The Annals of Applied Probability
1992, Vol. 2, No. 4, 767–818

CONVEX DUALITY IN CONSTRAINED PORTFOLIO OPTIMIZATION[1]

BY JAKŠA CVITANIĆ AND IOANNIS KARATZAS[2]

Columbia University

We study the stochastic control problem of maximizing expected utility from terminal wealth and/or consumption, when the portfolio is constrained to take values in a given closed, convex subset of $\mathscr{R}^d$. The setting is that of a continuous-time, Itô process model for the underlying asset prices. General existence results are established for optimal portfolio/consumption strategies, by suitably embedding the constrained problem in an appropriate family of unconstrained ones, and finding a member of this family for which the corresponding optimal policy obeys the constraints. Equivalent conditions for optimality are obtained, and explicit solutions leading to feedback formulae are derived for special utility functions and for deterministic coefficients. Results on incomplete markets, on short-selling constraints and on different interest rates for borrowing and lending are covered as special cases. The mathematical tools are those of continuous-time martingales, convex analysis and duality theory.

1. Introduction and summary. This paper develops a theory for the classical consumption/investment problem of mathematical economics, when the portfolio is constrained to take values in a given closed, convex, nonempty subset K of $\mathscr{R}^d$. We adopt a continuous-time, Itô process model for the financial market with one bond and d stocks [which goes back to Merton (1969) in the case of constant coefficients], and study in its framework the stochastic control problem of maximizing expected utility from terminal wealth and/or consumption, under the above-mentioned constraint.

The unconstrained version of this problem is, by now, well known and understood; compare with Karatzas, Lehoczky and Shreve (1987)—hereafter abbreviated KLS (1987)—as well as Karatzas (1989) and Cox, Huang (1989) and Pliska (1986). In very general terms, our approach for the constrained problem consists in "embedding" it into a suitable family of unconstrained problems, with the same objective but different random environments; one then tries to single out a member of this family, for which the optimal portfolio actually *obeys* the constraint (i.e., takes values in K), and thereby solves the original problem as well.

Received July 1991; revised November 1991.

[1]The results of this paper have been drawn from the first author's doctoral dissertation at Columbia. A preliminary version was presented as an invited lecture at the 20th Conference on Stochastic Processes and their Applications, Nahariya, Israel, June 1991.

[2]Research supported in part by NSF Grant DMS-90-22188.

AMS 1980 *subject classifications*. Primary 93E20, 90A09, 60H30; secondary 60G44, 90A16, 49N15.

Key words and phrases. Constrained optimization, convex analysis, duality, stochastic control, portfolio and consumption processes, martingale representations.

Such an approach was used by Karatzas, Lehoczky, Shreve and Xu (KLSX) (1991) in the context of the so-called *incomplete markets*—a special case, as it turns out, of the theory developed here. In KLSX (1991), the above-mentioned embedding arises naturally in the form of "fictitious completion" of the incomplete financial market. It was far from obvious to us, at the outset of this work, that such an embedding should exist, and should prove fruitful, in this general context as well.

One distinctive aspect of this approach is that it relates the original, or "primal", stochastic control problem to a certain "dual" one, in the sense that a solution to the primal problem induces a solution for the dual (and vice versa). This duality goes back to Bismut (1973), and was introduced in problems of this sort by Xu (1990), who treated in his doctoral dissertation the special case $K = [0, \infty)^d$. It was also exploited by KLSX (1991) and He and Pearson (1991) in the context of incomplete markets. It is of great importance here as well because, as it turns out, it is far easier to prove existence of optimal policies in the dual, rather than in the primal, problem.

The paper is organized as follows: the ingredients of the model are laid out in Sections 2–5, and Section 6 poses the unconstrained and constrained (primal) stochastic control problems. In Section 7 we review the solution to the former, and introduce the family of *auxiliary unconstrained problems* in Section 8. We tackle in Section 9 the controllability question of describing a class of random variables which can be obtained as terminal wealth levels by means of portfolios that take values in the set K. Section 10 lays out four equivalent conditions that a member of this family of auxiliary unconstrained problems has to satisfy, in order for its solution to coincide with that of the original *constrained* problem. The equivalence of these conditions is established in Theorem 10.1, which may be regarded as the focal point of the paper. In terms of these conditions one can solve straightaway, and very explicitly, for the optimal portfolio and consumption rules in the important special case of logarithmic utility functions (Section 11).

One of the equivalent conditions in Theorem 10.1 leads naturally to a *dual stochastic control problem*; this is formulated, and is related to the primal problem, in Section 12, whereas Section 13 settles the existence question of optimal processes for both the dual and the primal problem. This analysis culminates in Theorem 13.1, which is the second most important result in the paper. Examples and special cases are discussed in Sections 14 and 15. We present in Section 16 some extensions of the theory. A technical and lengthy argument in the proof of Theorem 10.1 is carried out in Appendix A. Finally, Appendix B applies the convex duality methodology developed in this paper to the important consumption/investment problem with a higher interest rate for borrowing.

The mathematical tools employed throughout are those of continuous-time martingales, duality theory, and convex analysis. In particular, the support function $\delta(x) \triangleq \sup_{\pi \in K}(-\pi^* x)$ of the convex set $-K$, and its effective domain $\tilde{K}$ (the barrier cone of $-K$), play a crucial role in the selection of the appropriate family of auxiliary unconstrained problems, in the formulation of duality and in the nature of the solution to the original, constrained problem.

2. The model. We consider a financial market $\mathscr{M}$ which consists of one *bond* and several (d) *stocks*. The prices $P_0(t)$, $\{P_i(t)\}_{1 \le i \le d}$ of these assets evolve according to the equations

$$dP_0(t) = P_0(t) r(t)\, dt, \qquad P_0(0) = 1, \tag{2.1}$$

$$dP_i(t) = P_i(t)\left[b_i(t)\, dt + \sum_{j=1}^{d} \sigma_{ij}(t)\, dW_j(t)\right], \tag{2.2}$$

$$P_i(0) = 1, i = 1, \dots, d.$$

Here $W = (W_1, \dots, W_d)^*$ is a standard Brownian motion in $\mathscr{R}^d$, defined on a complete probability space $(\Omega, \mathscr{F}, P)$, and we shall denote by $\{\mathscr{F}_t\}$ the P-augmentation of the filtration $\mathscr{F}_t^W = \sigma(W(s);\ 0 \le s \le t)$ generated by W. The *coefficients* of $\mathscr{M}$, that is, the processes $r(t)$ (scalar interest rate), $b(t) = (b_1(t), \dots, b_d(t))^*$ (vector of appreciation rates) and $\sigma(t) = \{\sigma_{ij}(t)\}_{1 \le i, j \le d}$ (volatility matrix), are assumed to be progressively measurable with respect to $\{\mathscr{F}_t\}$ and to satisfy

$$r(t) \ge -\eta, \qquad \forall\, 0 \le t \le T, \tag{2.3}$$

$$\xi^* \sigma(t) \sigma^*(t) \xi \ge \varepsilon \|\xi\|^2, \qquad \forall\, (t, \xi) \in [0, T] \times \mathscr{R}^d \tag{2.4}$$

almost surely, for given real constants $\varepsilon > 0$ and $\eta \ge 0$, as well as

$$E\int_0^T r(s)\, ds < \infty. \tag{2.5}$$

All processes encountered throughout the paper will be defined on the fixed, finite horizon $[0, T]$.

We shall assume throughout that the "relative risk" process

$$\theta(t) \triangleq \sigma^{-1}(t)[b(t) - r(t)\mathbf{1}], \tag{2.6}$$

where $\mathbf{1} = (1, \dots, 1)^*$, satisfies the finite-energy condition

$$E\int_0^T \|\theta(t)\|^2\, dt < \infty. \tag{2.7}$$

The exponential local martingale

$$Z_0(t) \triangleq \exp\left[-\int_0^t \theta^*(s)\, dW(s) - \tfrac{1}{2}\int_0^t \|\theta(s)\|^2\, ds\right], \tag{2.8}$$

the discount process

$$\gamma_0(t) \triangleq \exp\left\{-\int_0^t r(s)\, ds\right\} \tag{2.9}$$

and their product

$$H_0(t) \triangleq \gamma_0(t) Z_0(t) \tag{2.10}$$

will be employed quite frequently.

2.1 REMARK. It is a straightforward consequence of the strong nondegeneracy condition (2.4), that the matrices $\sigma(t), \sigma^*(t)$ are invertible, and that the norms of $(\sigma(t))^{-1}, (\sigma^*(t))^{-1}$ are bounded above and below by δ and $1/\delta$, respectively, for some $\delta \in (1, \infty)$; compare with Karatzas and Shreve (1988), page 372.

3. Portfolio and consumption processes. Consider an economic agent whose actions cannot affect market prices, and who can decide, at any time $t \in [0, T]$, (i) what proportion $\pi_i(t)$ of his wealth $X(t)$ to invest in the ith stock $(1 \le i \le d)$, and (ii) at what rate $c(t) \ge 0$ to withdraw money for consumption. Of course these decisions can only be based on the current information $\mathscr{F}_t$, without anticipation of the future. With $\pi(t) = (\pi_1(t), \dots, \pi_d(t))^*$ chosen, the amount $X(t)[1 - \sum_{i=1}^d \pi_i(t)]$ is invested in the bond. Thus, in accordance with the model set forth in (2.1) and (2.2), the wealth process $X(t)$ satisfies the linear stochastic equation

$$\begin{aligned} dX(t) &= \sum_{i=1}^{d} \pi_i(t) X(t) \left\{ b_i(t)\, dt + \sum_{j=1}^{d} \sigma_{ij}(t)\, dW_j(t) \right\} \\ &\quad + \left\{ 1 - \sum_{i=1}^{d} \pi_i(t) \right\} X(t) r(t)\, dt - c(t)\, dt \\ &= [r(t)X(t) - c(t)]\, dt + X(t)\pi^*(t)\sigma(t)\, dW_0(t), \\ &\qquad\qquad X(0) = x > 0, \end{aligned} \tag{3.1}$$

where we have set

$$W_0(t) \triangleq W(t) + \int_0^t \theta(s)\, ds. \tag{3.2}$$

We formalize the preceding considerations as follows.

3.1 DEFINITION. (i) An $\mathscr{R}^d$-valued, $\{\mathscr{F}_t\}$-progressively measurable process $\pi = \{\pi(t), 0 \le t \le T\}$ with $\int_0^T \|\pi(t)\|^2\, dt < \infty$, a.s., will be called a *portfolio process*.

(ii) A nonnegative, $\{\mathscr{F}_t\}$-progressively measurable process $c = \{c(t), 0 \le t \le T\}$ with $\int_0^T c(t)\, dt < \infty$, a.s., will be called a *consumption process*.

(iii) Given a pair (π, c) as previously, the solution $X \equiv X^{x, \pi, c}$ of the equation (3.1) will be called the *wealth process* corresponding to the portfolio/consumption pair (π, c) and initial capital $x \in (0, \infty)$.

3.2 DEFINITION. A portfolio/consumption process pair (π, c) is called *admissible* for the initial capital $x \in (0, \infty)$, if

$$X^{x,\pi,c}(t) \geq 0, \qquad \forall\, 0 \leq t \leq T, \tag{3.3}$$

holds almost surely. The set of admissible pairs (π, c) will be denoted by $\mathscr{A}_0(x)$.

In the notation of (2.8)–(2.10), the equation (3.1) leads to

$$\gamma_0(t)X(t) + \int_0^t \gamma_0(s)c(s)\,ds = x + \int_0^t \gamma_0(s)X(s)\pi^*(s)\sigma(s)\,dW_0(s), \tag{3.4}$$

as well as

$$\begin{aligned} &H_0(t)X(t) + \int_0^t H_0(s)c(s)\,ds \\ &\quad = x + \int_0^t H_0(s)X(s)[\sigma^*(s)\pi(s) - \theta(s)]^*\,dW(s) \end{aligned} \tag{3.5}$$

(from Itô's rule, applied to the product of $\gamma_0 X$ and Z_0). In particular, the process on the left-hand side of (3.5) is seen to be a continuous local martingale; if $(\pi, c) \in \mathscr{A}_0(x)$, this local martingale is also nonnegative, thus a supermartingale. Consequently,

$$E\left[H_0(T)X^{x,\pi,c}(T) + \int_0^T H_0(t)c(t)\,dt\right] \leq x, \qquad \forall\,(\pi, c) \in \mathscr{A}_0(x). \tag{3.6}$$

4. Convex sets and their support functions. We shall fix throughout a nonempty, closed, convex set K in $\mathscr{R}^d$, and denote by

$$\delta(x) \equiv \delta(x|K) \triangleq \sup_{\pi \in K}(-\pi^* x) \colon \mathscr{R}^d \to \mathscr{R} \cup \{+\infty\} \tag{4.1}$$

the support function of the convex set $-K$. This is a closed, positively homogeneous, proper convex function on $\mathscr{R}^d$ [Rockafellar (1970), page 114], finite on its *effective domain*

$$\begin{aligned} \tilde{K} &\triangleq \{x \in \mathscr{R}^d;\ \delta(x|K) < \infty\} \\ &= \{x \in \mathscr{R}^d;\ \exists\, \beta \in \mathscr{R} \text{ s.t. } -\pi^* x \leq \beta,\ \forall\, \pi \in K\}, \end{aligned} \tag{4.2}$$

which is a convex cone (called the *barrier cone* of $-K$). It will be assumed throughout this paper that

$$\text{the function } \delta(\cdot|K) \text{ is continuous on } \tilde{K} \tag{4.3}$$

and bounded below on $\mathscr{R}^d$:

$$\delta(x|K) \geq \delta_0, \qquad \forall\, x \in \mathscr{R}^d \text{ for some } \delta_0 \in \mathscr{R}. \tag{4.4}$$

4.1 REMARK. Clearly, (4.4) holds (with $\delta_0 = 0$) if K contains the origin. On the other hand, Theorem 10.2 in Rockafellar [(1970), page 84] guarantees that (4.3) is satisfied, in particular, if $\tilde{K}$ is locally simplicial.

4.2 REMARK. Condition (4.4) is a technical one, needed in the duality and existence proofs of Sections 12 and 13. In certain cases, such existence results can be established directly, even in situations when (4.4) does not hold (cf. Remark 14.10). This condition is *not* used in proving the equivalence of the various statements in Theorem 10.1.

We shall have occasion to use the subadditivity property

$$\delta(x+y) \le \delta(x) + \delta(y), \qquad \forall\, x \in \mathscr{R}^d,\, y \in \mathscr{R}^d, \tag{4.5}$$

of the support function $\delta(\cdot)$ in (4.1).

5. Utility functions. A function U: $(0,\infty) \to \mathscr{R}$ will be called a *utility function* if it is strictly increasing, strictly concave, of class C^1 and satisfies

$$U'(0+) \triangleq \lim_{x \downarrow 0} U'(x) = \infty, \qquad U'(\infty) \triangleq \lim_{x \to \infty} U'(x) = 0. \tag{5.1}$$

We shall denote by I the (continuous, strictly decreasing) inverse of the function U'; this function maps $(0,\infty)$ onto itself, and satisfies $I(0+) = \infty$, $I(\infty) = 0$. We also introduce the Legendre–Fenchel transform

$$\tilde{U}(y) \triangleq \max_{x>0} [U(x) - xy] = U(I(y)) - yI(y), \qquad 0 < y < \infty, \tag{5.2}$$

of $-U(-x)$; this function $\tilde{U}$ is strictly decreasing and strictly convex, and satisfies

$$\tilde{U}'(y) = -I(y), \qquad 0 < y < \infty, \tag{5.3}$$

$$U(x) = \min_{y>0} \left[\tilde{U}(y) + xy\right] = \tilde{U}(U'(x)) + xU'(x), \qquad 0 < x < \infty. \tag{5.4}$$

The useful inequalities

$$U(I(y)) \ge U(x) + y[I(y) - x], \tag{5.5}$$

$$\tilde{U}(U'(x)) + x[U'(x) - y] \le \tilde{U}(y), \tag{5.6}$$

valid for all $x > 0$, $y > 0$, are direct consequences of (5.2) and (5.4). It is also easy to check that

$$\tilde{U}(\infty) = U(0+), \qquad \tilde{U}(0+) = U(\infty) \tag{5.7}$$

hold; compare with KLSX (1991), Lemma 4.2.

5.1 REMARK. We shall have occasion, in the sequel, to impose the following conditions on our utility functions:

$$c \mapsto cU'(c) \text{ is nondecreasing on } (0,\infty), \tag{5.8}$$

$$\text{for some } \alpha \in (0,1),\, \gamma \in (1,\infty) \text{ we have } \quad \alpha U'(x) \ge U'(\gamma x), \qquad \forall\, x \in (0,\infty). \tag{5.9}$$

5.2 REMARK. Condition (5.8) is equivalent to

$$(5.8') \qquad y \mapsto yI(y) \text{ is nonincreasing on } (0,\infty)$$

and implies that

$$(5.8'') \qquad x \mapsto \tilde{U}(e^x) \text{ is convex on } \mathscr{R}.$$

[If U is of class C^2, then condition (5.8) amounts to the statement that $-cU''(c)/U'(c)$, the so-called *Arrow-Pratt measure of relative risk-aversion*, does not exceed 1.]

Similarly, condition (5.9) is equivalent to having

$$(5.9') \qquad I(\alpha y) \le \gamma I(y), \qquad \forall\, y \in (0,\infty) \text{ for some } \alpha \in (0,1), \gamma > 1.$$

Iterating (5.9′), we obtain the apparently stronger statement

$$(5.9'') \quad \forall\, \alpha \in (0,1), \exists\, \gamma \in (1,\infty) \quad \text{such that} \quad I(\alpha y) \le \gamma I(y), \quad \forall\, y \in (0,\infty).$$

6. The constrained and unconstrained optimization problems. We shall consider throughout a continuous function U_1: $[0,T] \times (0,\infty) \to \mathscr{R}$ such that, for any given $t \in [0,T]$, the function $U_1(t,\cdot)$ has all the properties of a utility function as in Section 5. We shall denote by $U_1'(t,\cdot)$ the derivative of $U_1(t,\cdot)$, by $I_1(t,\cdot)$ the inverse of $U_1'(t,\cdot)$ and by $\tilde{U}_1(t,\cdot)$ the function of (5.2). We shall also consider throughout a utility function U_2, as in Section 5.

Corresponding to any given pair (π, c) in the class $\mathscr{A}_0(x)$ of Definition 3.2, we have the total expected utility

$$(6.1) \qquad J(x;\pi,c) \triangleq E\int_0^T U_1(t,c(t))\,dt + EU_2(X^{x,\pi,c}(T)),$$

provided that the two expectations are well defined.

6.1 DEFINITION. The *unconstrained optimization problem* is to maximize the expression of (6.1) over the class $\mathscr{A}_0'(x)$ of pairs $(\pi,c) \in \mathscr{A}_0(x)$ that satisfy

$$(6.2) \qquad E\int_0^T U_1^-(t,c(t))\,dt + EU_2^-(X^{x,\pi,c}(T)) < \infty.$$

[Here and in the sequel, x^- denotes the negative part of the real number x: $x^- = \max(-x,0)$.] The value function of this problem will be denoted by

$$(6.3) \qquad V_0(x) \triangleq \sup_{(\pi,c)\in\mathscr{A}_0'(x)} J(x;\pi,c), \qquad x \in (0,\infty).$$

6.2 ASSUMPTION. $V_0(x) < \infty$, $\forall\, x \in (0,\infty)$.

6.3 DEFINITION. The *constrained optimization problem* is to maximize the expression of (6.1) over the class

$$(6.4) \quad \mathscr{A}'(x) \triangleq \{(\pi,c) \in \mathscr{A}_0'(x);\ \pi(t,\omega) \in K \text{ for } \ell \otimes P\text{-a.e. } (t,\omega)\},$$

where K is the closed, convex set of Section 4 and ℓ denotes Lebesgue measure. The value function of this problem will be denoted by

$$V(x) \triangleq \sup_{(\pi, c) \in \mathscr{A}'(x)} J(x; \pi, c), \qquad x \in (0, \infty). \tag{6.5}$$

Quite obviously,

$$V(x) \leq V_0(x) < \infty, \qquad \forall\, x \in (0, \infty), \tag{6.6}$$

from Assumption 6.2. It is also fairly straightforward that both functions $V_0(\cdot)$ and $V(\cdot)$ are increasing and concave on $(0, \infty)$.

6.4 Remark. It can be checked that the Assumption 6.2 is satisfied, if the processes $r(\cdot)$ and $\theta(\cdot)$ of (2.6) are bounded [uniformly in (t, ω)] and if the functions U_1, U_2 are nonnegative and satisfy the growth condition

$$0 \leq U_1(t, x), U_2(x) \leq \kappa(1 + x^{\alpha}), \qquad \forall\, (t, x) \in [0, T] \times (0, \infty), \tag{6.7}$$

for some constants $\kappa \in (0, \infty)$ and $\alpha \in (0, 1)$: compare with Xu (1990) or KLSX (1991) for details.

7. Solution of the unconstrained problem. The unconstrained problem of Definition 6.1 is by now well known and understood; compare with Karatzas, Lehoczky and Shreve (1987), Karatzas and Shreve (1988), Section 5.8.C and Cox and Huang (1989). For easy later reference and usage, we repeat here the nature of the solution.

7.1 Assumption. Suppose that the expectation

$$\mathscr{X}_0(y) \triangleq E\left[\int_0^T H_0(t) I_1(t, yH_0(t))\, dt + H_0(T) I_2(yH_0(T))\right] \tag{7.1}$$

is finite, for every $y \in (0, \infty)$.

Under this assumption, the function $\mathscr{X}_0$: $(0, \infty) \to (0, \infty)$ is continuous and strictly decreasing, with $\mathscr{X}(0+) = \infty$ and $\mathscr{X}(\infty) = 0$; we let $\mathscr{Y}_0$ denote its inverse and introduce the random variables

$$\xi_0 \triangleq I_2(\mathscr{Y}_0(x) H_0(T)), \tag{7.2}$$

$$c_0(t) \triangleq I_1(t, \mathscr{Y}_0(x) H_0(t)), \qquad 0 \leq t \leq T. \tag{7.3}$$

7.2 Lemma. *The quantities of* (7.2) *and* (7.3) *satisfy*

$$E\left[\int_0^T H_0(t) c_0(t)\, dt + H_0(T)\xi_0\right] = x, \tag{7.4}$$

$$E\int_0^T U_1^-(t, c_0(t))\, dt + EU_2^-(\xi_0) < \infty \tag{7.5}$$

and

$$J(x; \pi, c) \leq E\left[\int_0^T U_1(t, c_0(t))\, dt + U_2(\xi_0)\right], \qquad \forall\, (\pi, c) \in \mathscr{A}_0'(x). \tag{7.6}$$

PROOF. From (5.5) we have $U_1(t, c_0(t)) \geq U_1(t, 1) + \mathscr{Y}_0(x)H_0(t)[c_0(t) - 1]$ and $U_2(\xi_0) \geq U_2(1) + \mathscr{Y}_0(x)H_0(T)[\xi_0 - 1]$, whence

$$E\int_0^T U_1^-(t, c_0(t))\,dt + EU_2^-(\xi_0) \leq |U_2(1)| + \int_0^T |U_1(t,1)|\,dt + \mathscr{Y}_0(x)\left[EH_0(T) + \int_0^T EH_0(t)\,dt\right] < \infty$$

because $EH_0(t) \leq e^{\eta T}$ [from the supermartingale property of Z_0 and the condition (2.3)]. This establishes (7.5), whereas (7.4) is obvious from the definitions (7.1)–(7.3).

Now consider an arbitrary pair $(\pi, c) \in \mathscr{A}_0'(x)$: Using (5.5) again, we obtain

$$U_1(t, c_0(t)) \geq U_1(t, c(t)) + \mathscr{Y}_0(x)H_0(t)[c_0(t) - c(t)],$$
$$U_2(\xi_0) \geq U_2(X^{x,\pi,c}(T)) + \mathscr{Y}_0(x)H_0(T)[\xi_0 - X^{x,\pi,c}(T)],$$

almost surely, and therefore

$$E\left[\int_0^T U_1(t, c_0(t))\,dt + U_2(\xi_0)\right] \geq J(x;\pi,c) + \mathscr{Y}_0(x)\left\{x - E\left[\int_0^T H_0(t)c(t)\,dt + H_0(T)X^{x,\pi,c}(T)\right]\right\} \geq J(x;\pi,c),$$

thanks to (7.4) and (3.6). □

7.3 PROPOSITION. *Let $c(\cdot)$ be a consumption process and B a positive, $\mathscr{F}_T$-measurable random variable with*

$$x = E\left[\int_0^T H_0(t)c(t)\,dt + H_0(T)B\right] < \infty.$$

There exists a portfolio process π, such that $(\pi, c) \in \mathscr{A}_0(x)$ and $X^{x,\pi,c}(T) = B$ a.s.

PROOF. We introduce the continuous, positive process X via

$$X(t) \triangleq \frac{1}{H_0(t)}E\left[\int_t^T H_0(s)c(s)\,ds + H_0(T)B \,\Big|\, \mathscr{F}_t\right], \qquad 0 \leq t \leq T. \tag{7.7}$$

This process satisfies $X(0) = x$, $X(T) = B$ a.s. On the other hand, the martingale

$$\begin{aligned} M_0(t) &\triangleq E\left[\int_0^T H_0(s)c(s)\,ds + H_0(T)B \,\Big|\, \mathscr{F}_t\right] \\ &= H_0(t)X(t) + \int_0^t H_0(s)c(s)\,ds \end{aligned} \tag{7.8}$$

can be represented as a stochastic integral: $M_0(t) = x + \int_0^t \psi^*(s)\,dW(s)$ for a suitable $\{\mathscr{F}_t\}$-progressively measurable, $\mathscr{R}^d$-valued process ψ that satisfies $\int_0^T \|\psi(s)\|^2\, ds < \infty$ a.s. Comparing (7.8) with (3.5), we conclude that X is the wealth process corresponding to the pair (π, c), where the portfolio π is given by

$$\pi(t) = (\sigma^*(t))^{-1}\left[\frac{\psi(t)}{X(t)H_0(t)} + \theta(t)\right]. \tag{7.9}$$ □

Putting these two results together, we obtain the solution of the unconstrained problem (6.3).

7.4 THEOREM. *With ξ_0, c_0 given as in* (7.2) *and* (7.3), *there exists a portfolio process π_0 such that $(\pi_0, c_0) \in \mathscr{A}_0'(x)$, $X^{x,\pi_0,c_0}(T) = \xi_0$ a.s. and $V_0(x) = J(x; \pi_0, c_0)$.*

8. Auxiliary unconstrained optimization problems. Our purpose in this section is to introduce a family of auxiliary, unconstrained optimization problems [cf. (8.3), (8.4) and (8.12)] and to embed in this family the constrained problem of Definition 6.3 (Proposition 8.3).

Let $\mathscr{H}$ denote the space of $\{\mathscr{F}_t\}$-progressively measurable processes $\nu = \{\nu(t), 0 \le t \le T\}$ with values in $\mathscr{R}^d$ and

$$[\![\nu]\!]^2 \triangleq E\int_0^T \|\nu(t)\|^2\, dt < \infty.$$

$\mathscr{H}$ is a Hilbert space when endowed with the inner product

$$\langle \nu_1, \nu_2 \rangle \triangleq E\int_0^T \nu_1^*(t)\nu_2(t)\, dt.$$

We introduce also the class of processes

$$\mathscr{D} \triangleq \left\{\nu \in \mathscr{H};\ E\int_0^T \delta(\nu(t))\, dt \le \infty\right\}, \tag{8.1}$$

where $\delta(\cdot)$ is the support function in (4.1), and observe that $\nu \in \mathscr{D}$ implies

$$\nu(t,\omega) \in \tilde{K} \quad \text{for } \ell \otimes P\text{-a.e. } (t,\omega) \in [0,T] \times \Omega. \tag{8.2}$$

Here $\tilde{K}$ is the barrier cone of (4.2).

Corresponding to any given $\nu \in \mathscr{D}$, we introduce *a new financial market $\mathscr{M}_\nu$* with one bond and d stocks:

$$dP_0^{(\nu)}(t) = P_0^{(\nu)}(t)[r(t) + \delta(\nu(t))]\, dt, \tag{8.3}$$

$$dP_i^{(\nu)}(t) = P_i^{(\nu)}(t)\left[(b_i(t) + \nu_i(t) + \delta(\nu(t)))\, dt + \sum_{j=1}^d \sigma_{ij}(t)\, dW_j(t)\right], \quad 1 \le i \le d, \tag{8.4}$$

by analogy with (2.1) and (2.2). In this new market $\mathscr{M}_\nu$, the analogues of (2.6), (2.8)–(2.10) and (3.2) become

$$\begin{aligned}\theta_\nu(t) &\triangleq \sigma^{-1}(t)\big[b(t) + \nu(t) + \delta(\nu(t))\mathbf{1} - (r(t) + \delta(\nu(t)))\mathbf{1}\big]\\ &= \theta(t) + \sigma^{-1}(t)\nu(t),\end{aligned} \tag{8.5}$$

$$\gamma_\nu(t) \triangleq \exp\left[-\int_0^t \{r(s) + \delta(\nu(s))\}\,ds\right], \tag{8.6}$$

$$Z_\nu(t) \triangleq \exp\left[-\int_0^t \theta_\nu^*(s)\,dW(s) - \tfrac{1}{2}\int_0^t \|\theta_\nu(s)\|^2\,ds\right], \tag{8.7}$$

$$H_\nu(t) \triangleq \gamma_\nu(t) Z_\nu(t), \tag{8.8}$$

$$W_\nu(t) \triangleq W(t) + \int_0^t \theta_\nu(s)\,ds, \tag{8.9}$$

and the analogues of (2.3), (2.5) and (2.7) are satisfied.

The wealth process $X_\nu^{x,\pi,c}$, corresponding to a given portfolio/consumption process pair (π, c) in $\mathscr{M}_\nu$, satisfies

$$\begin{aligned}dX_\nu^{x,\pi,c}(t) &= \big[(r(t) + \delta(\nu(t)))X_\nu^{x,\pi,c}(t) - c(t)\big]\,dt + X_\nu^{x,c,\pi}(t)\pi^*(t)\sigma(t)\,dW_\nu(t)\\ &= \big[r(t)X_\nu^{x,\pi,c}(t) - c(t)\big]\,dt\\ &\quad + X_\nu^{x,\pi,c}(t)\big[\delta(\nu(t)) + \pi^*(t)\nu(t)\big]\,dt\\ &\quad + X_\nu^{x,\pi,c}(t)\pi^*(t)\sigma(t)\,dW_0(t), \qquad X_\nu^{x,\pi,c}(0) = x,\end{aligned} \tag{8.10}$$

or equivalently

$$\begin{aligned}&H_\nu(t) X_\nu^{x,\pi,c}(t) + \int_0^t H_\nu(s)c(s)\,ds\\ &\quad = x + \int_0^t H_\nu(s) X_\nu^{x,\pi,c}(s)(\sigma^*(s)\pi(s) - \theta_\nu(s))^*\,dW(s)\end{aligned} \tag{8.11}$$

by analogy with (3.1) and (3.5). We denote by $\mathscr{A}_\nu(x)$ the class of pairs (π, c) for which

$$X_\nu^{x,\pi,c}(t) \ge 0, \qquad \forall\, 0 \le t \le T, \tag{8.12}$$

holds almost surely, and define

$$\mathscr{A}_\nu'(x) \triangleq \left\{(\pi, c) \in \mathscr{A}_\nu(x);\ E\int_0^T U_1^-(t, c(t))\,dt + EU_2^-(X_\nu^{x,\pi,c}(T)) < \infty\right\}$$

(by analogy with Definitions 3.2 and 6.1). The *unconstrained optimization problem in* $\mathscr{M}_\nu$ consists of maximizing $J(x; \pi, c)$ over $(\pi, c) \in \mathscr{A}_\nu'(x)$; its value function will be denoted by

$$V_\nu(x) \triangleq \sup_{(\pi, c) \in \mathscr{A}_\nu'(x)} J(x; \pi, c), \qquad x \in (0, \infty). \tag{8.13}$$

8.1 Remark. For an arbitrary $(\pi, c) \in \mathscr{A}'(x)$, denote by $X \equiv X^{x,\pi,c}$ the wealth process corresponding to (π, c) and initial capital x in the original market $\mathscr{M}$; cf. (3.1). A comparison of (3.1) with (8.10) shows

$$X_\nu^{x,\pi,c}(t) \geq X^{x,\pi,c}(t) \geq 0, \qquad \forall\, 0 \leq t \leq T,$$

a.s. [recall (3.3); the fact that $\delta(\nu(t)) + \pi^*(t)\nu(t) \geq 0$ because $\pi(t) \in K$, $\ell \otimes P$-a.e.; and the explicit formulae for the solution of linear stochastic differential equations of Karatzas and Shreve (1988), page 361]. Therefore, $(\pi, c) \in \mathscr{A}_\nu'(x)$ and

$$EU_2(X^{x,\pi,c}(T)) \leq EU_2(X_\nu^{x,\pi,c}(T)).$$

We deduce

$$\mathscr{A}'(x) \subset \mathscr{A}_\nu'(x), \qquad V(x) \leq V_\nu(x), \qquad \forall\, \nu \in \mathscr{D}. \tag{8.14}$$

8.2 Definition. By analogy with (7.1), we introduce the function

$$\mathscr{X}_\nu(y) \triangleq E\left[\int_0^T H_\nu(t) I_1(t, yH_\nu(t))\, dt + H_\nu(T) I_2(yH_\nu(t))\right], \qquad 0 < y < \infty, \tag{8.15}$$

and consider the subclass of $\mathscr{D}$ given by

$$\mathscr{D}' \triangleq \{\nu \in \mathscr{D};\ \mathscr{X}_\nu(y) < \infty, \forall\, y \in (0,\infty)\}. \tag{8.16}$$

For every $\nu \in \mathscr{D}'$, the function $\mathscr{X}_\nu(\cdot)$ of (8.15) is continuous and strictly decreasing, with $\mathscr{X}_\nu(0+) = \infty$ and $\mathscr{X}_\nu(\infty) = 0$; we denote its inverse by $\mathscr{Y}_\nu(\cdot)$.

According to Section 7, the optimal consumption, level of terminal wealth, and corresponding optimal wealth process, for the problem of (8.13), are given as

$$c_\nu(t) \equiv c_\nu^x(t) \triangleq I_1(t, \mathscr{Y}_\nu(x) H_\nu(t)), \tag{8.17}$$

$$\xi_\nu \equiv \xi_\nu^x \triangleq I_2(\mathscr{Y}_\nu(x) H_\nu(T)) \tag{8.18}$$

and

$$X_\nu(t) \triangleq \frac{1}{H_\nu(t)} E\left[\int_t^T H_\nu(s) c_\nu(s)\, ds + H_\nu(T)\xi_\nu \,\middle|\, \mathscr{F}_t\right], \tag{8.19}$$

respectively, for any ν in the class $\mathscr{D}'$ of (8.16). The process X_ν of (8.19) satisfies then the equation (8.10), with $c \equiv c_\nu$ and an appropriate portfolio process $\pi \equiv \pi_\nu$:

$$\begin{aligned} dX_\nu(t) &= [r(t)X_\nu(t) - c_\nu(t)]\, dt \\ &\quad + X_\nu(t)[\delta(\nu(t)) + \pi_\nu^*(t)\nu(t)]\, dt \\ &\quad + X_\nu(t)\pi_\nu^*(t)\sigma(t)\, dW_0(t), \qquad X_\nu(0) = x. \end{aligned} \tag{8.20}$$

The pair (π_ν, c_ν) belongs to $\mathscr{A}_\nu'(x)$, and is optimal for the problem of (8.13).

8.3 PROPOSITION. *Suppose that, for some $\lambda \in \mathscr{D}'$ and with the notation established above, the following hold for $\ell \otimes P$-a.e. (t, ω):*

$$\pi_\lambda(t, \omega) \in K, \tag{8.21}$$

$$\delta(\lambda(t, \omega)) + \pi_\lambda^*(t, \omega)\lambda(t, \omega) = 0. \tag{8.22}$$

Then the pair (π_λ, c_λ) belongs to $\mathscr{A}'(x)$ of (6.4), *is optimal for the constrained optimization problem of* (6.5) *in the original market $\mathscr{M}$, and satisfies*

$$E\left[\int_0^T U_1(t, c_\lambda(t))\,dt + U_2(\xi_\lambda)\right] \le V_\nu(x), \qquad \forall\, \nu \in \mathscr{D}. \tag{8.23}$$

PROOF. Thanks to (8.21) and (8.22), the equation (8.20) with $\nu \equiv \lambda$ becomes

$$\begin{aligned} dX_\lambda(t) &= \left[r(t)X_\lambda(t) - c_\lambda(t)\right]dt + X_\lambda(t)\pi_\lambda^*(t)\sigma(t)\,dW_0(t), \\ &\qquad\qquad X_\lambda(0) = x,\ X_\lambda(T) = \xi_\lambda. \end{aligned} \tag{8.24}$$

Comparing (8.24) with (3.1), we see that X_λ is also the wealth process corresponding to (π_λ, c_λ) *in the original market $\mathscr{M}$*; furthermore, from this and (8.21) we conclude that $(\pi_\lambda, c_\lambda) \in \mathscr{A}'(x)$ and

$$V_\lambda(x) = E\left[\int_0^T U_1(t, c_\lambda(t))\,dt + U_2(\xi_\lambda)\right] \le V(x).$$

But we have the opposite inequality from (8.14), whence the optimality of (π_λ, c_λ) for the problem of (6.5).

On the other hand, let us fix an arbitrary $\nu \in \mathscr{D}$, and let $X_\nu^\lambda \equiv X_\nu^{x, \pi_\lambda, c_\lambda}$ be the wealth process corresponding to the pair (π_λ, c_λ) in the market $\mathscr{M}_\nu$. The equation (8.10) becomes

$$\begin{aligned} dX_\nu^\lambda(t) &= \left[r(t)X_\nu^\lambda(t) - c_\lambda(t)\right]dt + X_\nu^\lambda(t)\left[\delta(\nu(t)) + \pi_\lambda^*(t)\nu(t)\right]dt \\ &\quad + X_\nu^\lambda(t)\pi_\lambda^*(t)\sigma(t)\,dW_0(t), \qquad X_\nu^\lambda(0) = x. \end{aligned}$$

A comparison with (8.24) leads, just as in Remark 8.1, to

$$X_\nu^\lambda(t) \ge X_\lambda(t) > 0, \qquad \forall\, t \in [0, T],$$

almost surely. Thus $(c_\lambda, \pi_\lambda) \in \mathscr{A}_\nu'(x)$ and $V_\lambda(x) \le V_\nu(x)$; but this is (8.23). □

8.4 REMARK. Suppose that

$$\left\{\begin{array}{l} \text{both } U_2(\cdot) \text{ and } U_1(t, \cdot) \text{ satisfy condition (5.9) with the same} \\ \text{constants } \alpha \text{ and } \gamma, \text{ for all } t \in [0, T] \end{array}\right\}. \tag{8.25}$$

It is then easy to see, using (5.9″), that $\mathscr{X}_\nu(y) < \infty$ for some $y \in (0, \infty)$ implies $\nu \in \mathscr{D}'$.

8.5 REMARK. In the market $\mathscr{M}_\nu$ of (8.3) and (8.4), the discounted stock price and wealth processes $\gamma_\nu P_i^{(\nu)}$ and $\gamma_\nu X_\nu^{x,\pi,c}$ satisfy the equations

$$d\big(\gamma_\nu(t) P_i^{(\nu)}(t)\big) = -\gamma_\nu(t) P_i^{(\nu)}(t) \sum_{j=1}^{d} \sigma_{ij}(t)\, dW_{\nu j}(t), \qquad i = 1,\dots,d,$$

$$d\big(\gamma_\nu(t) X_\nu^{x,\pi,c}(t)\big) = -\gamma_\nu(t) c(t)\, dt + \big(\gamma_\nu(t) X_\nu^{x,\pi,c}(t)\big)\pi^*(t)\sigma(t)\, dW_\nu(t),$$

respectively. In particular, none of these two processes depends on the support function $\delta(\cdot)$ of (4.1).

9. Contingent claims attainable by constrained portfolios. Consider a portfolio/consumption process pair (π, c) in the class $\mathscr{A}(x)$ of Definition 3.2, with

$$\pi(t,\omega) \in K, \quad \text{for } \ell \otimes P\text{-a.e. } (t,\omega), \tag{9.1}$$

and recall the wealth process $X^{x,\pi,c}(\cdot)$ corresponding to (π, c) in $\mathscr{M}$ [(3.1)]. On the other hand, for an arbitrary $\nu \in \mathscr{D}$, the process $H_\nu(\cdot)$ of (8.8) satisfies the equation

$$dH_\nu(t) = -H_\nu(t)\big[(r(t) + \delta(\nu(t)))\, dt + \theta_\nu^*(t)\, dW(t)\big], \qquad H_\nu(0) = 1. \tag{9.2}$$

An application of the product rule to $H_\nu X^{x,\pi,c}$ leads then to the analogue of (3.5); namely, that

$$\begin{aligned} &H_\nu(t) X^{x,\pi,c}(t) \\ &\quad + \int_0^t H_\nu(s)c(s)\, ds + \int_0^t H_\nu(s) X^{x,\pi,c}(s)\big[\delta(\nu(s)) + \pi^*(s)\nu(s)\big]\, ds \\ &\quad = x + \int_0^t H_\nu(s) X^{x,\pi,c}(s)\big[\sigma^*(s)\pi(s) - \theta_\nu(s)\big]^*\, dW(s), \quad 0 \le t \le T, \end{aligned} \tag{9.3}$$

is a continuous, nonnegative local martingale, hence a supermartingale. In particular,

$$E\left[H_\nu(T) X^{x,\pi,c}(T) + \int_0^T H_\nu(s)c(s)\, ds\right] \le x, \qquad \forall\, \nu \in \mathscr{D}. \tag{9.4}$$

Based on these preliminary considerations, our next result provides an extension of Proposition 7.3 for the "hedging" of contingent claims by "constrained" portfolios of the form (9.1).

9.1 THEOREM. *Let c be a consumption process, B a positive $\mathscr{F}_T$-measurable random variable, and suppose there exists a process $\lambda \in \mathscr{D}$ such that*

$$\begin{aligned} &E\left[H_\nu(T)B + \int_0^T H_\nu(s)c(s)\, ds\right] \\ &\quad \le E\left[H_\lambda(T)B + \int_0^T H_\lambda(s)c(s)\, ds\right] =: x < \infty, \qquad \forall\, \nu \in \mathscr{D}. \end{aligned} \tag{9.5}$$

Then there exists a portfolio process π, such that the pair (π, c) belongs to the class $\mathscr{A}'(x)$ of (6.4) *and $X^{x,\pi,c}(T) = B$ a.s.*

Proof. By analogy with Proposition 7.3, there exists a portfolio process π such that the wealth process $X \equiv X_\lambda^{x,\pi,c}$, corresponding to (π, c) in $\mathscr{M}_\lambda$, is given by

$$H_\lambda(t)X(t) + \int_0^t H_\lambda(s)c(s)\,ds$$

$$= M_\lambda(t) \triangleq E\left[H_\lambda(T)B + \int_0^T H_\lambda(s)c(s)\,ds \middle| \mathscr{F}_t\right] \tag{9.6}$$

$$= x + \int_0^t H_\lambda(s)X(s)[\sigma^*(s)\pi(s) - \theta_\lambda(s)]^*\,dW(s)$$

and satisfies

$$dX(t) = [r(t)X(t) - c(t)]\,dt$$

$$+ X(t)[\{\delta(\lambda(t)) + \pi^*(t)\lambda(t)\}\,dt + \pi^*(t)\sigma(t)\,dW_0(t)], \tag{9.7}$$

$$X(0) = x,\ X(T) = B.$$

To conclude, we have to show that π satisfies both (9.1) and

$$\delta(\lambda(t,\omega)) + \pi^*(t,\omega)\lambda(t,\omega) = 0, \qquad \ell \otimes P\text{-a.e. } (t,\omega). \tag{9.8}$$

Step 1. Take an arbitrary but fixed $\nu \in \mathscr{D}$, consider a suitable sequence $\{\tau_n\}_{n=1}^\infty$ of stopping times that increase a.s. to T [cf. (9.13) for the precise definition] and, for every fixed $\varepsilon \in (0,1)$, $n \in \mathbb{N}$, introduce a new process $\lambda_{\varepsilon,n} \equiv \lambda_{\varepsilon,n}^{(\nu)} \in \mathscr{D}$ by

$$\lambda_{\varepsilon,n}^{(\nu)}(t) \triangleq \lambda(t) + \varepsilon[\nu(t) - \lambda(t)]1_{\{t \le \tau_n\}}. \tag{9.9}$$

Consider also the notation

$$x(\nu) \triangleq E\left[H_\nu(T)B + \int_0^T H_\nu(s)c(s)\,ds\right], \tag{9.10}$$

$$L(t) \equiv L^{(\nu)}(t) \triangleq \int_0^t \check{\delta}^{(\nu)}(\lambda(s))\,ds,$$

$$\text{where } \check{\delta}^{(\nu)}(\lambda(s)) \triangleq \begin{cases} -\delta(\lambda(s)), & \text{if } \nu \equiv 0, \\ \delta(\nu(s) - \lambda(s)), & \text{otherwise}, \end{cases} \tag{9.11}$$

$$N(t) \equiv N^{(\nu)}(t) \triangleq \int_0^t (\sigma^{-1}(s)(\nu(s) - \lambda(s))^*)\,dW_\lambda(s) \tag{9.12}$$

and define the stopping times $\{\tau_n\}_{n\in\mathbb{N}}$ as follows:

$$\tau_n \triangleq T \wedge \inf\Big\{ t\in[0,T];\quad |L^{(\nu)}(t)|\ge n, \text{ or } |N^{(\nu)}(t)|\ge n, \text{ or}$$

$$\int_0^t \|\sigma^{-1}(s)(\nu(s)-\lambda(s))\|^2\,ds \ge n, \text{ or}$$

$$\int_0^t \|\theta(s)+\sigma^{-1}(s)\lambda(s)\|^2\,ds \ge n, \tag{9.13}$$

$$\text{or } \int_0^t X^2(s)\gamma_\lambda^2(s)\|\sigma^{-1}(s)(\nu(s)-\lambda(s))$$

$$+(L_s+N_s)\sigma^*(s)\pi(s)\|^2\,ds \ge n\Big\}, \qquad n\in\mathbb{N}.$$

Below, we shall take $\nu\equiv\lambda+\rho$ (for arbitrary fixed $\rho\in\mathscr{D}$) and $\nu\equiv 0$. For both these choices, the preceding sequence satisfies $\lim_{n\to\infty}\uparrow\tau_n = T$ almost surely.

STEP 2. As we shall see below, for both choices $\nu\equiv\lambda+\rho$ ($\rho\in\mathscr{D}$) and $\nu\equiv 0$, and every $n\in\mathbb{N}$, we have

$$\limsup_{\varepsilon\downarrow 0}\frac{x(\lambda)-x\big(\lambda^{(\nu)}_{\varepsilon,n}\big)}{\varepsilon}$$

$$\le E\Big[H_\lambda(T)B\big(L^{(\nu)}_{\tau_n}+N^{(\nu)}_{\tau_n}\big)+\int_0^T H_\lambda(t)c(t)\big(L^{(\nu)}_{t\wedge\tau_n}+N^{(\nu)}_{t\wedge\tau_n}\big)\,dt\Big] \tag{9.14}$$

$$= E\int_0^{\tau_n} H_\lambda(t)X(t)\big[\pi^*(t)(\nu(t)-\lambda(t))\,dt + dL^{(\nu)}(t)\big]. \tag{9.15}$$

By assumption, the left-hand side of (9.14) is *nonnegative*, and thus so is the expression of (9.15).

STEP 3. In particular, with $\nu\equiv\lambda+\rho$ ($\rho\in\mathscr{D}$), this observation leads to

$$E\int_0^{\tau_n} X(t)H_\lambda(t)\big[\pi^*(t)\rho(t)+\delta(\rho(t))\big]\,dt \ge 0, \qquad \forall\, n\in\mathbb{N}, \tag{9.16}$$

and thence to

$$\phi(t;\rho)\triangleq \pi^*(t)\rho(t)+\delta(\rho(t))\ge 0, \qquad \ell\otimes P\text{-a.e.} \tag{9.17}$$

[Indeed, suppose that for some $\rho\in\mathscr{D}$ the inequality (9.17) fails on a set $A\subset[0,T]\times\Omega$ of positive product measure. Notice that $\phi(t;\eta\rho)=\eta\phi(t;\rho)$ for every $\eta>0$; replacing ρ by $\eta\rho$ on the set A and choosing $\eta>0$ large enough, we can then violate (9.16) with ρ replaced by $\tilde\rho=\rho 1_{A^c}+\eta\rho 1_A$.]

In particular, (9.17) implies that, for every $r\in\tilde K$,

$$-\pi^*(t,\omega)r\le\delta(r|K), \qquad \forall\,(t,\omega)\in A_r,$$

where $A_r \subset [0,T] \times \Omega$ is a set of full product measure. But then so is $A \triangleq \bigcap_{r \in \tilde{K},\, r \in Q^d} A_r$, and from the assumption (4.3):

$$-\pi^*(t,\omega) r \le \delta(r|K), \qquad \forall\, (t,\omega) \in A,\ r \in \tilde{K}. \tag{9.18}$$

Now fix $(t,\omega) \in A$; from (9.18), the fact that K is closed, and Theorem 13.1 in Rockafellar [(1970), page 112], we obtain (9.1).

On the other hand, for $\nu \equiv 0$, the nonnegativity of (9.15) leads to

$$E\int_0^{\tau_n} H_\lambda(t) X(t)\left[\pi^*(t)\lambda(t) + \delta(\lambda(t))\right] dt \le 0, \qquad \forall\, n \in \mathbb{N}.$$

In light of (9.16) (which is valid, in particular, with $\rho \equiv \lambda$), this implies (9.8).

STEP 4. *Proof of* (9.14). For either $\nu \equiv \lambda + \rho$ or $\nu \equiv 0$, we have

$$\delta(\lambda(s) + \varepsilon(\nu(s) - \lambda(s))) - \delta(\lambda(s)) \le \varepsilon \check{\delta}^{(\nu)}(\lambda(s)).$$

[Indeed, (4.5) and the positive homogeneity properties of $\delta(\cdot)$, give $\delta(\lambda(s) + \varepsilon(\nu(s) - \lambda(s))) - \delta(\lambda(s)) \le \varepsilon\delta(\nu(s) - \lambda(s))$ for $\nu \equiv \lambda + \rho$, whereas with $\nu \equiv 0$, $\delta((1-\varepsilon)\lambda(s)) - \delta(\lambda(s)) = -\varepsilon\delta(\lambda(s))$.] In either case,

$$\begin{aligned}\frac{H_{\lambda_{\varepsilon,n}}(t)}{H_\lambda(t)} &= \exp\left[-\int_0^{t\wedge\tau_n}\{\delta(\lambda(s) + \varepsilon(\nu(s) - \lambda(s))) - \delta(\lambda(s))\}\,ds\right.\\ &\qquad\left. -\varepsilon N_{t\wedge\tau_n} - \frac{\varepsilon^2}{2}\int_0^{t\wedge\tau_n}\left\|\sigma^{-1}(s)(\nu(s) - \lambda(s))\right\|^2 ds\right]\\ &\ge \exp\left[-\varepsilon(L_{t\wedge\tau_n} + N_{t\wedge\tau_n}) - \frac{\varepsilon^2}{2}\int_0^{t\wedge\tau_n}\left\|\sigma^{-1}(s)(\nu(s) - \lambda(s))\right\|^2 ds\right]\\ &\ge e^{-3\varepsilon n},\end{aligned} \tag{9.19}$$

from the construction of the stopping times τ_n in (9.13). On the other hand, we have

$$\begin{aligned}\frac{x(\lambda) - x(\lambda_{\varepsilon,n})}{\varepsilon} &= EQ_n^\varepsilon,\\ Q_n^\varepsilon &\triangleq H_\lambda(T)\frac{B}{\varepsilon}\left\{1 - \frac{H_{\lambda_{\varepsilon,n}}(T)}{H_\lambda(T)}\right\}\\ &\qquad + \int_0^T H_\lambda(t)\frac{c(t)}{\varepsilon}\left\{1 - \frac{H_{\lambda_{\varepsilon,n}}(t)}{H_\lambda(t)}\right\} dt.\end{aligned} \tag{9.20}$$

The family $\{Q_n^\varepsilon\}_{0<\varepsilon<1}$ is dominated by the random variable

$$Q_n \triangleq K_n\left[BH_\lambda(T) + \int_0^T c(t) H_\lambda(t)\,dt\right],$$

$$K_n \triangleq \sup_{0<\varepsilon<1}\frac{1 - e^{-3\varepsilon n}}{\varepsilon},$$

with expectation $EQ_n = K_n x(\lambda) < \infty$. Therefore, by Fatou's lemma

$$
\begin{aligned}
\limsup_{\varepsilon \downarrow 0} & \frac{x(\lambda) - x(\lambda_{\varepsilon,n})}{\varepsilon} \\
&\le E\Big(\limsup_{\varepsilon \downarrow 0} Q_n^\varepsilon\Big) \\
&= E\bigg[H_\lambda(T) B \limsup_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\bigg\{1 - \frac{H_{\lambda_{\varepsilon,n}}(T)}{H_\lambda(T)}\bigg\} \\
&\qquad + \int_0^T H_\lambda(t) c(t) \limsup_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\bigg\{1 - \frac{H_{\lambda_{\varepsilon,n}}}{H_\lambda(t)}\bigg\}\, dt \\
&\le E\bigg[H_\lambda(T) B(L_{\tau_n} + N_{\tau_n}) + \int_0^T H_\lambda(t) c(t) (L_{t\wedge\tau_n} + N_{t\wedge\tau_n})\, dt\bigg].
\end{aligned}
$$

STEP 5. *Proof of* (9.15). By analogy with (3.4), we have

$$
\begin{aligned}
R(t) &\triangleq \gamma_\lambda(t) X(t) + \int_0^t \gamma_\lambda(s) c(s)\, ds \\
&= x + \int_0^t \gamma_\lambda(s) X(s) \pi^*(s) \sigma(s)\, dW_\lambda(s).
\end{aligned}
\tag{9.21}
$$

According to the Girsanov and Novikov theorems [cf. Karatzas and Shreve (1988), Section 3.5], the process $\{W_\lambda(t \wedge \tau_n),\ 0 \le t \le T\}$ is Brownian motion under the probability measure $\tilde{P}_n(A) = E[Z_\lambda(\tau_n) 1_A]$.

Let us apply the product rule to $X(t)\gamma_\lambda(t)(L_t + N_t)$ to obtain, in conjunction with (9.21), (9.11) and (9.12),

$$
\begin{aligned}
d[X(t)&\gamma_\lambda(t)(L_t + N_t)] \\
&= d\bigg[\bigg(R(t) - \int_0^t \gamma_\lambda(s) c(s)\, ds\bigg)(L_t + N_t)\bigg] \\
&= X(t)\gamma_\lambda(t)(dL_t + dN_t) + (dR(t) - \gamma_\lambda(t) c(t)\, dt)(L_t + N_t) + d\langle R, N\rangle_t \\
&= X(t)\gamma_\lambda(t)(dL_t + dN_t) - \gamma_\lambda(t) c(t)(L_t + N_t)\, dt + (L_t + N_t)\, dR(t) \\
&\quad + X(t)\gamma_\lambda(t)\pi^*(t)(\nu(t) - \lambda(t))\, dt.
\end{aligned}
$$

In particular,

$$
\begin{aligned}
\gamma_\lambda(\tau_n) X(\tau_n)&(L_{\tau_n} + N_{\tau_n}) + \int_0^{\tau_n} \gamma_\lambda(t) c(t)(L_t + N_t)\, dt \\
&= \int_0^{\tau_n} \gamma_\lambda(t) X(t)\big[\sigma^{-1}(t)(\nu(t) - \lambda(t)) \\
&\qquad\qquad + (L_t + N_t)\sigma^*(t)\pi(t)\big]^*\, dW_\lambda(t) \\
&\quad + \int_0^{\tau_n} \gamma_\lambda(t) X(t)\big[\pi^*(t)(\nu(t) - \lambda(t))\, dt + dL_t\big].
\end{aligned}
\tag{9.22}
$$

Now let us take expectations with respect to the probability measure $d\tilde{P}_n = Z_\lambda(\tau_n)\,dP$; from the definition of the stopping time τ_n in (9.13), the expectation of the stochastic integral in (9.22) is equal to zero and thus

$$\begin{aligned}(9.23)\qquad & E\left[H_\lambda(\tau_n)X(\tau_n)(L_{\tau_n}+N_{\tau_n}) + \int_0^{\tau_n} H_\lambda(t)c(t)(L_t+N_t)\,dt\right]\\ &= E\int_0^{\tau_n} H_\lambda(t)X(t)\left[\pi^*(t)(\nu(t)-\lambda(t))\,dt + dL_t\right].\end{aligned}$$

The right-hand side is the expression that appears in (9.15); thus, it remains to show that the left-hand side is equal to the expression that appears in (9.14). Indeed, (9.6) gives

$$H_\lambda(\tau_n)X(\tau_n) = E\left[H_\lambda(T)B + \int_{\tau_n}^T H_\lambda(t)c(t)\,dt\,\middle|\,\mathscr{F}_{\tau_n}\right] \quad \text{a.s.},$$

and so the left-hand side of (9.23) is equal to

$$\begin{aligned}&E\left[(L_{\tau_n}+N_{\tau_n})\left\{H_\lambda(T)B + \int_{\tau_n}^T H_\lambda(t)c(t)\,dt\right\} + \int_0^{\tau_n} H_\lambda(t)c(t)(L_t+N_t)\,dt\right]\\ &\quad = E\left[H_\lambda(T)B(L_{\tau_n}+N_{\tau_n}) + \int_0^T H_\lambda(t)c(t)(L_{t\wedge\tau_n}+N_{t\wedge\tau_n})\,dt\right]. \qquad \square\end{aligned}$$

10. Equivalent optimality conditions. For a fixed initial capital $x > 0$, let $(\hat{\pi}, \hat{c})$ be a given portfolio/consumption process pair in the class $\mathscr{A}'(x)$ of (6.4), denote the corresponding wealth process in $\mathscr{M}$ by $\hat{X}(\cdot)$, and consider the statement that this pair is optimal for the constrained optimization problem of Definition 6.3:

(*A*) *Optimality of* $(\hat{\pi}, \hat{c})$. We have

$$\begin{aligned}(10.1)\qquad & E\left[\int_0^T U_1(t,c(t))\,dt + U_2(X^{x,c,\pi}(T))\right]\\ &\le E\left[\int_0^T U_1(t,\hat{c}(t))\,dt + U_2(\hat{X}(T))\right],\end{aligned}$$

for every $(\pi, c) \in \mathscr{A}'(x)$, as well as

$$(10.2)\qquad E\left[\int_0^T \hat{c}(t)U_1'(t,\hat{c}(t))\,dt + \hat{X}(T)U_2'(\hat{X}(T))\right] < \infty.$$

We shall characterize (A) in terms of the following conditions (B)–(E), which concern a given process λ in the class $\mathscr{D}'$ of (8.16).

(*B*) *Financibility of* (c_λ, ξ_λ). There exists a portfolio process $\hat{\pi}_\lambda$ such that $(\hat{\pi}_\lambda, c_\lambda) \in \mathscr{A}'(x)$ and

$$\begin{aligned} &\hat{\pi}_\lambda(t,\omega) \in K, \qquad \delta(\lambda(t,\omega)) + \hat{\pi}_\lambda^*(t,\omega)\lambda(t,\omega) = 0, \\ &\qquad\qquad X^{x,\hat{\pi}_\lambda,c_\lambda}(t,\omega) = X_\lambda(t,\omega) \end{aligned} \tag{10.3}$$

hold for $\ell \otimes P$-a.e. (t,ω).

(*C*) *Minimality of* λ. For every $\nu \in \mathscr{D}$, we have

$$E\left[\int_0^T U_1(t, c_\lambda(t))\,dt + U_2(\xi_\lambda)\right] = V_\lambda(x) \le V_\nu(x). \tag{10.4}$$

(*D*) *Dual optimality of* λ. For every $\nu \in \mathscr{D}$, we have

$$\begin{aligned} &E\left[\int_0^T \tilde{U}_1(t, \mathscr{Y}_\lambda(x)H_\lambda(t))\,dt + \tilde{U}_2(\mathscr{Y}_\lambda(x)H_\lambda(T))\right] \\ &\qquad \le E\left[\int_0^T \tilde{U}_1(t, \mathscr{Y}_\lambda(x)H_\nu(t))\,dt + \tilde{U}_2(\mathscr{Y}_\lambda(x)H_\nu(T))\right]. \end{aligned} \tag{10.5}$$

(*E*) *Parsimony of* λ. For every $\nu \in \mathscr{D}$, we have

$$E\left[\int_0^T H_\nu(t)c_\lambda(t)\,dt + H_\nu(T)\xi_\lambda\right] \le x. \tag{10.6}$$

It should be observed that the expectations in (10.5) are well defined. Indeed, (5.2) gives

$$\begin{aligned} U_1(t, c(t)) &\le \tilde{U}_1(t, yH_\nu(t)) + yH_\nu(t)c(t), \\ U_2(X^{x,\pi,c}(T)) &\le \tilde{U}_2(yH_\nu(T)) + yH_\nu(T)X^{x,\pi,c}(T) \end{aligned} \tag{10.7}$$

a.s. for every $x > 0$, $y > 0$, $(\pi, c) \in \mathscr{A}_\nu'(x)$. In conjunction with (9.4), this leads to

$$\begin{aligned} &E\left[\int_0^T \tilde{U}_1^-(t, yH_\nu(t))\,dt + \tilde{U}_2^-(yH_\nu(T))\right] \\ &\qquad \le E\left[\int_0^T U_1^-(t, c(t))\,dt + U_2^-(X^{x,\pi,c}(T))\right] + xy < \infty \end{aligned}$$

for every $y \in (0,\infty)$ and $\nu \in \mathscr{D}$.

10.1 THEOREM. *Conditions* (B)–(E) *are equivalent, and imply* (A) *with* $(\hat{\pi}, \hat{c}) = (\hat{\pi}_\lambda, c_\lambda)$. *Conversely, condition* (A) *implies the existence of* $\lambda \in \mathscr{D}'$ *that satisfies* (B)–(E) *with* $\hat{\pi}_\lambda \equiv \hat{\pi}$, *provided that* (5.8), (8.25) *and* (12.2) *hold for* $U_1(t,\cdot)$ *and* $U_2(\cdot)$.

This can be regarded as the focal result of the paper. Its condition (D) leads naturally to the introduction of a *dual stochastic control problem* in (12.1) of

Section 12, whereas convex duality theory can then be used to relate the value function and optimal process of this problem to those of the *primal* one (of Definition 6.3); cf. Propositions 12.1, 12.2 and Theorem 12.4. Under suitable conditions, one can also establish the *existence* of an optimal process for this dual problem and, based on the above-mentioned primal–dual relationships and on the implication (D) ⇒ (A) of Theorem 10.1, prove the *existence of an optimal pair* $(\hat{\pi}, \hat{c})$ *for the primal problem*; cf. Theorem 13.1.

PROOF OF THEOREM 10.1. The implication (B) ⇒ (E) is a consequence of (9.4). The implications (B) ⇒ (A) and (B) ⇒ (C) follow from Proposition 8.3, together with the observation

$$E\left[X_\lambda(T)U_2'(X_\lambda(T)) + \int_0^T c_\lambda(t)U_1'(t, c_\lambda(t))\,dt\right] = x\mathscr{Y}_\lambda(x) < \infty.$$

The implication (E) ⇒ (B) is a consequence of Theorem 9.1 with $c \equiv c_\lambda$ and $B \equiv \xi_\lambda$.

For the implication (E) ⇒ (D), write (5.6) with $x \to c_\lambda(t)$, $y \to \mathscr{Y}_\lambda(x)H_\nu(t)$ [respectively, $x \to \xi_\lambda$, $y \to \mathscr{Y}_\lambda(x)H_\nu(T)$] to obtain

$$\tilde{U}_1(t, \mathscr{Y}_\lambda(x)H_\nu(t)) \geq \tilde{U}_1(\mathscr{Y}_\lambda(x)H_\lambda(t)) + \mathscr{Y}_\lambda(x)[H_\lambda(t)c_\lambda(t) - H_\nu(t)c_\lambda(t)],$$

$$\tilde{U}_2(\mathscr{Y}_\lambda(x)H_\nu(T)) \geq \tilde{U}_2(\mathscr{Y}_\lambda(x)H_\lambda(T)) + \mathscr{Y}_\lambda(x)[H_\lambda(T)c_\lambda(T) - H_\nu(T)c_\lambda(T)],$$

respectively. Now integrate and add, to get

$$\begin{aligned} &E\left[\int_0^T \tilde{U}_1(t, \mathscr{Y}_\lambda(x)H_\nu(t))\,dt + \tilde{U}_2(\mathscr{Y}_\lambda(x)H_\nu(T))\right] \\ &\quad\geq E\left[\int_0^T \tilde{U}_1(t, \mathscr{Y}_\lambda(x)H_\lambda(t))\,dt + \tilde{U}_2(\mathscr{Y}_\lambda(x)H_\lambda(T))\right] \\ &\qquad + \mathscr{Y}_\lambda(x)\left\{x - E\left[\int_0^T H_\nu(t)c_\lambda(t)\,dt + H_\nu(T)\xi_\lambda\right]\right\}. \end{aligned}$$

This last expression, in braces, is nonnegative by (10.6), and (10.5) follows.

(D) ⇒ (B): Repeat the proof of Theorem 9.1 up to (9.14), with $c(t)$ replaced by $c_\lambda(t) = I_1(t, \mathscr{Y}_\lambda(x)H_\lambda(t))$, B replaced by $\xi_\lambda = I_2(\mathscr{Y}_\lambda(x)H_\lambda(T))$ and (9.5) by (10.5). It all then boils down to showing the analogue

$$\begin{aligned} &\limsup_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\left[E\left(\int_0^T \tilde{U}_1(t, yH_{\lambda_{\varepsilon,n}}(t))\,dt + \tilde{U}_2(yH_{\lambda_{\varepsilon,n}}(T))\right)\right. \\ &\qquad\qquad \left. - E\left(\int_0^T \tilde{U}_1(t, yH_\lambda(t))\,dt + \tilde{U}_2(yH_\lambda(T))\right)\right] \qquad (10.8) \\ &\quad \leq yE\left[\int_0^T c_\lambda(t)H_\lambda(t)(L_{t\wedge\tau_n} + N_{t\wedge\tau_n})\,dt + \xi_\lambda H_\lambda(T)(L_{\tau_n} + N_{\tau_n})\right] \end{aligned}$$

of (9.14), where we have set $y = \mathscr{Y}_\lambda(x)$. The rest of the proof follows without modification.

Now for any given $y \in (0, \infty)$, the family of random variables

$$(10.9)\quad \begin{aligned} Y_\varepsilon^{(n)} \triangleq \frac{1}{\varepsilon}\Bigg[&\left(\int_0^T \tilde{U}_1(t, yH_{\lambda_{\varepsilon,n}}(t))\,dt + \tilde{U}_2(yH_{\lambda_{\varepsilon,n}}(T))\right) \\ &-\left(\int_0^T \tilde{U}_1(t, yH_\lambda(t))\,dt + \tilde{U}_2(yH_\lambda(T))\right)\Bigg], \qquad \varepsilon \in (0,1), \end{aligned}$$

of the left-hand side of (10.8), is bounded from above by

$$Y^{(n)} \triangleq yK_n\left[\int_0^T H_\lambda(t) I_1(t, ye^{-3n}H_\lambda(t))\,dt + H_\lambda(T) I_2(ye^{-3n}H_\lambda(T))\right],$$

a random variable with expectation $yK_n\mathscr{X}_\lambda(ye^{-3n}) < \infty$ [here again, $K_n = \sup_{0<\varepsilon<1}(e^{3\varepsilon n} - 1)/\varepsilon$, and we have used (5.3), the fact that $I_2(\cdot)$, $I_1(t, \cdot\,)$ are decreasing, and (9.19)]. Therefore, from Fatou's lemma,

$$(10.10)\qquad \limsup_{\varepsilon \downarrow 0} E\big(Y_\varepsilon^{(n)}\big) \le E\Big(\limsup_{\varepsilon \downarrow 0} Y_\varepsilon^{(n)}\Big).$$

On the other hand, the random variables of (10.9) admit also the a.s. upper bound

$$(10.11)\quad \begin{aligned} Y_\varepsilon^{(n)} \le y\Bigg[&\int_0^T H_\lambda(t) I_1(t, ye^{-3\varepsilon n}H_\lambda(t))\Lambda_\varepsilon^{(n)}(t)\,dt \\ &+ H_\lambda(T) I_2(ye^{-3\varepsilon n}H_\lambda(T))\Lambda_\varepsilon^{(n)}(T)\Bigg] =: V_\varepsilon^{(n)}, \end{aligned}$$

where

$$\begin{aligned} \Lambda_\varepsilon^{(n)}(t) \triangleq \frac{1}{\varepsilon}\Bigg[1 - \exp\Bigg\{&-\varepsilon\big(L_{t\wedge\tau_n} + N_{t\wedge\tau_n}\big) \\ &- \frac{\varepsilon^2}{2}\int_0^{t\wedge\tau_n}\|\sigma^{-1}(s)(\nu(s) - \lambda(s))\|^2\,ds\Bigg\}\Bigg]. \end{aligned}$$

Quite clearly, $\Lambda_\varepsilon^{(n)}(t) \to_{\varepsilon\downarrow 0} L_{t\wedge\tau_n} + N_{t\wedge\tau_n}$ a.s. and

$$(10.12)\quad \begin{aligned} V_\varepsilon^{(n)} \to_{\varepsilon\downarrow 0} y\Bigg[&\int_0^T H_\lambda(t) I_1(t, yH_\lambda(t))\big(L_{t\wedge\tau_n} + N_{t\wedge\tau_n}\big)\,dt \\ &+ H_\lambda(T) I_2(yH_\lambda(T))\big(L_{\tau_n} + N_{\tau_n}\big)\Bigg], \quad \text{a.s.} \end{aligned}$$

Finally from (10.10)–(10.12), $\limsup_{\varepsilon\downarrow 0} E(Y_\varepsilon^{(n)}) \le E(\lim_{\varepsilon\downarrow 0} V_\varepsilon^{(n)}) = \text{RHS of}$ (10.8).

(C) ⇒ (D): For a fixed $\nu \in \mathscr{D}$ and arbitrary $(\pi, c) \in \mathscr{A}_\nu'(x)$, $y \in (0, \infty)$, we have from (10.7):

$$
\begin{aligned}
&E\left[\int_0^T U_1(t, c(t))\,dt + U_2(X^{x,\pi,c}(T))\right]\\
&\qquad \le E\left[\int_0^T \tilde{U}_1(t, yH_\nu(t))\,dt + \tilde{U}_2(yH_\nu(T))\right]\\
&\qquad\quad + yE\left[\int_0^T H_\nu(t)c(t)\,dt + H_\nu(T)X^{x,\pi,c}(T)\right]\\
&\qquad \le f_\nu(y) \triangleq E\left[\int_0^T \tilde{U}_1(t, yH_\nu(t))\,dt + \tilde{U}_2(yH_\nu(T))\right] + xy,
\end{aligned}
\tag{10.13}
$$

whence

$$
V_\nu(x) \le f_\nu(y), \qquad \forall\, y \in (0, \infty). \tag{10.14}
$$

Therefore, from (10.4) we obtain

$$
\begin{aligned}
f_\nu(\mathscr{Y}_\lambda(x)) \ge V_\nu(x) &\ge E\left[\int_0^T U_1(t, c_\lambda(t))\,dt + U_2(\xi_\lambda)\right]\\
&= E\left[\int_0^T \tilde{U}_1(t, \mathscr{Y}_\lambda(x)H_\lambda(t))\,dt + \tilde{U}_2(\mathscr{Y}_\lambda(x)H_\lambda(T))\right]\\
&\quad + \mathscr{Y}_\lambda(x)E\left[\int_0^T H_\lambda(t)I_1(t, \mathscr{Y}_\lambda(x)H_\lambda(t))\,dt\right.\\
&\qquad\qquad \left. + H_\lambda(T)I_2(\mathscr{Y}_\lambda(x)H_\lambda(T))\right]\\
&= E\left[\int_0^T \tilde{U}_1(t, \mathscr{Y}_\lambda(x)H_\lambda(t))\,dt + \tilde{U}_2(\mathscr{Y}_\lambda(x)H_\lambda(T))\right] + x\mathscr{Y}_\lambda(x),
\end{aligned}
$$

whence

$$
\begin{aligned}
&E\left[\int_0^T \tilde{U}_1(t, \mathscr{Y}_\lambda(x)H_\lambda(t))\,dt + \tilde{U}_2(\mathscr{Y}_\lambda(x)H_\lambda(T))\right]\\
&\qquad \le f_\nu(\mathscr{Y}_\lambda(x)) - x\mathscr{Y}_\lambda(x)\\
&\qquad = E\left[\int_0^T \tilde{U}_1(t, \mathscr{Y}_\lambda(x)H_\nu(t))\,dt + \tilde{U}_2(\mathscr{Y}_\lambda(x)H_\nu(T))\right].
\end{aligned}
$$

(A) ⇒ (B): Proved in Appendix A. (This implication is not used in any of the results of the paper.) □

10.2 Remark. The condition (10.2) of (A) becomes vacuous, if

$$
\int_0^T U_1(t, 0+)\,dt > -\infty, \qquad U_2(0+) > -\infty,
$$

because then, from concavity,

$$E\left[\int_0^T \hat{c}(t)U_1'(t,\hat{c}(t))\,dt + \hat{X}(T)U_2'(\hat{X}(T))\right]$$
$$\le E\left[\int_0^T U_1(t,\hat{c}(t))\,dt + U_2(\hat{X}(T))\right] - \left[\int_0^T U_1(t,0+)\,dt + U_2(0+)\right]$$
$$= V(x) - \left[\int_0^T U_1(t,0+)\,dt + U_2(0+)\right] < \infty.$$

The role of (10.2) is to allow us to consider utility functions which are not bounded from below, such as logarithmic.

11. The logarithmic case. If $U_1(t,x) = U_2(x) = \log x$, for $(t,x) \in [0,T] \times (0,\infty)$, we have $I_1(t,y) = I_2(y) = 1/y$, $\tilde{U}_1(t,y) = \tilde{U}_2(y) = -(1+\log y)$ and

$$\mathscr{X}_\nu(y) = \frac{T+1}{y}, \qquad \mathscr{Y}_\nu(x) = \frac{T+1}{x}, \tag{11.1}$$

$$c_\nu^x(t) = \frac{x}{T+1}\frac{1}{H_\nu(t)}, \qquad \xi_\nu^x = \frac{x}{T+1}\frac{1}{H_\nu(T)}, \tag{11.2}$$

for every $\nu \in \mathscr{D}$. In particular $\mathscr{D}' = \mathscr{D}$ in this case. Therefore,

$$\begin{aligned} &E\left[\int_0^T \tilde{U}_1(t,\mathscr{Y}_\lambda(x)H_\nu(t))\,dt + \tilde{U}_2(\mathscr{Y}_\lambda(x)H_\nu(T))\right] \\ &\quad = -(1+T)\left(1+\log\frac{1+T}{x}\right) \\ &\qquad + E\left(\log\frac{1}{H_\nu(T)} + \int_0^T \log\frac{1}{H_\nu(t)}\,dt\right). \end{aligned} \tag{11.3}$$

But

$$E\left(\log\frac{1}{H_\nu(t)}\right) = E\int_0^t\left[r(s) + \delta(\nu(s)) + \frac{1}{2}\|\theta(s) + \sigma^{-1}(s)\nu(s)\|^2\right]ds,$$

and thus condition (D) amounts to a pointwise minimization of the convex function $\delta(x) + \frac{1}{2}\|\theta(t) + \sigma^{-1}(t)x\|^2$ over $x \in \tilde{K}$, for every $t \in [0,T]$:

$$\lambda(t) = \arg\min_{x\in\tilde{K}}\left[2\delta(x) + \|\theta(t) + \sigma^{-1}(t)x\|^2\right]. \tag{11.4}$$

11.1 Remark. Measurable selection theorems of the so-called Dubins–Savage type [e.g., Schäl (1974), (1975)] show that the process λ defined by (11.4) is indeed $\{\mathscr{F}_t\}$-progressively measurable. On the other hand, (11.4) leads directly to

$$2E\int_0^T \delta(\lambda(t))\,dt + E\int_0^T \|\theta(t) + \sigma^{-1}(t)\lambda(t)\|^2\,dt \le E\int_0^T \|\theta(t)\|^2\,dt < \infty,$$

whence $\lambda \in \mathscr{D}$.

Furthermore, (11.2) and (8.19) give

$$H_\lambda(t)X_\lambda(t) = x\left(1 - \frac{t}{T+1}\right), \qquad 0 \le t \le T,$$

and substituting this expression into (8.11), with ν replaced by λ, we obtain $\sigma^*(t)\hat{\pi}(t) = \theta_\lambda(t)$, $\ell \otimes P$-a.e.

We conclude that the optimal portfolio is given by

$$\hat{\pi}(t) = (\sigma(t)\sigma^*(t))^{-1}[\lambda(t) + b(t) - r(t)\mathbf{1}] \tag{11.5}$$

in terms of the market coefficients and the process λ of (11.4). Finally, from (8.17), the optimal consumption process $c_\lambda(\cdot)$ is given by

$$c_\lambda(t) = \frac{x}{(T+1)H_\lambda(t)} = \frac{X_\lambda(t)}{1 + (T-t)}, \qquad 0 \le t \le T. \tag{11.6}$$

It is doubtful that such explicit and general a result should exist for utility functions other than logarithmic.

12. A dual problem. In addition to our original constrained optimization problem of Definition 6.3, we shall introduce here the so-called *dual control problem* with value function

$$\begin{aligned} \tilde{V}(y) &\triangleq \inf_{\nu \in \mathscr{D}} \tilde{J}(y;\nu), \\ \tilde{J}(y;\nu) &\triangleq E\left[\int_0^T \tilde{U}_1(t, yH_\nu(t))\,dt + \tilde{U}_2(yH_\nu(T))\right], \end{aligned} \tag{12.1}$$

for $0 < y < \infty$. This new value function maps $(0,\infty)$ into itself, provided we have

$$\forall\, y \in (0,\infty), \exists\, \nu \in \mathscr{D} \text{ such that } \tilde{J}(y;\nu) < \infty \tag{12.2}$$

(cf. Remark 12.5). We shall also impose the assumption

$$\inf_{0 \le t \le T} U_1(t, 0+) > -\infty, \qquad U_2(0+) > -\infty. \tag{12.3}$$

The motivation for introducing this dual problem comes, of course, from condition (D) of Theorem 10.1, which amounts to $\tilde{V}(y) = \tilde{J}(y;\lambda)$ for $y = \mathscr{Y}_\lambda(x)$, in the notation of (12.1).

For any given $x > 0$, $y > 0$ and $(\pi, c) \in \mathscr{A}'(x)$, recall the a.s. inequalities of (10.7), and observe that they hold as equalities, if and only if

$$c(t) = I_1(t, yH_\nu(t)), \qquad X^{x,\pi,c}(T) = I_2(yH_\nu(T)). \tag{12.4}$$

Taking expectations, and the supermartingale property of the process in (9.3) into account, we obtain from (10.7):

$$
\begin{aligned}
J(x;\pi,c) &= E\left[\int_0^T U_1(t,c(t))\,dt + U_2(X^{x,\pi,c}(T))\right]\\
&\le \tilde{J}(y;\nu) + yE\left[\int_0^T H_\nu(t)c(t)\,dt + H_\nu(T)X^{x,\pi,c}(T)\right]\\
&\le \tilde{J}(y;\nu) + xy - yE\int_0^T H_\nu(t)X^{x,\pi,c}(t)[\delta(\nu(t)) + \pi^*(t)\nu(t)]\,dt\\
&\le \tilde{J}(y;\nu) + xy.
\end{aligned}
\tag{12.5}
$$

We have equality in (12.5) *if and only if* (12.4) *and*

$$\delta(\nu(t,\omega)) + \pi^*(t,\omega)\nu(t,\omega) = 0, \quad \text{for } \ell\otimes P\text{-a.e. } (t,\omega), \tag{12.6}$$

$$E\left[\int_0^T H_\nu(t)c(t)\,dt + H_\nu(T)X^{x,\pi,c}(T)\right] = x \tag{12.7}$$

hold. In particular, (12.5) implies

$$V(x) \le \tilde{V}(y) + xy, \qquad \forall\,(x,y)\in(0,\infty)^2. \tag{12.8}$$

12.1 PROPOSITION. *Suppose that* (12.2) *and*

$$\forall\, y\in(0,\infty),\ \exists\,\lambda_y\in\mathscr{D}' \text{ such that } \tilde{V}(y) = \tilde{J}(y;\lambda_y) \tag{12.9}$$

hold. Then, for any given $y\in(0,\infty)$ *and with* $x = \mathscr{X}_{\lambda_y}(y)$, *there exists a pair* $(\hat{\pi},\hat{c})\in\mathscr{A}'(x)$ *which is optimal for the primal problem, and we have*

$$\tilde{V}(y) = \sup_{\xi>0}[V(\xi) - y\xi], \qquad 0<y<\infty. \tag{12.10}$$

In particular, $\tilde{V}(\cdot)$ *is convex.*

PROOF. With $x = \mathscr{X}_{\lambda_y}(y)$, the assumption $\tilde{J}(y;\lambda_y)\le\tilde{J}(y;\nu)$, $\forall\ \nu\in\mathscr{D}$ of (12.9) amounts to (10.5) with $\lambda=\lambda_y$. The implications (D) ⇒ (A) and (D) ⇒ (B) of Theorem 10.1 show that there exists then an optimal pair $(\hat{\pi},\hat{c})\in\mathscr{A}'(x)$ for the primal problem and

$$\hat{\pi}(t)\in K, \qquad \delta(\lambda_y(t)) + \hat{\pi}^*(t)\lambda_y(t) = 0, \qquad c(t) = I_1\big(t, \mathscr{Y}_{\lambda_y}(x)H_{\lambda_y}(t)\big),$$
$$X^{x,\hat{\pi},\hat{c}}(t) = X_{\lambda_y}(t)$$

hold $\ell\otimes P$-a.e. In particular, $X^{x,\hat{\pi},\hat{c}}(T) = \xi_{\lambda_y}$ a.s. We conclude from (12.4)–(12.7) that $J(x;\hat{\pi},\hat{c}) = \tilde{J}(y;\lambda_y) + xy$ holds, whence

$$\tilde{V}(y) = \tilde{J}(y;\lambda_y) = J(x;\hat{\pi},\hat{c}) - xy = V(x) - xy \le \sup_{\xi>0}[V(\xi) - y\xi].$$

The inequality in the opposite direction follows from (12.8). □

12.2 PROPOSITION. *Assume that* (12.2), (12.3), (12.9), (8.25) *and*

$$U_2(\infty) = \infty \tag{12.11}$$

hold. Then, for any given $x \in (0,\infty)$, *there exists a number* $y(x) \in (0,\infty)$ *that achieves* $\inf_{y>0}[\tilde{V}(y) + xy]$; *furthermore, this number satisfies*

$$x = \mathscr{X}_{\lambda_{y(x)}}(y(x)). \tag{12.12}$$

PROOF. From (2.3), (8.6)–(8.8) and the supermartingale property of $Z_\nu(\cdot)$, we have

$$\gamma_\nu(t) \le e^M, \quad EH_\nu(t) \le e^M, \quad \forall\, 0 \le t \le T \quad \text{where } M \triangleq T(\eta + |\delta_0|). \tag{12.13}$$

This observation, coupled with the convexity and decreasing property of $\tilde{U}_1(t,\cdot)$ and $\tilde{U}_2(\cdot)$, Jensen's inequality, and (12.11) and (5.7), yields

$$\begin{aligned}\tilde{J}(y;\nu) &\ge \int_0^T \tilde{U}_1(t, yEH_\nu(t))\,dt + \tilde{U}_2(yEH_\nu(T))\\ &\ge \int_0^T \tilde{U}_1(t, ye^M)\,dt + \tilde{U}_2(ye^M) \xrightarrow[y\downarrow 0]{} \infty,\end{aligned}$$

whence $\tilde{V}(0+) = \infty$. It follows that the convex function $f_x(y) \triangleq \tilde{V}(y) + xy$, $y \in (0,\infty)$, satisfies $f_x(0+) = f_x(\infty) = \infty$, and thus attains its infimum at some $y(x) \in (0,\infty)$. Then Assumption (12.9) with $y \equiv y(x)$ gives

$$\begin{aligned}&\inf_{\xi>0}\left[\xi y(x)x + \tilde{J}(\xi y(x); \lambda_{y(x)})\right]\\ &\quad= \inf_{y>0}\left[yx + \tilde{J}(y;\lambda_{y(x)})\right] \ge \inf_{y>0}\left[xy + \tilde{V}(y)\right]\\ &\quad= f_x(y(x)) = xy(x) + \tilde{V}(y(x)).\end{aligned} \tag{12.14}$$

We shall see (Lemma 12.3) that the function

$$\begin{aligned}G_y(\xi) &\triangleq \tilde{J}(y\xi;\lambda_y)\\ &= E\left[\int_0^T \tilde{U}_1(t, y\xi H_{\lambda_y}(t))\,dt + \tilde{U}_2(y\xi H_{\lambda_y}(T))\right], \qquad 0 < \xi < \infty,\end{aligned} \tag{12.15}$$

is well defined and finite, and continuously differentiable at $\xi = 1$ with

$$G_y'(1) = -y\mathscr{X}_{\lambda_y}(y), \tag{12.16}$$

for any given $y \in (0,\infty)$. Now (12.14) implies that the function $D_x(\xi) \triangleq \xi xy(x) + G_{y(x)}(\xi)$, $0 < \xi < \infty$, achieves its infimum over $(0,\infty)$ at $\xi = 1$; thus its derivative must vanish there, that is, $(d/d\xi)D_x(\xi)|_{\xi=1} = xy(x) - y(x)\mathscr{X}_{\lambda_{y(x)}}(y(x)) = 0$ from (12.6), proving (12.12). □

12.3 LEMMA. *Under the assumptions of Proposition* 12.2, *the function* $G_y(\cdot)$ *of* (12.15) *is well defined and finite, continuously differentiable at* $\xi = 1$ *and satisfies* (12.16).

PROOF. Same as in KLSX (1991), Lemma 11.7, using (8.25) and (12.2). □

We now can put together the various results of this section and arrive at the following conclusion.

12.4 THEOREM. *Under the assumptions of Proposition* 12.2, *for any given* $x > 0$ *there exists an optimal pair* $(\hat{\pi}, \hat{c}) \in \mathscr{A}'(x)$ *for the constrained optimization problem* (6.5).

It remains to establish conditions, under which the assumption (12.9) will be valid. This is the objective of the next section.

12.5 REMARK. Under the conditions of Remark 6.4, the requirement (12.2) is satisfied. Indeed, the condition (6.7) leads to

$$0 \le \tilde{U}_1(t,y), \tilde{U}_2(y) \le \tilde{\kappa}(1 + y^{-\rho}), \qquad \forall\, (t,y) \in [0,T] \times (0,\infty)$$

for some $\tilde{\kappa} \in (0,\infty)$ and $\rho = \alpha/(1-\alpha)$. With $\nu \equiv 0$ and arbitrary $y \in (0,\infty)$, this inequality leads to $0 \le \tilde{J}(y;0) \le \tilde{\kappa}[\int_0^T \{1 + y^{-\rho} E(H_0(t))^{-\rho}\}\,dt + \{1 + y^{-\rho}E(H_0(T))^{-\rho}\}] < \infty$.

13. Existence.

We establish here the fundamental existence result of this paper.

13.1 THEOREM. *Assume that* (5.8), (8.25), (12.2), (12.3) *and* (12.11) *are satisfied. Then* (12.9) *holds*; *in particular, for any given* $x \in (0,\infty)$, *there exists an optimal pair* $(\hat{\pi}, \hat{c}) \in \mathscr{A}'(x)$ *for the problem of* (6.5).

Let us begin by introducing the following subset of the Hilbert space $\mathscr{H}$ (cf. beginning of Section 8):

$$\mathscr{G} \triangleq \{\nu \in \mathscr{H};\ \nu \text{ satisfies } (8.2)\}. \tag{13.1}$$

Quite obviously from (8.1), $\mathscr{D} \subset \mathscr{G} \subset \mathscr{H}$. For any given $y \in (0,\infty)$, we defined in (12.1) the functional $\tilde{J}_y(\nu) \equiv \tilde{J}(y;\nu)$ for all ν in the class $\mathscr{D}$ of (8.1). We now extend this definition to the entirety of $\mathscr{H}$, by setting

$$\begin{aligned} \zeta_\nu(t) \triangleq &\int_0^t (\theta(s) + \sigma^{-1}(s)\nu(s))^* \, dW(s) \\ &+ \tfrac{1}{2}\int_0^t \|\theta(s) + \sigma^{-1}(s)\nu(s)\|^2 \, ds \end{aligned} \tag{13.2}$$

and

$$\tilde{J}_y(\nu) \triangleq \begin{cases} E\int_0^T \tilde{U}_1\Big(t, y\exp\Big\{-\int_0^t (r(s) + \delta(\nu(s)))\,ds - \zeta_\nu(t)\Big\}\Big)\,dt \\ \qquad + E\tilde{U}_2\big(y\exp\{-\int_0^T (r(s)+\delta(\nu(s)))\,ds - \zeta_\nu(T)\}\big), \qquad \nu \in \mathscr{G}, \\ \infty, \quad \nu \in \mathscr{H} \setminus \mathscr{G}. \end{cases} \tag{13.3}$$

13.2 PROPOSITION. *Under the assumptions of Theorem* 13.1, *the functional* $\tilde{J}_y(\cdot)\colon \mathscr{H} \to \mathscr{R} \cup \{+\infty\}$ *of* (13.3) *is* (i) *convex*; (ii) *coercive*: $\lim_{[\![\nu]\!]\to\infty} \tilde{J}_y(\nu) = \infty$; *and* (iii) *lower-semicontinuous*: *for every* $\nu \in \mathscr{H}$ *and* $\{\nu_n\}_{n\in\mathbb{N}} \subseteq \mathscr{H}$ *with* $[\![\nu_n - \nu]\!] \to 0$ *as* $n \to \infty$, *we have*

$$\tilde{J}_y(\nu) \le \liminf_{n\to\infty} \tilde{J}_y(\nu_n). \tag{13.4}$$

PROOF. (i) This follows easily from the convexity and decrease of $\tilde{U}_1(t, e^{\cdot})$ and $\tilde{U}_2(e^{\cdot})$ [recall (5.8″)] and the convexity of $\delta(\cdot)$ and $\zeta_{\cdot}(t)$.

(ii) Similar reasoning gives, in the notation of (12.13) and with the help of Jensen's inequality,

$$\begin{aligned}
\tilde{J}_y(\nu) &\ge E\int_0^T \tilde{U}_1\big(t, y\exp\{M - \zeta_\nu(t)\}\big)\,dt + \tilde{U}_2\big(y\exp\{M - \zeta_\nu(T)\}\big)\\
&\ge \int_0^T \tilde{U}_1\big(t, y\exp\{M - E\zeta_\nu(t)\}\big)\,dt + \tilde{U}_2\big(y\exp\{M - E\zeta_\nu(T)\}\big)\\
&= \int_0^T \tilde{U}_1\left(t, ye^M \exp\left\{-\tfrac{1}{2}E\int_0^t \|\theta(s) + \sigma^{-1}(s)\nu(s)\|^2\,ds\right\}\right)dt\\
&\qquad + \tilde{U}_2\big(ye^M\exp\{-\tfrac{1}{2}[\![\theta + \sigma^{-1}\nu]\!]^2\}\big).
\end{aligned}$$

This last expression tends to infinity as $[\![\nu]\!] \to \infty$, from Remark 2.1.

(iii) Because $\nu_n \to \nu$ in $L^2([0,T] \times \Omega)$, we also have this convergence $\ell \otimes P$-a.e. along a (relabelled) subsequence. Therefore, for any given $t \in [0,T]$ we get, from Fatou's lemma, $\int_0^t \delta(\nu(s,\omega))\,ds \le \liminf_{n\to\infty} \int_0^t \delta(\nu_n(s,\omega))\,ds$, for P-a.e. $\omega \in \Omega$. On the other hand, we can show, as in Xu (1990), that $Z_{\nu_n}(t,\omega) \to_{n\uparrow\infty} Z_\nu(t,\omega)$, $\ell \otimes P$-a.e. $(t,\omega) \in [0,t] \times \Omega$, as well as $Z_{\nu_n}(T,\omega) \to_{n\uparrow\infty} Z_\nu(T,\omega)$, P-a.e. $\omega \in \Omega$, again along a (relabelled) subsequence. Applying Fatou's lemma again, along with the lower boundedness and decreasing property of $\tilde{U}_1(t,\cdot), \tilde{U}_2(\cdot)$, we arrive at (13.4). □

13.3 REMARK. For every $\mathscr{H} \setminus \mathscr{D}$, we have $\tilde{J}_y(\nu) = \infty$. Indeed, Jensen's inequality gives

$$\begin{aligned}
\tilde{J}_y(\nu) \ge \int_0^T \tilde{U}_1\bigg(t, y\exp\bigg\{&-E\int_0^t r(s)\,ds - E\int_0^T \delta(\nu(s))\,ds\\
&-\tfrac{1}{2}E\int_0^T \|\theta(s) + \sigma^{-1}(s)\nu(s)\|^2\,ds\bigg\}\bigg)dt\\
+ \tilde{U}_2\bigg(y\exp\bigg\{&-E\int_0^T r(s)\,ds - E\int_0^T \delta(\nu(s))\,ds\\
&-\tfrac{1}{2}E\int_0^T \|\theta(s) + \sigma^{-1}(s)\nu(s)\|^2\,ds\bigg\}\bigg).
\end{aligned}$$

Obviously from $\tilde{U}_2(0+) = U_2(\infty) = \infty$ [recall condition (12.11)], this lower bound is equal to $+\infty$ if $E\int_0^T \delta(\nu(s))\,ds = \infty$ [or, for that matter, if $E\int_0^T r(s)\,ds = \infty$, whence the imposition of condition (2.5)].

PROOF OF THEOREM 13.1. From Proposition OJ13.2 here, and Proposition 2.1.2 in Ekeland and Temam (1976), it follows that $\tilde{J}_y(\lambda_y) = \inf_{\nu \in \mathscr{H}} \tilde{J}_y(\nu) < \infty$ for some $\lambda_y \in \mathscr{G}$. From Remark 13.3, we know that actually $\lambda_y \in \mathscr{D}$. It remains to show $\lambda_y \in \mathscr{D}'$; but from Remark 8.5, it suffices to prove $\mathscr{X}_{\lambda_y}(y) < \infty$. This is done exactly as in KLSX (1991) (proof of Theorem 12.3 there). □

It can be checked easily that utility functions of the form

$$U_1(t,c) = e^{-\beta t}\frac{1}{\alpha}c^{\alpha}, \quad U_2(c) = e^{-\beta T}\frac{1}{\alpha}c^{\alpha}, \quad 0 \le t \le T,\, 0 < c < \infty, \tag{13.5}$$

with $\beta \in [0,\infty)$, $\alpha \in (0,1)$, satisfy all the conditions of Theorem 13.1.

14. Examples. We consider in this section a few examples of closed, convex sets K that are of relevance in financial economics as expressing reasonable constraints on portfolio choice, and for which the support function $\delta(\cdot)$ of (4.1) can be calculated fairly explicitly. In all these examples, the conditions (4.3) and (4.4) are satisfied rather trivially.

14.1 EXAMPLE. Let $K = \{\pi \in \mathscr{R}^d;\ \pi_i \ge 0,\ \forall\ i = 1,\dots,d\}$. Then

$$\delta(x) = \begin{cases} 0, & \text{if } x \in K, \\ \infty, & \text{if } x \notin K, \end{cases} \quad \text{and} \quad \tilde{K} = K.$$

This is the case considered by Xu (1990), in which *short selling of stocks is prohibited*.

14.2 EXAMPLE. Let $K = \{\pi \in \mathscr{R}^d;\ \pi_i = 0,\ \forall\ i = m+1,\dots,d\}$ for some fixed $m \in \{1,\dots,d-1\}$. Then

$$\delta(x) = \begin{cases} 0, & x_1 = \cdots = x_m = 0, \\ \infty, & \text{otherwise} \end{cases}$$

and

$$\tilde{K} = \left\{x \in \mathscr{R}^d;\ x_i = 0, \forall\ i = 1,\dots,m\right\}.$$

This is the *incomplete market* case of KLSX (1991), where investment is restricted to the first m stocks only (or equivalently, where the number of stocks is strictly smaller than the dimension of the driving Brownian motion W).

14.3 EXAMPLE. With m as in Example 14.2, consider the set

$$K = \left\{\pi \in \mathscr{R}^d;\ \pi_i \ge 0, \forall\ i = 1,\dots,m \text{ and } \pi_j = 0, \forall\ j = m+1,\dots,d\right\}.$$

Then

$$\delta(x) = \begin{cases} 0, & \text{if } (x_1, \dots, x_m) \in [0, \infty)^m, \\ \infty, & \text{otherwise} \end{cases}$$

and

$$\tilde{K} = \{x \in \mathscr{R}^d;\, x_i \geq 0,\, \forall\, i = 1, \dots, m\}.$$

This is the case discussed in He and Pearson (1991). It is a combination of Examples 14.1 and 14.2, namely, an *incomplete market without short-selling of stocks*.

14.4 Example. More generally, let K be a closed convex *cone* in $\mathscr{R}^d$. Then

$$\delta(x) = \begin{cases} 0, & x \in \tilde{K}, \\ \infty, & x \notin \tilde{K}, \end{cases}$$

where $\tilde{K} \equiv \{x \in \mathscr{R}^d;\, \pi^* x \geq 0,\, \forall\, \pi \in K\}$ is the polar cone of $-K$. This is the case treated by Shreve (1991), in the case of constant coefficients (using analytical methods).

14.5 Example. In the (trivial) unconstrained case $K = \mathscr{R}^d$, we have

$$\delta(x) = \begin{cases} 0, & x = 0, \\ \infty, & \text{otherwise}, \end{cases} \qquad \tilde{K} = \{0\}.$$

14.6 Remark. In all the preceding examples, $\delta(\cdot) \equiv 0$ on $\tilde{K}$. In particular then, in the context of *logarithmic utility functions* (Section 11), the problem of determining the process $\lambda \in \mathscr{D}$ of conditions (B)–(E) reduces to that of minimizing (pointwise) a simple quadratic form, over $\tilde{K}$:

$$\lambda(t) = \arg\min_{\nu \in \tilde{K}} \left\| \theta(t) + \sigma^{-1}(t)\nu \right\|^2. \tag{14.1}$$

In the most "extreme" case, that is, that of Example 14.5, we get $\lambda(t) \equiv 0$ from (14.1), and recover from (11.5) the unconstrained optimal portfolio [as in Karatzas (1989)]:

$$\hat{\pi}_0(t) \triangleq (\sigma^*(t))^{-1}\theta(t) = (\sigma(t)\sigma^*(t))^{-1}[b(t) - r(t)\mathbf{1}]. \tag{14.2}$$

On the other hand, let us consider the case of Example 14.2, and take for simplicity

$$\sigma(t) = \begin{bmatrix} \Sigma(t) \\ \rho(t) \end{bmatrix},$$

where $\Sigma(t, \omega)$ is an $(m \times d)$ matrix of full (row) rank and $\rho(t, \omega)$ is an $(n \times d)$ matrix with orthonormal rows that span the kernel of $\Sigma(t, \omega)$, for every (t, ω). In particular, $\rho(t)\rho^*(t) = I_n$ and $\Sigma(t)\rho^*(t) = 0$. Here, $n = d - m$. Then with

$B(t) \triangleq (b_1(t), \dots, b_m(t))^*$, $a(t) = (b_{m+1}(t), \dots, b_d(t))^*$ and $\Theta(t) \triangleq \Sigma^*(t)(\Sigma(t)\Sigma^*(t))^{-1}[B(t) - r(t)\mathbf{1}_m]$, we have $\theta(t) = \Theta(t) + \rho^*(t)[a(t) - r(t)\mathbf{1}_n]$, and for any $\nu \in \tilde{K}$ [necessarily of the form $\nu = \begin{pmatrix} \mathbf{0}_m \\ N \end{pmatrix}$ for some $N \in \mathscr{R}^m$],

$$\|\theta(t) + \sigma^{-1}(t)\nu\|^2 = \|\Theta(t) + \rho^*(t)(a(t) - r(t)\mathbf{1}_n + N)\|^2$$
$$= \|\Theta(t)\|^2 + \|\rho^*(t)(a(t) - r(t)\mathbf{1}_n + N)\|^2$$

because the two random vectors $\Theta(t), \rho^*(t)(a(t) - r(t)\mathbf{1}_n + N)$ are orthogonal. Thus, the minimization of (14.1) is achieved by the random vector $\lambda(t) = \begin{bmatrix} \mathbf{0}_m \\ \Lambda(t) \end{bmatrix}$, where $\Lambda(t) = r(t)\mathbf{1}_n - a(t)$. Back into (11.5), this leads to the optimal portfolio

$$\hat{\pi}(t) = \begin{bmatrix} (\Sigma(t)\Sigma^*(t))^{-1}[B(t) - r(t)\mathbf{1}_m] \\ \mathbf{0}_n \end{bmatrix}$$

of KLSX (1991), for incomplete markets with logarithmic utility.

14.7 EXAMPLE. *Rectangular constraints.* Consider the case $K = \times_{i=1}^d K_i$ where $K_i = [\alpha_i, \beta_i]$ for some fixed numbers $-\infty \le \alpha_i \le 0 \le \beta_i \le \infty$, with the understanding that the interval K_i is open to the right (left) if $b_i = \infty$ (respectively, if $\alpha_i = -\infty$). Then

$$\delta(x) = \sum_{i=1}^d \beta_i x_i^- - \sum_{i=1}^d \alpha_i x_i^+$$

and $\tilde{K} = \mathscr{R}^d$ if all the α_i, β_i are real. In general,

$$\tilde{K} = \{x \in \mathscr{R}^d;\, x_i \ge 0, \forall\, i \in \mathscr{I}_+ \text{ and } x_j \le 0, \forall\, j \in \mathscr{I}_-\},$$

where

$$\mathscr{I}_+ \triangleq \{i = 1, \dots, d | \beta_i = \infty\},$$
$$\mathscr{I}_- \triangleq \{j = 1, \dots, d | \alpha_j = -\infty\}.$$

14.8 EXAMPLE. *Rectangular constraints and logarithmic utility.* Consider the setting of Section 11, with the set K as in Example 14.7 and $\mathscr{I}_+ = \mathscr{I}_- = \varnothing$. With $d = 1$, $K = [\alpha, \beta]$, for fixed $-\infty < \alpha \le 0 \le \beta < \infty$, and $\delta(x) = \beta x^- - \alpha x^+$, the process of (11.4) becomes

$$\lambda(t) = \begin{cases} \sigma(t)[\sigma(t)\beta - \theta(t)], & \text{if } \sigma(t)\beta < \theta(t), \\ \sigma(t)[\sigma(t)\alpha - \theta(t)], & \text{if } \sigma(t)\alpha > \theta(t), \\ 0, & \text{otherwise.} \end{cases}$$

Consequently, the optimal portfolio $\hat{\pi}(\cdot)$ of (11.5) is given as

$$\hat{\pi}(t) = \begin{cases} \beta, & \text{if } \sigma^{-1}(t)\theta(t) > \beta, \\ \alpha, & \text{if } \sigma^{-1}(t)\theta(t) < \alpha, \\ \sigma^{-1}(t)\theta(t), & \text{otherwise.} \end{cases}$$

In other words, $\hat{\pi}(t)$ agrees with the optimal unconstrained portfolio $\pi_0(t)$ of (14.2), as long as this latter is in the interval $[\alpha, \beta]$; when this is not the case, $\hat{\pi}(t)$ selects the boundary point closest to $\pi_0(t)$.

The situation is more complicated in several dimensions. Let us study the preceding problem with $d = 2$, $\alpha \equiv (0,0)^*$, $\beta \equiv (1,1)^*$, $\theta = (1,2)^*$ and

$$\sigma = \begin{bmatrix} 1 & -10 \\ -1 & 1 \end{bmatrix}.$$

In the unconstrained case, the optimal portfolio is given by $(\sigma^*)^{-1}\theta = (-1/3, -4/3)^*$. It does not suffice now to take the coordinates of the optimal constrained portfolio $\hat{\pi}$ to be the closest ones to the unconstrained optimal portfolio, such that $\hat{\pi}$ takes values in K. That would give the portfolio $(0,0)^*$. However, the minimization of

$$f(x) = \tfrac{1}{2}\|\theta + \sigma^{-1}x\|^2 - \alpha^* x^+ + \beta^* x^-, \qquad x \in \mathscr{R}^2,$$

leads to the optimal dual process $\lambda = (13.5, 0)^*$, and the optimal portfolio $\hat{\pi}$ is given by

$$\hat{\pi} = (\sigma^*)^{-1}(\theta + \sigma^{-1}\lambda) = (0, 1/2)^*;$$

that is, do not invest in the first stock and invest half of the wealth in the second stock.

14.9 Example. *Constraints on borrowing*. From the point of view of applications, an interesting example is the one in which the total proportion $\sum_{i=1}^{d}\pi_i(t)$ of wealth invested in stocks is bounded from above by some real constant $a > 0$. For example, if we take $a = 1$, we exclude borrowing; with $a \in (1,2)$, we allow borrowing up to a fraction $a - 1$ of wealth. If we take $a = 1/2$, we have to invest at least half of the wealth in the riskless bond.

To illustrate what happens in this situation, let again $U_2(x) = \log x$, $U_1(t,x) \equiv 0$ and, for the sake of simplicity, $d = 2$, $\sigma =$ unit matrix and the constraints on the portfolio be given by

$$K = \{x \in \mathscr{R}^2;\, x_1 \geq 0,\, x_2 \geq 0,\, x_1 + x_2 \leq a\}$$

for some $a \in (0,1]$. (Obviously, we also exclude short-selling with this choice of K.)

We have here $\delta(x) \equiv a \max\{x_1^-, x_2^-\}$ and thus $\tilde{K} = \mathscr{R}^2$. By some elementary calculus and/or by inspection, and omitting the dependence on t, we can see that the optimal dual process λ that minimizes $\frac{1}{2}\|\theta + \nu\|^2 + \delta(\nu)$, and the optimal portfolio $\pi = \theta + \lambda$, are given, respectively, by:

$$\lambda = -\theta, \qquad \pi = (0,0)^* \quad \text{if } \theta_1, \theta_2 \leq 0$$

(do not invest in stocks if the bond rate is larger than the stocks' appreciation rates),

$$\begin{aligned}
&\lambda = (0, -\theta_2)^*, \quad &&\pi = (\theta_1, 0)^* \quad &&\text{if } \theta_1 \geq 0,\, \theta_2 \leq 0,\, a \geq \theta_1,\\
&\lambda = (a - \theta_1, -\theta_2)^*, \quad &&\pi = (a, 0)^* \quad &&\text{if } \theta_1 \geq 0,\, \theta_2 \leq 0,\, a < \theta_1,\\
&\lambda = (-\theta_1, 0)^*, \quad &&\pi = (0, \theta_2)^* \quad &&\text{if } \theta_1 \leq 0,\, \theta_2 \geq 0,\, a \geq \theta_2,\\
&\lambda = (-\theta_1, a - \theta_2)^*, \quad &&\pi = (0, a)^* \quad &&\text{if } \theta_1 \leq 0,\, \theta_2 \geq 0,\, a < \theta_2
\end{aligned}$$

(do not invest in the stock whose rate is less than the bond rate; invest $X \min\{a, \theta_i\}$ in the ith stock whose rate is larger than the bond rate),

$$\lambda = (0,0)^*, \qquad \pi = \theta \quad \text{if } \theta_1, \theta_2 \geq 0,\, \theta_1 + \theta_2 \leq a$$

(invest $\theta_i X$ in the respective stocks—as in the unconstrained case—whenever the optimal portfolio of the unconstrained case happens to take values in K),

$$\begin{aligned}
&\lambda = (a - \theta_1, -\theta_2)^*, \quad &&\pi = (a, 0)^* \quad &&\text{if } \theta_1, \theta_2 \geq 0,\, a \leq \theta_1 - \theta_2,\\
&\lambda = (-\theta_1, a - \theta_2)^*, \quad &&\pi = (0, a)^* \quad &&\text{if } \theta_1, \theta_2 \geq 0,\, a \leq \theta_2 - \theta_1
\end{aligned}$$

(with both θ_1, $\theta_2 \geq 0$ and $\theta_1 + \theta_2 > a$ do not invest in the stock whose rate is smaller; invest aX in the other one if the absolute value of the difference of the stock rates is larger than a),

$$\lambda_1 = \lambda_2 = \frac{a - \theta_1 - \theta_2}{2}, \qquad \pi_1 = \frac{a + \theta_1 - \theta_2}{2}, \qquad \pi_2 = \frac{a + \theta_2 - \theta_1}{2}$$

if θ_1, $\theta_2 \geq 0$, $\theta_1 + \theta_2 > a > |\theta_1 - \theta_2|$ [if none of the previous conditions is satisfied, invest the amount $(a/2)X$ in the stocks, corrected by the difference of their rates].

NOTE. Some regularity results on the value function of the problem with $d = 1$, $K = [0,1]$, constant coefficients and $U_1 \equiv 0$, were obtained in the doctoral dissertation of Zariphopoulou (1989) using mostly analytical techniques.

14.10 REMARK. In the setting of Example 4.8, *even with* $0 \notin [\alpha_i, \beta_i]$ *for some* $i \in \{1, \dots, d\}$, the function $f(x; t, \omega) \triangleq 2\delta(x) + \|\theta(t,\omega) + \sigma^{-1}(t,\omega)x\|^2$ appearing in (11.4) is bounded from below and satisfies $\lim_{|x| \to \infty} f(x; t, \omega) = \infty$, for every (t, ω). Thus an optimal dual process exists and is given by (11.4), even if (4.4) does not hold.

15. Deterministic coefficients and feedback formulae. Let us now consider briefly the case where the coefficients $r(\cdot), b(\cdot), \sigma(\cdot)$ of the market model are deterministic functions on $[0, T]$, which we shall take for simplicity to be bounded and continuous. Then there is a formal Hamilton–Jacobi–Bellman (HJB) equation associated with the dual optimization problem of (12.1), namely,

$$\begin{aligned}
(15.1)\qquad & Q_t + \inf_{x \in \tilde{K}} \left[\tfrac{1}{2} y^2 Q_{yy} \|\theta(t) + \sigma^{-1}(t) x\|^2 - y Q_y \delta(x) \right] \\
& \quad - y Q_y r(t) + \tilde{U}_1(t, y) = 0, \quad \text{in } [0, T) \times (0, \infty),
\end{aligned}$$

$$(15.2)\qquad Q(T, y) = \tilde{U}_2(y), \qquad y \in (0, \infty).$$

If there exists a classical solution $Q \in C^{1,2}([0,T) \times (0,\infty))$ of this equation, which satisfies appropriate growth conditions, then standard verification theorems in stochastic control [e.g., Fleming and Rishel (1975)] lead to the representation

$$\tilde{V}(y) = Q(0,y), \qquad 0 < y < \infty, \tag{15.3}$$

for the dual value function of (12.1).

15.1 Example. Suppose that $\delta \equiv 0$ on $\tilde{K}$ (as in Examples 14.1–14.5). Then

$$\lambda(t) = \arg\min_{x \in \tilde{K}} \|\theta(t) + \sigma^{-1}(t)x\|^2 \tag{15.4}$$

is *deterministic*, the same for all $y \in (0,\infty)$, and (15.1) becomes

$$Q_t + \tfrac{1}{2}\|\theta_\lambda(t)\|^2 y^2 Q_{yy} - r(t)yQ_y + \tilde{U}_1(t,y) = 0, \quad \text{in } [0,T) \times (0,\infty). \tag{15.5}$$

Standard theory [e.g., Friedman (1964)] guarantees then the existence and uniqueness of a classical solution for this equation.

In the case of constant coefficients, this solution can even be computed explicitly. Indeed, let us take $U_1(t,x) = e^{-\beta t}u_1(x)$ and $U_2(x) = e^{-\beta T}u_2(x)$, where $\beta > 0$ and u_1, u_2 are utility functions of class C^3, such that

$$\lim_{x \downarrow 0} \frac{(u_i'(x))^2}{u_i''(x)} \text{ exists}, \quad \lim_{x \to \infty} \frac{(u_i'(x))^\gamma}{u_i''(x)} = 0 \text{ for some } \gamma > 2,\ u_i(0) = 0,\ i = 1,2.$$

These conditions are satisfied for utility functions of the form (13.5). Let $\kappa = \frac{1}{2}\|\theta + \sigma^{-1}\lambda\|^2$, denote by ρ_+ (ρ_-) the positive (respectively, negative) root of $\kappa\rho^2 - (r - \beta - \kappa)\rho - r = 0$ and let $J_i \triangleq (u_i')^{-1}$,

$$J_+(y) \triangleq \int_0^{J_1(y)} (u_1'(\eta))^{-\rho_+}\, d\eta, \qquad J_-(y) \triangleq \int_0^{J_1(y)} (u_1'(\eta))^{-\rho_-}\, d\eta,$$

$$h(y) \triangleq \frac{u_1(J_1(y))}{\beta} - \frac{yJ_1(y)}{r} + \frac{1}{\kappa(\rho_+ - \rho_-)}\left[\frac{y^{1+\rho_+}}{\rho_+(1+\rho_+)}J_+(y) - \frac{y^{1+\rho_-}}{\rho_-(1+\rho_-)}J_-(y)\right],$$

$$\Phi(z) \triangleq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-u^2/2}\, du,$$

$$\mu(t,y,\xi) \triangleq \frac{1}{\sqrt{2\kappa t}}\left[\log\left(\frac{y}{\xi}\right) + (\beta - r \pm \kappa)t\right], \qquad y > 0,\ \xi > 0,$$

$$v(t,y,\xi) \triangleq \begin{cases} \xi e^{-\beta(T-t)}\Phi(-\mu_-(T-t,y,\xi)) \\ \quad -ye^{-r(T-t)}\Phi(-\mu_+(T-t,y,\xi)), & 0 \le t < T, \\ (\xi - y)^+, & t = T. \end{cases}$$

Then the solution Q of the Cauchy problem (15.5), (15.2) is given by

$$(15.6)\qquad Q(t,y) = e^{-\beta t}\left[h(y) + \int_0^\infty (\tilde{u}_2(\xi) - h(\xi))'' v(t,y,\xi)\, d\xi\right];$$

refer to KLS (1987), Section 7, for details.

15.2 Example. Consider the case $U_1(t,x) = U_2(x) = x^\alpha/\alpha$, $(t,x) \in [0,T] \times (0,\infty)$ for some $\alpha \in (0,1)$. Then $\tilde{U}_1(t,y) = \tilde{U}_2(y) = (1/\rho)y^{-\rho}$, $0 < y < \infty$, with $\rho \triangleq \alpha/(1-\alpha)$, and the solution of the Cauchy problem (15.1), (15.2) is of the form

$$Q(t,y) = \frac{1}{\rho} y^{-\rho} v(t), \qquad (t,y) \in [0,T] \times (0,\infty).$$

Here $v(\cdot)$ is the solution of $\dot{v}(t) + h(t)v(t) + 1 = 0$, $v(T) = 1$, with

$$h(t) \triangleq \rho \inf_{x\in\tilde{K}}\left[\frac{1+\rho}{2}\|\theta(t) + \sigma^{-1}(t)x\|^2 + \delta(x)\right] + r(t)\rho,$$

namely, $v(t) = \exp(\int_t^T h(s)\,ds)(1 + \int_t^T \exp(-\int_\theta^T h(u)\,du)\,d\theta)$. Again, the process $\lambda(\cdot)$ is *deterministic*, namely,

$$(15.7)\qquad \lambda(t) = \arg\min_{x\in\tilde{K}}\left[\|\theta(t) + \sigma^{-1}(t)x\|^2 + 2(1-\alpha)\delta(x)\right],$$

and is the same for all $y \in (0,\infty)$.

We conclude with a computation of the optimal portfolio and consumption processes in feedback form (in terms of current wealth), when the processes $r(\cdot)$, $b(\cdot)$, $\sigma(\cdot)$ and $\lambda(\cdot)$ are deterministic. In such a setting, define the function $\mathscr{X}(\cdot,\cdot)$ by

$$(15.8)\qquad \mathscr{X}(t,y) \triangleq \frac{1}{y}E\left[\int_t^T Y_s^{(t,y)} I_1(s, Y_s^{(t,y)})\,ds + Y_T^{(t,y)} I_2(Y_T^{(t,y)})\right],$$

where

$$dY_s^{(t,y)} = -Y_s^{(t,y)}\left[(r(s) + \delta(\lambda(s)))\,ds + (\theta(s) + \sigma^{-1}(s)\lambda(s))^*\,dW(s)\right], \qquad t \le s \le T,$$

$$Y_t^{(t,y)} = y.$$

Obviously $Y_s^{(0,y)} = yH_\lambda(s)$. For every $t \in [0,T]$, the function $\mathscr{X}(t,\cdot)$ is continuous and strictly decreasing on $(0,\infty)$, with $\mathscr{X}(t,0+) = \infty$ and $\mathscr{X}(t,\infty) = 0$. Denote its inverse by $\mathscr{Y}(t,\cdot)$. Assume also that $\mathscr{X}(t,\cdot)$ is continuously differentiable and that

$$y^2\frac{\partial}{\partial y}\mathscr{X}(t,y) = E\left[\int_t^T (Y_s^{(t,y)})^2 I_1'(s, Y_s^{(t,y)})\,ds + (Y_T^{(t,y)})^2 I_2'(Y_T^{(t,y)})\right],$$

where we denote by $I_1'(t,\cdot)$ the derivative of $I_1(t,\cdot)$. A sufficient condition for this is that $U_1 \in C^{0,2}$, $U_2 \in C^2$ and that U_2'', $U_1''(t,\cdot)$ be nondecreasing functions on $(0,\infty)$; compare with Proposition 4.4 in KLS (1987).

15.3 THEOREM. *Suppose that $r(\cdot)$, $b(\cdot)$ and $\sigma(\cdot)$ are deterministic, that there exists a deterministic $\lambda(\cdot) \in \mathscr{D}$, which achieves the infimum in* (12.1) *for all $y \in (0,\infty)$, and that*

$$I_1(t,y) + I_2(y) + |I_1'(t,y)| + |I_2'(y)| \le K(y^\alpha + y^{-\beta}), \qquad 0 < y < \infty,$$

holds for some real $\alpha > 0$, $\beta > 0$ and $K > 0$. With the notation and assumptions of the previous paragraph, the optimal portfolio/consumption process pair $(\pi_\lambda, c_\lambda) \in \mathscr{A}'(x)$ for the problem of (6.5) *is given by*

$$c_\lambda(t) = I_1(t, \mathscr{Y}(t, X_\lambda(t))), \tag{15.9}$$

$$\pi_\lambda(t) = -(\sigma(t)\sigma^*(t))^{-1}[b(t) - r(t)\mathbf{1} + \lambda(t)]\frac{\mathscr{Y}(t, X_\lambda(t))}{X_\lambda(t)\mathscr{Y}_x(t, X_\lambda(t))}, \tag{15.10}$$

in feedback form on the (optimal) current level of wealth $X_\lambda(t)$.

The proof follows along the lines of Ocone and Karatzas (1991) and KLS (1987) and is thus omitted. Notice that the assumption of deterministic $\lambda(\cdot) \equiv \lambda_y(\cdot)$ is satisfied for both Examples 15.1 and 15.2; in the case of the latter, the formulae (15.9) and (15.10) become

$$c_\lambda(t) = \frac{1}{2v(0)} X_\lambda(t),$$

$$\pi_\lambda(t) = \frac{1}{1-\alpha}(\sigma(t)\sigma^*(t))^{-1}[b(t) - r(t)\mathbf{1} + \lambda(t)].$$

15.2 EXAMPLE (Continued). For $1 \le i \ne j \le d$, this last formula gives

$$\frac{\pi_{\lambda_\alpha}^{(i)}(t)}{\pi_{\lambda_\alpha}^{(j)}(t)} = \frac{\left((\sigma(t)\sigma^*(t))^{-1}[b(t) - r(t)\mathbf{1} + \lambda_\alpha(t)]\right)^{(i)}}{\left((\sigma(t)\sigma^*(t))^{-1}[b(t) - r(t)\mathbf{1} + \lambda_\alpha(t)]\right)^{(j)}}. \tag{15.11}$$

Here $\lambda_\alpha(\cdot)$ is the function of (15.7), which in general [i.e., unless $\delta(\cdot) \equiv 0$ on $\tilde{K}$] *will depend on $\alpha \in (0,1)$, as will then the ratio of* (15.11).

In other words, for a general convex set K, the ratio $(\hat{\pi}^{(i)}(t))/(\hat{\pi}^{(j)}(t))$ of the optimal proportions in two different stocks will depend on the utility function *even in the case of constant coefficients*. This is in contrast to the unconstrained case [cf. Remark 4.5 in Ocone and Karatzas (1991)] or to the case where K is a cone.

16. Extensions and ramifications.

1. The theory that has been developed thus far goes through without change if one formally sets $U_1 \equiv 0$ and $c \equiv 0$ throughout, and considers the problem of *maximizing expected utility from terminal wealth only*.

2. Certain changes are required, however, in the case where one formally takes $U_2 \equiv 0$ and considers the problem of *maximizing expected utility from consumption* $E\int_0^T U_1(t, c(t))\,dt$. Then one only requires

$$\int_0^t c(s)\,ds + \int_0^t \|\pi(s)\|^2\,ds < \infty \quad \text{a.s.},$$

for every $t \in [0, T)$ in Definition 3.1, changes the inner product and norm of the Hilbert space $\mathscr{H}$ to

$$\langle \mu, \nu \rangle = E\int_0^T (T-s)\mu^*(s)\nu(s)\,ds,$$

$$[\![\nu]\!] = \sqrt{E\int_0^T (T-t)\|\nu(t)\|^2\,dt},$$

respectively, changes (8.1) to

$$\mathscr{D} = \left\{\nu \in \mathscr{H};\ E\int_0^T (T-t)\delta(\nu(t))\,dt < \infty\right\}$$

and defines the processes $Z_\nu(\cdot)$, $H_\nu(\cdot)$, $W_\nu(\cdot)$, $c_\nu(\cdot)$ and $X_\nu(\cdot)$ of (8.7)–(8.9), (8.17) and (8.19) on $[0, T)$ (note that the event $\{\lim_{t\uparrow T} Z_\nu(t) = 0\} = \{\int_0^T \|\theta(t) + \sigma^{-1}(t)\nu(t)\|^2\,dt = \infty\}$ may now have positive probability). With such modifications, as well as obvious changes in notation (such as ignoring statements and terms pertaining to terminal wealth), Theorems 9.1 and 10.1 continue to hold, as do the results of Section 11 in the logarithmic case. The duality and existence theories also go through, provided $\tilde{U}_1$ is of the form $\tilde{U}_1(t, y) = \phi(t)\tilde{U}(y)$, $0 \le t \le T$, $y \in (0, \infty)$, where $\phi\colon [0, T] \to [a, b]$ is a continuous function, $0 < a < b < \infty$, and $\tilde{U}$ is the function of (5.2) for a utility function U that satisfies (5.8), (5.9) and $U(0+) > -\infty$. As they are not hard to check, these claims are left to the reader's care.

3. Let $\mathscr{K} = \{K_t(\omega); (t, \omega) \in [0, T] \times \Omega\}$ be a family of closed, convex, nonempty subsets of $\mathscr{R}^d$, such that the corresponding family of support functions $\{\delta(\cdot|K_t(\omega)); (t, \omega) \in [0, T] \times \Omega\}$ is uniformly bounded from below [by some real constant δ_0, as in (4.4)]. Suppose also that

$$(16.1)\qquad \left\{\begin{array}{l}\text{for every } \{\mathscr{F}_t\}\text{-progressively measurable process } \nu(t),\\ 0 \le t \le T, \text{ such that } \nu_t(\omega) \in \tilde{K}_t(\omega) \text{ for } \ell \otimes P\text{-a.e. } (t, \omega), \text{ the}\\ \text{process } \delta(\nu_t|K_t),\ 0 \le t \le T, \text{ is } \{\mathscr{F}_t\}\text{-progressively measurable}\end{array}\right\}.$$

Now we consider the constrained optimization problem of Definition 6.3 with admissibility condition $\pi(t, \omega) \in K_t(\omega)$, $\ell \otimes P$-a.e. on $[0, T] \times \Omega$ in (6.4) and the dual problem of (12.1) with $\mathscr{D} = \{\nu \in \mathscr{H};\ E\int_0^T \delta(\nu_t|K_t)\,dt < \infty\}$. The entire theory goes through, then, under the assumption (16.2), which strengthens the one of (4.3):

$$(16.2)\qquad \left\{\begin{array}{l}\text{There exists a sequence } \{\nu_n\}_{n\in\mathbb{N}} \subset \mathscr{D} \text{ such that, for } \ell \otimes P\text{-a.e.}\\ (t, \omega) \in [0, T] \times \Omega \text{ and for every (nonrandom) vector } \nu \in\\ \tilde{K}_t(\omega), \text{ there exists a subsequence } \{\nu_{n_k}\}_{k\in\mathbb{N}} \text{ with}\\ \lim_{k\to\infty} \nu_{n_k}(t, \omega) = \nu \text{ and } \lim_{k\to\infty} \delta(\nu_{n_k}(t, \omega)) = \delta(\nu)\end{array}\right\}.$$

A sufficient condition for assumption (16.2) is that the convex cone $\tilde{K}_t(\omega) \equiv \tilde{K}$ be the same for all (t, ω), that $\delta(\cdot|K_t(\omega))$ be continuous on $\tilde{K}$ for all (t, ω) and that all constant $\tilde{K}$-valued processes $\nu(t, \omega) \equiv \nu$ belong to $\mathscr{D}$ [i.e., $E\int_0^T \delta(\nu|K_t)\,dt < \infty,\ \forall\ \nu \in \tilde{K}$]. In such a case, we may take $\{\nu_n\}_{n \in \mathbb{N}}$ to be a sequence of constant vectors, forming a dense subset of $\tilde{K}$.

These conditions, as well as assumption (16.1), are satisfied in the case of the random parallelepiped $K_t(\omega) = \times_{i=1}^d [\alpha_i(t, \omega), \beta_i(t, \omega)]$ where (by analogy with Example 14.7) the $\{\mathscr{F}_t\}$-progressively measurable processes α_i, β_i are square-integrable and take values in $(-\infty, 0]$ and $[0, \infty)$, respectively. In this case, $\tilde{K}_t(\omega) \equiv \mathscr{R}^d$.

APPENDIX A

The purpose of this section is to establish the implication (A) $\Rightarrow$ (B) in the proof of Theorem 10.1, under the assumptions (5.8), (8.25) and (12.2) on the utility functions $U_1(t, \cdot\,)$ and $U_2(\cdot)$. Our proof uses these assumptions heavily, but we do not have counterexamples suggesting their necessity for the validity of the implication.

Let us denote by $(\hat{\pi}, \hat{c})$ the optimal portfolio/consumption process pair in $\mathscr{A}'(x)$, whose existence is being assumed in (A). According to (3.5), the corresponding wealth process $\hat{X}(\cdot)$ satisfies

$$\text{(A.1)} \qquad \begin{aligned} &H_0(t)\hat{X}(t) + \int_0^t H_0(s)\hat{c}(s)\,ds \\ &\quad = x - \int_0^t H_0(s)\hat{X}(s)\mu^*(s)\,dW(s), \qquad 0 \le t \le T, \end{aligned}$$

where

$$\text{(A.2)} \qquad \mu \triangleq \theta - \sigma^*\hat{\pi}.$$

A.1 Lemma. *We have* $P[\hat{X}(t) > 0, \forall\ 0 \le t \le T] = 1$.

Proof. From (A.1), $H_0\hat{X}$ is a nonnegative, continuous supermartingale. To show that it is a.s. positive on $[0, T]$, it suffices to prove that the event $B \triangleq \{\hat{X}(T) = 0\}$ has probability 0 [cf. Karatzas and Shreve (1988), Problem 1.3.29].

Let us notice that the solution of the linear stochastic equation (A.1) is given by

$$\text{(A.3)} \qquad H_0(t)\hat{X}(t) = Z_\nu(t)\left[x - \int_0^t \frac{H_0(s)}{Z_\nu(s)}\hat{c}(s)\,ds\right], \qquad 0 \le t \le T,$$

with $\nu = \sigma\sigma^*\pi$, and define $\hat{c}_\varepsilon(t) \triangleq (1-\varepsilon)\hat{c}(t)$, $\hat{X}_\varepsilon(t) \triangleq X^{x, \hat{\pi}, \hat{c}_\varepsilon}(t)$, for $0 < \varepsilon < 1$. We have

$$\text{(A.4)} \qquad \hat{X}_\varepsilon(t) = \hat{X}(t) + \varepsilon\frac{Z_\nu(t)}{H_0(t)}\int_0^t \frac{H_0(s)}{Z_\nu(s)}\hat{c}(s)\,ds \ge \hat{X}(t) \ge 0, \qquad 0 \le t \le T.$$

In particular $(\hat{\pi}, \hat{c}_\varepsilon) \in \mathscr{A}'(x)$ and on the event $B = \{\hat{X}(T) = 0\}$ we have

$$\int_0^T \frac{H_0(s)}{Z_\nu(s)} \hat{c}(s)\, ds = x, \qquad \hat{X}_\varepsilon(T) = \varepsilon x \frac{Z_\nu(T)}{H_0(T)}. \tag{A.5}$$

The optimality of $(\hat{\pi}, \hat{c})$ gives

$$\begin{aligned} 0 &\geq E\bigg[\int_0^T \{U_1(t, (1-\varepsilon)\hat{c}(t)) - U_1(t, \hat{c}(t))\}\, dt \\ &\qquad + \big\{U_2\big(\hat{X}_\varepsilon(T)\big) - U_2\big(\hat{X}(T)\big)\big\}\bigg] \\ &= -\varepsilon E\int_0^T \hat{c}(t) U_1'(t, \eta_\varepsilon(t))\, dt \\ &\qquad + E\big[\big(\hat{X}_\varepsilon(T) - \hat{X}(T)\big) U_2'(\rho_\varepsilon)\big], \end{aligned} \tag{A.6}$$

where $(1-\varepsilon)\hat{c}(t) \leq \eta_\varepsilon(t) \leq \hat{c}(t)$ and $\hat{X}(T) \leq \rho_\varepsilon \leq \hat{X}_\varepsilon(T)$. In particular, using the property (5.8) on $U_1(t, \cdot\,)$, we obtain from (A.6), (A.5):

$$\begin{aligned} \frac{1}{1-\varepsilon} E\int_0^T \hat{c}(t) U_1'(t, \hat{c}(t))\, dt &\geq E\int_0^T \hat{c}(t) U_1'(t, \eta_\varepsilon(t))\, dt \\ &\geq E\left[\frac{\hat{X}_\varepsilon(T) - \hat{X}(T)}{\varepsilon} U_2'(\rho_\varepsilon) 1_B\right] \\ &\geq xE\left[\frac{Z_\nu(T)}{H_0(T)} U_2'\left(\varepsilon x \frac{Z_\nu(T)}{H_0(T)}\right) 1_B\right]. \end{aligned}$$

Suppose $P(B) > 0$; then letting $\varepsilon \downarrow 0$ above, we obtain from Fatou's lemma

$$xU_2'(0+) E\left[\frac{Z_\nu(T)}{H_0(T)} 1_B\right] \leq E\int_0^T \hat{c}(t) U_1'(t, \hat{c}(t))\, dt < \infty,$$

a contradiction, since $U_2'(0+) = \infty$. □

Our program is to show that there exists a process $\lambda \in \mathscr{D}'$, such that the positive process $\hat{X}$ can be represented as

$$H_\lambda(t)\hat{X}(t) = E\left[\int_t^T H_\lambda(s) c_\lambda(s)\, ds + H_\lambda(T)\xi_\lambda \middle| \mathscr{F}_t\right], \qquad 0 \leq t \leq T \tag{A.7}$$

and the requirements

$$\hat{c}(t) = c_\lambda(t), \tag{A.8}$$

$$\delta(\lambda(t)) + \lambda^*(t)\hat{\pi}(t) = 0 \tag{8.22}$$

hold $\ell \otimes P$-a.e. Then from (8.19) and (A.7) we shall have $X^{x, \hat{\pi}, c_\lambda}(\cdot) = X_\lambda(\cdot)$ and (B) will be established.

We shall consider the integrable, nondecreasing process

$$A(t) \triangleq \int_0^t \hat{c}(s) U_1'(s, \hat{c}(s))\, ds, \qquad 0 \le t \le T, \tag{A.9}$$

and the continuous martingale

$$M(t) \triangleq E\big[A(T) + \hat{X}(T) U_2'(\hat{X}(T)) \big| \mathscr{F}_t\big] = y_0 + \int_0^t \psi^*(s)\, dW(s), \tag{A.10}$$

where $y_0 = EM(T) < \infty$ [by the assumption (10.2)] and $\psi(\cdot)$ is a suitable $\mathscr{R}^d$-valued and $\{\mathscr{F}_t\}$-progressively measurable process (from the martingale representation theorem).

A.2 Lemma. *We have, $\ell \otimes P$-a.e., the identity*

$$\hat{c}(t) = I_1\left(t, \frac{M(t) - A(t)}{\hat{X}(t)}\right). \tag{A.11}$$

Proof. Let us start by defining the nonnegative process

$$\hat{\beta}(t) \triangleq \hat{c}(t)/\hat{X}(t), \qquad 0 \le t \le T, \tag{A.12}$$

in terms of which the solution of the linear equation (3.4) (with $c = \hat{c}$, $\pi = \hat{\pi}$, $X = \hat{X}$) is given as

$$\begin{aligned}\gamma_0(t)\hat{X}(t) = x \exp\bigg[&-\int_0^t \Big(\hat{\beta}(s) + \tfrac{1}{2}\|\hat{\pi}^*(s)\sigma(s)\|^2\Big)\, ds \\ &+ \int_0^t \hat{\pi}^*(s)\sigma(s)\, dW_0(s)\bigg].\end{aligned} \tag{A.13}$$

Our method proceeds by a *small random perturbation* of the process $\hat{\beta}$ in (17.12). In particular, for an arbitrary but fixed $\{\mathscr{F}_t\}$-progressively measurable process $\rho(\cdot)$ with $|\rho(t)| \le 1 \wedge \hat{\beta}(t)$, $0 \le t \le T$ and $0 < \varepsilon < 1/2$, we define

$$\beta_\varepsilon(t) \triangleq \hat{\beta}(t) + \varepsilon\rho(t), \qquad 0 \le t \le T, \tag{A.14}$$

$$\begin{aligned} X_\varepsilon(t) &\triangleq \hat{X}(t) \exp\left\{-\varepsilon\int_0^t \rho(s)\, ds\right\}, \\ c_\varepsilon(t) &\triangleq X_\varepsilon(t)\beta_\varepsilon(t) = \big[\hat{c}(t) + \varepsilon\rho(t)\hat{X}(t)\big]\exp\left(-\varepsilon\int_0^t \rho(s)\, ds\right).\end{aligned} \tag{A.15}$$

Note that $X^{x,\hat{\pi},c_\varepsilon}(\cdot) \equiv X_\varepsilon(\cdot)$ and so $(\hat{\pi}, c_\varepsilon) \in \mathscr{A}'(x)$. On the other hand, we have

$$c_\varepsilon(t) \ge \tfrac{1}{2}\hat{c}(t)e^{-\varepsilon T}, \qquad |c_\varepsilon(t) - \hat{c}(t)| \le \varepsilon\, \text{const.}\, \hat{c}(t),$$

whence, in conjunction with (5.8),

$$\frac{1}{\varepsilon}\big|U_1(t, c_\varepsilon(t)) - U_1(t, \hat{c}(t))\big| \le \text{const.}\, \hat{c}(t) U_1'(t, \hat{c}(t)).$$

This last process is $\ell \otimes P$-integrable, thanks to (10.2). Similarly,

$$\frac{1}{\varepsilon}\left|U_2(X_\varepsilon(T)) - U_2(\hat{X}(T))\right| \le \text{const.}\ \hat{X}(T)U_2'(\hat{X}(T)) \quad \text{a.s.},$$

where again the right-hand side is integrable.

From these remarks and the Dominated Convergence Theorem, we obtain

$$\begin{aligned} 0 &\le \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} E\left[\int_0^T \{U_1(t,\hat{c}(t)) - U_1(t,c_\varepsilon(t))\}\,dt \right.\\ &\qquad \left. + \left\{U_2(\hat{X}(T)) - U_2(X_\varepsilon(T))\right\}\right] \\ &= E\left[\int_0^T U_1'(t,\hat{c}(t))\left\{\hat{c}(t)\int_0^t \rho(s)\,ds - \rho(t)\hat{X}(t)\right\}dt \right.\\ &\qquad \left. + U_2'(\hat{X}(T))\hat{X}(T)\int_0^T \rho(s)\,ds\right]. \end{aligned} \tag{A.16}$$

But in the notation of (A.9), (A.10), it is easy to see that

$$E\int_0^T\left(\int_0^t \rho(s)\,ds\right)dA(t) = E\int_0^T \rho(t)\{E[A(T)|\mathscr{F}_t] - A(t)\}\,dt, \tag{A.17}$$

$$\begin{aligned} &E\left[U_2'(\hat{X}(T))\hat{X}(T)\int_0^T \rho(s)\,ds\right] \\ &\qquad = E\int_0^T \rho(t)E\{U_2'(\hat{X}(T))\hat{X}(T)|\mathscr{F}_t\}\,dt. \end{aligned} \tag{A.18}$$

Back into (A.16), these computations lead to

$$E\int_0^T \rho(t)\left[M(t) - A(t) - \hat{X}(t)U_1'(t,\hat{c}(t))\right]dt \ge 0.$$

From the arbitrariness of $\rho(\cdot)$, we deduce

$$\hat{X}(t)U_1'(t,\hat{c}(t)) = M(t) - A(t), \qquad \ell \otimes P\text{-a.e.}, \tag{A.19}$$

which is equivalent to (A.11). □

A.3 Remark. The right-hand side of (A.11) defines an a.s. continuous process. Thus we may, and shall, assume henceforth that $\hat{c}(\cdot)$ is given in its continuous modification, so that (A.11) actually holds for all $0 \le t \le T$, almost surely.

A.4 Proposition. *The process*

$$\lambda(t) \triangleq -\sigma(t)\left[\mu(t) + \frac{\psi(t)}{M(t) - A(t)}\right], \qquad 0 \le t \le T \tag{A.20}$$

satisfies (8.22) *and* $\int_0^T \|\lambda(t)\|^2\,dt < \infty$, $\int_0^T \delta(\lambda(t))\,dt < \infty$ *a.s.*

PROOF. Take an arbitrary portfolio process $\eta(\cdot)$ with values in K, a number $0 < \varepsilon < 1$, a suitable increasing sequence $\{\tau_n\}_{n \in \mathbb{N}}$ of $\{\mathscr{F}_t\}$-stopping times with $\lim_{n \to \infty} \tau_n = T$ a.s. [cf. (A.25) for the precise definition] and create a *small random perturbation of the optimal portfolio* $\hat{\pi}(\cdot)$, according to

$$\pi_\varepsilon(t) = \begin{cases} (1-\varepsilon)\hat{\pi}(t) + \varepsilon\eta(t), & 0 \le t \le \tau_n, \\ \hat{\pi}(t), & \tau_n < t \le T, \end{cases} \tag{A.21}$$

for every $n \in \mathbb{N}$. Define also $X_\varepsilon(\cdot)$ and $c_\varepsilon(\cdot)$ via

$$\begin{aligned} \gamma_0(t)X_\varepsilon(t) &\triangleq x\exp\left\{-\int_0^t \left(\hat{\beta}(s) + \tfrac{1}{2}\|\pi_\varepsilon^*(s)\sigma(s)\|^2\right)ds \right. \\ &\qquad\qquad \left. + \int_0^t \pi_\varepsilon^*(s)\sigma(s)\,dW_0(s)\right\}, \\ c_\varepsilon(t) &\triangleq \hat{\beta}(t)X_\varepsilon(t), \end{aligned} \tag{A.22}$$

and notice that $X^{x,\pi_\varepsilon,c_\varepsilon}(\cdot) \equiv X_\varepsilon(\cdot)$,

$$\begin{aligned} X_\varepsilon(t) = \hat{X}(t)\exp\left[\varepsilon\int_0^{t\wedge\tau_n}(\eta(s) - \hat{\pi}(s))^*\sigma(s)\,d\hat{W}(s)\right. \\ \left. - \frac{\varepsilon^2}{2}\int_0^{t\wedge\tau_n}\|\sigma^*(s)(\eta(s) - \hat{\pi}(s))\|^2\,ds\right] \end{aligned} \tag{A.23}$$

where

$$\begin{aligned} \hat{W}(t) &\triangleq W(t) + \int_0^t \mu(s)\,ds, \\ N(t) &\triangleq \int_0^t (\eta(s) - \hat{\pi}(s))^*\sigma(s)\,d\hat{W}(s). \end{aligned} \tag{A.24}$$

If we define the $\{\mathscr{F}_t\}$-stopping time

$$\begin{aligned} \tau_n \triangleq T \wedge \inf\Big\{ t \in [0,T];\ &|N(t)| \ge n, \\ &\text{or } A(t) \ge n, \text{ or } |M(t)| \ge n, \\ &\text{or } \langle N\rangle_t \ge n, \text{ or } \int_0^t \|\psi(s)\|^2\,ds \ge n \Big\}, \end{aligned} \tag{A.25}$$

for every $n \in \mathbb{N}$, we have almost surely

$$\hat{X}(t)e^{-3\varepsilon n} \le X_\varepsilon(t) \le \hat{X}(t)e^{3\varepsilon n}, \qquad \hat{c}(t)e^{-3\varepsilon n} \le c_\varepsilon(t) \le \hat{c}(t)e^{3\varepsilon n},$$
$$\forall\, 0 \le t \le T,$$

and $\lim_{n\to\infty}\uparrow \tau_n = T$. In particular, $(\pi_\varepsilon, c_\varepsilon) \in \mathscr{A}'(x)$ and

$$\frac{1}{\varepsilon}\left|U_2(X_\varepsilon(T)) - U_2(\hat{X}(T))\right| \le \frac{1}{\varepsilon}\left|X_\varepsilon(T) - \hat{X}(T)\right| U_2'(\hat{X}(T)e^{-3\varepsilon n})$$
$$\le U_2'(\hat{X}(T)e^{-3\varepsilon n})\hat{X}(T)e^{-3\varepsilon n}\frac{e^{3\varepsilon n}-1}{\varepsilon}e^{3\varepsilon n}$$
$$\le K_n U_2'(\hat{X}(T))\hat{X}(T) \quad \text{a.s.,}$$

where $K_n \triangleq e^{3n}\sup_{0<\varepsilon<1}(e^{3\varepsilon n}-1)/\varepsilon$, again thanks to condition (5.8). Similarly,

$$\frac{1}{\varepsilon}\left|U_1(t, c_\varepsilon(t)) - U_1(t, \hat{c}(t))\right| \le K_n U_1'(t, \hat{c}(t))\hat{c}(t), \qquad 0 \le t \le T.$$

From these inequalities, the integrability of the random variable

$$\int_0^T U_1'(t, \hat{c}(t))\hat{c}(t)\,dt + U_2'(\hat{X}(T))\hat{X}(T),$$

the optimality of $(\hat{\pi}, \hat{c}) \in \mathscr{A}'(x)$ and the Dominated Convergence Theorem, we obtain,

$$0 \ge \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}E\bigg[\int_0^T\{U_1(t, c_\varepsilon(t)) - U(t, \hat{c}(t))\}\,dt + \{U_2(X_\varepsilon(T)) - U_2(\hat{X}(T))\}\bigg]$$

$$(\text{A.26}) \quad = E\bigg[\int_0^T U_1'(t, \hat{c}(t))\hat{c}(t)\bigg\{\int_0^{t\wedge\tau_n}(\eta(s) - \hat{\pi}(s))^*\sigma(s)\,d\hat{W}(s)\bigg\}\,dt + U_2'(\hat{X}(T))\hat{X}(T)\int_0^{\tau_n}(\eta(s) - \hat{\pi}(s))^*\sigma(s)\,dW(s)\bigg]$$

$$= E\bigg[\int_0^T N_n(t)\,dA(t) + (M(T) - A(T))N_n(T)\bigg], \qquad \forall\, n \in \mathbb{N},$$

in the notation of (A.9), (A.10), (A.24) and with $N_n(t) \triangleq N(t\wedge\tau_n)$. The product rule and the definition (A.25) give

$$-E\bigg[\int_0^T N_n(t)\,dA(t) - N_n(T)A(T)\bigg]$$

$$(\text{A.27}) \quad \begin{aligned} &= E\int_0^T A(t)\,dN_n(t) \\ &= E\int_0^{\tau_n} A(t)(\eta(t) - \hat{\pi}(t))^*\sigma(t)[dW(t) + \mu(t)\,dt] \\ &= E\int_0^{\tau_n}(\eta(t) - \hat{\pi}(t))^*\sigma(t)A(t)\mu(t)\,dt. \end{aligned}$$

On the other hand, we get by the same token

$$E[M(T)N_n(T)] = E[M(T)N(\tau_n)] = E[M(\tau_n)N(\tau_n)]$$

$$\text{(A.28)} \qquad = E\int_0^{\tau_n}(\eta(t) - \hat{\pi}(t))^* \sigma(t)[M(t)\mu(t) + \psi(t)]\,dt.$$

Substituting from (A.27) and (A.28) back into (A.26), we obtain

$$\text{(A.29)} \quad E\int_0^{\tau_n}(M(t) - A(t))(\hat{\pi}(t) - \eta(t))^* \lambda(t)\,dt \le 0, \qquad \forall\, n \in \mathbb{N},$$

where $\lambda(\cdot)$ is the process of (A.20).

It follows from (A.29) that, for any K-valued portfolio process η, there exists a set $A_\eta \subseteq [0, T] \times \Omega$ of zero product measure, such that

$$\lambda^*(t,\omega)\eta(t,\omega) \ge \lambda^*(t,\omega)\hat{\pi}(t,\omega), \qquad \forall\,(t,\omega) \notin A_\eta.$$

In particular,

$$\lambda^*(t,\omega)\pi \ge \lambda^*(t,\omega)\hat{\pi}(t,\omega), \qquad \forall\,(t,\omega) \notin A_\pi$$

for every $\pi \in K$ (taking $\eta \equiv \pi$). Now $A = \bigcup_{\pi \in K \cap Q^d} A_\pi$ has zero product measure and $\delta(x) = \sup_{\pi \in K \cap Q^d}(-x^*\pi)$; therefore,

$$\delta(\lambda(t,\omega)) \le -\lambda^*(t,\omega)\hat{\pi}(t,\omega), \qquad \forall\,(t,\omega) \notin A.$$

The opposite inequality is trivially true, since $\hat{\pi}$ takes values in K; this establishes (8.22). It is not hard to see that $\int_0^T \|\lambda(s)\|^2\,ds < \infty$ holds almost surely. Then $\int_0^T \delta(\lambda(s))\,ds < \infty$ a.s. follows from this, (8.22) and the Cauchy–Schwarz inequality. □

A.5 Remark. Let us denote by $\mathscr{L}$ the class of $\{\mathscr{F}_t\}$-progressively measurable processes $\nu(\cdot)$ with

$$\int_0^T \|\nu(s)\|^2\,ds < \infty, \qquad \int_0^T \delta(\nu(s))\,ds < \infty \quad \text{a.s.}$$

All processes $\gamma_\nu(\cdot)$, $Z_\nu(\cdot)$ and $H_\nu(\cdot)$ are well defined for every $\nu \in \mathscr{L}$, as are the functions $\mathscr{X}_\nu(\cdot)$ and $\tilde{J}(\cdot;\nu) = E[\int_0^T \tilde{U}_1(t; \cdot H_\nu(t))\,dt + \tilde{U}_2(\cdot H_\nu(T))]$.

Proof of (A.7) and (A.8). From (A.11) and the definitions (A.9) and (A.10) we obtain

$$\hat{X}(t)U_1'(t,\hat{c}(t)) = M(t) - A(t)$$

$$\text{(A.30)} \qquad = E\left[\int_t^T U_1'(s,\hat{c}(s))\hat{c}(s)\,ds + U_2'(\hat{X}(T))\hat{X}(T)\,\middle|\,\mathscr{F}_t\right],$$

as well as

$$\begin{aligned}
&d\big(\hat{X}(t)U_1'(t,\hat{c}(t))\big)\\
&\qquad = dM(t) - dA(t)\\
\text{(A.31)}\quad &\qquad = \psi^*(t)\,dW(t) - U_1'(t,\hat{c}(t))\hat{c}(t)\,dt\\
&\qquad = \hat{X}(t)U_1'(t,\hat{c}(t))\big[-\hat{\beta}(t)\,dt\\
&\qquad\qquad + \{\sigma^*(t)\hat{\pi}(t) - \theta(t) - \sigma^{-1}(t)\lambda(t)\}^*\,dW(t)\big].
\end{aligned}$$

On the other hand, we may rewrite (A.13) as

$$\gamma_\lambda(t)\hat{X}(t) = x\exp\bigg[-\int_0^t\Big(-\hat{\beta}(s) + \delta(\lambda(s)) + \tfrac{1}{2}\|\sigma^*(s)\hat{\pi}(s)\|^2\Big)\,ds + \int_0^t \hat{\pi}^*(s)\sigma(s)\,dW_0(s)\bigg],$$

whence

$$d\left(\frac{1}{\gamma_\lambda(t)\hat{X}(t)}\right) = \frac{1}{\gamma_\lambda(t)\hat{X}(t)}\Big[\Big(\hat{\beta}(t) + \delta(\lambda(t)) + \|\sigma^*(t)\hat{\pi}(t)\|^2\Big)\,dt - \hat{\pi}^*(t)\sigma(t)\,dW_0(t)\Big]. \tag{A.32}$$

From the product rule and (A.31), (A.32) we deduce

$$d\left(\frac{U_1'(t,\hat{c}(t))}{\gamma_\lambda(t)}\right) = \frac{U_1'(t,\hat{c}(t))}{\gamma_\lambda(t)}\big[\theta(t) + \sigma^{-1}(t)\lambda(t)\big]^*\,dW(t), \tag{A.33}$$

whence $U_1'(t,\hat{c}(t)) = y_* H_\lambda(t)$, $y_* \triangleq U_1'(0,\hat{c}(0))$.

This yields, in conjunction with (A.10) and (A.30),

$$U_2'\big(\hat{X}(T)\big) = \frac{M(T) - A(T)}{\hat{X}(T)} = U_1'(T,\hat{c}(T)) = y_* H_\lambda(T), \tag{A.34}$$

$$H_\lambda(t)\hat{X}(t) = E\left[\int_t^T H_\lambda(s)\hat{c}(s)\,ds + H_\lambda(T)\hat{X}(T)\,\Big|\,\mathscr{F}_t\right], \quad 0 \le t \le T. \tag{A.30$'$}$$

Evaluated at $t = 0$, this last expression yields, in conjunction with (A.33) and (A.34),

$$x = E\left[\int_0^T H_\lambda(s)I_1(s, y_* H_\lambda(s))\,ds + H_\lambda(T)I_2(y_* H_\lambda(T))\right] = \mathscr{X}_\lambda(y_*).$$

From this and Remark 8.5, we conclude

$$\mathscr{X}_\lambda(y) < \infty, \quad \forall\, y \in (0,\infty) \quad \text{and} \quad y_* = \mathscr{Y}_\lambda(x), \tag{A.35}$$

where $\mathscr{Y}_\lambda = \mathscr{X}_\lambda^{-1}$. Consequently, the expressions (A.33) and (A.34) become

$\hat{c}(t) = I_1(t, \mathscr{Y}_\lambda(x)H_\lambda(t)) \equiv c_\lambda(t)$ [i.e., (A.8)] and $\hat{X}(T) = I_2(\mathscr{Y}_\lambda(x)H_\lambda(T)) \equiv \xi_\lambda$, respectively, and thus (A.30′) becomes (A.7). □

PROOF OF $\lambda \in \mathscr{D}'$. In Proposition A.4 we showed that the process $\lambda(\cdot)$ of (A.20) belongs to the class $\mathscr{L}$ (Remark A.5) and satisfies the requirements of condition (B). Just as in Section 10, the same arguments that led to the implications (B) ⇒ (E) ⇒ (D) in the proof of Theorem 10.1, show here again that

$$\tilde{J}(y_*;\lambda) \le \tilde{J}(y_*;\nu), \qquad \forall\, \nu \in \mathscr{L}, \quad \text{where} \quad y_* = \mathscr{Y}_\lambda(x).$$

We wish to prove $\lambda \in \mathscr{D}$ [because this then implies $\lambda \in \mathscr{D}'$, in conjunction with (A.35), and we are done]. Clearly, in light of condition (12.2), it suffices to show

$$\text{(A.36)} \qquad \tilde{J}(y;\nu) = \infty, \qquad \forall\, \nu \in \mathscr{L} \setminus \mathscr{D}, y \in (0,\infty).$$

For this, consider first $\nu \in \mathscr{L}$ with $[\![\nu]\!]^2 = E\int_0^T \|\nu(t)\|^2\, dt = \infty$, define $\tau_n \triangleq T \wedge \inf\{t \in [0,T];\ \int_0^t \|\nu(s)\|^2\, ds \ge n\}$ for $n \in \mathbb{N}$ and notice that $\lim_{n\to\infty} \uparrow \tau_n = T$ a.s. Recall that we have again $\gamma_\nu(t) \le e^M$, $0 \le t \le T$, in the notation of (12.13). This, the fact that $\tilde{U}_2(\cdot)$ is decreasing, the convexity of $\tilde{U}_2(\cdot)$ and $\tilde{U}_2(e^{\cdot})$, Jensen's inequality, and the supermartigale property of $Z_\nu(\cdot)$, imply for any $n \in \mathbb{N}$,

$$\begin{aligned}
E\tilde{U}_2(yH_\nu(T)) &= E\Big[E\big\{\tilde{U}_2(y\gamma_\nu(T)Z_\nu(T))\,|\,\mathscr{F}_{\tau_n}\big\}\Big] \\
&\ge E\Big[E\big\{\tilde{U}_2(ye^M Z_\nu(T))\,|\,\mathscr{F}_{\tau_n}\big\}\Big] \ge E\tilde{U}_2\big(ye^M E\{Z_\nu(T)\,|\,\mathscr{F}_{\tau_n}\}\big) \\
&\ge E\tilde{U}_2\big(ye^M Z_\nu(\tau_n)\big) \\
&\ge \tilde{U}_2\left(y\exp\left\{M - E\int_0^{\tau_n}\theta_\nu^*(s)\,dW(s) - \tfrac{1}{2}E\int_0^{\tau_n}\|\theta_\nu(s)\|^2\,ds\right\}\right) \\
&= \tilde{U}_2\left(y\exp\left\{M - \tfrac{1}{2}E\int_0^{\tau_n}\|\theta(s) + \sigma^{-1}(s)\nu(s)\|^2\,ds\right\}\right),
\end{aligned}$$

and $E\tilde{U}_2(yH_\nu(T)) = \infty$ follows by letting $n \uparrow \infty$. Second, take $\nu \in \mathscr{L}$ with $E\int_0^T \|\nu(s)\|^2\, ds < \infty$ but $E\int_0^T \delta(\nu(s))\, ds = \infty$. For such ν, Remark 13.3 shows $E\tilde{U}_2(yH_\nu(T)) = \infty$. □

APPENDIX B

Consumption/investment with a higher interest rate for borrowing. We have considered so far a model in which one is allowed to borrow money and at an interest rate R equal to the bond rate r. The purpose of this section is to show that the convex duality approach of the present paper permits a complete treatment of the consumption/investment problem of Definition 6.1 (without portfolio constraints) in the general case $R \ge r$. More specifically, we shall assume that the process $R(\cdot)$ is progressively measurable with respect to $\{\mathscr{F}_t\}$ and satisfies

$$R(t,\omega) \geq r(t,\omega) \quad \text{for a.e. } (t,\omega), \tag{B.1}$$

$$E\int_0^T (R(t) - r(t))^2\, dt < \infty. \tag{B.2}$$

In this market $\mathscr{M}^*$, the wealth process $X = X^{x,\pi,c}$ corresponding to a given portfolio/consumption pair (π, c) and initial capital $X(0) = x$ satisfies the analogue

$$\begin{aligned} dX(t) &= \sum_{i=1}^d \pi_i(t)X(t)\left\{b_i(t)\,dt + \sum_{j=1}^d \sigma_{ij}(t)\,dW_j(t)\right\} - c(t)\,dt \\ &\quad + \left(1 - \sum_{i=1}^d \pi_i(t)\right)^+ X(t)r(t)\,dt - \left(1 - \sum_{i=1}^d \pi_i(t)\right)^- X(t)R(t)\,dt \\ &= [r(t)X(t) - c(t)]\,dt \\ &\quad + X(t)\Bigg[\pi^*(t)\sigma(t)\,dW_0(t) \\ &\qquad - (R(t) - r(t))\left(1 - \sum_{i=1}^d \pi_i(t)\right)^- dt\Bigg] \end{aligned} \tag{B.3}$$

of (3.1), and the stochastic control problem is that of Definition 6.1.

B.1 Remark. Condition (B.1) implies that it is not optimal to borrow money and to invest money in the bond at the same time. Therefore, we restrict ourselves to policies for which the relative amount borrowed at time t is equal to $(1 - \sum_{i=1}^d \pi_i(t))^-$.

Consider now the bounded subset

$$\mathscr{D} \triangleq \{\nu \in \mathscr{H};\ -(R - r) \leq \nu_1 = \cdots = \nu_d \leq 0,\ \ell \otimes P\text{-a.e.}\} \tag{B.4}$$

of the Hilbert space $\mathscr{H}$ (Section 8), set

$$\delta(\nu(t)) \triangleq -\nu_1(t) \quad \text{for every } \nu \in \mathscr{D},\ 0 \leq t \leq T, \tag{B.5}$$

and notice that $0 \leq E\int_0^T \delta(\nu(t))\,dt \leq E\int_0^T (R(t) - r(t))\,dt < \infty$, $\nu \in \mathscr{D}$, in accordance with (8.1). With these conventions, the auxiliary market $\mathscr{M}_\nu$, $\nu \in \mathscr{D}$, of (8.3) and (8.4) consists of *exactly the same stocks* [since $\delta(\nu(t)) + \nu_i(t) = 0$, $i = 1, \ldots, d$] and of *just one interest rate* for both borrowing and lending, namely,

$$r_\nu(t) = r(t) - \nu_1(t), \qquad 0 \leq t \leq T. \tag{B.6}$$

Notice that $r \leq r_\nu \leq R$. Just as before, one looks for a process $\lambda \in \mathscr{D}$, for which the solution to the consumption/investment problem in $\mathscr{M}_\lambda$ (unconstrained, single interest rate r_λ) induces a solution to the corresponding problem in $\mathscr{M}^*$ (unconstrained, interest rate $R \geq r$ for borrowing).

With these new interpretations, *all the results* proved in Sections 9–13 go through with only the obvious changes. For instance, (8.21) and (8.22) are replaced by

$$\Psi^{\lambda,\pi_\lambda}(t,\omega) = 0 \quad \text{for } \ell \otimes P\text{-a.e. } (t,\omega), \tag{B.7}$$

where, for any portfolio π and any $\nu \in \mathscr{D}$, $\Psi^{\nu,\pi}$ is the nonnegative process

$$\begin{aligned}\Psi^{\nu,\pi}(t) &\triangleq [R(t) - r(t) + \nu_1(t)]\left(1 - \sum_{i=1}^{d} \pi_i(t)\right)^- \\ &\quad - \nu_1(t)\left(1 - \sum_{i=1}^{d} \pi_i(t)\right)^+, \qquad 0 \leq t \leq T;\end{aligned} \tag{B.8}$$

the same holds for (10.3). Similarly, $\delta(\nu(t)) + \pi^*(t)\nu(t)$ has to be replaced by $\Psi^{\nu,\pi}(t)$ in (9.3), (9.7), (9.8), (12.5), (12.6) and so forth. In the proof of Theorem 9.1, we take $\nu(t) \triangleq \nu_1(t)\mathbf{1}$, where

$$\nu_1(t) \triangleq -[R(t) - r(t)]1_{\{\sum_{i=1}^d \pi_i(t) > 1\}},$$

and arrive at the analogue of (9.14):

$$\begin{aligned}0 &\leq \limsup_{\varepsilon \downarrow 0} \frac{x(\lambda) - x(\lambda_{\varepsilon,n}^{(\nu)})}{\varepsilon} \\ &\leq E\int_0^{\tau_n} H_\lambda(t)X(t)[\pi^*(t)(\nu(t) - \lambda(t)) - (\nu_1(t) - \lambda_1(t))]\,dt \\ &= -E\int_0^{\tau_n} H_\lambda(t)X(t)[\nu_1(t) - \lambda_1(t)]\left(1 - \sum_{i=1}^{d} \pi_i(t)\right) dt \\ &= -E\int_0^{\tau_n} H_\lambda(t)X(t)\Psi^{\lambda,\pi}(t)\,dt, \qquad \forall\, n \in \mathbb{N}.\end{aligned}$$

From this one obtains $\Psi^{\lambda,\pi} = 0$, $\ell \otimes P$-a.e., which is the analogue of (9.8).

In particular, under the conditions of Theorem 13.1, there exists an optimal process $\lambda_y \in \mathscr{D}$ for the dual problem of (12.1) and, for any given $x \in (0,\infty)$, there exists an *optimal portfolio/consumption process pair* $(\hat{\pi}, \hat{c})$ for the original control problem in $\mathscr{M}^*$.

B.2 Example. *General coefficients, logarithmic utilities.* In the special case $U_1(t,x) = U_2(x) = \log x$, $(t,x) \in [0,T] \times (0,\infty)$, we see from (11.3) that $\lambda(t) = \lambda_1(t)\mathbf{1}$, where

$$\lambda_1(t) = \arg \min_{r(t) - R(t) \leq x \leq 0} \left(-2x + \|\theta(t) + \sigma^{-1}(t)\mathbf{1}x\|^2\right).$$

With $A(t) \triangleq \operatorname{tr}[(\sigma^{-1}(t))^*(\sigma^{-1}(t))]$ and $B(t) \triangleq \theta^*(t)\sigma^{-1}(t)\mathbf{1}$, this minimization is achieved as follows:

$$\text{(B.9)}\quad \lambda_1(t) = \begin{cases} \dfrac{1 - B(t)}{A(t)}, & \text{if } 0 < B(t) - 1 < A(t)(R(t) - r(t)), \\ 0, & \text{if } B(t) \le 1, \\ r(t) - R(t), & \text{if } B(t) - 1 \ge A(t)(R(t) - r(t)). \end{cases}$$

From (11.5), the optimal portfolio is then computed as

$$\text{(B.10)}\quad \hat{\pi}(t) = \begin{cases} (\sigma(t)\sigma^*(t))^{-1}\left[b(t) - \left(r(t) + \dfrac{B(t) - 1}{A(t)}\right)\mathbf{1}\right], \\ \qquad\qquad \text{if } 0 < B(t) - 1 \le A(t)(R(t) - r(t)), \\ (\sigma(t)\sigma^*(t))^{-1}[b(t) - r(t)\mathbf{1}], \quad \text{if } B(t) \le 1, \\ (\sigma(t)\sigma^*(t))^{-1}[b(t) - R(t)\mathbf{1}], \\ \qquad\qquad \text{if } B(t) - 1 \ge A(t)(R(t) - r(t)). \end{cases}$$

With obvious minor modifications, the results of Section 15 carry over to this case as well; in particular, so do the feedback formulae (15.9) and (15.10) under the conditions of Theorem 15.3.

B.3 Example. *Deterministic coefficients, HARA Utilities.* In the case $U_1(t, x) = U_2(x) = x^\alpha/\alpha$, $(t, x) \in [0, T] \times (0, \infty)$ for some $\alpha \in (0, 1)$, we get $\lambda(t) = \lambda_1(t)\mathbf{1}$ from (15.7) as

$$\lambda_1(t) = \arg \min_{r(t) - R(t) \le x \le 0} \left[-2(1 - \alpha)x + \|\theta(t) + \sigma^{-1}(t)\mathbf{1}x\|^2\right]$$

$$\text{(B.11)}\qquad = \begin{cases} \dfrac{1 - \alpha - B(t)}{A(t)}, & \text{if } 0 < B(t) - 1 + \alpha < A(t)(R(t) - r(t)), \\ 0, & \text{if } B(t) \le 1 - \alpha, \\ r(t) - R(t), & \text{if } B(t) - 1 + \alpha \ge A(t)(R(t) - r(t)), \end{cases}$$

in the notation of Example B.1. The optimal portfolio is given as

$$\text{(B.12)}\quad \hat{\pi}(t) = \begin{cases} \dfrac{(\sigma(t)\sigma^*(t))^{-1}}{A(t)}\left[b(t) - \left(r(t) + \dfrac{B(t) - 1 + \alpha}{A(t)}\right)\mathbf{1}\right], \\ \qquad\qquad \text{if } 0 < B(t) - 1 + \alpha < A(t)(R(t) - r(t)), \\ \dfrac{(\sigma(t)\sigma^*(t))^{-1}}{1 - \alpha}[b(t) - r(t)\mathbf{1}], \quad \text{if } B(t) \le 1 - \alpha, \\ \dfrac{(\sigma(t)\sigma^*(t))^{-1}}{1 - \alpha}[b(t) - R(t)\mathbf{1}], \\ \qquad\qquad \text{if } B(t) - 1 + \alpha \ge A(t)(R(t) - r(t)). \end{cases}$$

B.4 Remark. The problem of maximizing expected utility from consumption, on an infinite horizon with discounting, was studied by Fleming and Zariphopoulou (1991) using analytical techniques for $d = 1$ and constant $R > r, b_1, \sigma_{11}$. Explicit formulae were obtained in the case of a HARA utility function.

B.5 Remark. It is also possible to study the constrained portfolio optimization problem of this paper in the presence of a higher interest rate for borrowing, using the "combined dual problem" of minimizing $\tilde{J}(y; \nu, \mu)$ over $(\nu, \mu) \in \mathscr{D}_1 \times \mathscr{D}_2$, where $\mathscr{D}_1$ and $\mathscr{D}_2$ are given by (8.1) and (B.4), respectively. Here $\tilde{J}(y; \nu, \mu)$ is defined as in (12.1) with $H_\nu(t)$ replaced by

$$H_{\nu,\mu}(t) \triangleq \exp\biggl[-\int_0^t \{r(s) + \delta(\nu(s)) - \mu_1(s)\}\,ds - \int_0^t \theta^*_{\nu,\mu}(s)\,dW(s) - \tfrac{1}{2}\int_0^t \|\theta_{\nu,\mu}(s)\|^2\,ds\biggr],$$

$$\theta_{\nu,\mu}(t) \triangleq \theta(t) + \sigma^{-1}(t)[\nu(t) + \mu(t)].$$

We leave the details of this development to the diligence of the reader.

Acknowledgment. We wish to thank Jérôme Detemple, Ralf Korn and the two anonymous referees, for their careful reading of the paper and their helpful suggestions.

REFERENCES

Bismut, J. M. (1973). Conjugate convex functions in optimal stochastic control. *J. Math. Anal. Appl.* **44** 384–404.

Cox, J. and Huang, C. F. (1989). Optimal consumption and portfolio policies when asset prices follow a diffusion process. *J. Economic Theory* **49** 33–83.

Ekeland, I. and Temam, R. (1976). *Convex Analysis and Variational Problems*. North-Holland, Amsterdam.

Fleming, W. H. and Rishel, R. W. (1975). *Deterministic and Stochastic Optimal Control*. Springer, New York.

Fleming, W. and Zariphopoulou, Th. (1991). An optimal investment/consumption model with borrowing. *Math. Oper. Res.* **16** 802–822.

Friedman, A. (1964). *Partial Differential Equations of Parabolic Type*. Prentice-Hall, Englewood Cliffs, N.J.

He, H. and Pearson, N. (1991). Consumption and portfolio policies with incomplete markets and short-sale constraints: The infinite-dimensional case. *J. Economic Theory* **54** 259–304.

Karatzas, I. (1989). Optimization problems in the theory of continuous trading. *SIAM J. Control Optimization* **27** 1221–1259.

Karatzas, I., Lehoczky, J. P. and Shreve, S. E. (1987). Optimal portfolio and consumption decisions for a "small investor" on a finite horizon. *SIAM J. Control Optimization* **25** 1557–1586.

Karatzas, I., Lehoczky, J. P., Shreve, S. E. and Xu, G. L. (1991). Martingale and duality methods for utility maximization in an incomplete market. *SIAM J. Control Optimization* **29** 702–730.

Karatzas, I. and Shreve, S. E. (1988). *Brownian Motion and Stochastic Calculus*. Springer, New York.

MERTON, R. C. (1969). Lifetime portfolio selection under uncertainty: the continuous-time case. *Rev. Econom. Statist.* **51** 247–257.

OCONE, D. L. and KARATZAS, I. (1991). A generalized Clark representation formula, with application to optimal portfolios. *Stochastics* **34** 187–220.

PLISKA, S. R. (1986). A stochastic calculus model of continuous trading: optimal portfolio. *Math. Oper. Res.* **11** 371–382.

ROCKAFELLAR, R. T. (1970). *Convex Analysis*. Princeton Univ. Press.

SCHÄL, M. (1974). A selection theorem for optimization problems. *Arch. Math.* **25** 219–224.

SCHÄL, M. (1975). Conditions for optimality in dynamic programming, and for the limit of n-state optimal policies to be optimal. *Z. Wahrsch. Verw. Gebiete* **32** 179–196.

SHREVE, S. E. (1991). Nôtes for a talk entitled "Constant coefficients, portfolio cone constraints," given at Columbia University, 8 February 1991. Personal communication.

XU, G. L. (1990). A duality method for optimal consumption and investment under short-selling prohibition. Ph.D. dissertation, Carnegie Mellon Univ.

ZARIPHOPOULOU, TH. (1989). Optimal Investment/Consumption Models with Constraints. Ph.D. dissertation, Brown Univ.

DEPARTMENT OF STATISTICS
COLUMBIA UNIVERSITY
618 MATH BUILDING, BOX 10
NEW YORK, NEW YORK 10027