

CONVEX BODIES, ECONOMIC CAP COVERINGS, RANDOM POLYTOPES

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§1. *Introduction.* Let K be a convex compact body with nonempty interior in the d -dimensional Euclidean space R^d and let x_1, \dots, x_n be random points in K , independently and uniformly distributed. Define $K_n = \text{conv} \{x_1, \dots, x_n\}$. Our main concern in this paper will be the behaviour of the deviation of $\text{vol } K_n$ from $\text{vol } K$ as a function of n , more precisely, the expectation of the random variable $\text{vol}(K \setminus K_n)$. We denote this expectation by $E(K, n)$.

There are few results known about $E(K, n)$, mainly when $d = 2$. (The case $d = 1$ is trivial.) Rényi and Sulanke [18, 19] proved that for smooth enough convex bodies $K \subset R^2$

$$E(K, n) \approx \text{const}(K)n^{-2/3}, \tag{1.1}$$

where $\text{const}(K)$ denotes a constant depending on K only and the notation $f(n) \approx g(n)$ means that f and g are asymptotically equal, i.e., $\lim f(n)/g(n) = 1$ when $n \rightarrow \infty$. This has been extended to smooth enough convex bodies $K \subset R^3$ by Wieacker [24], who obtained

$$E(K, n) \approx \text{const}(K)n^{-1/2}. \tag{1.2}$$

Buchta [4], see also Renyi and Sulanke [18], proved that for a convex polygon $P \subset R^2$

$$E(P, n) \approx \text{const}(P)n^{-1} \log n. \tag{1.3}$$

Little is known in R^d . Wieacker [24] determined $E(B^d, n)$ where B^d denotes the unit ball of R^d , obtaining

$$E(B^d, n) \approx \text{const}(d)n^{-2/(d+1)}. \tag{1.4}$$

Groemer [11] proved that for a convex compact body $K \subset R^d$ with $\text{vol } K = \text{vol } B^d$

$$E(K, n) \leq E(B^d, n), \tag{1.5}$$

with equality, if, and only if, K is an ellipsoid.

Until quite recently, nothing has been known about the case of general d -polytopes when $d > 2$. Buchta [5] proved $E(T, n) \approx \frac{3}{4}n^{-1}(\log n)^2$ where T denotes the three dimensional simplex. Dwyer, Kannan and Lovász [14] proved that

$$E(P, n) \leq \text{const}(P)n^{-1}(\log n)^d$$

for a polytope $P \subset R^d$. This was improved later by Dwyer [8] to

$$E(P, n) \leq \text{const}(P)n^{-1}(\log n)^{d-1}. \tag{1.6}$$

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They also proved that for a polytope P having a simple vertex (i.e., a vertex where exactly d facets meet)

$$E(P, n) \geq \text{const}(P)n^{-1}(\log n)^{d-1}. \tag{1.7}$$

For further information on the expectation of the number of vertices, surface area, mean width, etc. of K_n we refer the reader to Buchta [5], Dwyer [8], Gruber [12], Schneider [22].

We are going to relate $E(K, n)$ to another quantity which we now describe. First, define a map $v: K \rightarrow R$ as

$$v(x) = \min \{ \text{vol}(K \cap H) : x \in H, H \text{ a halfspace} \}.$$

Next, for $\varepsilon > 0$ define

$$K(v \leq \varepsilon) = \{ x \in K : v(x) \leq \varepsilon \}.$$

Sometimes we will write $K(\varepsilon)$ as a shorthand for $K(v \leq \varepsilon)$. Here our main result is

THEOREM 1. *Assume K is a convex compact body in R^d with $\text{vol } K = 1$. Then, for $n \geq n_0(d)$ we have*

$$\text{const vol } K(1/n) \leq E(K, n) \leq \text{const}(d) \text{ vol } K(1/n). \tag{1.8}$$

Theorem 1 means that $E(K, n)$ is of the same order of magnitude as $\text{vol } K(1/n)$. We will write this as $\text{vol } K(1/n) \sim E(K, n)$ so the notation $f(n) \sim g(n)$ means that $\liminf f(n)/g(n) > 0$ and $\liminf g(n)/f(n) > 0$. This notation implies two constants that are independent of n . We mention that in Theorem 1 the constants are independent of K as well. Actually, one of them is universal and the other depends on d only.

Theorem 1 can be used to determine the order of magnitude of $E(K, n)$ for different classes of convex bodies in R^d . First we prove a general upper bound for $\text{vol } K(1/n) \sim E(K, n)$.

THEOREM 2. *Let $K \subset R^d$ be a convex compact body with $\text{vol } K = 1$ and let $\varepsilon > 0$. Then*

$$\text{const}(d)\varepsilon(\log(1/\varepsilon))^{d-1} \leq \text{vol } K(\varepsilon). \tag{1.9}$$

This theorem is best possible (apart from the constant) as shown by the polytopes.

THEOREM 3. *Let $P \subset R^d$ be a polytope with $\text{vol } P = 1$ and let $\varepsilon \geq 0$. Then*

$$\text{vol } P(\varepsilon) \leq \text{const}(P)\varepsilon(\log(1/\varepsilon))^{d-1}. \tag{1.10}$$

Theorem 2 and 3 show that $\text{vol } P(\varepsilon) \sim \varepsilon(\log(1/\varepsilon))^{d-1}$ with the implied constant depending on P . This, together with Theorem 1 proves that for the class of polytopes $E(P, n) \sim n^{-1}(\log n)^{d-1}$. This result has been obtained independently by Dwyer [8]. The other extreme class of convex bodies is that of the smooth ones. We state an asymptotic result for this class without proof (see Leichweiss [26]).

THEOREM. For a \mathcal{C}^3 convex body $K \subset R^d$ with $\text{vol } K > 0$ and positive Gaussian curvature κ and for $\varepsilon > 0$ we have

$$\text{vol } K(\varepsilon) \approx \text{const } (d) \left(\int_{\delta K} \kappa^{1/(d+1)} dS \right) \varepsilon^{2/(d+1)}, \tag{1.11}$$

where the integration is taken on the boundary, δK , of K .

This theorem was also proved by Buchta, Gruber, Müller [6]. They noticed that the right-hand side here is a constant multiple of the affine surface area of K (cf. Blaschke [3]) and so Blaschke’s affine isoperimetric inequality implies that among all \mathcal{C}^3 convex bodies of unit volume $\text{vol } K(\varepsilon)$ is the largest for the ellipsoids.

Theorem 1, the Theorem above and Groemer’s result (1.5) show that for a \mathcal{C}^3 convex body $K \subset R^d$, $E(K, n) \sim n^{-2/(d+1)}$ with the implied constants depending on K . We are going to prove a theorem that will also yield this. Some preparations are needed. We write $B(\rho, x)$ for the ball of radius ρ and with centre $x \in R^d$. Let p be a point on the boundary δK of the convex compact set $K \subset R^d$. Assume there is a unique outer normal a (with $|a|=1$) to K at p . Then we call the point p ρ -circular if $\rho > 0$ and

$$K \subset B(\rho, p - \rho a).$$

The set of points that are ρ -circular for some $\rho > 0$ are called circular.

THEOREM 4. If the set of circular points of the boundary of K has positive $(d - 1)$ -dimensional measure in δK , then

$$\text{vol } K(\varepsilon) \geq \text{const } (K) \varepsilon^{2/(d+1)}.$$

It is clear that for smooth enough (\mathcal{C}^3 , say) convex bodies the conditions of Theorem 4 are satisfied. Thus for smooth convex bodies K we have from Theorem 1, 4 and (1.5) that $E(K, n) \sim n^{-2/(d+1)}$.

What happens between these two extreme classes is not a mystery: it is the usual unpredictable behaviour. Using (1.5), (1.10) and a general theorem of Gruber [13] (see Schneider [22] for a similar application) one can show this.

THEOREM 5. Assume $\omega(n) \rightarrow 0$ and $\Omega(n) \rightarrow \infty$ as $n \rightarrow \infty$. Then for most (in the Baire-category sense) convex bodies $K \subset R^d$ with $\text{vol } K = 1$ one has, for infinitely many n

$$E(K, n) < n^{-1}(\log n)^{d-1}\Omega(n),$$

and also, for infinitely many n

$$E(K, n) > n^{-2/(d+1)}\omega(n).$$

§2. Economic cap coverings. One of the main tools in proving Theorem 1 will be the construction of an economic cap covering of $K(v \leq \varepsilon)$. For a closed halfspace H the set $K \cap H$ is called a cap of K whenever it is nonempty.

THEOREM 6. *Assume a convex body $K \subset R^d$ is given with $\text{vol } K > 0$. Take ε with $0 < \varepsilon < \varepsilon_0(d) = (2d)^{-2d}$. Then there are caps K_1, \dots, K_m of K and pairwise disjoint subsets $K'_1, \dots, K'_m, K'_i \subset K_i$ ($i = 1, \dots, m$) such that*

- (i) $\bigcup_1^m K'_i \subset K(v \leq \varepsilon) \subset \bigcup_1^m K_i,$
- (ii) $\text{vol } K_i \leq 6^d \varepsilon \quad (i = 1, \dots, m),$
- (iii) $\text{vol } K'_i \geq (6d)^{-d} \varepsilon \quad (i = 1, \dots, m).$

This is what is called economic cap covering in the title. In Ewald, Larman, Rogers [9] there is another economic cap covering theorem for the inner parallel body of K (instead of $K(v \leq \varepsilon)$). Our proof of Theorem 6 is an adaptation of the one in Ewald, Larman, Rogers [9].

Actually, $K \setminus K(v \leq \varepsilon)$ is a certain kind of "inner parallel" body to K . One may wonder then if its volume is a convex function of ε or not, or, what is the same, if $\text{vol } K(v \leq \varepsilon)$ is a concave function of ε or not. Maybe the d -th root of $\text{vol } K(v \leq \varepsilon)$ is concave. We do not know the answer to these questions. However we can prove some concavity type property of $\text{vol } K(v \leq \varepsilon)$ that will be useful.

THEOREM 7. *Under the assumptions of Theorem 6 one has, for $\lambda \geq 1$,*

$$\text{vol } K(v \leq \varepsilon) \geq \text{const } (d) \lambda^{-d} \text{vol } K(v \leq \lambda \varepsilon). \tag{2.1}$$

We mention a Heilbronn type consequence of Theorem 6.

THEOREM 8. *Assume $P \subset R^d$ is a convex polytope with n vertices and $\text{vol } P > 0$. Then P has $(d + 1)$ vertices x_0, x_1, \dots, x_d such that*

$$\text{vol } (\text{conv } \{x_0, \dots, x_d\}) / \text{vol } P \leq \text{const } (d) n^{-(d+1)/(d-1)}. \tag{2.2}$$

This is a Heilbronn type result (cf. [15, 20]) because it says that among n points in convex position in R^d there is a simplex with small relative volume. This result is known in the plane in a sharper form, see Rényi, Sulanke [19].

Theorem 8 is related to a theorem of Arnold [1] (when $d = 2$) and Konyagin, Sevastyanov [16] (when $d > 2$) which states that for a lattice polytope $P \subset R^d$ with n vertices and positive volume one has

$$\text{const } (d) n^{(d+1)/(d-1)} \leq \text{vol } P. \tag{2.3}$$

Theorem 8 can be regarded as an extension of (2.3) to the case of general (non-lattice) polytopes. Actually, (2.2) implies (2.3) if the lattice polytope P has no $d + 1$ vertices on a hyperplane because then in the left-hand side of (2.2) the volume of the simplex is at least $1/(d!)$. In fact, the results of Arnold and Konyagin, Sevastyanov are contained in the results of G. E. Andrews [25].

§3. Notation, definitions, basic properties. A cap C of K is a set $C = K \cap H$ where H is a closed halfspace and $K \cap H$ is nonempty. Then $H = \{x \in R^d : a \cdot x \leq \alpha\}$ for some $a \in R^d$ with $|a| = 1$ and $\alpha \in R^1$. Here $a \cdot x$ denotes the scalar product of a and $x \in R^d$. It will be convenient to write $H = H(a, t)$ with $t = h(a) - \alpha$ where $h(a) = \max \{a \cdot x : x \in K\}$ is the support

function of K . With this notation t is the width of the cap C in direction a . In the same spirit we write $H(a, t_1, t_2)$ for the strip between the hyperplanes $H(a, t_1)$ and $H(a, t_2)$.

For a cap $C = K \cap H(a, t)$ a point $z \in C$ is called the centre of C if $a \cdot z = h(a)$. A cap may have several centres but we think of a cap as having a fixed centre, say, the centre of gravity of all centres.

For a cap $C = K \cap H(a, t)$ with centre z we define (when $\lambda \geq 0$)

$$C^\lambda = z + \lambda(C - z). \tag{3.1}$$

Obviously $C^1 = C$. It is easy to see that for $\lambda \geq 1$ one has

$$C^\lambda \supset K \cap H(a, \lambda t). \tag{3.2}$$

When $x \in K$, a minimal cap is defined as a cap $C(x)$ with $x \in C(x)$ and

$$\text{vol } C(x) = v(x) = \min \{ \text{vol } H \cap K : x \in H \text{ a halfspace} \}.$$

Let us write $H(a = t)$ for the bounding hyperplane of the halfspace $H(a, t)$. A standard variational argument shows that for a minimal cap

$$C(x) = K \cap H(a, t)$$

the point x is the centre of gravity of the section $K \cap H(a = t)$.

For $x \in K$ and $\lambda > 0$ we call the set

$$M(x, \lambda) = M_K(x, \lambda) = x + \lambda \{ (K - x) \cap (x - K) \} \tag{3.3}$$

a Macbeath region. Such regions were studied by A. M. Macbeath [17] and Ewald, Larman, Rogers [9]. A Macbeath region is obviously convex and centrally symmetric with centre x . We will write $M(x) = M_K(x) = M(x, 1)$ when convenient. Define a map $u : K \rightarrow R$ as

$$u(x) = \text{vol } M(x).$$

Macbeath [17] has shown that the set $K(u \geq \varepsilon) = \{x \in K : u(x) \geq \varepsilon\}$ is convex. The convexity of the set $K(v \geq \varepsilon)$ is trivial because it is the intersection of closed halfspaces. It turns out that $K(v \geq \varepsilon)$ is ‘‘close’’ to $K(u \geq \varepsilon)$.

THEOREM 9. *Assume $0 < \varepsilon < \varepsilon_1(d)$. Then*

$$K(v \leq \varepsilon) \subset K(u \leq 2\varepsilon) \subset K(v \leq 2(3d)^d \varepsilon)$$

and

$$\text{vol } K(v \leq \varepsilon) \leq \text{vol } K(u \leq 2\varepsilon) \leq c_1(d) \text{vol } K(v \leq \varepsilon)$$

where $\varepsilon_1(d)$ and $c_1(d)$ are constants depending on d only.

Here one can take

$$\varepsilon_1(d) = \frac{1}{2}(12d^3)^{-d}. \tag{3.4}$$

We will postpone the proof of this theorem till the last section because we will not use it in the paper.

Denote by $B(r)$ or $B^d(r)$ the ball of radius r and centre 0 in R^d . Throughout the paper we will assume that the given compact convex body $K \subset R^d$ (with

$\text{vol } K > 0$) is in “standard form”, i.e.,

$$B(r) \subset K \subset B(R) \quad \text{and} \quad dr \geq R. \tag{3.5}$$

It is well-known (see [7] for instance) that any convex compact body can be transformed by a volume preserving affine transformation into a body K in standard form. Further, it is clear that such a transformation does not change the quantities $\text{vol } K(v \leq \varepsilon)$, $\text{vol } K(u \leq \varepsilon)$ or $E(K, n)$ when $\text{vol } K = 1$.

The assumption $\text{vol } K = 1$ in the theorems is made for convenience. What we really need is $\text{vol } K > 0$. At some points we will have to consider sets K with $\text{vol } K \neq 1$. Then $\text{vol } K(v \leq \varepsilon)$ is not affine invariant and it is better to consider instead

$$\text{vol } K(v \leq \varepsilon \text{ vol } K) / \text{vol } K \tag{3.6}$$

which is affine invariant.

§4. *Proof of Theorem 6.* We start with two lemmas.

LEMMA 1. $u(x) \leq 2v(x)$.

Proof. Take a halfspace H with $x \in H$. Then

$$u(x) = \text{vol } M(x) \leq 2 \text{vol } (M(x) \cap H) \leq 2 \text{vol } (K \cap H),$$

so

$$u(x) \leq 2 \min \{ \text{vol } (K \cap H) : x \in H \} = 2v(x).$$

LEMMA 2. $v(x) \leq (3d)^d u(x)$ if $v(x) \leq (2d)^{-2}d$ or if $u(x) \leq (12d^3)^{-d}$.

Proof. We prove first that $v(x) \leq (2d)^{-2}d$ implies $v(x) \leq (3d)^d u(x)$.

Take a minimal cap $C(x) = K \cap H(a, t)$. As we mentioned earlier x is the centre of gravity of the section $K \cap H(a = t)$. Then, by Lemma 2 of Ewald, Larman, Rogers [9],

$$C(x) \subset M(x, 3d) \tag{4.1}$$

provided $B(r/2) \cap H(a, t)$ is empty and $t \leq r/4$.

Assume now that (4.1) fails. Then either $B(r/2) \cap H(a, t)$ is nonempty or $t \geq r/4$. We show now that both cases contradict the condition $v(x) \leq (2d)^{-2}d$.

In the first case, i.e., when $B(r/2) \cap H(a, t) \neq \emptyset$, the set $B(r) \cap H(a, t)$ contains a cap C_r of $B(r)$ whose width is $r/2$. Moreover, by (3.5),

$$C = K \cap H(a, t) \supset B(r) \cap H(a, t) \supset C_r$$

so $\text{vol } C(x) \geq \text{vol } C_r$. A simple computation shows now that $\text{vol } C_r \geq (2d)^{-d}$.

In the second case when $t \geq r/4$, i.e., the width of $C(x)$ in direction a is at least $r/4$, let z be the centre of $C(x)$. Consider the cone L with apex z whose base is the intersection of $B(r)$ with the hyperplane through 0 and orthogonal to a . The height of this cone is $h(a) \leq R$ and its volume is

$$\text{vol } L = (1/d)r^{d-1}\omega_{d-1}h(a)$$

where ω_{d-1} is the volume of B^{d-1} , the unit ball of R^{d-1} . The cap $C(x)$ contains

the part of this cone L lying in the strip $H(a, 0, t)$. The volume of this part is

$$\begin{aligned} \left(\frac{t}{h(a)}\right)^d \text{vol } L &= \left(\frac{t}{h(a)}\right)^{d-1} \frac{t}{d} r^{d-1} \omega_{d-1} \geq \left(\frac{r}{4R}\right)^{d-1} \frac{r}{4d} r^{d-1} \omega_{d-1} \\ &\geq \left(\frac{1}{4d}\right)^{d-1} \frac{1}{4d} d^d r^d \omega_d (\omega_{d-1}/\omega_d) d^{-d} \\ &\geq (4d)^{-d} R^d \omega_d (\omega_{d-1}/\omega_d) d^{-d} \geq (2d)^{-2d} (\omega_{d-1}/\omega_d) \\ &\geq (2d)^{-2d}. \end{aligned}$$

So $\text{vol } C(x) \geq (2d)^{-2d}$. This contradiction shows that (4.1) holds. Then obviously, $v(x) \leq (3d)^d u(x)$.

To finish the proof of the lemma we prove now that $u(x) \leq (12d^3)^{-d}$ implies $v(x) \leq (2d)^{-2d}$. To see this we claim that

$$K(v \geq (2d)^{-2d}) \subset K(u \geq (12d^3)^{-d}).$$

Both sets here are convex (the second by Macbeath's result [17]) and both of them contain the origin. When x is a point on the boundary of $K(v \geq (2d)^{-2d})$, i.e., when $v(x) = (2d)^{-2d}$, then by the first part of this proof,

$$u(x) \geq (3d)^{-d} v(x) = (12d^3)^{-d},$$

i.e., $x \in K(u \geq (12d^3)^{-d})$.

Now we turn to the proof of Theorem 6. Consider the set $K(v \geq \varepsilon)$ and choose a maximal system of points x_1, x_2, \dots, x_m from the boundary $\partial K(v \geq \varepsilon)$ of the set $K(v \geq \varepsilon)$ subject to the condition that

$$M(x_i, \frac{1}{2}) \cap M(x_j, \frac{1}{2}) = \emptyset \quad \text{when} \quad x_i \neq x_j. \tag{4.2}$$

This maximal system is indeed finite because the sets $M(x_i, \frac{1}{2})$ are pairwise disjoint, all of them lie in K and

$$\text{vol } M(x_i, \frac{1}{2}) = 2^{-d} \text{vol } M(x_i, 1) = 2^{-d} u(x_i) \geq (6d)^{-d} v(x_i) = (6d)^{-d} \varepsilon \tag{4.3}$$

according to Lemma 2.

CLAIM 1. $K(v \leq \varepsilon) \subset \bigcup \{M(x_i, 5) : i = 1, \dots, m\}$.

Proof. Consider any point $y'' \in K(v \leq \varepsilon)$. As $0 \in \text{int } K(v \geq \varepsilon)$, the halfline stemming from 0 in direction y'' intersects the boundary of the convex set $K(v \geq \varepsilon)$ and K at the points y and y' , respectively. Now x_1, \dots, x_m form a maximal system in $\delta K(v \geq \varepsilon)$ with respect to (4.2) and $y \in \delta K(v \geq \varepsilon)$. So there is an i such that

$$M(x_i, \frac{1}{2}) \cap M(y, \frac{1}{2}) \neq \emptyset.$$

Then, by Lemma 1 of Ewald, Larman, Rogers [9],

$$M(y, 1) \subset M(x_i, 5). \tag{4.4}$$

We will prove now that $y' \in M(y, 1)$. This will show that the line segment $[y, y']$ and, consequently, the point $y'' \in [y, y']$ lie in $M(y, 1)$ and this will prove the Claim.

Assume $y' \notin M(y, 1)$. On the line through 0 and y let z be the point at distance r from 0 and such that 0 lies between z and y . Then $z \in B(r) \subset K$ and so $y' \notin M(y, 1)$ implies

$$|z - y| < |y - y'|.$$

Consider now the minimal cap $C = C(y) = K \cap H$. Clearly, H cannot contain 0 for otherwise C would contain "half" of the ball $B(r)$ which has volume $\frac{1}{2}r^d \omega_d \geq \frac{1}{2}d^{-d} > \varepsilon = \text{vol } C$. Then H must contain y' . Then H must contain "half" of the cone L whose apex is y' and whose base is the intersection of the set $\text{conv}(\{y\} \cup B(r))$ with the halfspace orthogonal to, and passing through, the vector y . Computing volumes again

$$\begin{aligned} \text{vol } C &\geq \frac{1}{2} \text{vol } L \geq \frac{1}{2} \frac{1}{d} |y'| r^{d-1} \omega_{d-1} \left(\frac{|y - y'|}{|y'|} \right)^d \\ &\geq \frac{1}{2d} r^d \omega_{d-1} \left(\frac{|z - y|}{|y'|} \right)^d \geq \frac{1}{2d} r^d \omega_d (r/R)^d (\omega_{d-1}/\omega_d) > (2d)^{-2d} \geq \varepsilon. \end{aligned}$$

Now we have an economic cap covering of $K (v \leq \varepsilon)$ by Macbeath regions. We are going to turn it into a covering by caps.

For this end consider the minimal cap $C_i = C(x_i) = K \cap H(a_i, t_i)$, for $i = 1, \dots, m$. Define

$$\begin{aligned} K_i &= K \cap H(a_i, 6t_i), \\ K'_i &= M(x_i, \frac{1}{2}) \cap H(a_i, t_i). \end{aligned}$$

We claim that the sets K_i, K'_i satisfy the requirements of the theorem. First, as the sets $M(x_i, \frac{1}{2})$ are pairwise disjoint, so are the sets K'_i . According to (4.1), $C_i \subset M(x_i, 3d)$ so

$$\begin{aligned} \text{vol } K'_i &= \frac{1}{2} \text{vol } M(x_i, \frac{1}{2}) = \frac{1}{2} (6d)^{-d} \text{vol } M(x_i, 3d) \\ &\geq \frac{1}{2} (6d)^{-d} \text{vol } C_i = \frac{1}{2} (6d)^{-d} \varepsilon. \end{aligned}$$

One can get $\text{vol } K'_i \geq (6d)^{-d} \varepsilon$ from here by observing that the central symmetry of $M(x_i, 3d)$ and (4.1) imply $2 \text{vol } C_i \leq \text{vol } M(x_i, 3d)$.

Notice that $M(x_i, 1)$ lies in the strip $H(a_i, 0, 2t_i)$. Then $M(x_i, 5)$ lies in the strip $H(a_i, -4t_i, 6t_i)$ as the centre of $M(x_i, \lambda)$ is on the hyperplane $H(a_i = t_i)$. Thus

$$K \cap M(x_i, 5) \subset K \cap H(a_i, -4t_i, 6t_i) = K \cap H(a_i, 6t_i) = K_i$$

and indeed

$$K(v \leq \varepsilon) \subset \bigcup_1^m K_i.$$

According to (3.1) and (3.2)

$$\text{vol } K_i \leq 6^d \text{vol } C_i = 6^d \varepsilon.$$

Finally, $K'_i \subset K_i$ is evident.

§5. *Proof of Theorem 7.* Let K_1, \dots, K_m be the economic cap covering of $K(v \leq \varepsilon)$ from Theorem 6. We will prove that the union of $K_i^{d\lambda}$ ($i = 1, \dots, m$) covers $K(v \leq \lambda\varepsilon)$.

So we take a point $x \in K(v \leq \lambda\varepsilon)$. We have to show that

$$x \in K_1^{d\lambda} \cup \dots \cup K_m^{d\lambda},$$

thus we may assume that $x \notin K_1 \cup \dots \cup K_m$.

The minimal cap $C(x) = K \cap H(a, t)$ has centre z (say), and the line segment $[x, z]$ intersects the boundary of $K(v \leq \varepsilon)$ at the point y . Clearly $v(y) = \varepsilon$. Let t' be the distance of y from the hyperplane $H(a, 0)$ (which supports K at z). Then $y \in H(a = t')$ and

$$\begin{aligned} \varepsilon = v(y) &\leq \text{vol}(K \cap H(a, t')) = \int_0^{t'} \text{vol}_{d-1}(K \cap H(a = \tau)) d\tau \\ &\leq t' \max \{ \text{vol}_{d-1}(K \cap H(a = \tau)) : 0 \leq \tau \leq t' \} \\ &\leq t' \max \{ \text{vol}_{d-1}(K \cap H(a = \tau)) : 0 \leq \tau \leq t \}. \end{aligned}$$

On the other hand

$$\lambda\varepsilon \geq v(x) = \text{vol} K \cap H(a, t) \geq \frac{1}{d} t \max \{ \text{vol}_{d-1}(K \cap H(a = \tau)) : 0 \leq \tau \leq t \},$$

where the last inequality follows from the fact that the double cone whose base is the maximal section $K \cap H(a = \tau)$ is contained in $C(x)$. Now $t/t' = |z - x|/|z - y|$ and so we get

$$|z - x| \leq \lambda d |z - y|.$$

Consider now the cap $K_i = K \cap H(a_i, t_i)$ from the cap covering that contains y . Let z_i be the centre of K_i and write y_i for the intersection $[z_i, x] \cap H(a_i = t_i)$. The line L through z and x intersects the hyperplanes $H(a_i = 0)$ and $H(a_i = t_i)$ at the points z' and y' , respectively. It is easy to check that the points z', z, y, y', x come on L in this order. Then

$$\frac{|x - z_i|}{|y_i - z_i|} = \frac{|x - z'|}{|y' - z'|} \leq \frac{|x - z'|}{|y - z'|} = \frac{|x - z| + |z - z'|}{|y - z| + |z - z'|} \leq \frac{|x - z|}{|y - z|} \leq \lambda d.$$

So indeed $x \in K_1^{d\lambda} \cup \dots \cup K_i^{d\lambda}$. Now

$$\begin{aligned} \text{vol} K(v \leq \lambda\varepsilon) &\leq \sum_{i=1}^m \text{vol} K_i^{d\lambda} \leq (\lambda d)^d \sum_{i=1}^m \text{vol} K_i \leq (\lambda d)^d 6^d m\varepsilon \\ &\leq (6\lambda d)^d (6d)^d \sum_{i=1}^m \text{vol} K'_i \leq (36\lambda d^2)^d \text{vol} K(v \leq \varepsilon). \end{aligned}$$

§6. *Proof of Theorem 1.* To establish the lower bound let $x \in K$ and let $C(x)$ be the corresponding minimal cap. Then

$$\text{Prob}(x \notin K_n) \geq \text{Prob}(C(x) \cap K_n = \emptyset) = (1 - v(x))^n.$$

Consequently, for $\varepsilon > 0$, we get

$$\begin{aligned} E(K, n) &= \int_K \text{Prob}(x \notin K_n) \geq \int_K (1 - v(x))^n \geq \int_{K(v \leq \varepsilon)} (1 - v(x))^n \\ &\geq \int_{K(v \leq \varepsilon)} (1 - \varepsilon)^n = (1 - \varepsilon)^n \text{vol } K(v \leq \varepsilon). \end{aligned}$$

Choosing now $\varepsilon = 1/n$ (and assuming $n \geq 3$) we have

$$\frac{1}{4} \text{vol } K \left(v \leq \frac{1}{n} \right) \leq E(K, n).$$

Proving the upper bound is more involved. First we use an idea from Bárány and Füredi [2]. Let x_1, \dots, x_n be randomly chosen points and write $N(x) = \{x_1, \dots, x_n\} \cap M(x)$ when $x \in K$. Further, denote by $n(x)$ the cardinality of $N(x)$. Now

$$\begin{aligned} \text{Prob}(x \notin K_n) &= \sum_{m=0}^n \text{Prob}(x \notin K_n | n(x) = m) \text{Prob}(n(x) = m) \\ &\leq \sum_{m=0}^n \text{Prob}(x \notin \text{conv } N(x) | n(x) = m) \text{Prob}(n(x) = m). \end{aligned} \tag{6.1}$$

According to a theorem of Wendel [24] (cf. Füredi [10] as well)

$$\text{Prob}(x \notin \text{conv } N(x) | n(x) = m) = 2^{-(m-1)} \sum_{i=1}^{d-1} \binom{m}{i}.$$

Using this

$$\begin{aligned} \text{Prob}(x \notin K_n) &\leq 2 \sum_{m=0}^n 2^{-m} \sum_{i=0}^{d-1} \binom{m}{i} [u(x)]^m [1 - u(x)]^{n-m} \binom{n}{m} \\ &= 2 \sum_{i=0}^{d-1} \sum_{m=i}^n \binom{m}{i} \binom{n}{m} [u(x)/2]^m [1 - u(x)]^{n-m} \\ &= 2 \sum_{i=0}^{d-1} \sum_{m=i}^n \binom{n}{i} \binom{n-i}{m-i} [u(x)/2]^m [1 - u(x)]^{n-m} \\ &= 2 \sum_{i=0}^{d-1} \binom{n}{i} \sum_{m=i}^m \binom{n-i}{m-i} [u(x)/2]^m [1 - u(x)]^{n-m} \\ &= 2 \sum_{i=0}^{d-1} \binom{n}{i} \sum_{k=0}^{n-i} \binom{n-i}{k} [u(x)/2]^{k+i} [1 - u(x)]^{n-i-k} \\ &= 2 \sum_{i=0}^{d-1} \binom{n}{i} [u(x)/2]^i [1 - \frac{1}{2}u(x)]^{n-i}. \end{aligned} \tag{6.2}$$

Now we integrate

$$\begin{aligned}
 E(K, n) &= \int \text{Prob}(x \notin K_n) dx \\
 &\leq 2 \sum_{i=0}^{d-1} \binom{n}{i} \int_K [\frac{1}{2}u(x)]^i [1 - \frac{1}{2}u(x)]^{n-i} dx \\
 &= 2 \sum_{i=0}^{d-1} \binom{n}{i} \sum_{\lambda=1}^n \int_{[(\lambda-1)/n] \leq u(x) \leq (\lambda/n)} [\frac{1}{2}u(x)]^i [1 - \frac{1}{2}u(x)]^{n-i} dx \\
 &< 2 \sum_{i=0}^{d-1} \binom{n}{i} \sum_{\lambda=1}^n \int_{[(\lambda-1)/n] \leq u(x) \leq (\lambda/n)} \left(\frac{\lambda}{2n}\right)^i \left(1 - \frac{\lambda-1}{2n}\right)^{n-i} \\
 &\leq 2 \sum_{i=0}^{d-1} \binom{n}{i} \sum_{\lambda=1}^n \left(\frac{\lambda}{2n}\right)^i \left(1 - \frac{\lambda-1}{2n}\right)^{n/2} \text{vol } K\left(u \leq \frac{\lambda}{n}\right). \tag{6.3}
 \end{aligned}$$

Here

$$2 \binom{n}{i} \left(\frac{\lambda}{n}\right)^i \leq \lambda^i / (2^i i!) \leq \lambda^{d-1}$$

because $i \geq d - 1$. Moreover, when $n \geq 2d$,

$$\left(1 - \frac{\lambda-1}{2n}\right)^{n-i} \leq \left(1 - \frac{\lambda-1}{2n}\right)^{n-i} \leq e^{-(\lambda-1)/4}.$$

According to Lemma 2

$$K(u \leq \lambda/n) \subset K(v \leq (3d)^d \lambda/n)$$

provided $\lambda/n < (12d^3)^{-d}$. We set $\Lambda = \lfloor (12d^3)^{-d} n \rfloor$ and continue (6.3).

$$\begin{aligned}
 E(K, n) &\leq \sum_{\lambda=1}^{\Lambda} d\lambda^{d-1} e^{-(\lambda-1)/4} \text{vol } K(v \leq (3d)^d \lambda/n) \\
 &\quad + \sum_{\lambda=\Lambda+1}^n d\lambda^{d-1} e^{-(\lambda-1)/4}, \tag{6.4}
 \end{aligned}$$

as $\text{vol } K(v \leq \varepsilon) \leq \text{vol } K = 1$ for every $\varepsilon \geq 0$. Here in the first sum

$$\text{vol } K(v \leq (3d)^d \lambda/n) \leq [36\lambda(3d)^d d^2]^d \text{vol } K(v \leq 1/n),$$

by Theorem 7, so the first sum is at most

$$\text{vol } K\left(v \leq \frac{1}{n}\right) \sum_{\lambda=1}^{\infty} d\lambda^{d-1} e^{-(\lambda-1)/4} \lambda^d [36d^2(3d)^d]^d < \text{const}(d) \text{vol } K\left(v \leq \frac{1}{n}\right). \tag{6.5}$$

To estimate the second sum in (6.4) observe that it is less than

$$\sum_{\lambda=1}^{\infty} d\lambda^{d-1} e^{-(\lambda-1)/8} e^{-(12d^3)^{-d}(n/8)} < c_1(d) e^{-(12d^3)^{-d}(n/8)},$$

where $c_1(d)$ is a constant depending on d only. We need a lower bound on

$\text{vol } K(v \leq 1/n)$. We could use Theorem 2, but we prefer the very simple $\text{vol } K(v \leq \varepsilon) \geq \varepsilon$ inequality, which follows from the fact that $C(x) \subset K(v \leq \varepsilon)$ for any x with $v(x) = \varepsilon$. Using this the second sum in (6.4) is less than

$$c_1(d)e^{-(12d^3)^{-d}(n/8)} < c_2(d)\frac{1}{n} \leq c_2(d) \text{vol } K(v \leq 1/n). \tag{6.6}$$

With (6.5) and (6.6) we get from (6.4) that indeed

$$E(K, n) \leq \text{const } (d) \text{vol } K(v \leq 1/n).$$

We mention here that a byproduct of (6.2) is that:

$\text{Prob}(x \notin K_n)$

$$\leq 2 \text{Prob}(\text{less than } d \text{ points from } \{x_1, \dots, x_n\} \text{ lie in } M(x) \cap C(x)).$$

§7. *Proof of Theorem 2.* We start with some notation. Fix $a \in R^d, |a| = 1$ somehow and let $H(a = t_0)$ be the hyperplane whose intersection with K has the largest $(d - 1)$ -dimensional volume among all hyperplanes $H(a = t)$. Assume the width of K in direction a is at most $2t_0$. If this were not the case we would take $-a$ instead of a . As a will be fixed throughout this proof we will write $H(t) = H(a = t)$. Define, further,

$$Q(t) = H(t) \cap K \quad \text{and} \quad q(t) = \text{vol}_{d-1} Q(t).$$

Our choice for t_0 insures that

$$q(t) \geq (t/t_0)^{d-1} q(t_0) \quad \text{for } 0 \leq t \leq t_0, \tag{7.1}$$

$$2t_0 q(t_0) \geq \text{vol } K = 1. \tag{7.2}$$

LEMMA 3. For $\varepsilon > 0$ and $0 < t < t_0$

$$K(u_K \leq \varepsilon) \cap H(t) \supset Q(t)(u_{Q(t)} \leq \varepsilon/2t).$$

Proof. We are going to show that $x \in H(t) \cap K$ implies $u_K(x) \leq 2tu_{Q(t)}(x)$. This will prove the lemma.

Notice, first that $M(x)$ lies in the strip $H(a, 0, 2t)$. Then

$$u(x) = \int_0^{2t} \text{vol}_{d-1}(M(x) \cap H(\tau)) d\tau \leq 2t \text{vol}_{d-1}(M(x) \cap H(t))$$

because $M(x)$ is centrally symmetric so its largest section is the middle one, $M(x) \cap H(t)$. Next

$$\begin{aligned} M(x) \cap H(t) &= \{x + [(K - x) \cap (x - K)]\} \cap H(t) \\ &= x + \{[(K \cap H(t)) - x] \cap [x - (K \cap H(t))]\} \\ &= x + [(Q(t) - x) \cap (x - Q(t))] \\ &= M_{Q(t)}(x). \end{aligned}$$

Then

$$u(x) \leq 2t \operatorname{vol}_{d-1} M_{Q(t)}(x) = 2tu_{Q(t)}(x).$$

We will now show that for $0 < \varepsilon \leq 1$

$$\operatorname{vol} K(u \leq \varepsilon) \geq \operatorname{const}(d) \varepsilon (\log(1/\varepsilon))^{d-1}. \tag{7.3}$$

Recalling Lemma 2 this proves that for $\varepsilon \leq (2d)^{-2d}$

$$\operatorname{vol} K(v \leq \varepsilon) \geq \operatorname{vol} K(u \leq (3d)^{-\varepsilon} \varepsilon) \geq \operatorname{const}(d) \varepsilon (\log(1/\varepsilon))^{d-1}.$$

When $\varepsilon > (2d)^{-2d}$, the statement of the theorem follows from the fact that $\operatorname{vol} K(v \leq \varepsilon)$ is an increasing function of ε .

We prove (7.3) by induction on d . The case $d = 1$ is trivial. We will need the induction hypothesis in the invariant form (3.6): for $Q \subset \mathbb{R}^{d-1}$ compact, convex with $\operatorname{vol}_{d-1} Q > 0$ and for $0 < \eta \leq 1$

$$\operatorname{vol} Q(u_Q \leq \eta \operatorname{vol} Q) / \operatorname{vol} Q \geq c_{d-1} \eta (\log(1/\eta))^{d-2}.$$

Assuming this holds we prove (7.3). Write

$$\begin{aligned} \operatorname{vol} K(u \leq \varepsilon) &= \operatorname{vol} [K(u \leq \varepsilon) \cap H(a, 0, t_0)] = \int_0^{t_0} \operatorname{vol}_{d-1} [K(u \leq \varepsilon) \cap H(t)] dt \\ &\geq \int_0^{t_0} \operatorname{vol}_{d-1} Q(t) (u_{Q(t)} \leq \varepsilon/2t) dt, \end{aligned} \tag{7.4}$$

according to Lemma 3. Define $\eta = \eta(t) = \varepsilon/(2tq(t))$ and let t_1 be the unique solution to $\eta(t) = 1$ between 0 and t_0 . Then, by the induction hypothesis, for $t_1 \leq t \leq t_0$

$$\begin{aligned} \operatorname{vol}_{d-1} Q(t) (u_{Q(t)} \leq \eta q(t)) &\geq c_{d-1} q(t) \eta (\log(1/\eta))^{d-2} \\ &= c_{d-1} \frac{\varepsilon}{2t} \left[\log \left(\frac{2tq(t)}{\varepsilon} \right) \right]^{d-2} \\ &\geq c_{d-1} \frac{\varepsilon}{2t} \left[\log \left(\frac{2t}{\varepsilon} \left(\frac{t}{t_0} \right)^{d-1} q(t_0) \right) \right]^{d-2}, \end{aligned}$$

where the last inequality follows from (7.1). We continue (7.4).

$$\operatorname{vol} K(u \leq \varepsilon) \geq \int_{t_1}^{t_0} c_{d-1} \frac{\varepsilon}{2t} \left[\log \left(\frac{2t^d q(t_0)}{\varepsilon t_0^{d-1}} \right) \right]^{d-2}. \tag{7.5}$$

Define α by $\alpha^d = 2q(t_0)/(\varepsilon t_0^{d-1})$ and let $t_2 = 1/\alpha$. Then, by (7.1) again, $t_1 < t_2 < t_0$. Substitute now $\tau = \alpha t$ with $\tau_i = \alpha t_i$, $i = 0, 2$. Continue (7.5).

$$\begin{aligned} \operatorname{vol} K(u \leq \varepsilon) &\geq \int_{\tau_2}^{\tau_0} c_{d-1} \frac{\varepsilon}{2} \frac{1}{\tau} (\log \tau)^{d-2} d\tau = \frac{\varepsilon c_{d-1}}{2(d-1)} (\log \tau)^{d-1} \Big|_{\tau=\tau_2}^{\tau=\tau_0} \\ &= \frac{\varepsilon c_{d-1}}{2(d-1)} \left[\log \left(\frac{t_0 (2q(t_0))^{1/d}}{(\varepsilon t_0^{d-1})^{1/d}} \right) \right]^{d-1} \geq \frac{\varepsilon c_{d-1}}{2(d-1)} \left(\frac{1}{d} \log \frac{1}{\varepsilon} \right)^{d-1} \end{aligned}$$

where the last inequality follows from (7.2).

§8. *Proof of Theorem 3.* We prove this theorem for simplices first and then for general polytopes. We may take any simplex $S \subset R^d$ because for a nonsingular linear transformation A one clearly has

$$\text{vol } S(v_S \leq \varepsilon) / \text{vol } S = \text{vol } AS(v_{AS} \leq |\det A| \varepsilon) / \text{vol } AS. \tag{8.1}$$

We take a regular simplex $S = \text{conv} \{y_0, \dots, y_d\}$ with $\text{vol } S = 1$.

LEMMA 4. Assume $z \in \text{int } S$ and the nearest vertex to z is y_0 . Then

$$C(z) \supset M(\frac{1}{2}(z + y_0)).$$

Proof. Let $C(z) = S \cap H(a, t)$ be the minimal cap for z . Recall the definition:

$$H(a, t) = \{x \in R^d : a \cdot x = h(a) - t\} \text{ with } h(a) = \max \{a \cdot x : x \in S\}.$$

We know that $z \in H(a = t)$ and that z is the centre of gravity of the section $S \cap H(a = t)$.

Obviously, $h(a) = a \cdot y_i$ for some vertex y_i . Consider

$$x \in M(\frac{1}{2}(z + y_i)) = S \cap (z + y_i - S).$$

Then $x = z + y_i - y$ with $y \in S$, so

$$a \cdot x = a \cdot z + a \cdot y_i - a \cdot y \geq h(a) - t.$$

This shows that

$$M(\frac{1}{2}(z + y_i)) \subset S \cap H(a, t) = C(z)$$

for some vertex, y_i , of S .

Assume now that z is closer to y_j than to y_k . We will prove then that $M(\frac{1}{2}(z + y_k)) \subset C(z)$ does not hold. This will prove the lemma.

Consider the reflection, z' , of z to the hyperplane bisecting the line segment $[y_j, y_k]$. We show that $z' \notin C(z)$ and $z' \in M(\frac{1}{2}(z + y_k))$. By the symmetry of the regular simplex we have $v(z) = v(z')$. Now $z' \in \text{int } C(z)$ would imply $v(z') < v(z)$, a contradiction. And if z' were on the bounding hyperplane of $C(z)$, then $C(z) = C(z')$ must hold. But this cannot be the case because both z and z' cannot be the centre of gravity of the section $S \cap H(a = t)$. So $z' \notin C(z)$. On the other hand $z' = z + \alpha(y_k - y_j) \in S$ for some $\alpha \in (0, 1)$. Then

$$z' = z + \alpha(y_k - y_j) = z + y_k - [(1 - \alpha)y_k + \alpha y_j] \in z + y_k - S.$$

Thus $z' \in M(\frac{1}{2}(z + y_k))$ and then $M(\frac{1}{2}(z + y_k)) \subset C(z)$ is indeed impossible.

Define $T_i = \{x \in S : |x - y_i| = \min \{|x - y_j| : j = 0, \dots, d\}\}$. Then

$$S(v \leq \varepsilon) = \bigcup_{i=0}^d (T_i \cap S(v \leq \varepsilon)) \subset \bigcup_{i=0}^d \{x \in T_i : u(\frac{1}{2}(x + y_i)) \leq \varepsilon\},$$

by Lemma 4. Thus

$$\text{vol } S(v \leq \varepsilon) \leq (d + 1) \text{vol} \{x \in T_0 : u(\frac{1}{2}(x + y_0)) \leq \varepsilon\}.$$

Define now an affine transformation $A : R^d \rightarrow R^d$ with $Ay_0 = 0$ and $Ay_i = e_i$ ($i = 1, \dots, d$) where e_1, \dots, e_d form an orthonormal basis of R^d . Write

$Ax = (\xi_1, \dots, \xi_d) = \xi_1 e_1 + \dots + \xi_d e_d$. Then $x \in T_0$, $u_S(\frac{1}{2}(x + y_0)) \leq \varepsilon$ imply $\xi_1 \dots \xi_d \leq |\det A| \varepsilon$ and $\max \{\xi_i: i = 1, \dots, d\} \leq 1$.

Similarly as in (8.1) and (3.6)

$$\frac{\text{vol} \{x \in T_0: u(\frac{1}{2}(x + y_0)) \leq \varepsilon\}}{\text{vol } T_0} \leq \frac{\text{vol} \{\xi \in R^d: \xi_1 \dots \xi_d \leq |\det A| \varepsilon, 0 \leq \xi_i \leq 1\}}{\text{vol } AT_0}.$$

A simple induction argument shows that for $0 < \eta \leq 1$

$$\text{vol} \{\xi \in R^d: \xi_1 \dots \xi_d \leq \eta, 0 < \xi_i < 1\} = \eta \sum_{j=0}^{d-1} \frac{1}{j!} \left(\log \frac{1}{\eta}\right)^j.$$

But $\det A$ is a constant depending on d only so for $\varepsilon < \varepsilon_0(d)$ we get

$$\text{vol } S(v \leq \varepsilon) \leq (d + 1) \text{vol} (T_0 \cap S(v \leq \varepsilon)) \leq \text{const} (d) \varepsilon (\log (1/\varepsilon))^{d-1}.$$

Now we prove the theorem for general polytopes $P \subset R^d$. Take a triangulation of P into simplices S_1, \dots, S_m using vertices of P only. Then

$$P(v_P \leq \varepsilon) \subset \bigcup_{i=1}^m S_i(v_{S_i} \leq \varepsilon).$$

With suitable (nonsingular) affine transformations $A_i: R^d \rightarrow R^d$ such that $A_i S_i = S$ we have

$$\begin{aligned} \text{vol } P(v_P \leq \varepsilon) &\leq \sum_{i=1}^m \text{vol } S_i(v_{S_i} \leq \varepsilon) = \sum_{i=1}^m \frac{\text{vol } S_i}{\text{vol } AS_i} \text{vol } AS_i(v_{AS_i} \leq |\det A_i| \varepsilon) \\ &\leq \varepsilon \sum_{i=1}^m (\log (1/(|\det A_i| \varepsilon)))^{d-1} \leq \text{const} (P) \varepsilon \left(\log \frac{1}{\varepsilon}\right)^{d-1}. \end{aligned}$$

We mention here that there is an alternative proof for this theorem using the arguments of the proof of Theorem 2.

§9. *Proof of Theorem 4.* It is clear that for some $\rho > 0$, $\delta > 0$ the set of (ρ, δ) -circular points Ω form a set of positive measure in ∂K . Take $p \in \Omega$ and consider $z = p - \alpha q$ where q is the outer unit normal to K at P . Assume

$$\alpha \leq \min \left(\frac{\delta}{d}, \varepsilon^{2/(d+1)} \rho^{-(d-1)/(d+1)} \omega_{d-1}^{-(d-1)/(d+1)} \right).$$

CLAIM 2. $z \in K(v \leq \varepsilon)$.

Proof. Assume this is false. Then for a minimal cap $C(z) = K \cap H(a, t)$ one has $\text{vol } C(z) > \varepsilon$. Take a chord $[x, y]$ through z of K lying in the bounding hyperplane $H(a = t)$ of $C(z)$. Consider a minimal cap $C'(z)$ of the ball $B^\rho = B(\rho, p - \rho a)$. As p is ρ -circular, one of the endpoints of the chord $[x, y]$, x (say), lies in the cap $C'(z)$. But z is the centre of gravity of the section $K \cap H(a = t)$ and so, according to a well-known result (see [7], e.g.)

$$(d - 1)|z - x| \geq |y - z|.$$

This shows that y lies in the minimal cap $C'(z')$ of B^p where $z' = p - d\alpha a$. As y is an arbitrary point of the section $K \cap H(a = t)$, we have that $C(z) \subset C'(z')$. Then

$$\text{vol } C(z) \leq \text{vol } C'(z') \leq (\alpha d)^{(d+1)/2} (2\rho)^{(d-1)/2} \omega_{d-1}.$$

By the choice of α this is less than ε . A contradiction.

We claim now that

$$\text{vol } K(v \leq \varepsilon) \geq \text{const}(K) \alpha \text{vol}_{d-1} \Omega. \tag{9.1}$$

This will prove the theorem for $\alpha = \text{const } \varepsilon^{2/(d+1)}$ if ε is small enough. Define first

$$L_s = \{p \in \delta K : K \supset B(s, p - sa)\}.$$

It is well-known [27] that $\text{vol}_{d-1}(\delta K \setminus L_s) \rightarrow 0$ as $s \rightarrow 0$. Choose $s > 0$ so that $\text{vol}_{d-1}(\Omega \cap L_s) \geq \frac{1}{2} \text{vol}_{d-1} \Omega$. Now to see (9.1) we use the proof of the cap covering theorem (Theorem 6). So choose a maximal system of points x_1, \dots, x_m from $\delta K(v \geq \varepsilon)$ subject to the conditions:

$$M(x_i, \frac{1}{2}) \cap M(x_j, \frac{1}{2}) = \emptyset; \tag{9.2}$$

$$\text{the centre of the minimal cap } C(x_i) \text{ lies in } \Omega \cap L_s. \tag{9.3}$$

So let $C(x_i) = K \cap H(a_i, t_i)$ with centre $p_i \in \Omega \cap L_s$. We know from the proof of the cap covering theorem that

$$\Omega \cap L_s \subset \bigcup_{i=1}^m M(x_i, 5) \subset \bigcup_{i=1}^m K_i$$

where $K_i = K \cap H(a_i, 6t_i)$. According to Claim 2, the width of the cap $C(x_i)$ is at least α , so the width of K_i is at least 6α . Then

$$\begin{aligned} \text{vol } K(v \leq \varepsilon) &\geq \sum_{i=1}^m \frac{1}{2} \text{vol } M(x_i, \frac{1}{2}) \\ &\geq \text{const}(d) \sum_{i=1}^m \text{vol } K_i \\ &\geq \text{const}(d) \frac{6\alpha}{d} \sum_{i=1}^m \text{vol}_{d-1}(K \cap H(a_i = 6t_i)). \end{aligned}$$

Now $\text{vol}_{d-1}(K \cap H(a_i = 6t_i)) \geq \text{const}(d, \rho, s) \text{vol}_{d-1}(\partial K \cap K_i)$. This follows from the fact that the outer normals to K at the points of $\partial K \cap K_i$ cannot differ much from a_i (if ε is small enough) because p_i is in L_s . Using this we get

$$\begin{aligned} \text{vol } K(v \leq \varepsilon) &\geq \text{const}(d, \rho, s) \alpha \sum_{i=1}^m \text{vol}_{d-1}(\partial K \cap K_i) \\ &\geq \text{const}(K) \alpha \text{vol}_{d-1}(\Omega \cap L_s) \geq \text{const}(K) \frac{1}{2} \alpha \text{vol}_{d-1}(\Omega). \end{aligned}$$

§10. *Proof of Theorem 8.* Let $P \subset R^d$ be a convex polytope having n vertices and assume $\text{vol } P = 1$. Set

$$\varepsilon = c_0(d) n^{-(d+1)/(d-1)}$$

where the constant $c_0(d)$ is to be determined later.

We assume that n is large enough to ensure that $\varepsilon < (2d)^{-2d}$. Then Theorem 6 applies: there are caps K_1, \dots, K_m and subsets K'_1, \dots, K'_m satisfying (i), (ii) and (iii) of Theorem 6. Then

$$\begin{aligned} m(6d)^{-d}\varepsilon &\leq \sum_{i=1}^m \text{vol } K'_i \leq \text{vol } P(v \leq \varepsilon) \\ &\leq 4E(P, [1/\varepsilon]) \leq 4c(d)([1/\varepsilon])^{-2/(d+1)} \leq 8c(d)\varepsilon^{2/(d+1)} \end{aligned}$$

where the third inequality follows from Theorem 1 and the fourth from (1.5) and (1.4) with a suitable constant $c(d)$. This shows that

$$m \leq 8(6d)^d c(d) \varepsilon^{-(d-1)/(d+1)} \leq 8(6d)^d c(d) c_0(d)^{-(d-1)/(d+1)} n \leq \frac{1}{d+1} n$$

if we choose $c_0(d)$ large enough.

Now the caps K_1, \dots, K_m cover $P(v \leq \varepsilon)$ so they cover the boundary of P as well. Then there is a cap, K_i say, containing at least $n/m \geq d+1$ vertices, y_0, \dots, y_d of P . Consequently $\text{conv}\{y_0, \dots, y_d\} \subset K$ and

$$\text{vol conv}\{y_0, \dots, y_d\} \leq \text{vol } K_i \leq 6^d \varepsilon \leq \text{const}(d) n^{-(d+1)/(d-1)}.$$

§11. *Proof of Theorem 9.* Lemma 1 implies $K(v \leq \varepsilon) \subset K(u \leq 2\varepsilon)$. By Lemma 2, if $\varepsilon \leq \varepsilon_1(d) = \frac{1}{2}(12d^3)^{-d}$, then $K(u \leq 2\varepsilon) \leq K(v \leq 2(3d)^d \varepsilon)$ so indeed

$$K(v \leq \varepsilon) \subset K(u \leq 2\varepsilon) \subset K(v \leq 2(3d)^d \varepsilon).$$

Computing volumes here and applying Theorem 7 gives

$$\text{vol } K(v \leq \varepsilon) \leq \text{vol } K(u \leq 2\varepsilon) \leq c_1(d) \text{vol } K(v \leq \varepsilon).$$

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