ &= \sum_j \chi_\omega(\gamma_j)\, \Phi_{\xi_S}(p_j^{-1} g_S)\\ &= \sum_j \chi_\omega(\gamma_j)\, U_{\xi_S \otimes \xi^0}((p_j^{-1} g_S, 1)).\end{aligned}$$

Since $U_{\xi_S \otimes \xi^0}$ is left invariant under $P'_n(k)$ and ξ^0 transforms by χ_ω under $K_0(\mathfrak{n})$ we have

$$\begin{aligned}\chi_\omega(\gamma_j)\, U_{\xi_S \otimes \xi^0}((p_j^{-1} g_S, 1)) &= \chi_\omega(\gamma_j)\, U_{\xi_S \otimes \xi^0}((g_S, p_j))\\ &= U_{\xi_S \otimes \Pi^S(\gamma_j)\, \xi^0}((g_S, p_j))\\ &= U_{\xi_S \otimes \Pi^S(a_j)\, \xi^0}((g_S, 1^S)).\end{aligned}$$

Thus

$$(\mathscr{E}_t\, \Phi_{\xi_S})\,(g_S) = U_{\xi_S \otimes \Pi^S(\widetilde{\Phi}_t)\, \xi^0}((g_S, 1^S)).$$

As noted above, ξ^0 is an eigenfunction for $\mathscr{H}(G^T, K^T)$ with eigencharacter Λ. Thus

$$\begin{aligned}(\mathscr{E}_t\, \Phi_{\xi_S})\,(g_S) &= \Lambda(\widetilde{\Phi}_t)\, U_{\xi_S \otimes \xi^0}((g_S, 1^S))\\ &= \Lambda(\widetilde{\Phi}_t)\, \Phi_{\xi_S}(g_S). \quad \square\end{aligned}$$

We now have that Π_S is an irreducible subrepresentation of $\mathscr{A}(\Gamma_0(\mathfrak{n})\backslash G_S; \omega_S, \chi_\omega^{-1})$ which consists of Hecke eigenvectors for the Hecke algebra $\mathscr{H}_c^T$ with eigencharacter Λ. We may then apply Theorem A of the appendix to conclude that there exists an irreducible automorphic representation Π' of $GL_n(\mathbf{A})$ such that $\Pi'_S \simeq \Pi_S$ (since Π_S is irreducible) and such that Π'^T is the unique irreducible representation of G^T with eigencharacter Λ for $\mathscr{H}(G^T, K^T)$. Thus $\Pi'^T \simeq \Pi^T$ and Π' satisfies the conclusions of Theorem 2. □

Proof of Corollary 1. — We begin with the representation Π' from Theorem 2. Π' is an automorphic representation with the desired properties, but it may be only a subquotient of the space of automorphic forms. The fact that our original representation Π is generic will allow us to pass from Π' to a proper automorphic representation, that is, a subrepresentation of the space of automorphic forms. Since our original representation Π was generic, then Π' is *quasi-generic* in the sense that $\Pi' = \bigotimes \Pi'_v$ is irreducible and for almost all v, Π'_v is generic. Hence to complete the corollary as stated, it is enough to prove the following result.

Proposition **10.5.** — *Let Π' be an irreducible automorphic quasi-generic representation of $GL_n(\mathbf{A})$. Then there exists an irreducible proper automorphic representation Π'' which is quasi-isomorphic to Π'. Moreover, $\Pi'_v \simeq \Pi''_v$ for all non-archimedean places v where Π'_v is both generic and unramified.*

To prove this we will use the following well-known fact.

Lemma **10.3.** — *Let Π_v be an irreducible admissible unramified generic representation of $GL_n(k_v)$ over a non-archimedean local field k_v. Then there exist unramified characters $\chi_{1,v}, \ldots, \chi_{n,v}$ of $GL_1(k_v) = k_v^\times$ such that*

$$\Pi_v = \mathrm{Ind}_{B_n(k_v)}^{GL_n(k_v)}(\chi_{1,v} \otimes \ldots \otimes \chi_{n,v}).$$

Proof. — By the theory of spherical functions [8] we know that there are unramified characters $\chi_{1,v}, \ldots, \chi_{n,v}$ of $GL_1(k_v)$ such that Π_v is the unique unramified constituent of $\mathrm{Ind}_{B_n(k_v)}^{GL_n(k_v)}(\chi_{1,v} \otimes \ldots \otimes \chi_{n,v})$. Without loss of generality we may write each $\chi_{i,v}(x) = |\, x\,|_v^{u_i}$ with the $u_i \in \mathbf{C}$ and assume $\mathrm{Re}(u_1) \geqslant \ldots \geqslant \mathrm{Re}(u_n)$. Following Jacquet [20], if we group the characters into families with $\mathrm{Re}(u_i)$ equal and induce these up to the appropriate GL_s we get a sequence of quasi-tempered representations $\tau_{1,v}, \ldots, \tau_{r,v}$. Since these induced representations $\tau_{i,v}$ are irreducible [19] we may use induction in stages to get

$$\mathrm{Ind}_{B_n(k_v)}^{GL_n(k_v)}(\chi_{1,v} \otimes \ldots \otimes \chi_{n,v}) = \mathrm{Ind}_{Q_v}^{GL_n(k_v)}(\tau_{1,v} \otimes \ldots \otimes \tau_{r,v})$$

for an appropriate parabolic Q_v. Then this induced representation is actually an induced representation of Langlands type. As Jacquet observed in [20] Π_v is in fact the Langlands quotient of this representation. Since Π_v is generic we know by Jacquet and Shalika [26] that this induced representation of Langlands type must actually be irreducible and hence

$$\Pi_v = \mathrm{Ind}_{Q_v}^{GL_n(k_v)}(\tau_{1,v} \otimes \ldots \otimes \tau_{r,v}) = \mathrm{Ind}_{B_n(k_v)}^{GL_n(k_v)}(\chi_{1,v} \otimes \ldots \otimes \chi_{n,v}). \quad \square$$

Proof of the Proposition. — Since Π' is automorphic, then by Langlands [34] there exists a partition $(r_1, \ldots, r_m)$ of n and irreducible cuspidal representations σ_i of $GL_{r_i}(\mathbf{A})$ such that Π' is a subquotient of $\Xi = \mathrm{Ind}_{Q(\mathbf{A})}^{GL_n(\mathbf{A})}(\sigma_1 \otimes \ldots \otimes \sigma_m)$, where Q is the standard parabolic subgroup associated to the partition.

Let v be a non-archimedean place where Π'_v is both generic and unramified. By Lemma 10.3, there exist unramified characters $\chi_{1,v}, \ldots, \chi_{n,v}$ of $GL_1(k_v)$ such that $\Pi'_v = \mathrm{Ind}_{B_n(k_v)}^{GL_n(k_v)}(\chi_{1,v} \otimes \ldots \otimes \chi_{n,v})$. On the other hand, Π'_v is a generic unramified constituent of $\Xi_v = \mathrm{Ind}_{Q(k_v)}^{GL_n(k_v)}(\sigma_{1,v} \otimes \ldots \otimes \sigma_{m,v})$. By [3, Lemma 2.24] and Rodier [40], each $\sigma_{i,v}$ must be generic and unramified. By Lemma 10.3, each $\sigma_{i,v}$ must be fully induced from unramified characters of $B_{r_i}(k_v)$ and, by transitivity of unitary induction, there are unramified characters $\mu_{1,v}, \ldots, \mu_{n,v}$ of $GL_n(k_v)$ such that $\Xi_v = \mathrm{Ind}_{B_n(k_v)}^{GL_n(k_v)}(\mu_{1,v} \otimes \ldots \otimes \mu_{n,v})$. Since Π'_v and Ξ_v are both fully induced off the Borel and have a common constituent, namely Π'_v, by [4] they have the same Jordan-Hölder constituents. But Π'_v is irreducible. Hence, so is Ξ_v and $\Xi_v = \Pi'_v$. Since Ξ_v is now irreducible at almost all places and has a finite composition series at the remaining finite number of places, we see that the global representation Ξ will have a finite composition series and each composition factor will be admissible.

Using the theory of Einstein series, at least one constituent of Ξ embeds into the space $\mathscr{A}(GL_n(k)\backslash GL_n(\mathbf{A}))$ as a proper automorphic representation. In fact, Lemma 7 of Langlands [34] gives a non-zero intertwining of a subrepresentation of Ξ to the space of automorphic forms. Taking any irreducible submodule of the image gives a constituent of Ξ embedded as a proper automorphic representation. Let Π'' be this component. At all v where Ξ_v is irreducible, we must have $\Pi''_v = \Xi_v$. In particular $\Pi''_v = \Pi'_v$ at all non-archimedean v where Π'_v is generic and unramified. $\square$

This completes the proof of Corollary 1. $\square$

Proof of Corollary 2. — Take Π' from the conclusion of Theorem 2. As in the proof of Proposition 10.5 we have that there exists a partition $(r_1, \ldots, r_m)$ of n and irreducible cuspidal representations σ_i of $GL_{r_i}(\mathbf{A})$ such that Π' is a subquotient of

$$\Xi = \mathrm{Ind}_{Q(\mathbf{A})}^{GL_n(\mathbf{A})}(\sigma_1 \otimes \ldots \otimes \sigma_m).$$

By [25] the components $\Pi'_v \simeq \Pi_v$ for non-archimedean v where Π_v is unramified completely determine the partition and the σ_i, so this data is completely determined by Π. Furthermore, as in the proof of Proposition 10.5, at the places where Π_v is unramified Ξ_v is irreducible and $\Pi_v \simeq \Pi'_v \simeq \Xi_v$. Set $\Pi''_v = \Xi_v$ at these places.

Now consider any other non-archimedean place v. Since the σ_i are cuspidal, they are generic and the same is true of their local components. Hence at any finite place, Ξ_v has a unique generic constituent. Let Π''_v be this constituent.

At those places $v \in S$, let $\Pi''_v = \Pi'_v \simeq \Pi_v$. This is a generic constituent of Ξ_v.

Let $\Pi'' = \otimes \Pi''_v$. Then Π'' is the unique generic constituent of Ξ subject to $\Pi''_v \simeq \Pi_v$ for $v \in S$. By Langlands' result [34] Π'' is automorphic. This is the desired representation. □

11. A third converse theorem

In the next version of the converse theorem we relax the condition that Π be generic. The cost is that we can no longer guarantee that the automorphic representation Π' we produce agrees with Π at the places $v \in S$. We now repeat the statement, already given in Section 2, of the precise result:

Theorem 3. — Let $n \geqslant 3$. Let Π be an irreducible admissible representation of $GL_n(\mathbf{A})$ *whose central character ω_Π is invariant under $k^\times$ and whose L-function $L(\Pi, s)$ is absolutely convergent in some half-plane. Let S be a non-empty finite set of places of k, containing all archimedean places, such that the S-class number of k is one. Suppose that for every m with $1 \leqslant m \leqslant n-1$ and every $\tau \in \Omega^0_S(m)$ the L-function $L(\Pi \times \tau, s)$ is nice. Then there exists an irreducible automorphic representation Π' of* $GL_n(\mathbf{A})$ *such that $\Pi'_v \simeq \Pi_v$ for all non-archimedean places v where Π_v is unramified.*

This is only a mild modification of Theorem 2.

Proof. — For each v let Ξ_v be the representation of Langlands type having Π_v as its unique irreducible quotient. Each Ξ_v is of the form

$$\Xi_v = \mathrm{Ind}_{Q_v}^{GL_n(k_v)}(\rho_{1,v}\,|\;|^{u_{1,v}} \otimes \ldots \otimes \rho_{m_v,v}\,|\;|^{u_{m_v,v}})$$

where Q_v is a standard parabolic subgroup associated to a partition $(r_{1,v}, \ldots, r_{m_v,v})$ of n, $\rho_{i,v}$ is an irreducible tempered representation of $GL_{r_{i,v}}(k_v)$ and the $u_{i,v}$ are real numbers satisfying $u_{1,v} > \ldots > u_{m_v,v}$. As we noted in the proof of Theorem 1, each Ξ_v has the same central character as Π_v and each Ξ_v is an induced of Whittaker type and hence injects into its Whittaker model. By the local theory of L-functions for non-generic representations [24, 27] we have by definition

$$L(\Pi_v \times \tau_v, s) = L(\Xi_v \times \tau_v, s)$$

$$\varepsilon(\Pi_v \times \tau_v, s, \psi_v) = \varepsilon(\Xi_v \times \tau_v, s, \psi_v)$$

for all irreducible admissible τ_v of $GL_m(k_v)$ with $1 \leqslant m \leqslant n-1$.

Now if we form the representation $\Xi = \otimes \Xi_v$, then Ξ is a global induced representation of $GL_n(\mathbf{A})$ of Whittaker type having an automorphic central character and such that the L-function $L(\Xi \times \tau, s)$ is nice for every $\tau \in \Omega^0_S(m)$ with $1 \leqslant m \leqslant n-1$.

To proceed as in Theorem 2 and embed Ξ_S into a space of classical automorphic forms we need to choose a standard vector in each Ξ_v for all $v \notin S$. For each $v \notin S$ for which Π_v is unramified, Ξ_v must also be and it must have a unique $K_v = GL_n(\mathfrak{o}_v)$-fixed vector ξ_v^0 which projects to the distinguished K_v-fixed vector of Π_v. Since Π_v is the unique irreducible quotient, Ξ_v must be cyclic and generated by ξ_v^0. Since ξ_v^0 is the unique K_v-fixed vector in Ξ_v, it must transform by a character Λ_v under the local Hecke algebra $\mathscr{H}(GL_n(k_v), K_v)$ of compactly supported K_v-bi-invariant functions on $GL_n(k_v)$. Since the quotient map $\Xi_v \to \Pi_v$ is intertwining, the image of ξ_v^0 in Π_v will also transform by this character and Π_v is the unique irreducible unramified representation of $GL_n(k_v)$ associated to this character.

For the places v not in S where Π_v is not unramified, if we let $\mathscr{W}(\Xi_v, \psi)$ be the Whittaker model of Ξ_v then by Jacquet and Shalika [26] the restriction of the functions in $\mathscr{W}(\Xi_v, \psi_v)$ to the mirabolic $P_{n,v}$ contains all smooth functions on $P_{n,v}$ which are left quasi-invariant under $N_{n,v}$, i.e., the space of $\mathrm{Ind}_{N_{n,v}}^{P_{n,v}}(\psi_v)$. Choose a function W_v' which is fixed by $K_v \cap P_{n,v}$. The corresponding function W_v^0 in $\mathscr{W}(\Xi_v, \psi_v)$ will have a stabilizer containing $K_1(\mathfrak{p}_v^{m_v})$ for some $m_v \geqslant 0$. We take the corresponding vector ξ_v^0 as our standard vector at this place.

If we let $\xi^0 = \bigotimes_{v \notin S} \xi_v^0$, then $\xi^0 \in \Xi^S$ and ξ^0 is fixed by $K_1(\mathfrak{n})$ where $\mathfrak{n} = \prod_{v \in T'} \mathfrak{p}_v^{m_v}$. The argument of Section 8 still gives that ξ^0 transforms by the character χ_ω under $K_0(\mathfrak{n})$ even though Ξ is not irreducible since Ξ has central character $\omega = \omega_\Pi$.

We now proceed as in Theorem 2. For each $\xi_S \in \Xi_S$ we form the functions $U_{\xi_S \otimes \xi^0}(g)$ and $V_{\xi_S \otimes \xi^0}(g)$. From the methods of Theorem 1 and 2,

$$U_{\xi_S \otimes \xi^0}((g_S, 1^S)) = V_{\xi_S \otimes \xi^0}((g_S, 1^S)) = \Phi_{\xi_S}(g_S)$$

for $g_S \in G_S$ and the map $\xi_S \mapsto \Phi_{\xi_S}(g_S)$ embeds Ξ_S into $\mathscr{A}(\Gamma_0(\mathfrak{n})\backslash G_S; \omega_S, \chi_\omega^{-1})$. Since Ξ_S has Π_S as its unique irreducible quotient, if we take a vector $\xi_S \in \Xi_S$ which has a non-zero projection to Π_S then ξ_S must be a cyclic generator for Ξ_S. Hence the image of Ξ_S in $\mathscr{A}(\Gamma_0(\mathfrak{n})\backslash G_S; \omega_S, \chi_\omega^{-1})$ is cyclic with a generator f_0.

As noted before, for all places $v \notin T$ (as before T is the smallest set of places containing S outside of which Π_v is unramified) ξ_v^0 is a Hecke eigenvector. Hence our standard vector ξ^0 is an eigenvector for $\mathscr{H}(G^T, K^T)$ with eigencharacter $\Lambda = \bigotimes_{v \notin T} \Lambda_v$. Then Proposition 10.4 shows that for every $\xi_S \in \Xi_S$ the function Φ_{ξ_S} is a Hecke eigenfunction for $\mathscr{H}_c^T$ with eigencharacter Λ.

We now have that Ξ_S is a cyclic subrepresentation of $\mathscr{A}(\Gamma_0(\mathfrak{n})\backslash G_S; \omega_S, \chi_\omega^{-1})$ which consists of Hecke eigenvectors for the Hecke algebra $\mathscr{H}_c^T$ with eigencharacter Λ. Applying Theorem A of the appendix, we conclude that there exists an irreducible automorphic representation Π' of $GL_n(\mathbf{A})$ such that Π_S' is a constituent of Ξ_S and Π'^T is the unique representation of G^T with eigencharacter Λ. But as we have seen above, Π^T is also the unique representation of G^T with eigencharacter Λ. Therefore $\Pi_v' \simeq \Pi_v$ for all non-archimedean v where Π_v is unramified. □

12. Our final converse theorem

Theorems 2 and 3 have the drawback that the automorphic representation Π' associated to Π need not be cuspidal nor unique. However it is possible to associate to Π a unique collection of cuspidal representations σ_i on general linear groups $\mathrm{GL}_{r_i}(\mathbf{A})$ with $(r_1, \ldots, r_m)$ a partition of n.

Theorem 4. — Let Π be an irreducible admissible representation of $\mathrm{GL}_n(\mathbf{A})$ satisfying the hypotheses of Theorem 3. Then there exists a partition $(r_1, \ldots, r_m)$ of n and irreducible cuspidal representations σ_i of $\mathrm{GL}_{r_i}(\mathbf{A})$ such that for all non-archimedean places where Π_v is unramified we have that $\sigma_{i,v}$ is unramified and $\mathrm{L}(\Pi_v, s) = \prod_i \mathrm{L}(\sigma_{i,v}, s)$. Moreover, the sequence $(\sigma_1, \ldots, \sigma_m)$ is unique up to permutation.

Proof. — By either Theorem 2 or Theorem 3 we have associated to Π an automorphic representation Π' such that $\Pi_v \simeq \Pi'_v$ for all non-archimedean places v where Π_v is unramified. By Langlands [34] there is a partition $(r_1, \ldots, r_m)$ of n and irreducible cuspidal representations σ_i of $\mathrm{GL}_{r_i}(\mathbf{A})$ such that Π' is a constituent of $\Xi = \mathrm{Ind}_{\mathrm{Q}(\mathbf{A})}^{\mathrm{GL}_n(\mathbf{A})}(\sigma_1 \otimes \ldots \otimes \sigma_m)$ where Q is the standard parabolic associated to the partition. By Jacquet and Shalika [25] this sequence of cuspidal representations is unique up to ordering. Moreover, the sequence is uniquely determined by the unramified constituent of Ξ_v at those places where Ξ_v is unramified. But this unramified constituent is exactly Π_v. Hence the sequence $(\sigma_1, \ldots, \sigma_m)$ is uniquely determined by Π.

In computing the local L-factors, at the places where a representation is unramified, the local L-function can be computed from the unramified vector using Hecke theory. At those places where Ξ_v is unramified, each $\sigma_{i,v}$ must also be and Ξ_v has a unique unramified vector which in turn projects to the unramified vector in its unramified quotient, namely Π'_v. Thus when Π'_v is unramified

$$\mathrm{L}(\Pi'_v, s) = \mathrm{L}(\Xi_v, s) = \prod \mathrm{L}(\sigma_{i,v}, s).$$

But when Π_v is unramified, $\Pi_v \simeq \Pi'_v$. Thus

$$\mathrm{L}(\Pi_v, s) = \prod \mathrm{L}(\sigma_{i,v}, s)$$

as desired.

Next, suppose that $(r'_1, \ldots, r'_k)$ is another partition of n and $\sigma'_1, \ldots, \sigma'_k$ are cuspidal automorphic forms on the $\mathrm{GL}_{r'_i}(\mathbf{A})$ such that for all v where Π_v is unramified we have

$$\mathrm{L}(\Pi_v, s) = \prod \mathrm{L}(\sigma'_{i,v}, s) = \prod \mathrm{L}(\sigma_{i,v}, s).$$

Then let

$$\Xi' = \mathrm{Ind}_{\mathrm{Q}'(\mathbf{A})}^{\mathrm{GL}_n(\mathbf{A})}(\sigma'_1 \otimes \ldots \otimes \sigma'_k)$$

where Q′ is the standard parabolic subgroup associated to the partition $(r'_1, \ldots, r'_k)$. At the places v where Π_v is unramified we then have $L(\Xi_v, s) = L(\Xi'_v, s)$. For GL_n, the L-function of an unramified representation π_v completely determines the Satake parameter $t_{\pi_v} \in GL_n(\mathbf{C})$ of π_v since

$$L(\Pi_v, s) = \det(\mathbf{I}_n - t_{\pi_v} q_v^{-s})^{-1}.$$

Thus we see that for these places Ξ_v and Ξ'_v must have the same unramified constituent. Hence again Jacquet and Shalika [25] let us conclude that $n = m$, $r_i = r'_i$, and $\sigma_i \simeq \sigma'_i$ after reordering. □

APPENDIX

We retain the notation and conventions of Sections 7-10. In particular, k is a global field, S is a non-empty finite set of places of k containing all archimedean places, and $\mathfrak{o}_S$ is the ring of S-integers of k.

Let ω be a character of $\mathbf{A}^\times$ which is trivial on $k^\times$. Let $\mathscr{A}(\mathrm{GL}_n(k)\backslash\mathrm{GL}_n(\mathbf{A});\omega)$ denote the space of automorphic forms on $\mathrm{GL}_n(\mathbf{A})$ which transform under the center $\mathrm{Z}_n(\mathbf{A})$ by the character ω, i.e., $f(zg) = \omega(z)f(g)$ for $g \in \mathrm{GL}_n(\mathbf{A})$ and $z \in \mathrm{Z}_n(\mathbf{A})$. The purpose of this appendix is to explain the connection between a space of classical automorphic forms with Nebentypus $\mathscr{A}(\Gamma_0(\mathfrak{n})\backslash\mathrm{G}_S;\omega_S,\chi_\omega^{-1})$ and the subspace of the adelic automorphic forms $\mathscr{A}(\mathrm{GL}_n(k)\backslash\mathrm{GL}_n(\mathbf{A});\omega)$ which are fixed by $\mathrm{K}_1(\mathfrak{n})$. The automorphic forms in $\mathscr{A}(\Gamma_0(\mathfrak{n})\backslash\mathrm{G}_S;\omega_S,\chi_\omega^{-1})$ are analogous to the functions in $\mathscr{A}(\Gamma_0(\mathrm{N})\backslash\mathrm{SL}_2(\mathbf{R});\chi)$ which are obtained by lifting classical modular forms on the upper half-plane $\mathfrak{H}$ with respect to $\Gamma_0(\mathrm{N})$ and Nebentypus character χ to functions on the group $\mathrm{SL}_2(\mathbf{R})$. For this reason, we will refer to the functions in $\mathscr{A}(\Gamma_0(\mathfrak{n})\backslash\mathrm{G}_S;\omega_S,\chi_\omega^{-1})$ as "classical" automorphic forms. The functions in $\mathscr{A}(\mathrm{GL}_n(k)\backslash\mathrm{GL}_n(\mathbf{A});\omega)$ we will refer to as "adelic" automorphic forms. In the case of class number one fields, and forms without Nebentypus, this is explained in [6]. For the convenience of the reader, we recall the extensions to the S-arithmetic case, still assuming the S-class number is one.

1. Relation between automorphic forms

Assume that the S-class number of k is one. One consequence of this is that $\mathbf{A}^\times = k^\times k_S^\times \mathfrak{u}^S$. As a consequence of strong approximation for SL_n [30] and the fact that $\det(\mathrm{K}_1(\mathfrak{n})) = \mathfrak{u}^S$ we have that $\mathrm{GL}_n(\mathbf{A})$ may be decomposed as

$$\mathrm{GL}_n(\mathbf{A}) = \mathrm{GL}_n(k)\,\mathrm{G}_S\,\mathrm{K}_1(\mathfrak{n}) \tag{A.1}$$

as in [6]. Since $\mathrm{K}_1(\mathfrak{n}) \subset \mathrm{K}_0(\mathfrak{n})$ we also have

$$\mathrm{GL}_n(\mathbf{A}) = \mathrm{GL}_n(k)\,\mathrm{G}_S\,\mathrm{K}_0(\mathfrak{n}). \tag{A.2}$$

From the decomposition (A.1) we have

$$\mathscr{A}(\mathrm{GL}_n(k)\backslash\mathrm{GL}_n(\mathbf{A}))^{\mathrm{K}_1(\mathfrak{n})} \simeq \mathscr{A}(\Gamma_1(\mathfrak{n})\backslash\mathrm{G}_S) \tag{A.3}$$

where the isomorphism associates to each $\mathrm{K}_1(\mathfrak{n})$-invariant automorphic form f on $\mathrm{GL}_n(\mathbf{A})$ the classical form f_c given by $f_c(g_S) = f((g_S, 1^S))$.

For our purposes we need to keep track of the central character. Let us suppose that $\mathscr{A}(GL_n(k)\backslash GL_n(\mathbf{A}); \omega)^{K_1(\mathfrak{n})}$ is non-empty. Write $\mathfrak{n} = \prod_{v \notin S} \mathfrak{p}_v^{m_v}$ with $m_v \geqslant 0$ and $m_v = 0$ for almost all v. Let $T' = \{ v \mid m_v \neq 0 \}$ and $T = S \cup T'$. So $m_v = 0$ for $v \notin T$. Let $f(g)$ be a non-zero function in this space. If $v \notin T$ then for any local unit $u_v \in \mathfrak{o}_v^\times$ we have $u_v I_n \in GL_n(\mathfrak{o}_v) = K_{1,v}(\mathfrak{p}_v^{m_v})$ which is naturally embedded in $K_1(\mathfrak{n})$ and so $\omega_v(u_v) f(g) = f(u_v I_n g) = f(g)$ so that $\omega_v(u_v) = 1$. Similarly, if $v \in T'$ and u_v is a local unit of the form $1 + \mathfrak{p}_v^{m_v}$ then $\omega_v(u_v) = 1$. So ω_v is unramified at $v \in T$ and has conductor at least $\mathfrak{p}_v^{m_v}$ at the places $v \in T'$. Since $(\mathfrak{o}_S/\mathfrak{n})^\times \simeq \prod_{v \notin S}(\mathfrak{o}_v/\mathfrak{p}_n^{m_v})^\times$, ω defines a character χ_ω of $(\mathfrak{o}_S/\mathfrak{n})^\times$ via this isomorphism by $\chi_\omega = \prod_{v \notin S} \omega_v$.

The central character ω allows us to define a character $\chi_\omega = \prod \chi_v$ of $K_0(\mathfrak{n})$ as in Section 8. The construction there was not dependent on the space of $K_1(\mathfrak{n})$-fixed vectors being one, just on the existence of $K_1(\mathfrak{n})$-fixed vectors and the central character. Since the second construction of this character in Section 8 is through the character χ_ω of $(\mathfrak{o}_S/\mathfrak{n})^\times$, we see that χ_ω also defines a character of $\Gamma_0(\mathfrak{n})$ through the quotient map $\Gamma_0(\mathfrak{n})/\Gamma_1(\mathfrak{n}) \simeq (\mathfrak{o}_S/\mathfrak{n})^\times$ as in Section 10.

Now let $\mathscr{A}(\Gamma_0(\mathfrak{n})\backslash G_S; \omega_S, \chi_\omega^{-1})$ be the space of classical automorphic forms f_c on G_S satisfying

$$(1) \qquad f_c(\gamma g_S) = \chi_\omega^{-1}(\gamma) f_c(g_S) \quad \text{for } \gamma \in \Gamma_0(\mathfrak{n}) \subset G_S$$

$$(2) \qquad f_c(z_S\, g_S) = \omega_S(z_S) f_c(g_S) \quad \text{for } z_S \in Z_n(k_S) \simeq k_S^\times.$$

Then from the decomposition (A.2) we have

$$(A.4) \qquad \mathscr{A}(GL_n(k)\backslash GL_n(\mathbf{A}); \omega)^{K_1(\mathfrak{n})} \simeq \mathscr{A}(\Gamma_0(\mathfrak{n})\backslash G_S; \omega_S, \chi_\omega^{-1})$$

where the isomorphism associates to every $K_1(\mathfrak{n})$ invariant automorphic form f on $GL_n(\mathbf{A})$ the classical form f_c on G_S given by $f_c(g_S) = f((g_S, 1^S))$ and to a classical form f_c on G_S with Nebentypus character χ_ω^{-1} the adelic form given by $f(\gamma g_S k_0) = f_c(g_S)\, \chi_\omega(k_0)$ where $\gamma \in GL_n(\mathbf{A})$ and $k_0 \in K_0(\mathfrak{n})$ as in the decomposition in (A.2).

2. Comparison of Hecke algebras

Both the spaces in (A.3) and (A.4) have natural Hecke algebras which act on them. We will describe these algebras and compare their actions.

If G is any locally compact totally disconnected topological group and K is an open compact subgroup of G we will let $\mathscr{H}(G, K)$ denote the space of K-bi-invariant compactly supported functions on G. This space is an algebra under convolution: the *Hecke algebra of* G *with respect to* K.

The space $\mathscr{A}(GL_n(k)\backslash GL_n(\mathbf{A}))^{K_1(\mathfrak{n})}$ is most naturally a module for the Hecke algebra $\mathscr{H}(G^S, K_1(\mathfrak{n})) = \bigotimes_{v \notin S} \mathscr{H}(G_v, K_{1,v}(\mathfrak{p}_v^{m_v}))$ acting by right convolution. Since $K_1(\mathfrak{n})$ is not the maximal compact subgroup of G^S, the algebra $\mathscr{H}(G^S, K_1(\mathfrak{n}))$ is not necessarily commutative. However $\mathscr{H}(G^T, K^T) = \bigotimes_{v \notin T} \mathscr{H}(G_v, K_v)$ is commutative since $K^T = \prod_{v \notin T} K_v$ and K_v is the maximal compact subgroup at all places $v \notin T$. The algebra $\mathscr{H}(G^T, K^T)$ is naturally a subalgebra of $\mathscr{H}(G^S, K_1(\mathfrak{n}))$ by the embedding $\Phi \mapsto \Phi_{K_{T'}} \otimes \Phi$ where $\Phi_{K_{T'}}$ is the normalized characteristic function of $K_{T'} = \prod_{v \in T'} K_1(\mathfrak{p}_v^{m_v}) \subset G_{T'}$, i.e., the characteristic function of $K_{T'}$ divided by the volume of $K_{T'}$. Then $\mathscr{H}(G^T, K^T)$ is the subalgebra of functions whose support lies in $K_{T'}\, G^T$. For any $g \in G^S$ let Φ_g denote the characteristic function of the double coset $K_1(\mathfrak{n})\, g K_1(\mathfrak{n})$ divided by the volume of $K_1(\mathfrak{n})$.

The commutative algebra $\mathscr{H}(G^T, K^T)$ acts naturally on $\mathscr{A}(GL_n(k)\backslash GL_n(\mathbf{A}))^{K_1(\mathfrak{n})}$ by right convolution

$$(\Phi * \varphi)(g) = \int_{G^T} \Phi(h)\, \varphi(gh)\, dh$$

for $\Phi \in \mathscr{H}(G^T, K^T)$ and $\varphi \in \mathscr{A}(GL_n(k)\backslash GL_n(\mathbf{A}))^{K_1(\mathfrak{n})}$.

When keeping track of the actions of $K_0(\mathfrak{n})$ and $\Gamma_0(\mathfrak{n})$ it is most convenient to use an intermediate algebra between $\mathscr{H}(G^S, K_1(\mathfrak{n}))$ and $\mathscr{H}(G^T, K^T)$. Let us set

$$G^S(\mathfrak{n}) = \Big(\prod_{v \in T'} K_{0,v}(\mathfrak{p}_v^{m_v})\Big)\, G^T.$$

Then $G^S \supset G^S(\mathfrak{n}) \supset G^T$ and $G^S(\mathfrak{n}) \supset K_0(\mathfrak{n}) \supset K_1(\mathfrak{n})$. Since $K_0(\mathfrak{n})$ normalizes $K_1(\mathfrak{n})$ and has as quotient $K_0(\mathfrak{n})/K_1(\mathfrak{n}) \simeq \prod_{v \in T'} K_0(\mathfrak{p}_v^{m_v})/K_1(\mathfrak{p}_v^{m_v}) \simeq \prod_{v \in T'} (\mathfrak{o}_v/\mathfrak{p}_v^{m_v})^\times \simeq (\mathfrak{o}_S/\mathfrak{n})^\times$ we see that the double coset algebra $\mathscr{H}(G^S(\mathfrak{n}), K_1(\mathfrak{n}))$ is naturally isomorphic to $\mathbf{C}[(\mathfrak{o}_S/\mathfrak{n})^\times] \otimes \mathscr{H}(G^T, K^T)$ and is therefore again commutative and contains $\mathscr{H}(G^T, K^T)$. Let us set $\mathscr{H}(\mathfrak{n}) = \mathscr{H}(G^S(\mathfrak{n}), K_1(\mathfrak{n}))$.

As before, the commutative algebra $\mathscr{H}(\mathfrak{n})$ acts on $\mathscr{A}(GL_n(k)\backslash GL_n(\mathbf{A}); \omega)^{K_1(\mathfrak{n})}$ by right convolution

$$(\Phi * \varphi)(g) = \int_{G^S(\mathfrak{n})} \Phi(h)\, \varphi(gh)\, dh$$

for $\Phi \in \mathscr{H}(\mathfrak{n})$ and $\varphi \in \mathscr{A}(GL_n(k)\backslash GL_n(\mathbf{A}); \omega)^{K_1(\mathfrak{n})}$.

There is a corresponding classical Hecke algebra, which we will denote by $\mathscr{H}_c(\mathfrak{n})$, which acts on $\mathscr{A}(\Gamma_0(\mathfrak{n})\backslash G_S; \omega_S, \chi_\omega^{-1})$. To describe $\mathscr{H}_c(\mathfrak{n})$, let $M = GL_n(k) \cap G_S\, G^S(\mathfrak{n})$. The group M consists of those rational matrices $\gamma \in GL_n(k)$ such that for all $v \in T'$ the v-component γ_v lies in $K_{0,v}(\mathfrak{p}_v^{m_v})$. We may view M as a subgroup of $GL_n(k)$ and hence of both G_S and G^S. Then $\Gamma_1(\mathfrak{n}) = GL_n(k) \cap G_S\, K_1(\mathfrak{n}) \subset M$.

Lemma **A.1.** — *The map* $\alpha : \Gamma_1(\mathfrak{n})\backslash M/\Gamma_1(\mathfrak{n}) \to K_1(\mathfrak{n})\backslash G^S/K_1(\mathfrak{n})$ *which is given by* $\Gamma_1(\mathfrak{n})\, t\Gamma_1(\mathfrak{n}) \mapsto K_1(\mathfrak{n})\, tK_1(\mathfrak{n})$ *is injective with image* $K_1(\mathfrak{n})\backslash G^S(\mathfrak{n})/K_1(\mathfrak{n})$. *Furthermore, if we have the decomposition into right cosets* $\Gamma_1(\mathfrak{n})\, t\Gamma_1(\mathfrak{n}) = \coprod a_j\, \Gamma_1(\mathfrak{n})$ *then also* $K_1(\mathfrak{n})\, tK_1(\mathfrak{n}) = \coprod a_j\, K_1(\mathfrak{n})$.

Proof. — This argument is modeled after that of Shimura [41], who proved a similar result for GL_2.

Let V be the n-dimensional vector space on which $GL_n(k)$ acts having basis $\{e_1, \ldots, e_n\}$ with respect to which K and $K_1(\mathfrak{n})$ are defined. If Λ is the space of $\mathfrak{o}_S$-lattices in V then there is a natural action of both $GL_n(k)$ and G^S on Λ [37]. Let $L_0 = \mathfrak{o}_S e_1 + \ldots + \mathfrak{o}_S e_n$ be the free $\mathfrak{o}_S$-lattice such that $GL_n(\mathfrak{o}^S)$ is the stabilizer of L_0 in G^S and $GL_n(\mathfrak{o}_S)$ is the stabilizer of L_0 in $GL_n(k)$. Set $L_1 = \mathfrak{o}_S e_1 + \ldots + \mathfrak{o}_S e_{n-1} + \mathfrak{n}e_n$, so that $K_0(\mathfrak{n})$ is the set of $g \in G^S$ such that $gL_0 = L_0$ and $gL_1 = L_1$, and $K_1(\mathfrak{n})$ is the subgroup of elements g which act trivially on L_0/L_1. We define $\Gamma_1(\mathfrak{n})$ in $GL_n(k)$ by the same conditions.

Let $u \in K_1(\mathfrak{n})$ and $t \in M$. We first claim that there exists $\gamma \in \Gamma_1(\mathfrak{n})$ and $u_1 \in K_1(\mathfrak{n})$ such that $ut = \gamma t u_1$. To see this, consider the lattices $L_2 = tL_0$ and $L_3 = utL_0$. After scaling by an element of $Z_n(k)$ if necessary, which will not effect our conclusion, we may assume $L_2, L_3 \subset L_0$. By the theory of invariant factors, Theorem 81.11 of [37], there exists a basis $x_1, \ldots, x_n$ of V and $\mathfrak{o}_S$-ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ and fractional ideals $\mathfrak{b}_1, \ldots, \mathfrak{b}_n$ such that

$$L_0 = \mathfrak{a}_1 x_1 + \ldots + \mathfrak{a}_n x_n$$

$$L_2 = \mathfrak{a}_1 \mathfrak{b}_1 x_1 + \ldots + \mathfrak{a}_n \mathfrak{b}_n x_n$$

and $L_3 = uL_2$.

Consider a place $v \in T'$. Then $u_v \in K_{1,v}(\mathfrak{p}_v^{m_v})$ and $t_v \in K_{0,v}(\mathfrak{p}_v^{m_v})$. Therefore $L_{0,v} = L_{2,v} = L_{3,v}$. Hence $u_v^1 L_{2,v} = L_{3,v}$ with $u_v^1 = 1_v$.

Consider a place $v \notin T$. Then $L_{3,v} = u_v L_{2,v} = u_v(\Sigma(\mathfrak{a}_i)_v (\mathfrak{b}_i)_v x_i)$. Now write $u_v = u_v^1 d_v$ where $\det(u_v^1) = 1$ and d_v is the diagonal matrix $\mathrm{diag}(\det(u_v), 1, \ldots, 1)$ with respect to the basis $\{x_1, \ldots, x_n\}$. Then $d_v L_{0,v} = L_{0,v}$ so that $d_v \in K_v$ and hence $u_v^1 \in K_v$. Also $u_v L_{2,v} = u_v^1 d_v(\Sigma(\mathfrak{a}_i)_v (\mathfrak{b}_i)_v x_i) = u_v^1 L_{2,v}$. Hence $u_v^1 L_{2,v} = L_{3,v}$.

Let $u^1 = \Pi u_v^1$. Then $u^1 \in K_1(\mathfrak{n}) \cap SL_n(\mathbf{A}^S)$ is such that $u^1 L_2 = L_3$. Let $c \in k^\times$ be such that $L_1, L_2, L_3 \supset cL_0$. Then by strong approximation for SL_n there exists $\gamma \in SL_n(k)$ such that $\gamma \equiv u^1 \pmod{c\mathfrak{o}_S}$. Then $\gamma L_2 = L_3$, $\gamma L_0 = L_0$, and since $\gamma_v \equiv 1 \pmod{c\mathfrak{o}_v}$ for $v \in T'$, $\gamma_v L_{1,v} = L_{1,v}$ and γ acts trivially on L_0/L_1. Hence $\gamma \in SL_n(k) \cap G_S K_1(\mathfrak{n}) \subset \Gamma_1(\mathfrak{n})$. We now have $\gamma t L_0 = \gamma L_2 = L_3 = utL_0$. Hence there exists u_1 in the stabilizer of L_0 in G^S, namely K^S, such that $ut = \gamma t u_1$. Since t, u, and γ are all in $K_0(\mathfrak{n})$, we must have $u_1 \in K_0(\mathfrak{n})$ as well. However, since $u_1 = t^{-1}\gamma^{-1} ut$ we see that u_1 acts trivially on L_0/L_1 and so $u_1 \in K_1(\mathfrak{n})$. Thus $ut = \gamma t u_1$ with $\gamma \in \Gamma_1(\mathfrak{n})$ and $u_1 \in K_1(\mathfrak{n})$.

We are now ready to prove injectivity. Let $t_1, t_2 \in M$ be such that

$$K_1(\mathfrak{n})\, t_1\, K_1(\mathfrak{n}) = K_1(\mathfrak{n})\, t_2\, K_1(\mathfrak{n}).$$

Then there exist $u_1, u_2 \in K_1(\mathfrak{n})$ such that $u_1 t_1 = t_2 u_2$. Write $u_1 t_1 = \gamma_1 t_1 u_3$ with $\gamma_1 \in \Gamma_1(\mathfrak{n})$ and $u_3 \in K_1(\mathfrak{n})$. Then $\gamma_1 t_1 u_3 = t_2 u_2$. Hence $t_2^{-1} \gamma t_1 \in GL_n(k) \cap G_S K_1(\mathfrak{n}) = \Gamma_1(\mathfrak{n})$. Thus $\gamma_1 t_1 = t_2 \gamma_2$ and $\Gamma_1(\mathfrak{n})\, t_1\, \Gamma_1(\mathfrak{n}) = \Gamma_1(\mathfrak{n})\, t_2\, \Gamma_1(\mathfrak{n})$. Thus the map α is injective. The fact that the image is $K_1(\mathfrak{n})\backslash G^S(\mathfrak{n})/K_1(\mathfrak{n})$ is clear.

Now suppose that for $t \in M$ we have $\Gamma_1(\mathfrak{n})\, t\Gamma_1(\mathfrak{n}) = \coprod_{j=1}^{m} a_j\, \Gamma_1(\mathfrak{n})$. Multiplying by $K_1(\mathfrak{n})$ on the right we have $\Gamma_1(\mathfrak{n})\, tK_1(\mathfrak{n}) = \bigcup a_j\, K_1(\mathfrak{n})$ and it is easy to see that $a_i\, K_1(\mathfrak{n}) = a_j\, K_1(\mathfrak{n})$ implies that $a_i\, \Gamma_1(\mathfrak{n}) = a_j\, \Gamma_1(\mathfrak{n})$ so that the right hand side is a disjoint union. But as we have seen above, any ut with $u \in K_1(\mathfrak{n})$ and $t \in M$ can be written as $ut = \gamma t u_1$ with $\gamma \in \Gamma_1(\mathfrak{n})$ and $u_1 \in K_1(\mathfrak{n})$. Hence $K_1(\mathfrak{n})\, tK_1(\mathfrak{n}) = \Gamma_1(\mathfrak{n})\, tK_1(\mathfrak{n})$. This finishes the proof. □

We will also need the following result on the choice of coset representatives.

Lemma **A.2.** — *For* $t \in M$ *there exists a decomposition* $\Gamma_1(\mathfrak{n})\, t\Gamma_1(\mathfrak{n}) = \coprod a_j\, \Gamma_1(\mathfrak{n})$ *with* $a_j \in P'_n(k)\, \Gamma_0(\mathfrak{n})$.

Proof. — This lemma is a consequence of the class number one assumption.

To better illustrate this, let us first consider the case where there is no level, so $\mathfrak{n} = \mathfrak{o}_S$ and $\Gamma = \Gamma_1(\mathfrak{n}) = \Gamma_0(\mathfrak{n}) = GL_n(\mathfrak{o}_S)$, and remove the class number assumption for the moment. Then $M = GL_n(k)$. We claim that $| P'_n(k)\backslash GL_n(k)/\Gamma | = h_S$.

To prove this, let us first recall some facts about the classification of lattices over the Dedekind domain $\mathfrak{o}_S$ [28]. If L is a $\mathfrak{o}_S$-lattice of rank n then L has the form

$$L = \mathfrak{a}_1 x_1 + \dots + \mathfrak{a}_n x_n$$

with $\mathfrak{a}_i$ fractional $\mathfrak{o}_S$-ideals. The group $GL_n(k)$ acts on these lattices and this action has a complete invariant, namely the Steinitz invariant

$$\mathrm{St}(L) = \mathrm{cl}(\mathfrak{a}_1 \dots \mathfrak{a}_n)$$

where $\mathrm{cl}(\mathfrak{b})$ represents the ideal class of the fractional ideal $\mathfrak{b}$. So, given two rank n lattices L_1 and L_2 there exists an element $\gamma \in GL_n(k)$ such that $\gamma L_1 = L_2$ if and only if $\mathrm{St}(L_1) = \mathrm{St}(L_2)$ [28, Theorem 10.14]. Since $\Gamma = GL_n(\mathfrak{o}_S)$ is the stabilizer in $GL_n(k)$ of the standard lattice

$$L_0 = \mathfrak{o}_S\, e_1 + \dots + \mathfrak{o}_S\, e_n$$

then the set $GL_n(k)/\Gamma$ is in one-to-one correspondence with the set Λ_0 of all rank n $\mathfrak{o}_S$-lattices with trivial Steinitz invariant.

Now consider the action of $P'_n(k)$ on the space Λ_0. Geometrically $P'_n(k)$ is the subgroup of $GL_n(k)$ which preserves the subspace $\langle e_1, \dots, e_{n-1} \rangle$ spanned by the first $n-1$ of the standard basis vectors. It has the structure of a semi-direct product of $GL_{n-1}(k) \times GL_1(k)$ acting on k^{n-1}. If $L_1 \in \Lambda_0$ then we may associate to L_1 the rank $n-1$ sublattice $L'_1 = L_1 \cap \langle e_1, \dots, e_{n-1} \rangle$. We claim that the Steinitz invariant of L'_1, i.e., $\mathrm{St}(L_1 \cap \langle e_1, \dots, e_{n-1} \rangle)$ is a complete invariant of the action of $P'_n(k)$ on Λ_0.

Suppose that $L_1, L_2 \in \Lambda_0$ and $L_1 = pL_2$ with $p \in P'_n(k)$. Let

$$L'_i = L_i \cap \langle e_1, \dots, e_{n-1} \rangle.$$

Then by Theorem 81.3 of O'Meara [37] there exists $y_i = \Sigma a_{i,j} e_j$ with $a_{i,n} \neq 0$ and fractional ideals $\mathfrak{a}_i$ such that

$$\text{(A.5)} \qquad L_i = L_i' + \mathfrak{a}_i y_i.$$

Let the action of p on $\langle e_1, \dots, e_{n-1} \rangle$ be given by the element $A \in GL_{n-1}(k)$ then from $L_1 = pL_2$ we find $L_1' + \mathfrak{a}_1 y_1 = AL_2' + \mathfrak{a}_2 p y_2$. Since neither y_1 nor py_2 lie in $\langle e_1, \dots, e_{n-1} \rangle$ we find

$$pL_2 \cap \langle e_1, \dots, e_{n-1} \rangle = L_1 \cap \langle e_1, \dots, e_{n-1} \rangle = L_1' = AL_2'$$

and hence $St(L_2 \cap \langle e_1, \dots, e_{n-1} \rangle) = St(L_2') = St(AL_2') = St(pL_2 \cap \langle e_1, \dots, e_{n-1} \rangle)$. Hence $St(L \cap \langle e_1, \dots, e_{n-1} \rangle)$ is a $P_n'(k)$ orbit invariant.

Now suppose that $L_1, L_2 \in \Lambda_0$ are such that

$$St(L_1 \cap \langle e_1, \dots, e_{n-1} \rangle) = St(L_2 \cap \langle e_1, \dots, e_{n-1} \rangle).$$

Let $L_i' = L_i \cap \langle e_1, \dots, e_{n-1} \rangle$. Then there exists $A \in GL_{n-1}(k)$ such that $L_1' = AL_2'$. Write each L_i as $L_i = L_i' + \mathfrak{a}_i y_i$ as in (A.5). Since $St(L_1) = St(L_2)$ and $St(L_1') = St(L_2')$ we see that $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are in the same ideal class. So, modifying y_2 by a non-zero scalar if necessary, we may assume $L_i = L_i' + \mathfrak{a} y_i$. Since each $y_i = \Sigma a_{i,j} e_j$ with $a_{i,n} \neq 0$ we may solve the equation

$$\begin{pmatrix} A & b \\ 0 & d \end{pmatrix} y_2 = y_1$$

(with $A \in GL_{n-1}(k)$ as above) for the $(n-1) \times 1$ vector b and the non-zero scalar d. Then $p = \begin{pmatrix} A & b \\ 0 & d \end{pmatrix} \in P_n'(k)$ and $pL_2 = L_1$, so that L_1 and L_2 lie in the same $P_n'(k)$-orbit.

Hence $St(L \cap \langle e_1, \dots, e_{n-1} \rangle)$ is a complete invariant for the action of $P_n(k)$ on $\Lambda_0 \simeq GL_n(k)/\Gamma$. Since this invariant can take on any ideal class as a value, we see that $| P_n'(k) \backslash GL_n(k)/\Gamma | = h_S$ as desired. In the case $n = 2$ this is the usual proof that the number of cusps for the full Hilbert modular group is equal to the class number of the underlying field.

Now let us return to the class number one case, i.e., we again assume $h_S = 1$. Then the above argument gives that $GL_n(k) = P_n'(k)\, GL_n(\mathfrak{o}_S)$ which implies the lemma when there is no level $\mathfrak{n}$. In the case of level, we claim that $M = (P_n'(k) \cap M)\, \Gamma_0(\mathfrak{n})$ from which the lemma follows. Of course, we have $M \supset (P_n'(k) \cap M)\, \Gamma_0(\mathfrak{n})$ so we need only prove the opposite inclusion. Let $m \in M$. Since $h_S = 1$ we may write $m = p\gamma$ with $p \in P_n'(k)$ and $\gamma \in GL_n(\mathfrak{o}_S)$. View $GL_n(\mathfrak{o}_S)$ as $GL_n(k) \cap G_S K^S$. Then for $v \in T'$ we have $m_v = p_v \gamma_v$ or $p_v = m_v^{-1} \gamma_v$. Since $m_v \in K_{0,v}(\mathfrak{p}_v^{m_v})$ and $\gamma_v \in GL_n(\mathfrak{o}_v)$ we see that $p_v \in GL_n(\mathfrak{o}_v)$ at these places. But $P_n'(k_v) \cap GL_n(\mathfrak{o}_v) \subset K_{0,v}(\mathfrak{p}_v^{m_v})$. Hence $p_v \in K_{0,v}(\mathfrak{p}_v^{m_v})$ for $v \in T'$ and $p \in P_n'(k) \cap M$. Now consider γ. Since $\gamma \in GL_n(\mathfrak{o}_S) = GL_n(k) \cap G_S K^S$ we know that for all $v \notin S$ we have $\gamma_v \in K_v$. Now if $v \in T'$ we have $\gamma_v = p_v^{-1} m_v \in K_{0,v}(\mathfrak{p}_v^{m_v})$.

Hence $\gamma \in \mathrm{GL}_n(k) \cap \mathrm{G_S}\, \mathrm{K}_0(\mathfrak{n}) = \Gamma_0(\mathfrak{n})$. Thus $\mathrm{M} = (\mathrm{P}'_n(k) \cap \mathrm{M})\, \Gamma_0(\mathfrak{n})$. This then proves the lemma for $h_{\mathrm{S}} = 1$. □

Let $\mathscr{H}_c(\mathfrak{n})$ denote the $\mathbf{C}$-span of the double cosets $\Gamma_1(\mathfrak{n})\backslash \mathrm{M}/\Gamma_1(\mathfrak{n})$. Then the map α induces a $\mathbf{C}$-linear bijection $\alpha : \mathscr{H}_c(\mathfrak{n}) \to \mathscr{H}(\mathfrak{n})$ which takes the double coset $\Gamma_1(\mathfrak{n})\, t\Gamma_1(\mathfrak{n})$ to the normalized characteristic function Φ_t. The algebra structure on $\mathscr{H}_c(\mathfrak{n})$ is the pull back of that of $\mathscr{H}(\mathfrak{n})$ via α. If $\Gamma_1(\mathfrak{n})\, t\Gamma_1(\mathfrak{n}) \in \mathscr{H}_c(\mathfrak{n})$ then the classical Hecke operator $\mathscr{E}_t$ acting on $\mathscr{A}(\Gamma_0(\mathfrak{n})\backslash \mathrm{G_S}\,;\, \omega_s, \chi_\omega^{-1})$ is defined as follows. For $f \in \mathscr{A}(\Gamma_0(\mathfrak{n})\backslash \mathrm{G_S}\,;\, \omega_s, \chi_\omega^{-1})$ and $\Gamma_1(\mathfrak{n})\, t\Gamma_1(\mathfrak{n}) = \coprod a_j\, \Gamma_1(\mathfrak{n})$ the action is

$$(\mathscr{E}_t f)\,(g_{\mathrm{S}}) = \sum_j f(a_j^{-1}\, g_{\mathrm{S}}).$$

If we recall that $\mathscr{H}(\mathfrak{n})$ acts on $\mathscr{A}(\mathrm{GL}_n(k)\backslash \mathrm{GL}_n(\mathbf{A})\,;\, \omega)^{\mathrm{K}_1(\mathfrak{n})}$ by convolution

$$(\Phi * \varphi)\,(g) = \int_{\mathrm{G^S}(\mathfrak{n})} \Phi(h)\, \varphi(gh)\, dh$$

for $\Phi \in \mathscr{H}(\mathfrak{n})$ and $\varphi \in \mathscr{A}(\mathrm{GL}_n(k)\backslash \mathrm{GL}_n(\mathbf{A})\,;\, \omega)^{\mathrm{K}_1(\mathfrak{n})}$, then we have the following result.

Proposition **A.1.** — *The bijection*

$$\mathscr{A}(\mathrm{GL}_n(k)\backslash \mathrm{GL}_n(\mathbf{A})\,;\, \omega)^{\mathrm{K}_1(\mathfrak{n})} \xrightarrow{\sim} \mathscr{A}(\Gamma_0(\mathfrak{n})\backslash \mathrm{G_S}\,;\, \omega_{\mathrm{S}}, \chi_\omega^{-1})$$

given in (A.4) *is an isomorphism of Hecke modules under the identification of algebras given by* $\alpha^{-1} : \mathscr{H}(\mathfrak{n}) \xrightarrow{\sim} \mathscr{H}_c(\mathfrak{n})$.

3. Comparison of automorphic representations

We would now like to compare certain automorphic subrepresentations of $\mathscr{A}(\Gamma_0(\mathfrak{n})\backslash \mathrm{G_S}\,;\, \omega_{\mathrm{S}}, \chi_\omega^{-1})$ which consist of Hecke eigenfunctions for the subalgebra $\mathscr{H}_c^{\mathrm{T}}$ of $\mathscr{H}_c(\mathfrak{n})$ which corresponds via α with the subalgebra $\mathscr{H}(\mathrm{G^T}, \mathrm{K^T})$ of $\mathscr{H}(\mathfrak{n})$, with the representations they generate in $\mathscr{A}(\mathrm{GL}_n(k)\backslash \mathrm{GL}_n(\mathbf{A})\,;\, \omega)$. For the sake of envisioned applications we work in the context of cyclic representations rather than irreducible ones.

Theorem A. — *Let* Π_{S} *be a cyclic automorphic subrepresentation of* $\mathscr{A}(\Gamma_0(\mathfrak{n})\backslash \mathrm{G_S}\,;\, \omega, \chi_\omega^{-1})$ *which consists of Hecke eigenvectors for* $\mathscr{H}_c^{\mathrm{T}} \simeq \mathscr{H}(\mathrm{G^T}, \mathrm{K^T})$ *with eigencharacter* Λ. *Then there exists an irreducible automorphic representation* Π' *of* $\mathrm{GL}_n(\mathbf{A})$ *such that* Π'_{S} *is a constituent of* Π_{S} *and* Π'^{T} *is the unique irreducible representation of* $\mathrm{G^T}$ *with eigencharacter* Λ.

Proof. — Using the isomorphism of (A.4) we may embed Π_{S} as a $\mathrm{G_S}$-invariant subspace of $\mathscr{A}(\mathrm{GL}_n(k)\backslash \mathrm{GL}_n(\mathbf{A})\,;\, \omega)$ consisting of $\mathrm{K}_1(\mathfrak{n})$-fixed vectors. Let f_0 be a cyclic generator of Π_{S} in $\mathscr{A}(\mathrm{GL}_n(k)\backslash \mathrm{GL}_n(\mathbf{A})\,;\, \omega)$. Let (Π_1, V_1) be the $\mathrm{GL}_n(\mathbf{A})$ subrepresentation generated by Π_{S}. Then Π_1 will also be cyclic, generated by f_0.

Let Π^{T} be the unique irreducible admissible $\mathrm{G^T}$-module associated to the character Λ of $\mathscr{H}(\mathrm{G^T}, \mathrm{K^T})$ [3].

Since f_0 is a Hecke eigenfunction for $\mathscr{H}_c^T$ with eigenfunctional Λ, then as an element of $\mathscr{A}(GL_n(k)\backslash GL_n(\mathbf{A});\omega)$ it is an eigenfunction for $\mathscr{H}(G^T, K^T)$ as well.

Let U be a maximal $GL_n(\mathbf{A})$-invariant subspace of V_1 not containing f_0 (such a U exists by Zorn's lemma). Then V_1/U is a non-zero irreducible subquotient of the space of automorphic forms and hence admissible by [6], paragraphs 4.5 and 4.6. Call the representation of $GL_n(\mathbf{A})$ on this quotient Π'. Then Π' is an irreducible automorphic representation and $\Pi' = \otimes \Pi'_v$. Since Π' is irreducible and contains a K^T fixed vector with eigenfunctional Λ, namely the image of f_0, we see that $\Pi'^T \simeq \Pi^T$. Now consider Π'_S. Since the map $V_1 \to \Pi'$ is intertwining, we see that Π'_S is an irreducible quotient of V_1. Since Π_1 was generated by the G_S module Π_S, Π'_S must be isomorphic to an irreducible constituent of Π_S. □

REFERENCES

[1] H. Bass, K-theory and stable algebra, *Publ. Math. IHES*, **22** (1964), 5-60.

[2] H. Bass, J. Milnor, and J.-P. Serre, Solution of the congruence subgroup problem for $SL_n(n \geqslant 3)$ and $Sp_{2n}(n \geqslant 2)$, *Publ. Math. IHES*, **33** (1967), 59-137.

[3] J. N. Bernstein and A. V. Zelevinsky, Representations of the group GL(n, F) where F is a non-archimedean local field, *Russian Math. Surveys*, **31:3** (1976), 1-68.

[4] J. N. Bernstein and A. V. Zelevinsky, Induced representations of reductive $\mathfrak{p}$-adic groups, I, *Ann. scient. Éc. Norm. Sup.*, 4e série, **10** (1977), 441-472.

[5] A. Borel, Automorphic L-functions, *Proc. Symp. Pure Math.*, **33**, part 2 (1979), 27-61.

[6] A. Borel and H. Jacquet, Automorphic forms and representations, *Proc. Symp. Pure Math.*, **33**, part 1 (1979), 189-202.

[7] A. Borel and N. Wallach, Continuous Cohomology, Discrete Subgroups, and Representations of Reductive groups, *Annals of Mathematics Study*, No. 94, Princeton, Princeton University Press, 1980.

[8] P. Cartier, Representations of $\mathfrak{p}$-adic groups: a survey, *Proc. Symp. Pure math.*, **33**, part 1 (1979), 111-155.

[9] W. Casselman, *Introduction to the theory of admissible representations of $\mathfrak{p}$-adic reductive groups*, manuscript, 1975,

[10] W. Casselman, Canonical extensions of Harish Chandra modules, *Can. J. Math.*, **XLI** (1989), 385-438.

[11] L. Clozel, Représentations galoisiennes associées aux représentations automorphes autoduales de GL(n), *Publ. Math. IHES*, **73** (1991), 97-145.

[12] D. Flath, Decomposition of representations into tensor products, *Proc. Symp. Pure Math.*, **33**, part 1 (1979). 179-183.

[13] I. M. Gelfand, M. I. Graev and I. I. Piatetski-Shapiro, *Representation Theory and Automorphic Functions*, Boston, Academic Press, 1990.

[14] R. Godement, Introduction à la théorie de Langlands, *Séminaire Bourbaki*, 1966-1967, No. 321.

[15] H. Hamburger, Über die Riemannsche Funktionalgleichung der ζ-Funktion, *Math. Zeit.*, **10** (1921), 240-254; **11** (1922), 224-245; **13** (1922), 283-311.

[16] G. Harder, Minkowskische Reductionstheorie über Funktionenköpern, *Inv. Math.*, **7** (1969), 33-54.

[17] E. Hecke, Über die Bestimung Dirichletscher Reihen durch ihre Funktionalgleichung, *Math. Ann.*, **112** (1936), 664-699.

[18] E. Hecke, *Mathematische Werke*, Göttingen, Vandenhoeck und Ruprecht, 1959.

[19] H. Jacquet, Generic representations, in *Non-commutative Harmonic Analysis*, Lecture Notes in Mathematics No. 587, Berlin-Heidelberg-New York, Springer Verlag, 1977, p. 91-101.

[20] H. Jacquet, Principal L-functions of the linear group, *Proc. Symp. Pure Math.*, **33**, part 1 (1979), 111-155.

[21] H. Jacquet and R. P. Langlands, *Automorphic Forms on* GL(2), Lecture Notes in Mathematics No. 114, Berlin-Heidelberg-New York, Springer Verlag, 1970.

[22] H. Jacquet, I. I. Piatetski-Shapiro and J. Shalika, Automorphic forms on GL(3), I & II, *Ann. of Math.*, **109** (1979), 169-258.

[23] H. Jacquet, I. I. Piatetski-Shapiro, and J. Shalika, Conducteur des représentations du groupe linéaire, *Math. Ann.*, **256** (1981), 199-214.

[24] H. Jacquet, I. I. Piatetski-Shapiro and J. Shalika, Rankin-Selberg Convolutions, *Am. J. Math.*, **105** (1983), 367-464.

[25] H. Jacquet and J. Shalika, On Euler products and the classification of automorphic representations, I, *Am. J. Math.*, **103** (1981), 449-558; II, **103** (1981), 777-815.

[26] H. Jacquet and J. Shalika, The Whittaker models of induced representations, *Pacific J. Math.*, **109** (1983), 107-120.

[27] H. Jacquet and J. Shalika, Rankin-Selberg Convolutions: Archimedean Theory, *Festschrift in Honor of I. I. Piatetski-Shapiro*, part I, Rehovot, Weizmann Science Press, 1990, p. 125-207.

[28] N. Jacobson, *Basic Algebra II*, San Francisco, W. H. Freeman and Co., 1980.

[29] A. Knapp, Local Langlands correspondence: Archimedean case, *Proc. Symp. Pure Math.*, **55**, part 2 (1994), 393-410.

[30] M. Kneser, Strong approximation, *Proc. Symp. Pure Math.*, **9** (1966), 187-196.

[31] B. Kostant, On Whittaker vectors and representation theory, *Inv. Math.*, **48** (1978), 101-184.

[32] J.-P. Labesse and J. Schwermer, On liftings and cusp cohomology of arithmetic groups, *Inv. Math.*, **83** (1986), 383-401.

[33] R. P. Langlands, *On the Functional Equation Satisfied by Eisenstein Series*, Lecture Notes in Mathematics No. 544, New York, Springer Verlag, 1976.

[34] R. P. Langlands, On the notion of an automorphic representation, *Proc. Symp. Pure Math.*, **33**, part 1 (1979), 203-207.

[35] H. Maass, Über eine neue Art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen, *Math. Ann.*, **121** (1944), 141-183.

[36] C. Moeglin and J.-L. Waldspurger, *Décomposition spectrale et Séries d'Einstein. Une paraphrase de l'écriture*, Boston, Birkhäuser, 1994.

[37] O. T. O'Meara, *Introduction to Quadratic Forms*, Berlin-Göttingen-Heidelberg, Springer Verlag, 1963.

[38] I. I. Piatetski-Shapiro, *Zeta-functions of* GL(n), Preprint, University of Maryland, 1976.

[39] I. I. Piatetski-Shapiro, The converse theorem for GL(n), *Festschrift in Honor of I. I. Piatetski-Shapiro*, part II, Rehovot, Weizmann Science Press, 1990, p. 185-195.

[40] F. Rodier, Whittaker models for admissible representations of reductive $\mathfrak{p}$-adic split groups, *Proc. Symp. Pure Math.*, **26** (1973), 425-430.

[41] G. Shimura, On Dirichlet series and Abelian varieties attached to automorphic forms, *Ann. Math.*, **76** (1962), 237-294.

[42] A. Weil, Über die Bestimung Dirichletscher Reihen durch Funktionalgleichungen, *Math. Ann.*, **168** (1967), 149-156.

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