Geophys. J. R. astr. Soc. (1967) **13**, 71–101.

Converting Vector and Tensor Equations to Scalar Equations in Spherical Coordinates*

George E. Backus

Summary

We show how to represent a vector field in a spherical shell in terms of three scalar fields, and a second-order tensor field in terms of nine scalar fields. We derive the scalar representations of the most common algebraic and differential operations on vector and second-order tensor fields, and apply the results to obtain scalar formulations of various tensor problems in the continuum mechanics of the Earth's mantle. We give an exhaustive catalogue of all possible equilibrium stress fields in the mantle, we deduce the scalar equations of elastic-gravitational oscillation of a transversely isotropic, radially stratified, spherically symmetrical Earth, and we give the scalar convection equations for a viscous, self-gravitating, spherically symmetrical mantle with radially variable viscosity.

1. Introduction

A Scalar Representation Theorem for tangent vector fields on the surfaces of spheres has been known for some time (Ahlfors & Sario 1960). Recently an analogous theorem for second-order tangent tensor fields has been proved (Backus 1966). It is the purpose of the present paper to show that these two theorems, Gibbs' dyadic notation (Gibbs & Wilson 1925) for tensor calculus, and Einstein's index conventions (Bergmann 1947) can be combined to give an economical formalism for dealing with vector and tensor fields in spherically symmetrical physical configurations. The formalism will be illustrated by applications to several problems involving the stress tensor in the Earth's solid mantle.

To describe the sort of problem in which the formalism is useful, let V be a spherical shell with centre O and boundary ∂V. Let $\mathscr{D}$ be a linear partial differential operator on vector or second-order tensor fields $\mathbf{T}$ in V, while $\mathscr{B}$ is a linear partial differential operator on similar fields $\mathbf{T}$ defined on ∂V. Suppose that $\mathscr{D}$ and $\mathscr{B}$ commute with all rigid rotations about the point O. Then the vector or tensor boundary value problem

$$\mathscr{D}\mathbf{T}=\mathbf{v} \text{ in } V, \tag{1.1}$$

$$\mathscr{B}\mathbf{T}=\mathbf{w} \text{ on } \partial V, \tag{1.2}$$

with given vector or second order tensor fields $\mathbf{v}$ and $\mathbf{w}$, can be reduced, with the aid of the formalism presented here, to a set of problems of the same form as (1.1) and (1.2), but with inhomogeneities v and w, solutions T, and differential operators $\mathscr{D}$ and $\mathscr{B}$ which are scalars. The scalar operators $\mathscr{D}$ and $\mathscr{B}$ of the reduced problem commute with all rigid rotations about O. Therefore in this scalar problem derivatives

* This work has been supported by the National Science Foundation under grant NSF–GP–4096 (Oscillations).

with respect to colatitude θ and longitude ϕ can only appear in the rotationally invariant combination

$$\nabla_S^2 = (\operatorname{cosec}\theta)\partial_\theta(\sin\theta)\partial_\theta + (\operatorname{cosec}^2\theta)\partial_\phi^2,$$

where ∂_θ means the partial derivative with respect to θ. On the surface of the unit sphere the surface spherical harmonics form a complete set of eigenfunctions for ∇_S^2. Thus the radial and angular variables in the scalar forms of (1.1) and (1.2) are separable, and the radial equations for scalars with the angular dependence of different surface spherical harmonics are decoupled. The net result of the formalism is to reduce a spherically symmetrical partial differential equation for a vector or tensor field to a decoupled set of ordinary differential equations for scalar fields.

We will develop the formalism in the following order: in Section 2 we combine the invariant tensor notation of Gibbs with Einstein's index conventions, so as to be able to deal economically with three dimensional vector and tensor fields on two-dimensional surfaces in Euclidean three-space.

In Section 3 we use the notation developed in Section 2 to calculate the results of algebraic and differential operations on vector fields and second-order tensor fields in generalized spherical co-ordinates. In Section 4 we state the Scalar Representation Theorems for tangent vector fields and second-order tangent tensor fields, show how these theorems lead to scalar representations of three-dimensional vector fields and second-order tensor fields, and calculate what operations on scalars correspond to the various algebraic and differential operations on the vector and tensor fields represented by those scalars.

In Section 5 the formalism is applied to various problems in the continuum mechanics of the Earth's mantle. The problems considered are: (1) to produce a catalogue of all possible equilibrium stress fields in the Earth's mantle, (2) to produce a simple derivation of the scalar equations of elastic-gravitational vibration of the most general perfectly elastic, spherically symmetrical Earth (such an Earth is transversely isotropic about all radii, but not necessarily completely isotropic), and (3) to obtain the scalar equations describing slow viscous convective yielding of the mantle, when the physical parameters vary with depth and the viscosity may be transversely isotropic rather than completely isotropic. Problems (2) and (3) can both be solved without using the formalism developed here, but at the cost of very heavy algebra. I do not know how to solve problem (1) without using the formalism developed here.

For readers familiar with Gibbs' invariant tensor notation and the notions of covariance and contravariance in differential geometry, the present paper is intended to be self-contained. In particular, the Scalar Representation Theorems for tangent vector fields and second-order tangent tensor fields are stated in full, although the proofs (Backus 1966) are not repeated here.

2. A notation for the differential geometry of surfaces

A. *Covariance, contravariance, and the metric tensor*

Let S be an oriented two-dimensional surface in three-dimensional Euclidean space. Let $\hat{n}(p)$ denote the positive unit normal to S at the point p on S. Let x^1 and x^2 be any curvilinear co-ordinates in an open region A of S. Choose a point O, not necessarily on S, and let $\mathbf{r}(x^1, x^2)$ be the position vector, relative to O, of the point on S with co-ordinates x^1, x^2. Let ∂_i denote the ordinary partial derivative (*not* the covariant derivative) with respect to x^i. We consider only right-handed nonsingular co-ordinate systems; that is, we assume that everywhere on A, $\hat{n}.(\partial_1\mathbf{r}\times\partial_2\mathbf{r})>0$.

A vector field $\mathbf{v}$ on S will be called a tangent vector field if at every point p of S, $\hat{n}(p).\mathbf{v}(p)=0$. A second-order tensor field $\mathbf{T}$ on S will be called a tangent tensor field if at every point p of S, $\hat{n}(p).\mathbf{T}(p)$ and $\mathbf{T}(p).\hat{n}(p)$ are both the zero vector. If

$\mathbf{I}$ is the three-dimensional identity tensor (the second order tensor such that for any vector $\mathbf{u}$, $\mathbf{u}.\mathbf{I}=\mathbf{I}.\mathbf{u}=\mathbf{u}$), then the equation

$$\mathbf{I}_S(p)=\mathbf{I}-\hat{n}(p)\hat{n}(p)$$

defines a second-order tangent tensor field on S, called the surface identity tensor. If $\mathbf{v}$ is a vector tangent to S at p,

$$\mathbf{v}.\mathbf{I}_S(p)=\mathbf{I}_S(p).\mathbf{v}=\mathbf{v}.$$

Let ∇ be the ordinary gradient operator in Euclidean three-space. The surface gradient ∇_S at a point p on S is defined as the perpendicular projection of ∇ onto S:

$$\nabla_S=\mathbf{I}_S.\nabla=\nabla-\hat{n}(p)[\hat{n}(p).\nabla]. \tag{2.1}$$

Since ∇_S involves differentiations only in directions tangent to S, ∇_S can be applied to any field defined on S, whether that field is defined elsewhere or not.

Let Q be a scalar, vector or tensor field on S. As we move from a point p on S, with position vector $\mathbf{r}$ and co-ordinates x^1, x^2, to a nearby point p' on S, with position vector $\mathbf{r}+d\mathbf{r}$ and co-ordinates x^1+dx^1, x^2+dx^2, the field Q changes by an amount dQ which can be calculated either as $dQ=d\mathbf{r}.\nabla_S Q$ or as $dQ=dx^i(\partial_i Q)$. Therefore

$$d\mathbf{r}.\nabla_S Q=dx^i(\partial_i Q). \tag{2.2}$$

In equation (2.2) and hereafter, all italic indices take the values 1 and 2. If an equation contains an italic index only once, that equation is understood to hold for both values of the index. If an index appears twice in one term, once as a subscript and once as a superscript, that term represents the sum over both possible values of the index.

In (2.2) if we set $Q=x^j$ we have $dx^j=d\mathbf{r}.(\nabla_S x^j)$. Therefore, for any Q, we have $d\mathbf{r}.[\nabla_S Q-(\nabla_S x^i)\partial_i Q]=0$. But from (2.1), $\hat{n}(p).[\nabla_S Q-(\nabla_S x^i)\partial_i Q]=0$. Since $d\mathbf{r}$ is an arbitrary vector tangent to S at p, it follows that

$$\left.\begin{aligned} &\nabla_S Q-(\nabla_S x^i)\partial_i Q=0,\\ \text{or}\quad &\nabla_S=(\nabla_S x^i)\partial_i. \end{aligned}\right\} \tag{2.3}$$

Since $d\mathbf{r}=(\partial_i\mathbf{r})dx^i$, (2.2) implies that $\partial_i Q=(\partial_i\mathbf{r}).\nabla_S Q$. In particular, if Q is x^j, $\partial_i x^j=(\partial_i\mathbf{r}).(\nabla_S x^j)$, or

$$(\partial_i\mathbf{r}).(\nabla_S x^j)=\delta_{ij}, \tag{2.4}$$

where δ is the Kronecker delta: $\delta_{11}=\delta_{22}=1$, $\delta_{12}=\delta_{21}=0$.

If $\mathbf{v}$ is a tangent vector field and $\mathbf{T}$ a tangent tensor field on S, then with respect to the co-ordinate system x^1, x^2 the covariant components v_i and T_{ij} of $\mathbf{v}$ and $\mathbf{T}$, the contravariant components v^i and T^{ij}, and the mixed components $T_i{}^j$ and $T^i{}_j$ are defined by the following equations:

$$\mathbf{v}=v_i(\nabla_S x^i)=v^i(\partial_i\mathbf{r}); \tag{2.5}$$

$$\begin{aligned}\mathbf{T}&=T_{ij}(\nabla_S x^i)(\nabla_S x^j)=T^{ij}(\partial_i\mathbf{r})(\partial_j\mathbf{r})\\ &=T_i{}^j(\nabla_S x^i)(\partial_j\mathbf{r})=T^i{}_j(\partial_i\mathbf{r})(\nabla_S x^j).\end{aligned} \tag{2.6}$$

From (2.4), (2.5) and (2.6),

$$v_i=\mathbf{v}.(\partial_i\mathbf{r}),\quad v^i=\mathbf{v}.(\nabla_S x^i), \tag{2.7}$$

and

$$\left.\begin{aligned} T_{ij}&=(\partial_i\mathbf{r}).\mathbf{T}.(\partial_j\mathbf{r}),\quad T^{ij}=(\nabla_S x^i).\mathbf{T}.(\nabla_S x^j),\\ T_i{}^j&=(\partial_i\mathbf{r}).\mathbf{T}.(\nabla_S x^j),\quad T^i{}_j=(\nabla_S x^i).\mathbf{T}.(\partial_j\mathbf{r}). \end{aligned}\right\} \tag{2.8}$$

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For historical reasons the components of the surface identity tensor $\mathbf{I}_S$ are usually written as g. Thus, from (2.8) and the definition of $\mathbf{I}_S$,

$$\left.\begin{aligned} g_{ij}&=(\partial_i\mathbf{r}).(\partial_j\mathbf{r}), \quad g^{ij}=(\nabla_S x^i).(\nabla_S x^j),\\ g_i{}^j&=(\partial_i\mathbf{r}).(\nabla_S x^j), \quad g^i{}_j=(\nabla_S x^i).(\partial_j\mathbf{r}).\end{aligned}\right\} \tag{2.9}$$

Equation (2.4) implies that $g_i{}^j$ and $g^i{}_j$ are everywhere numerically equal to δ_{ij} in any co-ordinate system. Therefore

$$\mathbf{I}_S=(\partial_i\mathbf{r})(\nabla_S x^i)=(\nabla_S x^i)(\partial_i\mathbf{r})=\nabla_S\mathbf{r}.$$

If (2.7) is applied to the tangent vector fields $\partial_i\mathbf{r}$ and $\nabla_S x^i$, (2.9) implies

$$(\partial_i\mathbf{r})=g_{ij}(\nabla_S x^j), \quad (\nabla_S x^i)=g^{ij}(\partial_j\mathbf{r}). \tag{2.10}$$

Therefore the symmetric matrix g_{ij} is the multiplicative inverse of the symmetric matrix g^{ij}:

$$g_{ik}g^{jk}=g_i{}^j. \tag{2.11}$$

Equation (2.11) is a special case of the general rule for calculating covariant components from contravariant or vice versa, the index-raising and -lowering rule. From (2.5), (2.6) and (2.10),

$$\begin{aligned} v_i&=g_{ij}v^j, \quad v^i=g^{ij}v_j,\\ T_{ij}&=g_{ik}T^k{}_j=g_{ik}g_{jl}T^{kl},\\ T^{ij}&=g^{ik}T_k{}^j=g^{ik}g^{jl}T_{kl}.\end{aligned}$$

The distance ds between the points $\mathbf{r}(x^1, x^2)$ and $\mathbf{r}(x^1+dx^1, x^2+dx^2)$ on S is given by $(ds)^2=d\mathbf{r}.d\mathbf{r}=[(\partial_i\mathbf{r})dx^i].[(\partial_j\mathbf{r})dx^j]=g_{ij}dx^i dx^j$. Therefore g_{ij} and g^{ij} are the covariant and contravariant components of the metric tensor on S in the co-ordinates x^1, x^2. Then the metric tensor is the surface identity tensor, and $g_{ij}dx^i dx^j$ is the first fundamental form on S.

B. *Covariant differentiation on S*

For any i, $\partial_i\mathbf{r}$ is a tangent vector field on S. For any i and j, $\partial_i\partial_j\mathbf{r}$ is a vector field on S, but not necessarily a tangent vector field. At any point p on S, $\partial_i\partial_j\mathbf{r}$ is a linear combination of the three linearly independent vectors $\partial_1\mathbf{r}$, $\partial_2\mathbf{r}$, $\hat{n}(p)$:

$$\partial_i\partial_j\mathbf{r}=\begin{pmatrix}k\\ij\end{pmatrix}(\partial_k\mathbf{r})+f_{ij}\hat{n}. \tag{2.12}$$

The symbols $\begin{pmatrix}k\\ij\end{pmatrix}$ and f_{ij} are defined as the coefficients in the linear combination (2.12). Clearly

$$f_{ij}=f_{ji}, \quad \begin{pmatrix}k\\ij\end{pmatrix}=\begin{pmatrix}k\\ji\end{pmatrix}.$$

From (2.12),

$$f_{ij}=\hat{n}.(\partial_i\partial_j\mathbf{r}). \tag{2.13}$$

Therefore f_{ij} are the covariant components, with respect to co-ordinates x^1, x^2, of the tangent tensor field

$$\mathbf{F}=(\nabla_S\nabla_S\mathbf{r}).\hat{n}=(\nabla_S\mathbf{I}_S).\hat{n}.$$

We will call $\mathbf{F}$ the 'extrinsic curvature tensor'. The perpendicular distance d^2n of the point $\mathbf{r}(x^1+dx^1, x^2+dx^2)$ from the plane tangent to S at $\mathbf{r}(x^1, x^2)$ is

$$d^2n = \tfrac{1}{2} f_{ij}\, dx^i\, dx^j,$$

d^2n being positive in the direction of $\hat{n}$. Therefore $\frac{1}{2} f_{ij}\, dx^i dx^j$ is the second fundamental form on S.

Since $\hat{n}\,.\,\hat{n}=1$, then $\hat{n}\,.\,\partial_i \hat{n}=0$. Therefore $\partial_i \hat{n}$ is a tangent vector field on S. Since $\hat{n}\,.\,(\partial_j \mathbf{r})=0$, $\partial_i \hat{n}\,.\,(\partial_j \mathbf{r})+\hat{n}\,.\,(\partial_i \partial_j \mathbf{r})=0$. Then from (2.7) and (2.13)

$$\partial_i \hat{n} = -f_{ij}(\nabla_S x^j) \tag{2.14}$$

so $\mathbf{F} = -\nabla_S \hat{n}$.

At any point p on S, the vector $\partial_i(\nabla_S x^j)$ is a linear combination of the three linearly independent vectors $(\nabla_S x^1)$, $(\nabla_S x^2)$ and $\hat{n}(p)$. By differentiating with respect to x^i the two identities $\hat{n}\,.\,(\nabla_S x^j)=0$ and $(\partial_k \mathbf{r})\,.\,(\nabla_S x^j)=g_k{}^j$, and appealing to (2.12), (2.14) and the constancy of $g_k{}^j$ on S, we easily deduce that

$$\partial_i(\nabla_S x^j) = -\begin{Bmatrix} j \\ ik \end{Bmatrix}(\nabla_S x^k) + f_i{}^j \hat{n}. \tag{2.15}$$

To calculate $\begin{Bmatrix} k \\ ij \end{Bmatrix}$ we note from (2.9) that

$$\partial_k g_{ij} = (\partial_k \partial_i \mathbf{r})\,.\,(\partial_j \mathbf{r}) + (\partial_i \mathbf{r})\,.\,(\partial_j \partial_k \mathbf{r}).$$

Then from (2.9) and (2.12)

$$\partial_k g_{ij} = (j, ki) + (i, kj)$$

where (k, ij) is an abbreviation for $g_{kl}\begin{Bmatrix} l \\ ij \end{Bmatrix}$. Permuting indices gives

$$\partial_i g_{jk} = (k, ij) + (j, ik),$$

$$\partial_j g_{ki} = (i, jk) + (k, ji).$$

Since $g_{ij}=g_{ji}$ and $(k, ij)=(k, ji)$, the three foregoing equations are three linear equations in the three unknowns (i, jk), (j, ki) and (k, ij). Their solution is

$$(k, ij) = \tfrac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}).$$

Hence

$$\begin{Bmatrix} k \\ ij \end{Bmatrix} = \tfrac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}). \tag{2.16}$$

Therefore (k, ij) are the Christoffel symbols of the first kind and $\begin{Bmatrix} k \\ ij \end{Bmatrix}$ are those of the second kind. Equation (2.12) is the equation of Gauss (see Willmore 1959, p. 112).

If q, $\mathbf{v}$, and $\mathbf{T}$ are respectively a scalar field, a tangent vector field, and a second-order tangent tensor field on S, we are now in a position to give explicitly the components of $\nabla_S q$, $\nabla_S(q\hat{n})$, $\nabla_S(q\hat{n}\hat{n})$, $\nabla_S \mathbf{v}$, $\nabla_S(\hat{n}\mathbf{v})$, $\nabla_S(\mathbf{v}\hat{n})$ and $\nabla_S \mathbf{T}$ in terms of q and the components of $\mathbf{v}$, $\mathbf{T}$, $\mathbf{I}_S$ and $\mathbf{F}$.

From (2.3), if q is a scalar field on S,

$$\nabla_S q = (D_i q)(\nabla_S x^i), \tag{2.17}$$

where

$$D_i q = \partial_i q. \tag{2.18}$$

Further, $\nabla_S(q\hat{n}) = (\nabla_S x^i)\partial_i(q\hat{n}) = (\nabla_S q)\hat{n} + q(\nabla_S \hat{n}) = (\nabla_S q)\hat{n} - q\mathbf{F}$, so

$$\nabla_S(q\hat{n}) = (D_i q)(\nabla_S x^i)\hat{n} - q\, f_{ij}(\nabla_S x^i)(\nabla_S x^j). \tag{2.19}$$

And $\nabla_S(q\hat{n}\hat{n}) = (\nabla_S q)\hat{n}\hat{n} + q(\nabla_S x^i)[(\partial_i \hat{n})\hat{n} + \hat{n}(\partial_i \hat{n})]$, so

$$\nabla_S(q\hat{n}\hat{n}) = (D_i q)(\nabla_S x^i)\hat{n}\hat{n} - q f_{ij}[(\nabla_S x^i)\hat{n}(\nabla_S x^j) + (\nabla_S x^i)(\nabla_S x^j)\hat{n}]. \tag{2.20}$$

From (2.3), (2.12) and (2.15), if $\mathbf{v}$ is a tangent vector field on S,

$$\left.\begin{aligned} \nabla_S \mathbf{v} &= (f_i{}^j v_j)(\nabla_S x^i)\hat{n} + (D_i v_j)(\nabla_S x^i)(\nabla_S x^j) \\ &= (f_{ij} v^j)(\nabla_S x^i)\hat{n} + (D_i v^j)(\nabla_S x^i)(\partial_j \mathbf{r}), \end{aligned}\right\} \tag{2.21}$$

where

$$\left.\begin{aligned} D_i v_j &= \partial_i v_j - \left(\begin{matrix} k \\ ij \end{matrix}\right) v_k, \\ D_i v^j &= \partial_i v^j + \left(\begin{matrix} j \\ ik \end{matrix}\right) v^k. \end{aligned}\right\} \tag{2.22}$$

Moreover, from (2.3),

$$\nabla_S(\hat{n}\mathbf{v}) = (\nabla_S \hat{n})\mathbf{v} + (\nabla_S x^i)\hat{n}(\partial_i \mathbf{v}) = -\mathbf{F}\mathbf{v} + (\nabla_S x^i)\hat{n}(\partial_i \mathbf{r}) . (\nabla_S \mathbf{v}).$$

Hence, from (2.21),

$$\nabla_S(\hat{n}\mathbf{v}) = (f_i{}^k v_k)(\nabla_S x^i)\hat{n}\hat{n} + (D_i v_k)(\nabla_S x^i)\hat{n}(\nabla_S x^k) - (f_{ij} v_k)(\nabla_S x^i)(\nabla_S x^j)(\nabla_S x^k). \tag{2.23}$$

Furthermore, from (2.3), $\nabla_S(\mathbf{v}\hat{n}) = (\nabla_S \mathbf{v})\hat{n} + (\nabla_S x^i)\mathbf{v}(\partial_i \hat{n})$. Hence, from (2.14) and (2.21),

$$\nabla_S(\mathbf{v}\hat{n}) = (f_i{}^k v_k)(\nabla_S x^i)\hat{n}\hat{n} + (D_i v_j)(\nabla_S x^i)(\nabla_S x^j)\hat{n} - (f_{ik} v_j)(\nabla_S x^i)(\nabla_S x^j)(\nabla_S x^k). \tag{2.24}$$

Finally, if $\mathbf{T}$ is a second-order tangent tensor field on S, then

$$\left.\begin{aligned} \nabla_S \mathbf{T} &= (f_i{}^j T_{jk})(\nabla_S x^i)\hat{n}(\nabla_S x^k) + (f_i{}^k T_{jk})(\nabla_S x^i)(\nabla_S x^j)\hat{n} \\ &\qquad + (D_i T_{jk})(\nabla_S x^i)(\nabla_S x^j)(\nabla_S x^k), \\ &= (f_i{}^j T_j{}^k)(\nabla_S x^i)\hat{n}(\partial_k \mathbf{r}) + (f_{ik} T_j{}^k)(\nabla_S x^i)(\nabla_S x^j)\hat{n} \\ &\qquad + (D_i T_j{}^k)(\nabla_S x^i)(\nabla_S x^j)(\partial_k \mathbf{r}), \\ &= (f_{ij}\, T^j{}_k)(\nabla_S x^i)\hat{n}(\nabla_S x^k) + (f_i{}^k T^j{}_k)(\nabla_S x^i)(\partial_j \mathbf{r})\hat{n} \\ &\qquad + (D_i T^j{}_k)(\nabla_S x^i)(\partial_j \mathbf{r})(\nabla_S x^k), \\ &= (f_{ij}\, T^{jk})(\nabla_S x^i)\hat{n}(\partial_k \mathbf{r}) + (f_{ik} T^{jk})(\nabla_S x^i)(\partial_j \mathbf{r})\hat{n} \\ &\qquad + (D_i T^{jk})(\nabla_S x^i)(\partial_j \mathbf{r})(\partial_k \mathbf{r}), \end{aligned}\right\} \tag{2.25}$$

where

$$\left.\begin{aligned} D_i T_{jk} &= \partial_i T_{jk} - \begin{pmatrix} l \\ ij \end{pmatrix} T_{lk} - \begin{pmatrix} l \\ ik \end{pmatrix} T_{jl}, \\ D_i T_j{}^k &= \partial_i T_j{}^k - \begin{pmatrix} l \\ ij \end{pmatrix} T_l{}^k + \begin{pmatrix} k \\ il \end{pmatrix} T_j{}^l, \\ D_i T^j{}_k &= \partial_i T^j{}_k + \begin{pmatrix} j \\ il \end{pmatrix} T^l{}_k - \begin{pmatrix} l \\ ik \end{pmatrix} T^j{}_l, \\ D_i T^{jk} &= \partial_i T^{jk} + \begin{pmatrix} j \\ il \end{pmatrix} T^{lk} + \begin{pmatrix} k \\ il \end{pmatrix} T^{jl}. \end{aligned}\right\} \quad (2.26)$$

It is clear from (2.18), (2.22), and (2.26) that D_i is the operator of covariant differentiation on S.

If $\mathbf{u}$ and $\mathbf{v}$ are tangent vector fields on S, $\nabla_S(\mathbf{uv})$ is *not* $(\nabla_S \mathbf{u})\mathbf{v} + \mathbf{u}(\nabla_S \mathbf{v})$. Rather

$$\nabla_S(\mathbf{uv}) = (\nabla_S \mathbf{u})\mathbf{v} + (\nabla_S x^i)\,\mathbf{u}(\partial_i \mathbf{r}).(\nabla_S \mathbf{v}).$$

From this equation it does follow that

$$D_i(u_j v_k) = (D_i u_j) v_k + u_j(D_i v_k),$$

a result also obtainable by direct calculation from (2.22) and (2.26). This product rule holds for covariant derivatives of two tensors of any order.

Since $\nabla_S \mathbf{I} = 0$ and $\mathbf{I}_S = \mathbf{I} - \hat{n}\hat{n}$, therefore $\nabla_S \mathbf{I}_S = -\nabla_S(\hat{n}\hat{n})$. Setting $q = 1$ in (2.20) gives

$$\nabla_S \mathbf{I}_S = f_{ij}(\nabla_S x^i)[(\nabla_S x^j)\hat{n} + \hat{n}(\nabla_S x^j)].$$

Comparison with (2.25) shows immediately that

$$D_i g_{jk} = D_i g_j{}^k = D_i g^j{}_k = D_i g^{jk} = 0, \quad (2.27)$$

a result which can also be deduced directly from (2.16) and (2.26).

C. *The surface rotator*

The tensor $-\mathbf{I}_S \times \hat{n} = -\mathbf{I} \times \hat{n}$ is a second-order tangent tensor on S, called the surface rotator on S. We will use h to denote its components in the co-ordinate system x^1, x^2. We have

$$\begin{aligned} h_{ij} &= (\partial_i \mathbf{r}).(-\mathbf{I}_S \times \hat{n}).(\partial_j \mathbf{r}) = -(\partial_i \mathbf{r} \times \hat{n}).(\partial_j \mathbf{r}) = \hat{n}.(\partial_i \mathbf{r} \times \partial_j \mathbf{r}) \\ &= (\partial_i \mathbf{r}).(-\hat{n} \times \partial_j \mathbf{r}) = (\partial_i \mathbf{r}).(-\hat{n} \times \mathbf{I}_S).(\partial_j \mathbf{r}), \end{aligned}$$

so $-\mathbf{I}_S \times \hat{n} = -\hat{n} \times \mathbf{I}_S$. Furthermore,

$$\begin{aligned} h_{ij} &= \hat{n}.(\partial_i \mathbf{r} \times \partial_j \mathbf{r}), \\ h_i{}^j &= \hat{n}.(\partial_i \mathbf{r} \times \nabla_S x^j), \\ h^i{}_j &= \hat{n}.(\nabla_S x^i \times \partial_j \mathbf{r}), \\ h^{ij} &= \hat{n}.(\nabla_S x^i \times \nabla_S x^j). \end{aligned}$$

If $\mathbf{v}$ is a tangent vector on S, $h_{ij} v^j$ are the covariant components of $(-\hat{n} \times \mathbf{I}_S).\mathbf{v} = -\hat{n} \times \mathbf{v}$. Thus as a linear operator, $-\hat{n} \times \mathbf{I}_S$ rotates tangent vectors through $-90°$ about the direction $\hat{n}$.

The length of $\partial_1 \mathbf{r}$ is $g_{11}{}^{\frac{1}{2}}$, that of $\partial_2 \mathbf{r}$ is $g_{22}{}^{\frac{1}{2}}$, and the cosine of θ, the angle between $\partial_1 \mathbf{r}$ and $\partial_2 \mathbf{r}$, is $g_{12} g_{11}{}^{-\frac{1}{2}} g_{22}{}^{-\frac{1}{2}}$. Hence $\sin\theta = g g_{11}{}^{-\frac{1}{2}} g_{22}{}^{-\frac{1}{2}}$ where g is the positive square root of the determinant of g_{ij}:

$$g = (\det g_{ij})^{\frac{1}{2}} = (\det g^{ij})^{-\frac{1}{2}}. \tag{2.28}$$

Since $\partial_1 \mathbf{r} \times \partial_2 \mathbf{r}$ is parallel to $\hat{n}$ and has length $g_{11}{}^{\frac{1}{2}} g_{22}{}^{\frac{1}{2}} \sin\theta$, or g, therefore $h_{12} = g$. Similarly $h^{12} = g^{-1}$, so

$$h_{ij} = g\varepsilon_{ij}, \quad h^{ij} = g^{-1}\varepsilon_{ij}, \tag{2.29}$$

where ε_{ij} is the two-dimensional alternating symbol:

$$\varepsilon_{11} = \varepsilon_{22} = 0, \quad \varepsilon_{12} = -\varepsilon_{21} = 1.$$

Since $\mathbf{I}_S = \mathbf{I} - \hat{n}\hat{n}$, therefore $\mathbf{I}_S \times \hat{n} = \mathbf{I} \times \hat{n}$, and

$$\partial_i(-\mathbf{I}_S \times \hat{n}) = -\mathbf{I} \times \partial_i \hat{n} = [(\partial_k \mathbf{r})(\nabla_S x^k) + \hat{n}\hat{n}] \times [f_{ij} \nabla_S x^j].$$

But $(\nabla_S x^k) \times (\nabla_S x^j) = h^{kj} \hat{n}$ and $\hat{n} \times (\nabla_S x^j) = h^{jk} \partial_k \mathbf{r}$. Consequently

$$\partial_i(-\mathbf{I}_S \times \hat{n}) = f_{ij} h^{jk} [\hat{n}(\partial_k \mathbf{r}) - (\partial_k \mathbf{r})\hat{n}],$$

and

$$\nabla_S(-\mathbf{I}_S \times \hat{n}) = f_{ij} h^{jk} (\nabla_S x^i) [\hat{n}(\partial_k \mathbf{r}) - (\partial_k \mathbf{r})\hat{n}].$$

Comparison with (2.25) shows that

$$D_i h_{jk} = D_i h_j{}^k = D_i h^j{}_k = D_i h^{jk} = 0, \tag{2.30}$$

a fact which can also be deduced directly from (2.16), (2.26) and (2.29).

Three very useful algebraic relations between h^{ij} and g^{ij} are these:

$$h^{ij} h^{kl} - g^{ik} g^{jl} + g^{il} g^{jk} = 0, \tag{2.31}$$

$$h_i{}^k h^j{}_k = g_i{}^j, \tag{2.32}$$

$$g^{ij} h^{kl} + g^{ik} h^{lj} + g^{il} h^{jk} = 0. \tag{2.33}$$

To prove equation (2.31), denote the left side by F^{ijkl}. Then

$$F^{ijkl} = -F^{jikl} = -F^{ijlk} = F^{klij}.$$

Thus $F^{ijkl} = 0$ unless $i \neq j$ and $k \neq l$, and we need only show that $F^{1212} = 0$. But $F^{1212} = h^{12} h^{12} - g^{11} g^{22} + g^{12} g^{21}$, which vanishes on account of (2.28) and (2.29). Equation (2.32) follows immediately from (2.11), (2.31) and the fact that g can be used to raise and lower indices on h. To prove equation (2.33), denote the left-hand side by G^{ijkl}. Then $G^{ijkl} = -G^{ikjl} = -G^{ilkj} = -G^{ijlk}$. Therefore $G^{ijkl} = 0$ unless j, k and l are all different. Since each of j, k and l must be 1 or 2, therefore $G^{ijkl} = 0$.

D. *The Riemann curvature tensor*

From (2.18) and (2.22), if q is a scalar field on S

$$(D_i D_j - D_j D_i) q = 0. \tag{2.34}$$

From (2.22) and (2.26), if $\mathbf{v}$ is a tangent vector field on S,

$$(D_i D_j - D_j D_i) v_k = K_{ijk}{}^l v_l \tag{2.35}$$

where

$$K_{ijk}{}^{l} = -\partial_i \begin{pmatrix} l \\ jk \end{pmatrix} + \partial_j \begin{pmatrix} l \\ ik \end{pmatrix} + \begin{pmatrix} m \\ ik \end{pmatrix}\begin{pmatrix} l \\ jm \end{pmatrix} - \begin{pmatrix} m \\ jk \end{pmatrix}\begin{pmatrix} l \\ im \end{pmatrix}. \tag{2.36}$$

Since $(D_i D_j - D_j D_i) v_k$ and v_l are covariant components of tangent tensor fields, (2.35) implies that $K_{ijk}{}^{l}$ are the mixed components of a fourth-order tangent tensor field $\mathbf{K}$ on S. This tensor field is the Riemann, or intrinsic, curvature tensor. By differentiating (2.12) once and using (2.14) and (2.36), we calculate that

$$(\partial_j \partial_i - \partial_i \partial_j) \partial_k \mathbf{r} = (K_{ijk}{}^{l} - f_{ik} f_j{}^{l} + f_{jk} f_i{}^{l}) \partial_l \mathbf{r} + (D_j f_{ik} - D_i f_{jk}) \hat{n}.$$

But ∂ is ordinary partial differentiation, so ∂_i, ∂_j and ∂_k commute and

$$(\partial_j \partial_i - \partial_i \partial_j) \partial_k \mathbf{r} = 0.$$

Therefore we have the Mainardi–Codazzi equations (Willmore 1959, p. 114):

$$D_i f_{jk} = D_j f_{ik};$$

and also, in terms of covariant rather than mixed components,

$$K_{ijkl} = f_{ik} f_{jl} - f_{jk} f_{il}. \tag{2.37}$$

This last is another equation of Gauss (Willmore 1959, p. 114). It implies that $K_{ijkl} = -K_{jikl} = -K_{ijlk} = K_{klij}$, so there is a scalar field K such that

$$K_{ijkl} = K h_{ij} h_{kl}. \tag{2.38}$$

From (2.28), (2.29), (2.37) and (2.38),

$$K = [\det(f_{ij})][\det(g_{ij})]^{-1}. \tag{2.39}$$

The scalar K is the Gaussian curvature of S.

If $\mathbf{v}$ is a tangent vector field on S, from (2.35) and (2.38)

$$(D_i D_j - D_j D_i) v_k = K h_{ij} h_k{}^{l} v_l. \tag{2.40}$$

E. *Spheres*

In three-dimensional Euclidean space let O be the fixed origin with reference to which position vectors are defined. Let S_r be the surface of the sphere of radius r whose centre is O. On S_r, $\mathbf{r}(x^1, x^2) = r\hat{n}(x^1, x^2)$ in any co-ordinate system x^1, x^2. Hence $\partial_i \mathbf{r} = r \partial_i \hat{n}$ and, from (2.10) and (2.14), $g_{ij}(\nabla_S x^j) = -r f_{ij}(\nabla_S x^j)$. It follows that on S_r

$$f_{ij} = -r^{-1} g_{ij}$$

and, from (2.39), $K = r^{-2}$. In particular, on S_1,

$$f_{ij} = -g_{ij} \tag{2.41}$$

and $K = 1$. If $\mathbf{v}$ is a tangent vector field on S_1, then (2.40) implies that:

$$(D_i D_j - D_j D_i) v_k = h_{ij} h_k{}^{l} v_l. \tag{2.42}$$

Equations (2.41) and (2.42) are largely responsible for the simplicity of spherical co-ordinates.

Let ∇_{S_r} be abbreviated as ∇_r:

$$\nabla_r = \nabla_{S_r}. \tag{2.43}$$

If $\hat{n}$ is the outward unit normal on S_1, and q, $\mathbf{v}$ and $\mathbf{T}$ are a scalar field, a tangent vector field, and a second-order tangent tensor field on S_1, equation (2.41) reduces equations (2.19), (2.20), (2.21), (2.23), (2.24) and (2.25) to the following:

$$\nabla_S(q\hat{n}) = (D_i q)(\nabla_S x^i)\hat{n} + q\mathbf{I}_S, \tag{2.44}$$

$$\nabla_S(q\hat{n}\hat{n}) = (D_i q)(\nabla_S x^i)\hat{n}\hat{n} + qg_{ij}[(\nabla_S x^i)\hat{n}(\nabla_S x^j) + (\nabla_S x^i)(\nabla_S x^j)\hat{n}], \tag{2.45}$$

$$\nabla_S \mathbf{v} = -\mathbf{v}\hat{n} + (D_i v_j)(\nabla_S x^i)(\nabla_S x^j), \tag{2.46}$$

$$\nabla_S(\hat{n}\mathbf{v}) = -\mathbf{v}\hat{n}\hat{n} + (D_i v_k)(\nabla_S x^i)\hat{n}(\nabla_S x^k) + \mathbf{I}_S \mathbf{v}, \tag{2.47}$$

$$\nabla_S(\mathbf{v}\hat{n}) = -\mathbf{v}\hat{n}\hat{n} + (D_i v_j)(\nabla_S x^i)(\nabla_S x^j)\hat{n} + g_{ik}(\nabla_S x^i)\mathbf{v}(\nabla_S x^k), \tag{2.48}$$

$$\nabla_S \mathbf{T} = -T_{ik}(\nabla_S x^i)\hat{n}(\nabla_S x^k) - T_{ji}(\nabla_S x^i)(\nabla_S x^j)\hat{n} + (D_i T_{jk})(\nabla_S x^i)(\nabla_S x^j)(\nabla_S x^k). \tag{2.49}$$

3. Generalized spherical coordinates

A. *Definition*

As in Section 2.E, choose a fixed origin O with reference to which position vectors will be defined. Let S_r be the surface of the sphere of radius r with centre at O. Take the positive unit normal $\hat{n}$ on S_r to be $\hat{r}$, the unit vector pointing away from O. Let x^1, x^2 be any right-handed, nonsingular system of curvilinear co-ordinates on S_1. Then x^1 and x^2 serve, via radial projection, as right-handed curvilinear co-ordinates on S_r. Any point in Euclidean three-space lies on exactly one S_r and is uniquely determined by its three co-ordinates r, x^1, x^2. A co-ordinate system in three-space will be called a system of generalized spherical co-ordinates if one of the three co-ordinates is r, the distance from O, and the other two are obtained by radial projection from a system of curvilinear co-ordinates x^1, x^2 on S_1.

Any scalar, vector, or tensor field on S_r can be projected radially, without change of magnitudes or directions, so as to be a field on S_1, and vice versa. The scalar $q(x^1, x^2)$, the vector $\mathbf{u}(x^1, x^2)$ or the tensor $\mathbf{T}(x^1, x^2)$ which is attached to the point (r, x^1, x^2) on S_r is simply thought of as attached to the point $(1, x^1, x^2)$ on S_1.

The position vector on S_1 is the unit vector $\hat{r}(x^1, x^2)$ and the position vector on S_r is $\mathbf{r}(x^1, x^2) = r\hat{r}(x^1, x^2)$. Hence

$$\partial_i \mathbf{r} = r\partial_i \hat{r}. \tag{3.1}$$

If ∇_r denotes the surface gradient on S_r, then

$$\nabla_r = r^{-1}\nabla_1. \tag{3.2}$$

The sense of this equation is that to obtain the surface gradient of a field on S_r, we can project that field radially onto S_1, take its surface gradient on S_1, divide by r, and project the result radially back onto S_r.

If $\tilde{g}_{ij}(x^1, x^2)$, $\tilde{g}^{ij}(x^1, x^2)$, $\tilde{g}_i{}^j(x^1, x^2)$ and $\tilde{g}^i{}_j(x^1, x^2)$ are the covariant, contravariant and mixed components of the two-dimensional metric tensor on S_1 with respect to the co-ordinates x^1, x^2, then the covariant, contravariant and mixed components of the three-dimensional metric tensor in the generalized spherical co-ordinate system (r, x^1, x^2) are as follows:

$$g_{rr} = 1, \quad g_{ri} = 0, \quad g_{ij} = r^2\tilde{g}_{ij};$$

$$g^{rr} = 1, \quad g^{ri} = 0, \quad g^{ij} = r^{-2}\tilde{g}^{ij};$$

$$g_r{}^r = 1, \quad g_r{}^i = 0, \quad g_i{}^j = \tilde{g}_i{}^j;$$

$$g^r{}_r = 1, \quad g^r{}_i = 0, \quad g^i{}_j = \tilde{g}^i{}_j.$$

B. *Spherical resolution of three-dimensional fields*

If $\mathbf{u}(r, x^1, x^2)$ is a three-dimensional vector field in Euclidean three-space, we can write

$$\mathbf{u}(r, x^1, x^2) = \hat{r} u_r(r, x^1, x^2) + \mathbf{u}_{S_r}(r, x^1, x^2),$$

where, for each fixed r, $\mathbf{u}_{S_r}$ is a tangent vector field on S_r. For each r, by radial projection $\mathbf{u}_{S_r}$ may be thought of as a tangent vector field on S_1; as such it will be denoted simply by $\mathbf{u}_S(r, x^1, x^2)$. Thus

$$\mathbf{u}(r, x^1, x^2) = \hat{r} u_r(r, x^1, x^2) + \mathbf{u}_S(r, x^1, x^2).$$

The function u_r and the field $\mathbf{u}_S$ are completely determined by $\mathbf{u}$. Under a change of generalized spherical co-ordinates u_r does not change, so we call it the 'scalar' part of $\mathbf{u}$; the field $\mathbf{u}_S$ is the 'tangent vector' part of $\mathbf{u}$. For any fixed r, $\tilde{u}_i$ and $\tilde{u}^i$ will denote the covariant and contravariant components of $\mathbf{u}_S$ regarded as a tangent vector field on S_1, not S_r. Then

$$\left.\begin{aligned} \mathbf{u} &= \hat{r} u_r + (\nabla_1 x^i) \tilde{u}_i \\ &= \hat{r} u_r + (\partial_i \hat{r}) \tilde{u}^i. \end{aligned}\right\} \quad (3.3)$$

In the generalized spherical co-ordinate system (r, x^1, x^2) the ordinary covariant components of $\mathbf{u}$ are, from (3.2), u_r, $r\tilde{u}_1$ and $r\tilde{u}_2$, while the ordinary contravariant components are, from (3.1), u_r, $r^{-1}\tilde{u}^1$ and $r^{-1}\tilde{u}^2$.

If $\mathbf{T}(r, x^1, x^2)$ is a second-order, three-dimensional tensor field in Euclidean three-space, then at any point (r, x^1, x^2) the tensor $\mathbf{T}$ is a linear combination of the nine linearly independent dyads $\hat{r}\hat{r}$, $\hat{r}(\nabla_1 x^i)$, $(\nabla_1 x^i)\hat{r}$ and $(\nabla_1 x^i)(\nabla_1 x^j)$. Therefore for each r we can write

$$\mathbf{T} = \hat{r}\hat{r} T_{rr} + \hat{r}\mathbf{T}_{rS} + \mathbf{T}_{S_r}\hat{r} + \mathbf{T}_{SS}$$

where, for any fixed r, T_{rr} is a scalar field on S_r, $\mathbf{T}_{rS}$ and $\mathbf{T}_{S_r}$ are tangent vector fields on S_r, and $\mathbf{T}_{SS}$ is a tangent tensor field on S_r. These four fields are uniquely determined by $\mathbf{T}$. When, for some fixed r, $\mathbf{T}_{rS}$ is regarded as a tangent vector field on S_1, its covariant and contravariant components will be written $\tilde{T}_{ri}$ and $\tilde{T}_r{}^i$. Similarly, $\tilde{T}_{ir}$ and $\tilde{T}^i{}_r$ will denote the covariant and contravariant components of $\mathbf{T}_{S_r}$, regarded for each fixed r as a tangent vector field on S_1. Finally, $\tilde{T}_{ij}$, $\tilde{T}^{ij}$, $\tilde{T}_i{}^j$ and $\tilde{T}^i{}_j$ will denote the covariant, contravariant and mixed components of $\mathbf{T}_{SS}$, regarded for each fixed r as a tangent tensor field on S_1. Then

$$\left.\begin{aligned} \mathbf{T} &= \hat{r}\hat{r} T_{rr} + \hat{r}(\partial_i \hat{r}) \tilde{T}_r{}^i + (\partial_i \hat{r})\hat{r} \tilde{T}^i{}_r + (\partial_i \hat{r})(\partial_j \hat{r}) \tilde{T}^{ij} \\ &= \hat{r}\hat{r} T_{rr} + \hat{r}(\nabla_1 x^i) \tilde{T}_{ri} + (\nabla_1 x^i)\hat{r} \tilde{T}_{ir} + (\nabla_1 x^i)(\nabla_1 x^j) \tilde{T}_{ij}. \end{aligned}\right\} \quad (3.4)$$

In the three-dimensional generalized spherical co-ordinate system (r, x^1, x^2) the contravariant components of $\mathbf{T}$ are, from (3.1), and (3.4), T_{rr}, $r^{-1}(\tilde{T}_r{}^i)$, $r^{-1}(\tilde{T}^i{}_r)$, $r^{-2}(\tilde{T}^{ij})$, and the covariant components are T_{rr}, $r\tilde{T}_{ri}$, $r\tilde{T}_{ir}$, $r^2 \tilde{T}_{ij}$. The mixed components are T_{rr}, $r^{-1} \tilde{T}_r{}^i$, $r\tilde{T}_{ir}$, $\tilde{T}_i{}^j$ and T_{rr}, $r\tilde{T}_{ri}$, $r^{-1} \tilde{T}^i{}_r$, $\tilde{T}^i{}_j$. The field T_{rr} is the scalar part of $\mathbf{T}$, $\mathbf{T}_{rS}$ and $\mathbf{T}_{S_r}$ are the tangent vector parts of $\mathbf{T}$, and $\mathbf{T}_{SS}$ is the tangent tensor part of $\mathbf{T}$. Formula (3.4) gives the 'spherical resolution' of $\mathbf{T}$ into its scalar, tangent vector and tangent tensor parts, just as formula (3.3) gives the spherical resolution of the vector field $\mathbf{u}$ into its scalar and tangent vector parts.

C. *Spherical resolution of algebraic operations*

If $\mathbf{u}$ and $\mathbf{v}$ are three-dimensional vector fields with spherical resolutions (3.3) then

$$\mathbf{u} \cdot \mathbf{v} = u_r v_r + \tilde{g}^{ij} \tilde{u}_i \tilde{v}_j \quad (3.5)$$

and

$$\mathbf{u}\times\mathbf{v}=(\tilde{h}^{ij}\tilde{u}_i\tilde{v}_j)\hat{r}+\tilde{h}^{ij}(u_r\tilde{v}_i-\tilde{u}_i v_r)(\partial_j\hat{r}), \tag{3.6}$$

where $\tilde{g}_{ij}$ are the covariant components of the metric tensor $\mathbf{I}_{S_1}$ on S_1 and $\tilde{h}^{ij}$ are the contravariant components of the surface rotator on S_1, $-\mathbf{I}_{S_1}\times\hat{r}$.

If $\mathbf{T}$ is a three-dimensional tensor field with spherical resolution (3.4), then

$$\mathbf{T}.\mathbf{u}=\hat{r}[T_{rr}u_r+\tilde{g}^{jk}\tilde{T}_{rj}\tilde{u}_k]+(\nabla_1 x^i)[\tilde{T}_{ir}u_r+\tilde{g}^{jk}\tilde{T}_{ij}\tilde{u}_k], \tag{3.7}$$

$$\mathbf{u}.\mathbf{T}=\hat{r}[u_r T_{rr}+\tilde{g}^{ij}\tilde{u}_i\tilde{T}_{jr}]+(\nabla_1 x^k)[u_r\tilde{T}_{rk}+\tilde{g}^{ik}\tilde{u}_i\tilde{T}_{jk}], \tag{3.8}$$

$$\begin{aligned}\mathbf{T}\times\mathbf{u}=\hat{r}\hat{r}[\tilde{h}^{jk}\tilde{T}_{rj}\tilde{u}_k]+\hat{r}(\nabla_1 x^j)[\tilde{h}_j{}^k(\tilde{T}_{rk}u_r-T_{rr}\tilde{u}_k)]\\+(\nabla_1 x^i)\hat{r}[\tilde{h}^{jk}\tilde{T}_{ij}\tilde{u}_k]+(\nabla_1 x^i)(\nabla_1 x^j)[\tilde{h}_j{}^k(\tilde{T}_{ik}u_r-\tilde{T}_{ir}\tilde{u}_k)],\end{aligned} \tag{3.9}$$

$$\begin{aligned}\mathbf{u}\times\mathbf{T}=\hat{r}\hat{r}[\tilde{h}^{ij}\tilde{u}_i\tilde{T}_{jr}]+\hat{r}(\nabla_1 x^j)[\tilde{h}^{ik}\tilde{u}_i\tilde{T}_{kj}]+(\nabla_1 x^i)\hat{r}[h_i{}^k(\tilde{u}_k T_{rr}-u_r\tilde{T}_{kr})]\\+(\nabla_1 x^i)(\nabla_1 x^j)[h_i{}^k(\tilde{u}_k\tilde{T}_{rj}-u_r\tilde{T}_{kj})].\end{aligned} \tag{3.10}$$

If $\mathbf{R}$ is also a three-dimensional tensor field with spherical resolution (3.4), and if $\mathbf{R}^T$ means the transpose of $\mathbf{R}$, then

$$\mathbf{R}^T=\hat{r}\hat{r}R_{rr}+\hat{r}\mathbf{R}_{Sr}+\mathbf{R}_{rS}\hat{r}+\mathbf{R}_{SS}^T. \tag{3.11}$$

Moreover,

$$\begin{aligned}\mathbf{R}^T.\mathbf{T}=\hat{r}\hat{r}(R_{rr}T_{rr}+\mathbf{R}_{Sr}.\mathbf{T}_{Sr})+\hat{r}(R_{rr}\mathbf{T}_{rS}+\mathbf{R}_{Sr}.\mathbf{T}_{SS})\\+(\mathbf{R}_{rS}T_{rr}+\mathbf{R}_{SS}^T.\mathbf{T}_{Sr})\hat{r}+(\mathbf{R}_{rS}\mathbf{T}_{rS}+\mathbf{R}_{SS}^T.\mathbf{T}_{SS}),\end{aligned}$$

so

$$\begin{aligned}\mathbf{R}^T.\mathbf{T}=\hat{r}\hat{r}(R_{rr}T_{rr}+\tilde{g}^{jk}\tilde{R}_{jr}\tilde{T}_{kr})+\hat{r}(\nabla_1 x^l)(R_{rr}\tilde{T}_{rl}+\tilde{g}^{jk}\tilde{R}_{jr}\tilde{T}_{kl})\\+(\nabla_1 x^i)\hat{r}(\tilde{R}_{ri}T_{rr}+\tilde{g}^{jk}\tilde{R}_{ji}\tilde{T}_{kr})+(\nabla_1 x^i)(\nabla_1 x^l)(\tilde{R}_{ri}\tilde{T}_{rl}+\tilde{g}^{jk}\tilde{R}_{ji}\tilde{T}_{kl}).\end{aligned} \tag{3.12}$$

Therefore

$$\mathrm{tr}(\mathbf{R}^T.\mathbf{T})=\mathrm{tr}(\mathbf{R}.\mathbf{T}^T)=R_{rr}T_{rr}+\mathbf{R}_{Sr}.\mathbf{T}_{Sr}+\mathbf{R}_{rS}.\mathbf{T}_{rS}+\mathrm{tr}(\mathbf{R}_{SS}^T.\mathbf{T}_{SS}),$$

and

$$\mathrm{tr}(\mathbf{R}^T.\mathbf{T})=R_{rr}T_{rr}+\tilde{g}^{ij}(\tilde{R}_{ir}\tilde{T}_{jr}+\tilde{R}_{ri}\tilde{T}_{rj})+\tilde{g}^{ik}\tilde{g}^{jl}(\tilde{R}_{ij}\tilde{T}_{kl}). \tag{3.13}$$

D. *Spherical resolution of differential operations*

In the present subsection we seek the spherical resolutions of the fields obtained by operating with ∇ on a scalar field q, and on vector and tensor fields $\mathbf{u}$ and $\mathbf{T}$ whose spherical resolutions are given. In particular, we seek the spherical resolutions of ∇q, $\nabla\mathbf{u}$, $\nabla.\mathbf{u}$, $\nabla\times\mathbf{u}$, $\nabla\mathbf{T}$, and $\nabla.\mathbf{T}$. In everything that follows, $\tilde{D}_i$ will denote covariant differentiation on S_1.

Equations (2.1) and (3.2) provide an expression for the three-dimensional gradient in terms of the surface gradient and the radial derivative:

$$\nabla=\hat{r}\partial_r+r^{-1}\nabla_1. \tag{3.14}$$

Then the spherical resolution of ∇q is simply

$$\nabla q=\hat{r}\partial_r q+r^{-1}\nabla_1 q. \tag{3.15}$$

For the vector field $\mathbf{u}$ of (3.3),

$$\nabla\mathbf{u}=(\hat{r}\partial_r+r^{-1}\nabla_1)(\hat{r}u_r+\mathbf{u}_S).$$

We can carry out the differentiations by recalling that $\nabla_1\hat{r}=\mathbf{I}_{S_1}$ and that (2.46) gives $\nabla_1\mathbf{u}_S$. Then

$$\nabla\mathbf{u}=\hat{r}\hat{r}(\partial_r u_r)+\hat{r}(\nabla_1 x^j)(\partial_r\tilde{u}_j)+(\nabla_1 x^i)\hat{r}r^{-1}(\tilde{D}_i u_r-\tilde{u}_i)$$
$$+(\nabla_1 x^i)(\nabla_1 x^j)r^{-1}(\tilde{D}_i\tilde{u}_j+u_r\tilde{g}_{ij}). \quad (3.16)$$

Equation (3.16) is the spherical resolution of $\nabla\mathbf{u}$. By definition, $\nabla.\mathbf{u}=\mathrm{tr}(\nabla\mathbf{u})$, so

$$\nabla.\mathbf{u}=(\partial_r+2r^{-1})u_r+r^{-1}\tilde{g}^{ij}\tilde{D}_i\tilde{u}_j. \quad (3.17)$$

Further, from (3.16)

$$\nabla\times\mathbf{u}=\hat{r}\times\hat{r}(\partial_r u_r)+\hat{r}\times(\nabla_1 x^j)(\partial_r\tilde{u}_j)+(\nabla_1 x^i)\times\hat{r}r^{-1}(\tilde{D}_i u_r-\tilde{u}_i)$$
$$+(\nabla_1 x^i)\times(\nabla_1 x^j)r^{-1}(\tilde{D}_i\tilde{u}_j+u_r\tilde{g}_{ij}).$$

Since $\nabla_1 x^i\times\nabla_1 x^j=\tilde{h}^{ij}\hat{r}$ and $\hat{r}\times\nabla_1 x^i=\tilde{h}^{ij}\partial_j\mathbf{r}$, therefore

$$\nabla\times\mathbf{u}=\hat{r}r^{-1}\tilde{h}^{ij}(\tilde{D}_i\tilde{u}_j)+(\nabla_1 x^i)r^{-1}\tilde{h}_i{}^j[\tilde{D}_j u_r-\partial_r(r\tilde{u}_j)]. \quad (3.18)$$

For the tensor field $\mathbf{T}$ of (3.4),

$$\nabla\mathbf{T}=(\hat{r}\partial_r+r^{-1}\nabla_1)(\hat{r}\hat{r}T_{rr}+\hat{r}\mathbf{T}_{rS}+\mathbf{T}_{Sr}\hat{r}+\mathbf{T}_{SS}).$$

Then from (2.45), (2.47), (2.48) and (2.49),

$$\nabla\mathbf{T}=\hat{r}\hat{r}\hat{r}[\partial_r T_{rr}]+\hat{r}\hat{r}(\nabla_1 x^k)[\partial_r\tilde{T}_{rk}]+\hat{r}(\nabla_1 x^j)\hat{r}[\partial_r\tilde{T}_{jr}]$$
$$+(\nabla_1 x^i)\hat{r}\hat{r}[r^{-1}(\tilde{D}_i T_{rr}-\tilde{T}_{ri}-\tilde{T}_{ir})]+\hat{r}(\nabla_1 x^j)(\nabla_1 x^k)[\partial_r\tilde{T}_{ij}]$$
$$+(\nabla_1 x^i)\hat{r}(\nabla_1 x^k)[r^{-1}(\tilde{D}_i\tilde{T}_{rk}-\tilde{T}_{ik}+T_{rr}\tilde{g}_{ik})]$$
$$+(\nabla_1 x^i)(\nabla_1 x^j)\hat{r}[r^{-1}(\tilde{D}_i\tilde{T}_{jr}-\tilde{T}_{ji}+T_{rr}\tilde{g}_{ij})]$$
$$+(\nabla_1 x^i)(\nabla_1 x^j)(\nabla_1 x^k)[r^{-1}(\tilde{D}_i\tilde{T}_{jk}+\tilde{g}_{ij}\tilde{T}_{rk}+\tilde{g}_{ik}\tilde{T}_{jr})]. \quad (3.19)$$

Then $\nabla.\mathbf{T}$ is obtained from (3.19) by contracting the first two members of each triad. Thus

$$\nabla.\mathbf{T}=\hat{r}\{r^{-1}[(r\partial_r+2)T_{rr}+\tilde{g}^{ij}\tilde{D}_i\tilde{T}_{jr}-\tilde{g}^{ij}\tilde{T}_{ij}]\}$$
$$+(\nabla_1 x^k)\{r^{-1}[(r\partial_r+2)\tilde{T}_{kr}+\tilde{T}_{rk}+\tilde{g}^{ij}\tilde{D}_i\tilde{T}_{jk}]\}. \quad (3.20)$$

In evaluating the right-hand sides of (3.16), (3.17), (3.18), (3.19) and (3.20) it must be remembered that u_r and T_{rr} are scalar fields on S_1, so $\tilde{D}_i u_r=\partial_i u_r$ and $\tilde{D}_i T_{rr}=\partial_i T_{rr}$; and that $\mathbf{T}_{rS}$ and $\mathbf{T}_{Sr}$ are tangent vector fields on S_1, so that $\tilde{D}_i\tilde{T}_{rj}$ and $\tilde{D}_i\tilde{T}_{jr}$ are given by (2.22); and that $\mathbf{T}_{SS}$ is a tangent tensor field on S_1, so that $\tilde{D}_i\tilde{T}_{jk}$ is given by (2.26).

In case $\mathbf{T}$ is symmetric, comparison of (3.4) and (3.11) shows that $\mathbf{T}_{rS}=\mathbf{T}_{Sr}$ and that $\mathbf{T}_{SS}$ is symmetric. Then (3.20) is slightly simplified:

$$\nabla.\mathbf{T}=\hat{r}\{r^{-1}[(r\partial_r+2)T_{rr}+\tilde{g}^{ij}\tilde{D}_i\tilde{T}_{rj}-\tilde{g}^{ij}\tilde{T}_{ij}]\}$$
$$+(\nabla_1 x^k)\{r^{-1}[(r\partial_r+3)\tilde{T}_{rk}+\tilde{g}^{ij}\tilde{D}_i\tilde{T}_{jk}]\}. \quad (3.21)$$

4. The vector and tensor representation theorems

A. *Statement of the surface theorems*

The spherical resolution of a vector field $\mathbf{u}$ resolves that field into a scalar part u_r and a tangent vector part $\mathbf{u}_S$. The spherical resolution of a second-order tensor field $\mathbf{T}$ resolves that field into a scalar part T_{rr}, two tangent vector parts $\mathbf{T}_{rS}$ and $\mathbf{T}_{Sr}$, and a tangent tensor part $\mathbf{T}_{SS}$. The fields $\mathbf{u}$ and $\mathbf{T}$ could be expressed entirely in terms of scalar fields if any tangent vector or second-order tangent tensor field could be so expressed. That such expressions exist is the content of the Tangent Vector and Tangent Tensor Representation Theorems (see Backus 1966).

As a preliminary to stating these theorems, consider the expansion of any scalar field q on S_1 in surface spherical harmonics:

$$q(x^1, x^2) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} q_n{}^m Y_n{}^m(x^1, x^2).$$

Here $Y_n{}^m$ is a normalized complex surface spherical harmonic of total order n and with ϕ (azimuth) dependence $e^{im\phi}$; and $q_n{}^m$ is a complex constant. Let $\mathscr{Y}_n$ denote the linear operator on scalar fields defined by the equation

$$\mathscr{Y}_n q = \sum_{m=-n}^{n} q_n{}^m Y_n^m. \tag{4.1}$$

Then $(\mathscr{Y}_n)^2 = \mathscr{Y}_n$ and $\mathscr{Y}_n \mathscr{Y}_{n'} = 0$ if $n \neq n'$. Furthermore, since $\nabla_1{}^2 Y_n{}^m = -n(n+1) Y_n{}^m$,

$$\nabla_1{}^2 \mathscr{Y}_n = -n(n+1) \mathscr{Y}_n. \tag{4.2}$$

Let $\mathscr{Q}_n$ be defined as

$$\mathscr{Q}_n = \mathscr{I} - \mathscr{Y}_n, \tag{4.3}$$

where $\mathscr{I}$ is the identity operator. Thus $\mathscr{Q}_n q = q - \mathscr{Y}_n q$. Then $(\mathscr{Q}_n)^2 = \mathscr{Q}_n$ and if $n \neq n'$ then $\mathscr{Q}_n \mathscr{Q}_{n'} = \mathscr{I} - \mathscr{Y}_n - \mathscr{Y}_{n'}$. Thus, applying $\mathscr{Q}_n$ to q simply removes from q the terms of order n in q's expansion in surface spherical harmonics.

The Tangent Vector Representation Theorem asserts that if $\mathbf{u}_S$ is any tangent vector field on S_1, there exist unique scalar fields V and W on S_1 such that

$$\mathscr{Y}_0 V = \mathscr{Y}_0 W = 0, \tag{4.4}$$

and

$$\mathbf{u}_S = \nabla_1 V - \hat{r} \times \nabla_1 W. \tag{4.5}$$

In curvilinear co-ordinates x^1, x^2 on S_1, the covariant components of (4.5) are

$$\tilde{u}_i = \tilde{D}_i V + \tilde{h}_i{}^j \tilde{D}_j W, \tag{4.6}$$

where, V and W being scalars, $\tilde{D}_i = \partial_i$. Note that in general if $\nabla_1 V - \hat{r} \times \nabla_1 W = 0$ we cannot assert that $V = W = 0$ but only that $\mathscr{Y}_0 V = \mathscr{Y}_0 W = 0$. That is, V and W may be non-zero constants. Similarly, if $\nabla_1 V - \hat{r} \times \nabla_1 W = \nabla_1 V' - \hat{r} \times \nabla_1 W'$, we have only $\mathscr{Y}_0 V = \mathscr{Y}_0 V'$, $\mathscr{Y}_0 W = \mathscr{Y}_0 W'$; it may be that $V - V'$ and $W - W'$ are non-zero constants.

The Tangent Tensor Representation Theorem asserts that if $\mathbf{T}_{SS}$ is any second-order tangent tensor field on S_1 there exist unique scalar fields H, L, M, and N such that

$$\mathscr{Y}_0 M = \mathscr{Y}_0 N = \mathscr{Y}_1 M = \mathscr{Y}_1 N = 0, \tag{4.7}$$

and

$$\mathbf{T}_{SS} = -H \mathbf{I}_{S_1} \times \hat{r} + (L - \nabla_1{}^2 M) \mathbf{I}_{S_1} + \mathscr{P}\{2 \nabla_1 \nabla_1 M - [\nabla_1 (\hat{r} \times \nabla_1) + (\hat{r} \times \nabla_1) \nabla_1] N\}. \tag{4.8}$$

In equation (4.8), $\nabla_1{}^2$ means tr $(\nabla_1 \nabla_1)$ and $\mathscr{P}\mathbf{T}$ means the purely tangential part of $\mathbf{T}$, the last term on the right in equations (3.4). In curvilinear co-ordinates x^1, x^2 on S_1, the covariant components of (4.8) are

$$\tilde{T}_{ij} = H\tilde{h}_{ij} + (L - \nabla_1{}^2 M)\tilde{g}_{ij} + 2\tilde{D}_i \tilde{D}_j M + (\tilde{h}_i{}^k \tilde{D}_j + \tilde{h}_j{}^k \tilde{D}_i)\tilde{D}_k N \tag{4.9}$$

where

$$\nabla_1{}^2 M = \tilde{g}^{ij} \tilde{D}_i \tilde{D}_j M = \tilde{g}^{ij}\left[\partial_i \partial_j M - \binom{k}{ij}\partial_k M\right]. \tag{4.10}$$

In general, if $\tilde{T}_{ij}$ and $\tilde{T}'_{ij}$ are both represented as in (4.9) and $\tilde{T}_{ij} = \tilde{T}'_{ij}$, we can conclude that $H = H'$ and $L = L'$, but only that $\mathscr{Q}_0 \mathscr{Q}_1 M = \mathscr{Q}_0 \mathscr{Q}_1 M'$ and $\mathscr{Q}_0 \mathscr{Q}_1 N = \mathscr{Q}_0 \mathscr{Q}_1 N'$. If we know beforehand that M, N, M' and N' satisfy equations (4.7) then $\tilde{T}_{ij} = \tilde{T}'_{ij}$ implies that $M = M'$ and $N = N'$.

B. *Scalar representation of vector and tensor fields*

If $\mathbf{u}$ is any three-dimensional vector field defined inside a spherical shell centred on the origin, and $\mathbf{u}$ is spherically resolved as in equation (3.3), the Tangent Vector Representation Theorem asserts the existence of unique scalar fields $V(r, x^1, x^2)$ and $W(r, x^1, x^2)$ such that for each fixed r equations (4.4) and (4.6) hold. Then

$$\mathbf{u} = \hat{r}U + (\nabla_1 x^i)[\tilde{D}_i V + \tilde{h}_i{}^j \tilde{D}_j W], \tag{4.11}$$

where $U = u_r$. Equation (4.11) will be called the scalar representation of $\mathbf{u}$.

If $\mathbf{T}$ is any second-order, three-dimensional tensor field defined inside a spherical shell centred on the origin, and $\mathbf{T}$ is spherically resolved as in equation (3.4), the Tangent Tensor Representation Theorem asserts the existence of unique scalar fields $H(r, x^1, x^2)$, $L(r, x^1, x^2)$, $M(r, x^1, x^2)$, and $N(r, x^1, x^2)$ such that for each fixed r equations (4.7) and (4.9) hold. The Tangent Vector Representation Theorem asserts the existence of unique scalars $Q_{rS}(r, x^1, x^2)$, $R_{rS}(r, x^1, x^2)$, $Q_{Sr}(r, x^1, x^2)$, and $R_{Sr}(r, x^1, x^2)$ satisfying (4.4) for each fixed r and such that

$$\left.\begin{aligned} \tilde{T}_{ri} &= \tilde{D}_i Q_{rS} - \tilde{h}_i{}^j \tilde{D}_j R_{rS}, \\ \tilde{T}_{ir} &= \tilde{D}_i Q_{Sr} + \tilde{h}_i{}^j \tilde{D}_j R_{Sr}. \end{aligned}\right\} \tag{4.12}$$

Then, setting $T_{rr} = P$, we can write

$$\begin{aligned} \mathbf{T} = \hat{r}\hat{r}P + \hat{r}(\nabla_1 x^i)[\tilde{D}_i Q_{rS} + \tilde{h}_i{}^j \tilde{D}_j R_{rS}] + (\nabla_1 x_i)\hat{r}[\tilde{D}_i Q_{Sr} + \tilde{h}_i{}^j \tilde{D}_j R_{Sr}] \\ + (\nabla_1 x^i)(\nabla_1 x^j)[H\tilde{h}_{ij} + (L - \nabla_1{}^2 M)\tilde{g}_{ij} + 2\tilde{D}_i \tilde{D}_j M + (\tilde{h}_i{}^k \tilde{D}_j + \tilde{h}_j{}^k \tilde{D}_i)\tilde{D}_k N]. \end{aligned} \tag{4.13}$$

Equation (4.13) will be called the scalar representation of $\mathbf{T}$. If $\mathbf{T}$ is symmetric, $Q_{rS} = Q_{Sr} = Q$, $R_{rS} = R_{Sr} = R$, and $H = 0$. Conversely, if these scalar relations hold, $\mathbf{T}$ is symmetric.

C. *Scalar representation of algebraic operations*

If $\mathbf{u}$ and $\mathbf{u}'$ are three-dimensional vector fields and $\mathbf{T}$ and $\mathbf{T}'$ are three-dimensional second-order tensor fields defined inside a spherical shell centred on the origin, then the nine equations (3.5) through (3.13) can be expressed in terms of the scalars in the representations (4.11) and (4.13). All that is involved is simple substitution and some applications of (2.32). Of the nine scalar expressions, here I quote only the three which appear to be most useful in continuum mechanics:

$$\begin{aligned} \mathbf{u}.\mathbf{u}' = UU' + \tilde{g}^{ij}[(\tilde{D}_i V)(\tilde{D}_j V') + (\tilde{D}_i W)(\tilde{D}_j W')] \\ + \tilde{h}^{ij}[(\tilde{D}_i V)(\tilde{D}_j W') + (\tilde{D}_i V')(\tilde{D}_j W)]; \end{aligned} \tag{4.14}$$

$$\mathbf{u}\times\mathbf{u}'=\hat{r}\{\tilde{g}^{ij}[(\tilde{D}_iV')(\tilde{D}_jW)-(\tilde{D}_iV)(\tilde{D}_jW')]+\tilde{h}^{ij}[(\tilde{D}_iV)(\tilde{D}_jV')+(\tilde{D}_iW)(\tilde{D}_jW')]\}$$
$$+(\nabla_1 x^i)\{[U(\tilde{D}_iW')-U'(\tilde{D}_iW)]+\tilde{h}_i{}^j[U'(\tilde{D}_jV)-U(\tilde{D}_jV')]\}; \quad (4.15)$$

$$\mathrm{tr}\,(\mathbf{T}^T.\mathbf{T}')=PP'+\tilde{g}^{ij}\,[(\tilde{D}_i\,Q_{rS})(\tilde{D}_j\,Q'_{rS})+(\tilde{D}_i\,R_{rS})(\tilde{D}_j\,R'_{rS})$$
$$+(\tilde{D}_i\,Q_{Sr})(\tilde{D}_j\,Q'_{Sr})+(\tilde{D}_i\,R_{Sr})(\tilde{D}_j\,R'_{Sr})]+\tilde{h}^{ij}[(\tilde{D}_i\,Q_{rS})(\tilde{D}_j\,R'_{rS})$$
$$+(\tilde{D}_i\,Q'_{rS})(\tilde{D}_j\,R_{rS})+(\tilde{D}_i\,Q_{Sr})(\tilde{D}_j\,R'_{Sr})+(\tilde{D}_i\,Q'_{Sr})(\tilde{D}_j\,R_{Sr})]$$
$$+2HH'+2LL'+2(\nabla_1{}^2M)(\nabla_1{}^2M')+2(\nabla_1{}^2N)(\nabla_1{}^2N')$$
$$-4\tilde{h}^{ij}\tilde{h}^{kl}[(\tilde{D}_i\,\tilde{D}_k\,M)(\tilde{D}_j\,\tilde{D}_l\,M')+(\tilde{D}_i\,\tilde{D}_k N)(\tilde{D}_j\,\tilde{D}_l N')]$$
$$+4\tilde{h}^{ij}\tilde{g}^{kl}[(\tilde{D}_i\,\tilde{D}_k\,M)(\tilde{D}_j\,\tilde{D}_l N')+(\tilde{D}_i\,\tilde{D}_k\,M')(\tilde{D}_j\,\tilde{D}_l N)]. \quad (4.16)$$

For most spherically symmetrical problems, expressions (4.14) and (4.16) can be greatly simplified. In most such problems, $\mathbf{u}.\mathbf{u}'$ or $\mathrm{tr}\,(\mathbf{T}^T.\mathbf{T}')$ will occur as the kernel f in a volume integral of the form

$$\int_V \psi f\, dV,$$

where V is the spherical shell in which the vector or tensor fields are defined and ψ is a weighting function which depends only on r. To evaluate such integrals it is sufficient to know the surface integral of f on each S_r in V. Therefore, when

$$\int_{S_r} (f_1-f_2)\,dA=0$$

for each S_r in V, we will say that f_1 and f_2 are spherically equivalent functions, and we will write

$$f_1\equiv f_2.$$

For each fixed r, $f_1(r, x^1, x^2)$ and $f_2(r, x^1, x^2)$ can be regarded as scalar fields on S_1. Then the condition that they be spherically equivalent is that for each fixed r in V

$$\int_{S_1} [f_1(r, x^1, x^2)-f_2(r, x^1, x^2)]\,dA=0.$$

Gauss's theorem for converting surface to line integrals is the source of the simpler expressions spherically equivalent to (4.14) and (4.16). Since S_1 has no boundary, if $\mathbf{u}_S$ is any continuously differentiable tangent vector field on S_1 Gauss's theorem asserts that

$$\int_{S_1} (\tilde{D}_i\tilde{u}_j)\,g^{ij}\,dA=0,$$

or $\tilde{g}^{ij}\,\tilde{D}_i\tilde{u}_j\equiv 0$. Therefore $\tilde{g}^{ij}\,\tilde{D}_i[V\tilde{D}_jV']\equiv 0$, so $\tilde{g}^{ij}(\tilde{D}_i\,V)(\tilde{D}_jV')\equiv -V(\nabla_1{}^2\,V')$. Similarly $\tilde{g}^{ij}(\tilde{D}_iV)(\tilde{D}_jV')\equiv -V'(\nabla_1{}^2\,V)$. This type of argument, together with the identities (2.31), (2.32), and (2.33), leads to the following conclusions:

$$\mathbf{u}.\mathbf{u}'\equiv UU'-V'(\nabla_1{}^2\,V)-W'(\nabla_1{}^2W); \quad (4.17)$$

$$\tfrac{1}{2}\mathrm{tr}\,(\mathbf{T}^T.\mathbf{T}')\equiv\tfrac{1}{2}PP'-\tfrac{1}{2}Q'_{rS}(\nabla_1{}^2\,Q_{rS})-\tfrac{1}{2}R'_{rS}(\nabla_1{}^2\,R_{rS})-\tfrac{1}{2}Q'_{Sr}(\nabla_1{}^2\,Q_{Sr})$$
$$-\tfrac{1}{2}R'_{Sr}(\nabla_1{}^2\,R_{Sr})+HH'+LL'+[\nabla_1{}^2\,M'][(\nabla_1{}^2+2)\,M]$$
$$+[\nabla_1{}^2N'][(\nabla_1{}^2+2)\,N]. \quad (4.18)$$

Since $\mathbf{u}.\mathbf{u}'=\mathbf{u}'.\mathbf{u}$ and $\mathrm{tr}(\mathbf{T}^T.\mathbf{T}')=\mathrm{tr}(\mathbf{T}'^T.\mathbf{T})$, the primed and unprimed scalars in (4.17) and (4.18) can be interchanged. If $\mathbf{T}$ and $\mathbf{T}'$ are symmetric, then $Q_{rS}=Q_{Sr}=Q$, $R_{rS}=R_{Sr}=R$, and $H=0$, and similarly for $\mathbf{T}'$. Then equation (4.18) becomes

$$\tfrac{1}{2}\mathrm{tr}(\mathbf{T}^T.\mathbf{T}')\equiv\tfrac{1}{2}PP'-Q'(\nabla_1{}^2 Q)-R'(\nabla_1{}^2 R)+LL' + [\nabla_1{}^2 M'][(\nabla_1{}^2+2)M]+[\nabla_1{}^2 N'][(\nabla_1{}^2+2)N]. \quad (4.19)$$

Another algebraic operation of some interest in continuum mechanics is the calculation of the trace $\mathrm{tr}\,\mathbf{T}$ and deviator $\mathscr{D}\mathbf{T}$ of a second-order tensor $\mathbf{T}$. From (4.13),

$$\mathrm{tr}\,\mathbf{T}=P+2L \quad (4.20)$$

while by its definition

$$\mathscr{D}\mathbf{T}=\mathbf{T}-\tfrac{1}{3}\mathbf{I}\,(\mathrm{tr}\,\mathbf{T}).$$

Since $\mathbf{I}=\hat{r}\hat{r}+\tilde{g}_{ij}(\nabla_1 x^i)(\nabla_1 x^j)$, equation (4.13) implies that $\mathscr{D}\mathbf{T}$ has the same scalars as $\mathbf{T}$, except that

$$\left.\begin{aligned} P_{\mathscr{D}\mathbf{T}}&=\tfrac{2}{3}(P-L),\\ L_{\mathscr{D}\mathbf{T}}&=\tfrac{1}{3}(L-P).\end{aligned}\right\} \quad (4.21)$$

Therefore $P_{\mathscr{D}\mathbf{T}}+2L_{\mathscr{D}\mathbf{T}}=0$, as expected from the fact that $\mathrm{tr}\,\mathscr{D}\mathbf{T}=0$.

For any second-order tensors $\mathbf{T}$ and $\mathbf{T}'$,

$$\mathrm{tr}(\mathbf{T}^T.\mathbf{T}')=\tfrac{1}{3}(\mathrm{tr}\,\mathbf{T})(\mathrm{tr}\,\mathbf{T}')+\mathrm{tr}\,[(\mathscr{D}\mathbf{T})^T.\mathscr{D}\mathbf{T}']. \quad (4.22)$$

From (4.20),

$$\tfrac{1}{3}(\mathrm{tr}\,\mathbf{T})(\mathrm{tr}\,\mathbf{T}')=\tfrac{1}{3}(P+2L)(P'+2L'). \quad (4.23)$$

From (4.21) and (4.19), if $\mathbf{T}$ and $\mathbf{T}'$ are symmetric

$$\tfrac{1}{2}\mathrm{tr}\,[(\mathscr{D}\mathbf{T})^T.\mathscr{D}\mathbf{T}']\equiv\tfrac{1}{3}(P-L)(P'-L')-Q'(\nabla_1{}^2 Q) -R'(\nabla_1{}^2 R)+[\nabla_1{}^2 M'][(\nabla_1{}^2+2)M]+[\nabla_1{}^2 N'][(\nabla_1{}^2+2)N]. \quad (4.24)$$

Therefore the energy densities of compression (4.23) and shear (4.24) are easy to separate in terms of the scalar representation of the stress tensor.

D. *Scalar representation of differential operations*

If $\mathbf{u}$ is a three-dimensional vector field and $\mathbf{T}$ a three-dimensional second-order tensor field inside a spherical shell centred on the origin, with the help of (4.11) and (4.13) we can express $\nabla\mathbf{u}$, $\nabla.\mathbf{u}$, $\nabla\times\mathbf{u}$, and $\nabla.\mathbf{T}$ in terms of scalars.

From equations (3.16) and (4.11) we can obtain a scalar expression for $\nabla\mathbf{u}$, but this expression is not quite in the form (4.13) for the scalar representation of a second-order tensor field. To put it in that form we note from (2.33) that

$$\tilde{h}_j{}^k\tilde{D}_i\tilde{D}_k-\tilde{h}_i{}^k\tilde{D}_j\tilde{D}_k=-\tilde{h}_{ij}\nabla_1{}^2.$$

Therefore

$$\begin{aligned}\nabla\mathbf{u}=\hat{r}\hat{r}[\partial_r U]&+\hat{r}(\nabla_1 x^i)[\tilde{D}_i(\partial_r V)+\tilde{h}_i{}^j\tilde{D}_j(\partial_r W)]\\ &+(\nabla_1 x^i)\hat{r}[\tilde{D}_i(r^{-1}U-r^{-1}V)+\tilde{h}_i{}^j\tilde{D}_j(-r^{-1}W)]\\ &+(\nabla_1 x^i)(\nabla_1 x^j)(2r)^{-1}\{-\tilde{h}_{ij}(\nabla_1{}^2 W)+[(2U+\nabla_1{}^2 V)-\nabla_1{}^2 V]\tilde{g}_{ij}\\ &+2\tilde{D}_i\tilde{D}_jV+(\tilde{h}_i{}^k\tilde{D}_j+\tilde{h}_j{}^k\tilde{D}_i)\tilde{D}_kW\}.\end{aligned} \quad (4.25)$$

If we contract $\nabla\mathbf{u}$ in (4.25) we obtain $\operatorname{tr}(\nabla\mathbf{u})$, or $\nabla.\mathbf{u}$:

$$\nabla.\mathbf{u}=r^{-1}[(r\partial_r+2)U+\nabla_1{}^2 V], \tag{4.26}$$

in agreement with (3.17) and (4.6). From either (4.25) or (3.18) and (4.6),

$$\nabla\times\mathbf{u}=-\hat{r}[r^{-1}\nabla_1{}^2 W]+(\nabla_1 x^i)r^{-1}\{\tilde{D}_i[\partial_r(rW)]+\tilde{h}_i{}^j\tilde{D}_j[U-\partial_r(rV)]\}. \tag{4.27}$$

From (3.20) and (4.13),

$$\nabla.\mathbf{T}=\hat{r}r^{-1}[(r\partial_r+2)P+\nabla_1{}^2 Q_{Sr}-2L]+(\nabla_1 x^i)r^{-1}\{\tilde{D}_i[(r\partial_r+2)Q_{rS}+Q_{Sr}]$$
$$+\tilde{h}_i{}^j\tilde{D}_j[(r\partial_r+2)R_{rS}+R_{Sr}]+\tilde{g}^{jk}\tilde{D}_j\tilde{T}_{ki}\}.$$

To put this equation in the form of the scalar representation (4.9) for the vector field $\nabla.\mathbf{T}$ we must evaluate $\tilde{g}^{jk}\tilde{D}_j\tilde{T}_{ki}$, $\tilde{T}_{ki}$ being given by (4.9). Several applications of (2.31), (2.32) and (2.33) lead to the result

$$\tilde{g}^{jk}\tilde{D}_j\tilde{T}_{ki}=\tilde{D}_i[L+(\nabla_1{}^2+2)M]+\tilde{h}_i{}^j\tilde{D}_j[-H+(\nabla_1{}^2+2)N]. \tag{4.28}$$

Therefore the scalar representation for the vector field $\nabla.\mathbf{T}$ is

$$\begin{aligned}\nabla.\mathbf{T}=\hat{r}r^{-1}&[(r\partial_r+2)P+\nabla_1{}^2 Q_{Sr}-2L]\\&+(\nabla_1 x^i)r^{-1}\{\tilde{D}_i[(r\partial_r+2)Q_{rS}+Q_{Sr}+L+(\nabla_1{}^2+2)M]\\&+\tilde{h}_i{}^j\tilde{D}_j[(r\partial_r+2)R_{rS}+R_{Sr}-H+(\nabla_1{}^2+2)N]\}.\end{aligned} \tag{4.29}$$

In case $\mathbf{T}$ is symmetric, $Q_{Sr}=Q_{rS}=Q$, $R_{Sr}=R_{rS}=R$, and $H=0$, so

$$\nabla.\mathbf{T}=\hat{r}r^{-1}[(r\partial_r+2)P+\nabla_1{}^2 Q-2L]+(\nabla_1 x^i)r^{-1}\{\tilde{D}_i[(r\partial_r+3)Q+L+(\nabla_1{}^2+2)M]$$
$$+\tilde{h}_i{}^j\tilde{D}_j[(r\partial_r+3)R+(\nabla_1{}^2+2)N]\}. \tag{4.30}$$

In using expressions (4.25), (4.26), (4.27), (4.29) and (4.30) it is important to remember that any function of r alone commutes with $\tilde{D}_i$.

5. Geophysical applications

A. *Equilibrium configurations in a nearly spherical, nearly hydrostatic Earth*

In this subsection we propose to catalogue all possible stress fields in the Earth's mantle which permit the Earth to be in static equilibrium. No hypothesis is made about the physical origin of the stress, so the catalogue is large. To restrict it, we note that any rotating equilibrium Earth model with small ellipticity can be obtained in a well-known way (Jeffreys 1959) by adding a small hydrostatic pressure field to some non-rotating equilibrium Earth model. Therefore we consider only non-rotating Earth models.

Of course the Earth's density structure has an important influence on its internal stress distribution, so we must admit the possibility of asphericity in the density, and in the upper and lower boundaries of the mantle. In order to keep the problem linear, and to imitate reality as far as it is known, we will assume that all such asphericities are small, and will neglect their products. No other approximations will be made.

Consider, then, a stationary (non-rotating) nearly spherical body. Choose an origin of co-ordinates near but not necessarily at its centre of mass. Let (r, x^1, x^2)

be generalized spherical co-ordinates centred on the chosen origin. Suppose the body to be fluid in a 'core' $0 \leqslant r \leqslant a[1+\alpha(x^1, x^2)]$, and solid in a 'mantle' $a[1+\alpha(x^1, x^2)] \leqslant r \leqslant b[1+\beta(x^1, x^2)]$; here $|\alpha| \ll 1$ and $|\beta| \ll 1$. A suitable choice of a and b permits us to assume without loss of generality that

$$\mathscr{Y}_0 \alpha = \mathscr{Y}_0 \beta = 0. \tag{5.1}$$

Suppose that in the core the density of the body is

$$\rho = \rho_0{}^C(r) + \rho_1{}^C(r, x^1, x^2),$$

and in the mantle the density is

$$\rho = \rho_0{}^M(r) + \rho_1{}^M(r, x^1, x^2),$$

where $|\rho_1{}^C| \ll \rho_0{}^C$ and $|\rho_1{}^M| \ll \rho_0{}^M$. Define $\rho_0(r)$ as $\rho_0{}^C(r)$ and $\rho_1(r)$ as $\rho_1{}^C(r)$ when $0 \leqslant r \leqslant a$; define $\rho_0(r)$ as $\rho_0{}^M(r)$ and $\rho_1(r)$ as $\rho_1{}^M(r)$ when $a < r \leqslant b$; define $\rho_0(r)$ and $\rho_1(r)$ as 0 when $b < r < \infty$. It will be necessary to continue smoothly $\rho_0{}^C$, $\rho_0{}^M$, $\rho_1{}^C$ and $\rho_1{}^M$ for short distances out of their domains of definition to obtain ρ_0 and ρ_1. A suitable choice of $\rho_0{}^C$ and $\rho_0{}^M$ permits us to assume without loss of generality that for each fixed r

$$\mathscr{Y}_0 \rho_1 = 0. \tag{5.2}$$

Now we define $\phi_0(r)$ as the gravitational potential of the spherically symmetric mass distribution ρ_0, and we define $g_0(r) = \partial_r \phi_0(r)$. Then the gravitational field of this spherical mass is $-g_0(r)\hat{r}$, and Poisson's equation for it is, for all r in $0 \leqslant r < \infty$,

$$(\partial_r + 2r^{-1}) g_0(r) = 4\pi G \rho_0(r). \tag{5.3}$$

Here G is Newton's universal constant of gravitation. Both ϕ_0 and g_0 are continuous for all r and vanish at $r=\infty$, while g_0 also vanishes at $r=0$.

In the slightly aspherical body which actually confronts us, the gravitational potential is $\phi_0(r) + \phi_1(r, x^1, x^2)$ where

$$\nabla^2 \phi_1 = 4\pi G(\rho - \rho_0)$$

and ϕ_1 and $\nabla\phi_1$ are continuous everywhere and vanish at infinity. Correct to the first order in the small asphericities α, β, ρ_1/ρ_0 we can obtain ϕ_1 by solving the equation

$$\nabla^2 \phi_1 = 4\pi G \rho_1 \tag{5.4}$$

for all r in $0 \leqslant r < \infty$, subject to the boundary conditions that ϕ_1 and $\partial_r \phi_1$ vanish at $r=\infty$ and be continuous everywhere, except that at $r=a$

$$[\partial_r \phi_1 + 4\pi G \rho_0 a\alpha]_-^+ = 0, \tag{5.5}$$

while at $r=b$

$$[\partial_r \phi_1 + 4\pi G \rho_0 b\beta]_-^+ = 0. \tag{5.6}$$

In the last two equations $[f(a)]_-^+$ means $\lim_{\varepsilon \to 0+} [f(a+\varepsilon) - f(a-\varepsilon)]$. The first order approximation ϕ_1 satisfies

$$\mathscr{Y}_0 \phi_1 = 0 \tag{5.7}$$

for each fixed r, because of (5.1) and (5.2). Inside the body, which is where we really want it, we can find ϕ_1 by solving (5.4) subject to the boundary condition (5.5) at $r=a$, and to the condition that at $r=b$ for $n=0, 1, 2, \ldots$, we have

$$(r\partial_r+n+1)\mathscr{Y}_n\phi_1+4\pi Gb^2\rho_0(b-)\mathscr{Y}_n\beta=0. \quad (5.8)$$

Still another formulation of the equations for ϕ_1 is obtained by defining

$$\rho'_1=\rho_1-a\alpha[\rho_0(a)]_-^+\delta(r-a)-b\beta[\rho_0(b)]_-^+\delta(r-b), \quad (5.9)$$

where δ is the Dirac delta function. Then equations (5.4), (5.5) and (5.6) are equivalent to the single equation

$$\nabla^2\phi_1=4\pi G\rho'_1, \quad (5.10)$$

valid in $0\leqslant r<\infty$.

If the spherical body whose gravitational potential is ϕ_0 were a fluid in hydrostatic equilibrium, the hydrostatic pressure $p_0(r)$ would be determined by the equation

$$\partial_r p_0+\rho_0 g_0=0$$

and the boundary condition

$$p_0(b)=0.$$

In the aspherical body which actually confronts us we define $p_0(r)$ by these equations, and write the stress tensor as $-p_0(r)\mathbf{I}+\mathbf{T}(r, x^1, x^2)$. We take p_0 and $\mathbf{T}$ to be 0 in $b<r<\infty$. We make no assumption that $\mathbf{T}/p_0$ is small, but we will show this to be so in the core. In the real Earth's mantle, the magnitude of the deviator of $\mathbf{T}$ is limited by the shear strength of the material, but this limitation does not enter the present discussion. The present theory imposes no *a priori* limits on $\mathbf{T}$ in the mantle.

If we neglect terms of second order in the asphericity, such as $\rho_1\nabla\phi_1$, the equilibrium conditions in the body are

$$\nabla.\mathbf{T}=\rho_0\nabla\phi_1+\rho_1\nabla\phi_0$$

while the boundary conditions are

$$[\hat{r}.\mathbf{T}+\hat{r}\rho_0 g_0 a\alpha]_-^+=0$$

at $r=a$ and

$$[\hat{r}.\mathbf{T}+\hat{r}\rho_0 g_0 b\beta]_-^++\mathbf{F}=0$$

at $r=b$. Here $\mathbf{F}$ is the external stress, if any, applied to the outer surface of the body. The integrals of $\mathbf{F}$ and $\mathbf{r}\times\mathbf{F}$ over the surface $r=b$ must vanish, because we cannot expect static equilibrium if there is a net external force or torque on the body. The equilibrium equation and boundary conditions are all included in the single equation

$$\nabla.\mathbf{T}+\mathbf{F}\delta(r-b)=\rho_0\nabla\phi_1+\rho'_1\nabla\phi_0, \quad (5.11)$$

valid for $0\leqslant r<\infty$ if ρ'_1 is defined by (5.9).

We can always write $\mathbf{F}$ in the form

$$\mathbf{F}=\hat{r}P'+\nabla_1 x^i[\tilde{D}_i Q'+\tilde{n}_i{}^j\tilde{D}_j R'], \quad (5.12)$$

where $\mathscr{Y}_0 Q'=\mathscr{Y}_0 R'=0$. The conditions that $\mathbf{F}$ produce no net torque or net force on the body are, respectively,

$$\mathscr{Y}_1 R'=0, \quad \mathscr{Y}_1(P'+2Q')=0, \quad (5.13)$$

equations which the reader can easily verify. We can always find unique scalars P, Q, R, L, M, N such that for every fixed r

$$\mathscr{Y}_0 Q = \mathscr{Y}_0 R = \mathscr{Y}_0 M = \mathscr{Y}_0 N = \mathscr{Y}_1 M = \mathscr{Y}_1 N = 0, \tag{5.14}$$

while

$$\mathbf{T} = \hat{r}\hat{r}P + [\hat{r}(\nabla_1 x^i) + (\nabla_1 x^i)\hat{r}][\tilde{D}_i Q + \tilde{h}_i{}^j \tilde{D}_j R] \\ + (\nabla_1 x^i)(\nabla_1 x^j)[(L - \nabla_1{}^2 M)\tilde{g}_{ij} + 2\tilde{D}_i \tilde{D}_j M + (\tilde{h}_i{}^k \tilde{D}_j + \tilde{h}_j{}^k \tilde{D}_i)\tilde{D}_k N]. \tag{5.15}$$

Then equation (5.11) has the scalar form

$$\left.\begin{aligned} (r\partial_r + 2)P + \nabla_1{}^2 Q - 2L + bP'\delta(r-b) &= r[\rho_0 \partial_r \phi_1 + \rho'_1 \partial_r \phi_0], \\ (r\partial_r + 3)Q + (\nabla_1{}^2 + 2)M + \mathscr{Q}_0 L + bQ'\delta(r-b) &= \rho_0 \phi_1, \\ (r\partial_r + 3)R + (\nabla_1{}^2 + 2)N + bR'\delta(r-b) &= 0. \end{aligned}\right\} \tag{5.16}$$

Outside the body, L, M, N, P, Q, R all vanish identically, while in the fluid core $\mathbf{T} = -p_1 \mathbf{I}$ for some perturbation pressure p_1, so $L = P = -p_1$ and $Q = R = M = N = 0$.

The foregoing equations constitute the static equilibrium conditions for any body whose density distribution is nearly spherical, and which has a fluid core and a solid mantle. Now we propose to exhibit explicitly all solutions of these equations, that is, all possible equilibrium Earth models.

First, we can choose a, b, and $\rho_0(r)$ arbitrarily. Then $\phi_0(r)$, $g_0(r)$ and $p_0(r)$ are completely determined and describe a spherically symmetrical mass of fluid in hydrostatic equilibrium. Such an object we will call a 'spherical hydrostatic Earth model'.

Next we can arbitrarily choose $\alpha(x^1, x^2)$, $\beta(x^1, x^2)$ and $\rho_1(r, x^1, x^2)$ in the mantle (but not the core), subject to (5.1) and (5.2).

In addition we can choose arbitrarily the value of the pressure perturbation p_1 at $r = 0$. We will now show that these choices serve to determine ρ_1 and p_1 throughout the core, and ϕ_1 everywhere. Since $\mathbf{T} = -p_1 \mathbf{I}$ in the core, $\mathbf{T}$ will be determined in the core. The resulting configuration, with $\rho_0, \phi_0, \rho_1, \phi_1, \alpha$ and β specified everywhere and $\mathbf{T} = -p_1 \mathbf{I}$ specified in the core will be called a 'core-hydrostatic aspherical Earth density model'.

The reason that we are not free to choose ρ_1 arbitrarily in the core is that hydrostatic equilibrium there is a severe constraint. In the core, since $L = P$ and Q, R, M and N vanish, we can write (5.16) as

$$\left.\begin{aligned} \partial_r P &= \rho_0 \partial_r \phi_1 + \rho_1 \partial_r \phi_0, \\ \mathscr{Q}_0 P &= \rho_0 \phi_1. \end{aligned}\right\} \tag{5.17}$$

The second of these equations asserts that there exists a function $q(r)$ of r alone such that

$$P = \rho_0 \phi_1 + q(r).$$

Then

$$\partial_r P = \partial_r q + \rho_1 \partial_r \phi_1 + \phi_1 \partial_r \rho_0.$$

Comparing this with the first of equations (5.17) we find

$$\partial_r q = \rho_1 \partial_r \phi_0 - \phi_1 \partial_r \rho_0.$$

From (5.2) and (5.7) it is clear that $\mathscr{Y}_0 \partial_r q = 0$. But $\mathscr{Y}_0 \partial_r q = \partial_r q$, so q is in fact a constant, independent of r, and $\rho_1 \partial_r \phi_0 = \phi_0 \partial_r \rho_0$. Therefore we can conclude that

$$\left.\begin{aligned} \rho_1 &= g_0{}^{-1}(\partial_r \rho_0)\phi_1, \\ P &= \rho_0 \phi_1 + P(0), \end{aligned}\right\} \tag{5.18}$$

where $P(0)$ is the value of P at $r=0$ [since ϕ_1 vanishes at $r=0$ on account of (5.7)]. Given (5.2) and (5.7), equations (5.17) and (5.18) are equivalent. But from (5.4) and the first member of (5.18) it follows that in $0 \leqslant r \leqslant a$

$$\nabla^2 \phi_1 = 4\pi G g_0^{-1} (\partial_r \rho_0) \phi_1. \tag{5.19}$$

Equation (5.4) in $a \leqslant r < \infty$ (where ρ_1 is given) and (5.19) in $0 \leqslant r \leqslant a$ (where ρ_1 is not given), together with boundary conditions (5.5) and (5.6) determine ϕ_1 everywhere. Then (5.18) determines ρ_1 and P in the core, and since $\mathbf{T} = P\mathbf{I}$ in the core, $\mathbf{T}$ is also determined there. Thus, starting with a particular spherical hydrostatic Earth model we can choose α, β, the mantle ρ_1, and $P(0)$ arbitrarily, subject to conditions (5.1), (5.2), $|\alpha| \ll 1$, $|\beta| \ll 1$, and $|\rho_1| \ll \rho_0$. These choices determine a unique core-hydrostatic aspherical Earth density model, in which ρ_1 and ϕ_1 are known everywhere and $\mathbf{T}$ is known in the core (and, of course, for $r > b$, where $\mathbf{T} = 0$).

Finally, given a core-hydrostatic aspherical Earth density model, we seek all mantle stress fields $\mathbf{T}$ which satisfy the equations and boundary conditions of mechanical equilibrium. In the mantle, equations (5.16) can be written

$$\left.\begin{aligned} 2L &= (r\partial_r + 2)\, P + \nabla_1{}^2 Q - r(\rho_0 \partial_r \phi_1 + \rho_1 \partial_r \phi_0), \\ (\nabla_1{}^2 + 2) M &= \rho_0 \phi_1 - (r\partial_r + 3)\, Q - \mathscr{Q}_0 L, \\ (\nabla_1{}^2 + 2) N &= -(r\partial_r + 3)\, R, \end{aligned}\right\} \tag{5.20}$$

while the boundary conditions are

$$[P + a\alpha g_0 \rho_0(a)]_-^+ = 0, \quad Q = R = 0 \tag{5.21}$$

at $r = a$, and

$$P + b\beta g_0 \rho_0(b-) = P', \quad Q = Q', \quad R = R' \tag{5.22}$$

at $r = b$. To solve these equations we choose P, Q, R in $a \leqslant r \leqslant b$ arbitrarily, subject to $\mathscr{Y}_0 Q = \mathscr{Y}_0 R = 0$ for each r and to the boundary conditions (5.21), (5.22). Then L, M and N are completely determined as the solutions of (5.20) and the auxiliary conditions $\mathscr{Y}_0 M = \mathscr{Y}_0 N = \mathscr{Y}_1 M = \mathscr{Y}_1 N = 0$. That such solutions L, M, N exist imposes four more restrictions on the arbitrariness of P, Q and R, namely that $\mathscr{Y}_0$ and $\mathscr{Y}_1$ must give zero when applied to the right-hand sides of the second and third of equations (5.20). That $\mathscr{Y}_0$ gives zero is not a new restriction, but follows from $\mathscr{Y}_0 \mathscr{Q}_0 = 0$ and the restrictions $\mathscr{Y}_0 Q = \mathscr{Y}_0 R = 0$, which have already been imposed. That $\mathscr{Y}_1$ gives zero is a new restriction, implying that P, Q and R must be so chosen that for each r in $a \leqslant r \leqslant b$ we have

$$\mathscr{Y}_1 [(r\partial_r + 2)(P + 2Q) - r g_0 \rho_1 - \rho_0 (r\partial_r + 2)\phi_1] = 0,$$

$$\mathscr{Y}_1 [(r\partial_r + 3)\, R] = 0.$$

The first of these $\mathscr{Y}_1$ restrictions is obtained by applying $\mathscr{Y}_1$ to the first two of equations (5.20), and eliminating L. The second restriction is obtained by applying $\mathscr{Y}_1$ to the third of equations (5.20). The first restriction can be simplified by an appeal to (5.3), (5.4) and the identity

$$\nabla^2 \mathscr{Y}_1 \phi_1 = (\partial_r{}^2 + 2r^{-1}\partial_r - 2r^{-2}) \mathscr{Y}_1 \phi_1.$$

The second restriction can be simplified in an obvious way, so that the two restrictions are

$$\partial_r \mathscr{Y}_1 [4\pi G r^2 (P + 2Q) - g_0 \partial_r (r^2 \phi_1)] = 0$$

$$\partial_r \mathscr{Y}_1 [r^3 R] = 0.$$

Thus there exist constants K_P and K_R such that if $a \leqslant r \leqslant b$,

$$\left.\begin{aligned} \mathscr{Y}_1[4\pi G r^2(P+2Q) - g_0 \partial_r(r^2 \phi_1)] &= K_P, \\ \mathscr{Y}_1[r^3 R] &= K_R. \end{aligned}\right\} \quad (5.23)$$

From (5.21) it is clear immediately that $K_R = 0$. In fact K_P also vanishes. To see this we note that from (5.10), (5.16) and $\mathscr{Y}_1 M = 0$ we can infer the validity of the first of equations (5.23) for $0 \leqslant r \leqslant b$. Thus we can evaluate K_P at $r = 0$, where obviously it vanishes. We finally conclude that the conditions on P, Q, R which permit (5.20) to be solved for an L, M, N satisfying $\mathscr{Y}_0 M = \mathscr{Y}_0 N = \mathscr{Y}_1 M = \mathscr{Y}_1 N$ are, in $a \leqslant r \leqslant b$,

$$\left.\begin{aligned} \mathscr{Y}_0 Q = \mathscr{Y}_0 R = \mathscr{Y}_1 R &= 0, \\ \mathscr{Y}_1[4\pi G r^2(P+2Q) - g_0 \partial_r(r^2 \phi_1)] &= 0. \end{aligned}\right\} \quad (5.24)$$

We assert that these conditions and the boundary conditions (5.21) and (5.22) can be satisfied if and only if the external stress **F** satisfies (5.13), that is, **F** exerts no net force or torque. The torque question is trivial. If we want $\mathscr{Y}_1 R = 0$ in $a \leqslant r \leqslant b$ and $R = R'$ at $r = b$, clearly we must have $\mathscr{Y}_1 R' = 0$. That is, **F** must exert no net torque. The force question we settle by appealing to (5.8) with $n = 1$ and to (5.22). These equations reduce the second of equations (5.24) at $r = b$ to the equation $\mathscr{Y}_1(P' + 2Q') = 0$, that is to the condition that **F** exert no net force.

In summary, we have a catalogue of equilibrium Earth models. To obtain a member of this catalogue, we make a number of arbitrary choices. We choose a, b and ρ_0, to construct a spherical hydrostatic Earth model in which ϕ_0 and g_0 are everywhere determined. Then we choose α, β, ρ_1 in the mantle, and P at $r = 0$, subject only to (5.1), (5.2), $|\alpha| \ll 1$, $|\beta| \ll 1$, and $|\rho_1| \ll \rho_0$; and we construct a core-hydrostatic, aspherical Earth density model in which ρ_1 and ϕ_1 are determined everywhere and **T** is determined in the core, $0 \leqslant r \leqslant a$. Finally we choose P, Q and R in $a \leqslant r \leqslant b$ arbitrarily, subject to the conditions (5.24) for each such r, and to the boundary conditions (5.21) and (5.22). These choices determine L, M and N and hence **T** uniquely throughout the mantle. From the method of construction it is clear that our catalogue contains every self-gravitating body which satisfies the following conditions:

(i) the body's density distribution is nearly spherically symmetric;

(ii) the body has a fluid core and a solid mantle;

(iii) the body is in static mechanical equilibrium under an external stress which exerts no net force or torque.

If $\alpha = \beta = 0$ and $\rho_1 = 0$ for all r, the discussion is exact, so the catalogue contains the exact descriptions of all possible equilibrium stress fields in any spherical shell in a spherically symmetric gravity field. If α, β and ρ_1/ρ_0 are small but not zero, the corresponding members of the catalogue are accurate only to the first order in these small quantities.

B. *Transversely isotropic elastic stress–strain relations*

The most general spherically symmetrical elastic model for the Earth's mantle is one in which the density and elastic constants depend only on r, and the stress–strain relation at any point p is invariant under all rotations about the radius through p. Such a stress–strain relation is not isotropic, but only transversely isotropic with a radial axis. It is the purpose of the present subsection to express such transversely isotropic stress–strain relations in spherical shells as relations between the six scalars P, Q, R, L, M, N of the symmetrical stress tensor **T** and the three scalars U, V, W of the displacement field **u**.

If x, y, z are Cartesian co-ordinates in a transversely isotropic medium whose axis of symmetry is the z axis, the relation between the strain tensor $\boldsymbol{\varepsilon}$ and the stress tensor $\mathbf{T}$ is (Stoneley 1949)

$$T_{33}=\beta\varepsilon_{33}+\lambda\varepsilon_{kk},$$
$$T_{i3}=T_{3i}=2\mu\varepsilon_{i3}=2\mu\varepsilon_{3i},$$
$$T_{ij}=(\lambda'\varepsilon_{kk}+\lambda\varepsilon_{33})\delta_{ij}+2\mu'\varepsilon_{ij},$$

where subscripts 1, 2 and 3 refer respectively to the x, y, and z axes and $\varepsilon_{kk}=\varepsilon_{11}+\varepsilon_{22}$. The five elastic constants of the material, λ, λ', μ, μ', and β, are independent. In an isotropic material, λ and μ are the Lamé parameters and $\lambda'=\lambda$, $\mu'=\mu$, $\beta=\lambda+2\mu$.

Now suppose we have a spherically symmetric elastic shell centred on the origin, O. Then the material must be transversely isotropic with axis of symmetry everywhere radial. The spherical resolution of the stress–strain relations leads to the following equations:

$$\left.\begin{aligned} T_{rr}&=\beta\varepsilon_{rr}+\lambda\,\mathrm{tr}\,\boldsymbol{\varepsilon}_{SS},\\ \mathbf{T}_{rS}&=\mathbf{T}_{Sr}=2\mu\boldsymbol{\varepsilon}_{rS}=2\mu\boldsymbol{\varepsilon}_{Sr},\\ \mathbf{T}_{SS}&=(\lambda'\,\mathrm{tr}\,\boldsymbol{\varepsilon}_{SS}+\lambda\varepsilon_{rr})\,\mathbf{I}_{S_1}+2\mu'\boldsymbol{\varepsilon}_{SS}.\end{aligned}\right\}\quad(5.25)$$

The elastic constants must be functions of r alone.

If $\mathbf{u}$ is the displacement field in the spherical shell, $2\boldsymbol{\varepsilon}=\nabla\mathbf{u}+(\nabla\mathbf{u})^T$. Here superscript T means transpose. From equation (3.16)

$$\boldsymbol{\varepsilon}=\hat{r}\hat{r}(\partial_r u_r)+[\hat{r}(\nabla_1 x^i)+(\nabla_1 x^i)\hat{r}](2r)^{-1}[(r\partial_r-1)\tilde{u}_i+\tilde{D}_i u_r]$$
$$+(\nabla_1 x^i)(\nabla_1 x^j)(2r)^{-1}[\tilde{D}_i\tilde{u}_j+\tilde{D}_j\tilde{u}_i+2u_r\tilde{g}_{ij}],\quad(5.26)$$

and

$$\mathrm{tr}\,\boldsymbol{\varepsilon}_{SS}=r^{-1}[\tilde{g}^{kl}\tilde{D}_k\tilde{u}_l+2u_r].\quad(5.27)$$

Therefore, from (5.25)

$$\left.\begin{aligned} T_{rr}&=\beta(\partial_r u_r)+\lambda r^{-1}(\tilde{g}^{kl}\tilde{D}_k\tilde{u}_l+2u_r),\\ T_{ri}&=T_{ir}=\mu r^{-1}[(r\partial_r-1)\tilde{u}_i+\tilde{D}_i u_r],\\ \tilde{T}_{ij}&=\tilde{g}_{ij}[\lambda' r^{-1}(\tilde{g}^{kl}\tilde{D}_k\tilde{u}_l+2u_r)+\lambda\partial_r u_r]+\mu' r^{-1}[\tilde{D}_i\tilde{u}_j+\tilde{D}_j\tilde{u}_i+2u_r\tilde{g}_{ij}].\end{aligned}\right\}\quad(5.28)$$

The stress tensor $\mathbf{T}$ has a scalar representation (5.15) and the displacement field $\mathbf{u}$ has a scalar representation (4.11). Then the scalar representation of the stress–strain relations (5.28) is

$$\left.\begin{aligned} P&=\beta\partial_r U+\lambda r^{-1}(\nabla_1^2 V+2U),\\ \tilde{D}_i Q+\tilde{h}_i^{\,j}\tilde{D}_j R&=\tilde{D}_i\{\mu r^{-1}[(r\partial_r-1)V+U]\}+\tilde{h}_i^{\,j}\tilde{D}_j[\mu(\partial_r-r^{-1})W],\\ (L-\nabla_1^2 M)\tilde{g}_{ij}&+2\tilde{D}_i\tilde{D}_j M+(\tilde{h}_i^{\,k}\tilde{D}_j+\tilde{h}_j^{\,k}\tilde{D}_i)\tilde{D}_k N\\ &=[\lambda\partial_r U+2(\lambda'+\mu')r^{-1}U+\lambda' r^{-1}\nabla_1^2 V]\tilde{g}_{ij}\\ &\quad+2\tilde{D}_i\tilde{D}_j(\mu' r^{-1}V)+(\tilde{h}_i^{\,k}\tilde{D}_j+\tilde{h}_j^{\,k}\tilde{D}_i)\tilde{D}_k(\mu' r^{-1}W).\end{aligned}\right\}\quad(5.29)$$

In deducing (5.29) we have used repeatedly the fact that $\tilde{D}_i$ commutes with any function of r alone.

From the discussion at the end of Section 4A it is clear how to obtain from (5.29) the relations between the scalars of **T** and those of **u**. Those relations are as follows:

$$\left.\begin{aligned} P&=\beta(\partial_r U)+\lambda r^{-1}(2U+\nabla_1{}^2 V),\\ Q&=\mu r^{-1}[(r\partial_r-1)V+\mathscr{D}_0 U],\\ R&=\mu r^{-1}(r\partial_r-1)W\\ L&=\lambda(\partial_r U)+(\lambda'+\mu')r^{-1}(2U+\nabla_1{}^2 V),\\ M&=\mu' r^{-1}\mathscr{D}_1 V,\\ N&=\mu' r^{-1}\mathscr{D}_1 W.\end{aligned}\right\}\quad(5.30)$$

In deriving (5.30) from (5.29) we have assumed that V, W, Q, R, M, and N are the unique scalars in (4.11) and (5.15) satisfying, for every r,

$$\mathscr{Y}_0 V=\mathscr{Y}_0 W=\mathscr{Y}_0 Q=\mathscr{Y}_0 R=\mathscr{Y}_0 M=\mathscr{Y}_0 N=\mathscr{Y}_1 M=\mathscr{Y}_1 N=0.$$

C. *Elastic equations of motion*

Suppose an elastic, self-gravitating, spherically symmetrical body of radius b is in static equilibrium and has an isotropic (but not homogeneous) stress tensor $-p_0(r)\,\mathbf{I}$, a density $\rho_0(r)$, and a gravitational potential $\phi_0(r)$. If the body is slightly disturbed, a time-dependent displacement field will be set up which will produce perturbations ρ_1 in the density, ϕ_1 in the gravitational potential, and $\mathbf{T}_E$ in the stress tensor at a fixed location in space. The exact equations of motion about equilibrium are

$$(\rho_0+\rho_1)\frac{D^2\mathbf{u}}{Dt^2}=\nabla.(-p_0\,\mathbf{I}+\mathbf{T}_E)-(\rho_0+\rho_1)\nabla(\phi_0+\phi_1),$$

where D/Dt is the substantial or moving derivative. Neglecting terms of second order in **u** and subtracting (5.4) we obtain

$$\rho_0\partial_t{}^2\mathbf{u}=\nabla.\mathbf{T}_E-\rho_0\nabla\phi_1-\rho_1\nabla\phi_0.$$

Boundary conditions on the outer and inner surfaces of the spherical shell and at interior spherical surfaces of discontinuity are boundary conditions on **u** and $\hat{r}.\mathbf{T}$, and therefore on the scalars U, V, W, P, Q, R. Hence it is convenient to use the stress–strain relations (5.30) in a form in which U, V, W, P, Q, and R are regarded as given, and L, M, N, $\partial_r U$, $\partial_r V$ and $\partial_r W$ are expressed in terms of them. That is, we want to solve (5.30) for L, M, N, $\partial_r U$, $\partial_r V$, and $\partial_r W$. The result is the following:

$$\left.\begin{aligned} \partial_r U&=\beta^{-1}P-\lambda(r\beta)^{-1}(2U+\nabla_1{}^2 V),\\ \partial_r V&=\mu^{-1}Q+r^{-1}(V-\mathscr{D}_0 U),\\ \partial_r W&=\mu^{-1}R+r^{-1}W,\\ L&=\lambda\beta^{-1}P+(\lambda'+\mu'-\lambda^2\beta^{-1})r^{-1}(2U+\nabla_1{}^2 V),\\ M&=\mu' r^{-1}\mathscr{D}_1 V,\\ N&=\mu' r^{-1}\mathscr{D}_1 W.\end{aligned}\right\}\quad(5.31)$$

In the same approximation the continuity equation is

$$\rho_1+\nabla.(\rho_0\mathbf{u})=0.$$

The stress tensor at a fixed location in space is $-p_0\mathbf{I}+\mathbf{T}_E$. To first order in $\mathbf{u}$, the stress tensor at a fixed material particle is $-p_0\mathbf{I}+\mathbf{T}=(1+\mathbf{u}.\nabla)(-p_0\mathbf{I}+\mathbf{T}_E)$; therefore, neglecting the second order term $\mathbf{u}.\nabla\mathbf{T}_E$,

$$\mathbf{T}=\mathbf{T}_E-(\mathbf{u}.\nabla p_0)\,\mathbf{I}.$$

It is $\mathbf{T}$, not $\mathbf{T}_E$, which is related to the strain by the elastic parameters. Therefore we rewrite the linearized equations of motion in the form

$$\rho_0\partial_t^2\mathbf{u}=\nabla(\mathbf{u}.\nabla p_0)+\nabla.\mathbf{T}-\rho_1\nabla\phi_0-\rho_0\nabla\phi_1. \tag{5.32}$$

In terms of the scalars used to represent $\mathbf{u}$ and $\mathbf{T}$ in (4.11) and (5.15), the continuity equation is

$$r\rho_1+(r\partial_r+2)(\rho_0 U)+\rho_0\nabla_1^2 V=0. \tag{5.33}$$

If we consider oscillations with angular frequency ω, so that $\partial_t=-i\omega$, then the equations of motion are

$$\left.\begin{aligned}(r\partial_r+2)P+\nabla_1^2 Q-2L+\rho_0(r\omega^2+4g_0)U+\rho_0 g_0\nabla_1^2 V-\rho_0 r g_1&=0,\\ (r\partial_r+3)Q+(\nabla_1^2+2)M+\rho_0\omega^2 rV+\mathcal{D}_0(L-\rho_0 g_0 U-\rho_0\phi_1)&=0,\\ (r\partial_r+3)R+(\nabla_1^2+2)N+\rho_0\omega^2 rW&=0,\end{aligned}\right\} \tag{5.34}$$

where g_0 is defined by (5.3) and g_1 by the equation

$$\partial_r\phi_1=g_1-4\pi G\rho_0 U. \tag{5.35}$$

In deducing (5.34) from (5.32), we have used (5.4) and (5.33). Therefore when we set $U=V=W=0$, then ρ_1, ϕ_1 and g_1 are forced to vanish, and equations (5.34) do not reduce to the general equilibrium equations (5.16) but only to the special case with $\rho_1=\phi_1=0$.

By using (5.33) and (5.35) we can write Poisson's equation for ϕ_1 in the following form:

$$\partial_r g_1=-2r^{-1}g_1-r^{-2}\nabla_1^2\phi_1-4\pi G\rho_0 r^{-1}\nabla_1^2 V. \tag{5.36}$$

The boundary conditions for free oscillation are that U, V, W, P, Q, R, ϕ_1 and g_1 must be continuous everywhere; at $r=b$ the boundary conditions are that P, Q, R and $\mathcal{Y}_n[rg_1+(n+1)\phi_1]$ should vanish for $n=0, 1, 2, \ldots$.

The equations for $\partial_r\phi_1$ and $\partial_r g_1$ in terms of $\phi_1, g_1, U, V, W, P, Q$ and R are equations (5.35) and (5.36). The equations for $\partial_r U$, $\partial_r V$, $\partial_r W$, $\partial_r P$, $\partial_r Q$, and $\partial_r R$ can be obtained directly from (5.31), (5.34) and the abbreviation $\gamma=\lambda'+\mu'-\lambda^2\beta^{-1}$. They are these:

$$\left.\begin{aligned}\partial_r U&=-\lambda(r\beta)^{-1}(2U+\nabla_1^2 V)+\beta^{-1}P,\\ \partial_r V&=r^{-1}(V-\mathcal{D}_0 U)+\mu^{-1}Q,\\ \partial_r W&=r^{-1}W+\mu^{-1}R\\ r\partial_r P&=(4r^{-1}\gamma-r\rho_0\omega^2-4\rho_0 g_0)U+(2r^{-1}\gamma-\rho_0 g_0)\nabla_1^2 V\\ &\qquad+2(\lambda\beta^{-1}-1)P-\nabla_1^2 Q+r\rho_0 g_1\\ r\partial_r Q&=(\rho_0 g_0-2r^{-1}\gamma)\mathcal{D}_0 U-r^{-1}[r^2\rho_0\omega^2+2\mu'+(\gamma+\mu')\nabla_1^2]V\\ &\qquad-\lambda\beta^{-1}\mathcal{D}_0 P-3Q+\rho_0\mathcal{D}_0\phi_1\\ r\partial_r R&=-[r\rho_0\omega^2+r^{-1}\mu'(\nabla_1^2+2)]W-3R.\end{aligned}\right\} \tag{5.37}$$

The system (5.35), (5.36), (5.37) separates into a pair of equations for R and W and six equations for ϕ_1, g_1, U, V, P and Q. Solutions of the second-order system, with ω chosen to make possible the fulfilment of the boundary conditions, are the toroidal normal modes. Solutions of the sixth-order system, with ω chosen to make possible the fulfilment of the boundary conditions, are the spheroidal normal modes. When the mantle is isotropic, $\lambda'=\lambda$, $\mu'=\mu$, $\beta'=\lambda+2\mu$, and equations (5.37) become the equations used by Alterman, Jarosch and Pekeris (1959) in their discussion of the elastic-gravitational normal modes of the Earth. Even the toroidal equations in a transversely isotropic Earth are different from the toroidal equations in an isotropic Earth, but at present the evidence from teleseismic travel times is consistent with an isotropic mantle, or at any rate a mantle whose anisotropy is small enough to be treated by first-order perturbation theory rather than the general theory presented here. Therefore the general form of equations (5.37) seems at present to be of limited physical interest.

The scalar fields U, V, W, L, M, N, P, Q, R, ϕ_1, g_1 can be expanded in series of surface spherical harmonics Y_n^m, the expansion coefficients being functions of r. For any particular Y_n^m the coefficients of the scalars U, V, W, P, Q, R, g_1, ϕ_1 separately satisfy (5.35), (5.36) and (5.37) with ∇_1^2 replaced by $-n(n+1)$. This remark proves that every elastic-gravitational normal mode of the system is a linear combination of normal modes obtained from surface spherical harmonics. That is, the solutions of (5.35), (5.36), (5.37) in the form

$$U=U_n^m(r)\,Y_n^m(x^1, x^2), \ldots, R=R_n^m(r)\,Y_n^m(x^1, x^2)$$

which satisfy the boundary conditions form a complete set of normal modes of elastic-gravitational oscillations of the Earth. As far as I know, in the geophysical literature to date it has only been shown that all such spherical harmonic solutions *are* normal modes, not that there are no other normal modes.

It should be noted that the foregoing proof of the completeness of the spherical harmonic normal modes depends only on the Scalar Representation Theorem for tangent vector fields on S_1, and not on the Representation Theorem for second-order tangent tensor fields. The latter only lightens the algebraic burden of deriving the scalar equations of motion (5.37).

An independent proof of the completeness of the normal modes can be obtained from the completeness of the vector spherical harmonics (see Blatt & Weisskopf 1952).

The normal modes with $n=0$ constitute a special case. Since $\mathscr{Y}_0 V=\mathscr{Y}_0 Q=0$, the equations for $\partial_r\phi_1$, $\partial_r g_1$, $\partial_r U$ and $\partial_r P$ involve only ϕ_1, g_1, U and P when $n=0$, and moreover $g_1=0$. Therefore the spheroidal system is a second-order system for U and P, together with a quadrature for ϕ_1.

The normal modes with $n=1$ lead to a sixth-order spheroidal and a second-order toroidal problem, and yet constitute a slightly special case, since $\mathscr{Y}_1 M=\mathscr{Y}_1 N=0$. That is, when $n=1$ the tangential part of the stress tensor is transversely isotropic.

For any n, the degeneracy of any normal frequency ω is $2n+1$, whether the mantle is isotropic or only transversely isotropic. This result is to be expected from the spherical symmetry of the problem and the structure of the irreducible, single-valued representations of the rotation group.

Equations (5.37) refer only to the mantle. In the fluid core $\mu'=\mu=0$ and $\lambda'=\beta=\lambda$, λ being the bulk modulus of the fluid. Therefore the stress–strain relations (5.30) become

$$P=\lambda r^{-1}[(r\partial_r+2)U+\nabla_1^2 V],$$

$$Q=R=0,$$

$$L=P,$$

$$M=N=0,$$

and the equations of motion (5.34) become

$$r\partial_r P+\rho_0(r\omega^2+4g_0)U+\rho_0 g_0 \nabla_1{}^2 V-\rho_0 r g_1=0,$$
$$\rho_0\omega^2 rV=\mathcal{Q}_0(\rho_0 g_0 U-P+\rho_0\phi_1)$$
$$W=0.$$

Then in the fluid core equations (5.37) must be replaced by the following:

$$\left.\begin{aligned}
&\partial_r U=-r^{-1}(2U+\nabla_1{}^2 V)+\lambda^{-1}P,\\
&\rho_0\omega^2 rV=\mathcal{Q}_0(\rho_0 g_0 U-P+\rho_0\phi_1),\\
&W=0,\\
&\partial_r P=-(\rho_0\omega^2+4r^{-1}\rho_0 g_0)U-r^{-1}\rho_0 g_0\nabla_1{}^2 V+\rho_0 g_1,\\
&Q=0,\\
&R=0.
\end{aligned}\right\} \quad (5.38)$$

In the core the toroidal oscillations vanish and the spheroidal oscillations are described by a fourth-order rather than a sixth-order system.

D. *A spherically symmetrical viscous fluid mantle*

In discussions of continental drift and mantle convection the mantle is often treated as a viscous fluid. This simplification has long been suspect among geophysicists, but has recently received some support (Gordon 1965) from the observation that at the typical high temperatures of at least the lower mantle the diffusion of vacancies through crystals or along their boundaries is the principal physical mechanism leading to shear of the mantle under arbitrarily small stresses. Such a deformation mechanism produces a linear relation between stress and strain-rate, but it is not clear at the time of writing whether the relation can be assumed to be isotropic or independent of the past history of the material. If the viscosity tensor should be transversely isotropic with respect to the local vertical (isotropy is a special case) and should depend only on depth, not on angular position, then the physical problem is invariant under all rigid rotations of the mantle about its centre and there will be convective flows of infinitesimal amplitude with the angular symmetry of the surface spherical harmonics. The techniques of the present paper provide a simple way of constructing these flows and showing that their superpositions include all physically possible infinitesimal flows.

In convection problems, heat transfer must be treated in some detail. Let H be the enthalpy per gram of mantle material, θ the local temperature, ρ the local density and p the local thermodynamic pressure (that obtained from ρ and θ via the equation of state for thermodynamic equilibrium). Then the internal energy equation can be written

$$\rho\frac{DH}{Dt}-\frac{Dp}{Dt}+\nabla.\mathbf{H}=\mathrm{tr}\,(\mathbf{T}.\nabla\mathbf{u})+q,$$

where D/Dt is $\partial_t+\mathbf{u}.\nabla$, the substantial derivative, $\mathbf{H}$ is the conductive heat flow vector, q is the radioactive or chemical heat production rate per unit volume, $\mathbf{T}$ is the viscous stress tensor, representable as in (5.15), and $\mathbf{u}$ is now the fluid velocity, not displacement, but still representable as in (4.11).

The thermodynamic equation connecting H with p and θ can be written

$$\frac{DH}{Dt}=c_p\frac{D\theta}{Dt}+(1-\alpha\theta)\tau\frac{Dp}{Dt},$$

where $\tau=\rho^{-1}$, c_p is the specific heat per gram at constant pressure, and α is the coefficient of thermal expansion at constant pressure. Therefore the energy equation is

$$\rho c_p \frac{D\theta}{Dt} - \alpha\theta \frac{Dp}{Dt} + \nabla . \mathbf{H} = \mathrm{tr}\,(\mathbf{T} . \nabla\mathbf{u}) + q. \tag{5.39}$$

The equation of state can be written

$$\tau^{-1} \frac{D\tau}{Dt} = \alpha \frac{D\theta}{Dt} - k_\theta \frac{Dp}{Dt}$$

where k_θ is the isothermal compressibility. The continuity equation is

$$\tau^{-1} \frac{D\tau}{Dt} = \nabla . \mathbf{u},$$

so

$$\alpha \frac{D\theta}{Dt} - k_\theta \frac{Dp}{Dt} = \nabla . \mathbf{u}. \tag{5.40}$$

Equations (5.39) and (5.40) can be solved for $D\theta/Dt$ and Dp/Dt, with the following results:

$$\left.\begin{aligned} \rho c_p k_s \frac{D\theta}{Dt} &= -\alpha\theta(\nabla . \mathbf{u}) + k_\theta[\mathrm{tr}\,(\mathbf{T} . \nabla\mathbf{u}) - \nabla . \mathbf{H} + q], \\ \rho c_p k_s \frac{Dp}{Dt} &= -\rho c_p(\nabla . \mathbf{u}) + \alpha[\mathrm{tr}\,(\mathbf{T} . \nabla\mathbf{u}) - \nabla . \mathbf{H} + q], \end{aligned}\right\} \tag{5.41}$$

where k_s is the adiabatic compressibility:

$$k_s = k_\theta - \alpha^2\theta(\rho c_p)^{-1}. \tag{5.42}$$

If we assume that

$$\rho = \rho_0(r) + \rho_1(r, x^1, x^2), \quad p = p_0(r) + p_1(r, x^1, x^2), \quad \theta = \theta_0(r) + \theta_1(r, x^1, x^2),$$

where ρ_0, p_0, and θ_0 represent a stationary equilibrium solution with a time-independent heat production rate q, while ρ_1, p_1 and θ_1 are proportional to $\mathbf{u}$, and if we neglect products of quantities of first order in $\mathbf{u}$, then equations (5.41) become

$$\left.\begin{aligned} \rho_0 c_p k_s(\partial_t\theta_1 + U\partial_r\theta_0) &= -\alpha\theta_0 \nabla . \mathbf{u} - k_\theta \nabla . \mathbf{H}_1, \\ \rho_0 c_p k_s(\partial_t p_1 + U\partial_r p_0) &= -\rho_0 c_p \nabla . \mathbf{u} - \alpha\nabla . \mathbf{H}_1 \end{aligned}\right\} \tag{5.43}$$

with

$$\mathbf{H}_1 = -\kappa_0 \nabla\theta_1 - \kappa_1 \nabla\theta_0. \tag{5.44}$$

In equations (5.43) and (5.44), c_p, k_s, k_θ, α and the thermal conductivity $\kappa_0(r)$ are functions of r alone, to be evaluated in the equilibrium state described by $\rho_0(r)$, $p_0(r)$ and $\theta_0(r)$, while $\kappa_1(r, x^1, x^2, t)$ is the change in thermal conductivity produced by the local changes p_1 and θ_1 in pressure and temperature.

Another form for the first-order equation of state is

$$\rho_1 = \rho_0(k_\theta p_1 - \alpha\theta_1) \tag{5.45}$$

and the first-order change $\phi_1(r, x^1, x^2)$ in the gravitational potential satisfies (5.10). The spherical resolutions and scalar representations of equations (5.43), (5.44), (5.45) and (5.10) can be written down immediately from (4.26) and the fact that

$$\nabla^2 = (\partial_r + 2r^{-1})\partial_r + r^{-2}\nabla_1{}^2.$$

The scalar representation of the relation between the viscous stress **T** and the strain rate calculated from the velocity **u** is (5.30) or (5.31), where λ, λ', μ, μ', and β are now viscosities. In an isotropic fluid, $\lambda'=\lambda$, $\mu'=\mu$, and $\beta=\lambda+2\mu$.

The linearized equation of motion is

$$\rho_0\,\partial_t\mathbf{u}=-\nabla p_1+\nabla.\mathbf{T}-\rho_0\nabla\phi_1-\rho_1\nabla\phi_0.$$

From (4.30), the scalar representation of this equation is

$$\left.\begin{aligned}\rho_0\,\partial_t U&=-\partial_r p_1-\rho_0\,\partial_r\phi_1-\rho_1\,\partial_r\phi_0+r^{-1}[(r\partial_r+2)P+\nabla_1{}^2 Q-2L],\\ r\rho_0\,\partial_t V&=-\mathscr{Q}_0(p_1+\rho_0\phi_1+L)+(r\partial_r+3)Q+(\nabla_1{}^2+2)M,\\ r\rho_0\,\partial_t W&=(r\partial_r+3)R+(\nabla_1{}^2+2)N.\end{aligned}\right\}\quad(5.46)$$

E. *An incompressible viscous Earth*

In the discussion of Section 5D, the incompressible fluid constitutes a singular case. The scalar representation of the constraint $\nabla.\mathbf{u}=0$ is

$$(r\partial_r+2)U+\nabla_1{}^2V=0.\qquad(5.47)$$

For any regular scalar field V there is a unique regular scalar field X which vanishes at $r=0$ and satisfies the equation

$$V=-(\partial_r+r^{-1})X.\qquad(5.48)$$

If $\mathscr{Y}_0V=0$ for each r, then $\mathscr{Y}_0X=0$ for each r. If U and V satisfy (5.47), then

$$\partial_r[r^2(U-r^{-1}\nabla_1{}^2X)]=0,$$

so $U=r^{-1}\nabla_1{}^2X+r^{-2}f(x^1,x^2)$. If U is regular at $r=0$, then f must vanish identically, so

$$U=r^{-1}\nabla_1{}^2X.\qquad(5.49)$$

Therefore the solenoidal velocity field **u** is described by the two scalars X and W. In fact

$$\mathbf{u}=\nabla\times(\mathbf{r}\times\nabla X)-(\mathbf{r}\times\nabla)W,$$

a representation of solenoidal vector fields in spheres which has already been proved (Backus 1958) in another connection.

University of California, San Diego,
California,
U.S.A.

References

Ahlfors, L. & Sario, L., 1960. *Riemann Surfaces*, Chapter 5, p. 283. Princeton University Press.

Alterman, Z., Jarosch, H. and Pekeris, C., 1959. Oscillations of the earth, *Proc. R. Soc.*, A, **252**, 80–95.

Backus, G., 1958. A class of self-sustaining dissipative spherical dynamos, *Ann. Phys.*, **4**, 381–384.

Backus, G., 1966. Potentials for tangent tensor fields on spheroids, *Archs ration. Mech. Analysis*, **22,** 210–252.

Bergmann, P., 1947. *Introduction to the Theory of Relativity*, Chapter 5, pp. 50–51. Prentice Hall, New York.

Blatt, J. and Weisskopf, V., 1952. *Theoretical Nuclear Physics*, pp. 796–799. Wiley, New York.

Gibbs, W. and Wilson, E., 1925. *Vector Analysis*. Yale University Press.

Gordon, R., 1965. Diffusion creep in the earth's mantle, *J. geophys. Res.*, **70,** 2413–2418.

Jeffreys, H., 1959. *The Earth* (4th edn.), Chapter 4, p. 147. Cambridge University Press.

Stoneley, R., 1949. The seismological implications of aeolotropy in continental structure, *Mon. Not. R. astr. Soc. geophys. Suppl.*, **5,** 222–232.

Willmore, T., 1959. *An Introduction to Differential Geometry*, Chapter 3. Oxford University Press.