Convex Dynamic Programming for Hybrid Systems

Hedlund, Sven; Rantzer, Anders

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$$
\begin{aligned}
N_3 \leq & \frac{\beta\, k_2 k_4 k_5 k}{1-k_*}\left(\frac{2\varepsilon_2}{\sqrt{1-k_*^2}} + |u_d| + k_2\,|u_d|\right) r_d^2 y_e^2 \\
& + \frac{\beta\, k_2 k_4 k_5 k}{1-k_*}\left(\frac{|\dot{u}_d|}{\sqrt{1-k_*^2}} + k_1\,|u_d|\right)|r_d|\, y_e^2 + \frac{\beta\, k_4 k_5 k_2 k}{1-k_*} \\
& \times \left(\frac{2k\,|u_d r_d|}{\sqrt{1-k_*^2}} + k^2\,|r_d v_d| + k_*^2 + k\,|r_d v_d| + k^2\,|u_d r_d v_d|\right) \\
& \times \frac{u_d^2 y_e^2}{\sqrt{1+x_e^2+y_e^2}} + \frac{\beta\, k_2 k_4 k_5}{1-k_*}\,\frac{k u_d^2}{2\varepsilon_2\sqrt{1-k_*^2}} v_e^2 \\
N_4 \leq & \frac{\beta\, k_4 k_5}{1-k_*}\,\frac{k u_d^2}{2\varepsilon_3} v_e^2 + \frac{\beta k_4 k_5 k u_d^2}{1-k_*}\,\frac{1}{4\varepsilon_1} x_e^2 + \frac{\beta\, k_4 k_5 k u_d^2 \varepsilon_1}{1-k_*} r_d^2 y_e^2 \\
& + \frac{\beta\, k_4 k_5 k\,|\dot{u}_d|}{(1-k_*)\sqrt{1-k_*^2}}\,\frac{|u_d|\, y_e^2}{\sqrt{1+x_e^2+y_e^2}} \\
& + \frac{\beta\, k_4 k_5 k}{1-k_*}\left(\frac{\varepsilon_3}{\sqrt{1-k_*^2}} + 2k_*^2\,|v_d| + 2k u_d^2\right. \\
& \left. + k_1 + k_2\,|r_d| + k_*^2\,|u_d| + k\,|u_d v_d| + 1\right)\frac{u_d^2 y_e^2}{\sqrt{1+x_e^2+y_e^2}}.
\end{aligned} \tag{48}
$$

Substituting (45)–(48) into (31) directly yields (34).

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Convex Dynamic Programming for Hybrid Systems

Sven Hedlund and Anders Rantzer

***Abstract*—A classical linear programming approach to optimization of flow or transportation in a discrete graph is extended to hybrid systems. The problem is finite dimensional if the state space is discrete and finite, but becomes infinite dimensional for a continuous or hybrid state space. It is shown how strict lower bounds on the optimal loss function can be computed by gridding the continuous state space and restricting the linear program to a finite-dimensional subspace. Upper bounds can be obtained by evaluation of the corresponding control laws.**

***Index Terms*—Convex optimization, dynamic programming, hybrid systems, linear program, optimal control.**

I. INTRODUCTION

For several decades, linear programming has been one of the main theoretical and computational tools for analysis and optimization of discrete systems. This includes problems of optimal transportation and optimal flow in a network [5], [9], [12]. The objective of this note is to extend the computational linear programming approach to hybrid systems, i.e., systems that involve interaction between discrete and continuous dynamics.

Practical control systems typically involve switching between several different modes, depending on the range of operation. Even if the dynamics in each mode is simple and well understood, automatic mode switching can give rise to unexpected phenomena. Moreover, many phenomena can be described either by a discrete model or a continuous one, depending on the context and purpose of the model [2]. Consider, for example, an asynchronous discrete-event driven thermostat, which discretizes temperature information as {too cold, normal, too hot}.

Basic aspects of hybrid systems were treated in [8] and [18]. For stability analysis, see [6] and the references therein. The reformulation of a nonlinear optimal control problem in terms of infinite-dimensional linear programming has previously been used for continuous-time systems in [15] and is closely connected to ideas of [14] and [19].

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The authors are with the Department of Automatic Control, Lund Institute of Technology, SE-22100 Lund, Sweden (e-mail: sven@control.lth.se; rantzer@control.lth.se).

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It should be noted that there is a close connection between optimal control and reachability. A control system can be extended with an extra state that integrates a cost along the trajectories. Hence, a certain control cost is achievable if and only if the corresponding state in the extended system is reachable. Conversely, reachability of a certain state can be investigated by solving the optimal control problem to get there in minimum time. Verification (reachability analysis) of discrete-event systems and timed automata is an extensively studied topic in computer science [1], [3], [11].

Lately, various efforts have been made to extend the classical optimal control methods to hybrid systems. Hybrid versions of the maximum principle have been presented in [13], [16], and [17]. Dynamic programming for hybrid systems is discussed in [4] and [7]. In this note, it is shown how strict lower bounds on the optimal-loss function can be computed by gridding the continuous state space and restricting the linear program (LP) to a finite-dimensional subspace. Upper bounds can be obtained by evaluation of the corresponding control laws. Computational examples are given with up to three dimensions in the continuous state space.

In the following, the set of all integers will be denoted $\mathbf{Z}$. The set of strictly positive real numbers will be denoted $\mathbf{R}^+$.

II. Problem Formulation

Definition 1: Let Q and Ω_μ be finite sets, while $X \subset \mathbf{R}^n$ and $\Omega_u \subset \mathbf{R}^m$. Let $\nu : X \times Q \times \Omega_\mu \mapsto Q$ and for every $q \in Q$ let $f_q : X \times \Omega_u \mapsto \mathbf{R}^n$. A *solution (trajectory)* of the hybrid system

$$\begin{cases} \dot{x}(t) = f_q(x, u) \\ q(t) = \nu(x(t), q(t^-), \mu(t)) \end{cases} \tag{1}$$

will be defined, given $u : [t_0, t_M] \mapsto \Omega_u$, a finite sequence of real numbers $t_0 < t_1 < t_2 < \cdots < t_M$, and $\mu : [t_0, t_M] \mapsto \Omega_\mu$ constant in each interval $[t_k, t_{k+1})$.

The pair (x, q), where $x : [t_0, t_M] \mapsto X$ is absolutely continuous and $q : [t_0, t_M] \mapsto Q$ is constant in each interval $[t_k, t_{k+1})$, $k = 0, 1, \ldots, M-1$, is called a trajectory of the hybrid system (1) if

$$\begin{cases} \dot{x}(t) = f_{q(t)}(x(t), u(t)), & \text{for almost all } t \in [t_0, t_M] \\ q(t) = \nu(x(t), q(t), \mu(t)) = q(t_k), & t \in (t_k, t_{k+1}) \\ q(t_{k+1}) = \nu(x(t_{k+1}), q(t_k), \mu(t_k)), & k = 0, 1, \ldots, M-1 \end{cases} \tag{2}$$

holds. □

Note that the second equation of (1) gives rise to *autonomous* switching in points (x, q) where $\nu(x, q, \mu) \neq q, \forall\, \mu \in \Omega_\mu$. The time argument t will often be omitted in the sequel.

The optimal control problem is to minimize the cost function

$$J(x_0, q_0, u(\cdot), \mu(\cdot), t_M, M) = \int_{t_0}^{t_M} l_{q(t)}(x(t), u(t))dt + \sum_{k=1}^{M} s(x(t_k), q(t_{k-1}), \nu(x(t_k), q(t_{k-1}), \mu(t_k))) \tag{3}$$

with respect to $u(\cdot), \mu(\cdot), t_M$, and M subject to (2) given an initial state (x_0, q_0) at time t_0, and a fixed set of possible final states, $(x, q)(t_M) \in Y_M \subset X \times Q$.

The function $s(x, q, r) > \varepsilon > 0$ is a cost for switching from discrete state q to r, the continuous part being x just before the switch. Note that $s(\cdot) > \varepsilon > 0$ limits the number of jumps in solutions close to optimality.

The framework developed in this note would also allow the number of continuous states to vary with the discrete mode according to $\dot{x}_q(t) = f_{q(t)}(x_q(t), u_q(t))$, where $x_q(t) \in X_q \subset \mathbf{R}^{n(q)}$, $u_q(t) \in \Omega_{u_q} \subset \mathbf{R}^{m(q)}$. The usage of the system description (1), however, will simplify notation.

Also, the possibility of *state jumps* [4], [7] has been omitted to keep the notational complexity at a reasonable level.

III. Hybrid Dynamic Programming

Proposition 1: Let $X = \cup_{k=1}^{N} X_k$ where $X_1, \ldots, X_N$ are closed polyhedra with disjoint interior, with $Q, \Omega_u, \Omega_\mu, f_q$, and ν defined as in Definition 1. Let $s : X \times Q \times Q \mapsto (0, \infty]$ and for $q \in Q$ let $l_q : X \times \Omega_u \mapsto [0, \infty]$. Suppose that $V_k \in \mathcal{C}^1(X_k \times Q, \mathbf{R})$ with $V_k(x, q) = V_j(x, q)$ for $x \in X_k \cap X_j$, $q \in Q$. Let $Y_M \subset Y \subseteq X \times Q$ and $V(x, q) = V_k(x, q)$ for $x \in X_k$, $q \in Q$. If for almost all $(x, q) \in Y \backslash Y_M$

$$0 \leq \frac{\partial V}{\partial x}(x, q) f_q(x, u) + l_q(x, u), \qquad u \in \Omega_u \tag{4}$$

$$0 \leq V(x, \nu(x, q, \mu)) - V(x, q) + s(x, q, \nu(x, q, \mu)), \quad \mu \in \Omega_\mu \tag{5}$$

$$0 \geq V(x_M, q_M) \qquad \forall\, (x_M, q_M) \in Y_M \tag{6}$$

then

$$\int_{t_0}^{t_M} l_{q(t)}(x(t), u(t))dt + \sum_{k=1}^{M} s(x(t_k), q(t_{k-1}), \nu(x(t_k), q(t_{k-1}), \mu(t_k))) \geq V(x(t_0), q(t_0))$$

for every solution to (1) that is contained in Y with $(x, q)(t_M) \in Y_M$. □

Proof: Let $\hat{u}(\cdot)$ and $\hat{\mu}(\cdot)$ be control signals that drive the system from the initial state $(x_0, q_0) \in Y$ at time t_0 to $(x_M, q_M) \in Y_M$ at time t_M. Let $x_k = x(t_k)$ and $q_k = q(t)$, $t_k \leq t < t_{k+1}$. Then

$$\begin{aligned} &J(x_0, q_0, \hat{u}(\cdot), \hat{\mu}(\cdot), t_M, M) \\ &\quad = \sum_{k=0}^{M-1} \int_{t_k}^{t_{k+1}} l_{q_k}(x, \hat{u})dt \\ &\qquad + \sum_{k=1}^{M} s(x_k, q_{k-1}, \nu(x_k, q_{k-1}, \hat{\mu}(t_k))) \\ &\quad \geq \sum_{k=0}^{M-1} \int_{t_k}^{t_{k+1}} -\frac{\partial V}{\partial x}(x, q_k) f_{q_k}(x, \hat{u})dt \\ &\qquad + \sum_{k=1}^{M} \{V(x_k, q_{k-1}) - V(x_k, q_k)\} \\ &\quad = \sum_{k=0}^{M-1} \{V(x_k, q_k) - V(x_{k+1}, q_k)\} \\ &\qquad + \sum_{k=1}^{M} \{V(x_k, q_{k-1}) - V(x_k, q_k)\} \\ &\quad = V(x_0, q_0) - V(x_M, q_M) = V(x_0, q_0). \end{aligned}$$

□

For the purely discrete case, the value function

$$V^\star(x, q) \equiv \min_{u(\cdot), \mu(\cdot), t_M, M} J(x_0, q_0, u(\cdot), \mu(\cdot), t_M, M)$$

satisfies the linear constraints (5)–(6), i.e., $\sup V = V^\star$. Continuous dynamics adds difficulty, however, and the aforementioned bound may in general not be tight, i.e., $\sup V(x, q) \leq V^\star(x, q)$.

For purely continuous systems, conditions for tightness have been derived in [19]. The theory needed, however, is quite advanced and an extension to the hybrid case falls outside the scope of this note.

IV. Discretization

Utilizing a computer to solve (4)–(6) for a specific control problem, a straight forward approach is to grid the state space and require the inequalities to be met at a set of uniformly distributed points in Y. This approximation will, however, not guarantee a lower bound on the optimal cost, unless the nature of f_q and V *between* the grid points is taken into consideration.

For uniform gridding of $\mathbf{R}^2$, let

$$x_{jk} = jhe_1 + khe_2, \qquad j,k \in \mathbf{Z}; \quad h \in \mathbf{R}^+ \tag{7}$$

where e_1 and e_2 are unit vectors along the coordinate axes, and h is the grid size. Also, let

$$\begin{aligned}
X^{jk} =& \{x_{jk} + \theta_1 he_1 + \theta_2 he_2 : -1 \le \theta_i \le 1\} && (8)\\
\left(\underline{f}_q^{jk}\right)_i =& \min_{x\in X^{jk}, u\in\Omega_u} (f_q(x,u))_i && (9)\\
\left(\overline{f}_q^{jk}\right)_i =& \max_{x\in X^{jk}, u\in\Omega_u} (f_q(x,u))_i && (10)\\
\underline{l}_q^{jk} =& \min_{x\in X^{jk}, u\in\Omega_u} l_q(x,u) && (11)\\
V_q^{jk} =& V(x_{jk},q) && (12)\\
\Delta_i V_q^{jk} =& \frac{V(x_{jk}+he_i,q) - V(x_{jk},q)}{h} && (13)\\
\Delta_{-i} V_q^{jk} =& \frac{V(x_{jk},q) - V(x_{jk}-he_i,q)}{h} && (14)
\end{aligned}$$

where $(\cdot)_i$ denotes the i:th vector component of $(\cdot)$.

For $A \subset \mathbf{R}^2 \times Q$, define the index set

$$I(A) = \{(j,k,q) | j,k \in \mathbf{Z},\ q \in Q,\ (x_{jk},q) \in A\}. \tag{15}$$

One possible finite approximation of (4)–(6) is then given by

$$\begin{aligned}
0 \le& \left(\lambda_q^{jk}\right)_1 + \left(\lambda_q^{jk}\right)_2 + \underline{l}_q^{jk}\\
(j,k,q) \in& I(Y\backslash Y_M) && (16)\\
\left(\lambda_q^{jk}\right)_{|i|} \le& \left(\underline{f}_q^{jk}\right)_{|i|} \Delta_i V_q^{jk}\\
i \in& \{-2,-1,1,2\},\ \ (j,k,q) \in I(Y\backslash Y_M) && (17)\\
\left(\lambda_q^{jk}\right)_{|i|} \le& \left(\overline{f}_q^{jk}\right)_{|i|} \Delta_i V_q^{jk}\\
i \in& \{-2,-1,1,2\},\ \ (j,k,q) \in I(Y\backslash Y_M) && (18)\\
0 \le& V_{\nu(x_{jk},q,\mu)}^{jk} - V_q^{jk} + s(x_{jk},q,\nu(x_{jk},q,\mu))\\
(j,k,q) \in& I(Y\backslash Y_M), \quad \mu \in \Omega_\mu && (19)\\
0 \ge& V_q^{jk} \quad (j,k,q) \in I(Y_M) && (20)
\end{aligned}$$

where $\lambda_q^{jk} \in \mathbf{R}^2$ for $(j,k,q) \in I(Y\backslash Y_M)$.

The constraints (16)–(18) form a combination of backward and forward difference approximations of (4) where the variable λ_q^{jk}, whose i:th component is an approximation of $(\partial V_q^{jk}/\partial x_i) f_q$, is used to preserve the lower bound property of the continuous inequality.

For $x = x_{jk} + \theta_1 he_1 + \theta_2 he_2$, where $0 \le \theta_i \le 1$, define the interpolating function

$$\begin{aligned}
V(x,q) =& (1-\theta_1)(1-\theta_2)V_q^{jk} + \theta_1(1-\theta_2)V_q^{(j+1)k}\\
&+(1-\theta_1)\theta_2 V_q^{j(k+1)} + \theta_1\theta_2 V_q^{(j+1)(k+1)}. && (21)
\end{aligned}$$

Theorem 1 (Discretization in $\mathbf{R}^2$): Define $Q, \Omega_u, \Omega_\mu, f_q, \nu, Y$, and Y_M as in Proposition 1. With definitions (7)–(15), and (21), if there exist $V_q^{jk} \in \mathbf{R}$ for $(j,k,q) \in I(Y)$ and $\lambda_q^{jk} \in \mathbf{R}^2$ for $(j,k,q) \in I(Y\backslash Y_M)$ that satisfy (16)–(20) then

$$\begin{aligned}
\int_{t_0}^{t_M} l_{q(t)}(x(t),u(t))dt + \sum_{k=1}^{M} s(x(t_k), q(t_{k-1})\\
\nu(x(t_k), q(t_{k-1}), \mu(t_k))) \ge V(x(t_0), q(t_0))
\end{aligned}$$

for every solution to (1) that is contained in Y with $(x,q)(t_M) \in Y_M$. □

Remark 1: Any function that meets the constraints, even the trivial choice $V(x,q) = 0$, is a lower bound on the true cost. Thus, to yield useful bounds, $V(x,q)$ needs to be maximized subject to (16)–(20). The maximization could be carried out in either several points in Y simultaneously (by maximizing the sum of the value function in several points $(x_{jk},q) \in Y$) or in one point $(x_0,q_0) \in Y$.

For the discretized problem, different choices of maximization criteria may lead to different results, and it would be interesting to construct an example where this difference is significant. Experience from examples shows, however, that the difference between the results of a single-point and a multipoint maximization is often small, making it possible to compute the value function in a large subset of Y solving *one* LP.

Remark 2: The restriction $(x,q)(t) \in Y$ in the optimal control problem is essential. It may happen that for some initial states x_0 there exist no admissible solutions inside X. The maximization of $V(x_0,q_0)$ can then lead to arbitrarily large values.

Remark 3: The theorem is easily extended to $\mathbf{R}^n$. Define $\mathbf{j} = (\mathbf{j}_1, \mathbf{j}_2, \ldots, \mathbf{j}_n)$ and exchange jk for the new multi-index $\mathbf{j}$ in the previous inequalities. The limits of all summations and enumerations should also be adjusted. Section VI shows an example in $\mathbf{R}^3$.

Proof: Assume that $x \in X^{jk}$. Noting that $\Delta_1 V_q^{jk} = \Delta_{-1} V_q^{(j+1)k}$, $\Delta_2 V_q^{jk} = \Delta_{-2} V_q^{j(k+1)}$, the inequalities (16)–(18) taken at grid points jk, $j(k+1)$, $(j+1)k$, and $(j+1)(k+1)$ give

$$\begin{aligned}
0 \le& f_{q1}(x,u)\Delta_1 V_q^{jk} + f_{q2}(x,u)\Delta_2 V_q^{jk} + l_q(x,u) && (22)\\
0 \le& f_{q1}(x,u)\Delta_1 V_q^{j(k+1)}\\
&+ f_{q2}(x,u)\Delta_2 V_q^{jk} + l_q(x,u) && (23)\\
0 \le& f_{q1}(x,u)\Delta_1 V_q^{jk}\\
&+ f_{q2}(x,u)\Delta_2 V_q^{(j+1)k} + l_q(x,u) && (24)\\
0 \le& f_{q1}(x,u)\Delta_1 V_q^{j(k+1)}\\
&+ f_{q2}(x,u)\Delta_2 V_q^{(j+1)k} + l_q(x,u). && (25)
\end{aligned}$$

The gradient of V is given by

$$\frac{\partial V_q}{\partial x} = \begin{bmatrix} (1-\theta_2)\Delta_1 V_q^{jk} + \theta_2 \Delta_1 V_q^{j(k+1)} \\ (1-\theta_1)\Delta_2 V_q^{jk} + \theta_1 \Delta_2 V_q^{(j+1)k} \end{bmatrix}^T$$

and, thus, adding (22)–(25) weighted with $(1-\theta_1)(1-\theta_2)$, $(1-\theta_1)\theta_2$, $\theta_1(1-\theta_2)$, and $\theta_1\theta_2$, respectively, proves that (4) is met for x. The inequality (5) holds since V is a convex combination of grid points that all meet (19), and (6) is the same condition as (20). □

Note that the minimization/maximization in (9)–(11) is in general not convex. However, Theorem 1 can be applied with any upper and lower bounds on f_q and l_q and such bounds are often easy to obtain.

Also, note a special case in which the burden of the local optimizations in Theorem 1 is lightened: if $f_q(x,u) = h_q(x) + g_q(x)u$ and $l_q(x,u) = o_q(x) + m_q(x)u$ while $\Omega_u = [-1,1]$, then u can be entirely eliminated from (16)–(18) by replacing $\underline{f}_q^{jk}$, $\overline{f}_q^{jk}$, and $\underline{l}_q^{jk}$ with $\underline{h}_q^{jk} \pm \underline{g}_q^{jk}$, $\overline{h}_q^{jk} \pm \overline{g}_q^{jk}$, and $\underline{o}_q^{jk} \pm \underline{m}_q^{jk}$, respectively. This will double the set of equations (16)–(18), but the functions h_q, g_q, o_q, and m_q are optimized over X^{jk} solely.

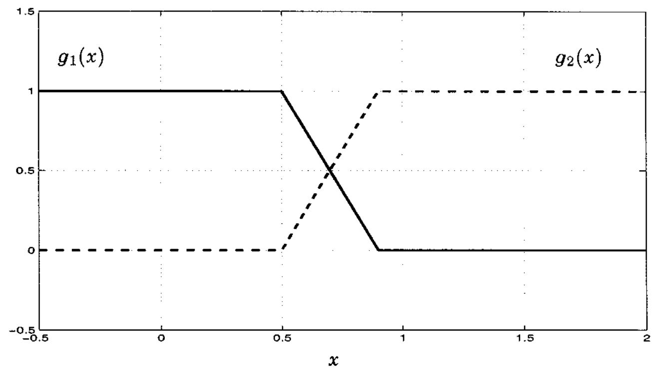

Fig. 1. Gear profiles for the truck.

V. Computing the Control Law

Provided that the lower bound V is a good enough approximation of the optimal cost, the optimal feedback control law can be calculated as

$$\begin{cases} u(x,q) = \underset{\hat{u}\in\Omega_u}{\operatorname{argmin}} \left\{ \frac{\partial V}{\partial x}(x,q) f_q(x,\hat{u}) + l_q(x,\hat{u}) \right\} \\ \mu(x,q) = \underset{\hat{\mu}\in\Omega_\mu | x\in S_{q,\nu}}{\operatorname{argmin}} \{ V(x,\nu) + s(x,q,\nu) \} \end{cases} \tag{26}$$

where $\nu = \nu(x,q,\hat{\mu})$. Note that the discrete input μ is chosen such that switching occur whenever there exist a discrete mode for which the value function has a lower value than the cost of the value function for the current mode minus the cost for switching there.

Consider the true optimal value function $V^\star$. For those (x,q,r) where the optimal trajectory requires mode switching, (4) will turn to equality, i.e., $V^\star(x,q) = V^\star(x,r) + s(x,q,r)$. A consequence of this is that for (26) to describe correct switching between the modes, $s(x,q,q)$ has to be defined as $s(x,q,q) = \varepsilon > 0$ (rather than the natural choice $s(x,q,q) = 0$). For $V^\star$, the proper control law is achieved as ε approaches 0. A small value of ε suffices, however, for numerical computations.

In practice, it is suitable to discretize u into $\hat{\Omega}_u = \{u_1, u_2, \ldots, u_a\} \subset \Omega_u$. Then, for each grid point x^{jk}, the problem is to find $u_q^{jk} \in \hat{\Omega}_u$ and $\mu_q^{jk} \in \Omega_\mu$ offline that minimize (26). Online, during control, $u(x,q)$ and $\mu(x,q)$ are obtained by multilinear interpolation analogously to (21).

The choice of $\hat{\Omega}_u$ may be crucial and is often a trade off between speed of computations and how close to optimal the result will be. For time optimal control problems where u enters affinely, however, the control signal only assumes its extremal values.

VI. Numerical Example—A Truck With a Flexible Transmission

The applicability of the theory is here illustrated by an example with three continuous states (see [10] for additional examples). Consider the system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \frac{1}{m}(-cx_2 + kx_3) \\ \dot{x}_3 = -x_2 + \frac{g_q(x_2)}{k} u, \qquad q = 1,2 \; -0.1 \le u \le 1.1. \end{cases} \tag{27}$$

The three continuous states of the system could be seen as position (x_1) and velocity (x_2) of a truck, and the rotational displacement of its transmission shaft (x_3). There are two discrete modes corresponding to different gears of the truck; the input throttle u is weighted by $g_q(x)$ which represents the efficiency of gear number q. The weighting functions are plotted in Fig. 1.

All the constants (the mass of the car m, the frictional damping c, and the spring constant of the transmission shaft k) are set to one.

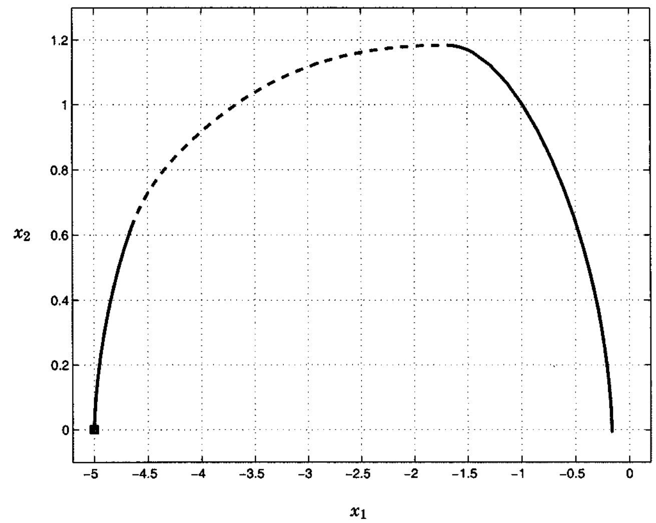

Fig. 2. Phase portrait of x_1 and x_2 under simulation. The solid line shows where gear number one has been used and the dashed line shows the second gear. The initial point is marked with a square.

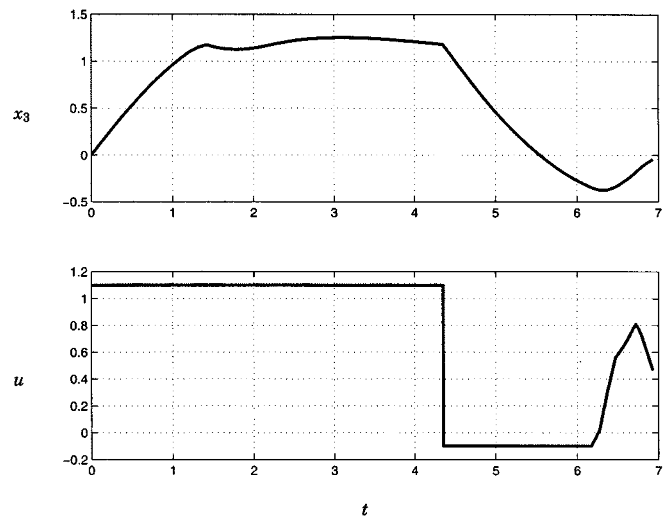

Fig. 3. Plot of spring tension (x_3) and the continuous valued control signal (u).

The objective is to bring (27) to $Y_M = \{(0,0)\}$ in minimum time. Torque losses when using the clutch call for an additional penalty for gear changes. Thus, the terms of (3) have been chosen as $l_1(x,u) = l_2(x,u) = 1$, $s(x,1,2) = s(x,2,1) = 0.8$.

Since it is difficult to visualize the three-dimensional value function, it is not shown here. A feedback control law is derived from the value function, however, and results from simulations using this law are shown in Fig. 2.

With the current cost function, it is obvious that whenever a gear switch is required, it is optimal to switch at the speed of equal efficiency between the gears $(x_2 = 0.7)$. This action can be noted in the figure when switching from the first gear to the second. The switch back to the first gear during the deceleration phase, however, occurs in the simulation at a much higher (nonoptimal) speed. This is a reasonable approximation error though, since the deceleration power is small $(u = -0.1)$. The difference in cost depending on how early the gear switch is made, is low compared to the total cost.

Fig. 3 shows how the rotational displacement of the transmission shaft varies with u. The spring tension builds up during the acceleration phase (approximately $0 \le t \le 4.3$) and is then released.

An upper bound is obtained by integrating the cost along the simulated trajectory, starting in $x_i = (-5, 0, 0)^T$, $q_i = 1$, is 8.5. The lower bound given by the value function is 7.9.

VII. Conclusion

This note presented an extended version of the Hamilton–Jacobi–Bellman (HJB) inequality to be used for optimal control of hybrid systems. The extended version constitutes a successful marriage between computer science and control theory, containing pure discrete-dynamic programming as well as pure continuous-dynamic programming as special cases.

The extended HJB inequality, which gives a lower bound on the value function, was discretized to a finite, computer-solvable LP that preserves the lower bound property. Based on the value function, an approximation of the optimal control feedback law was derived.

A problem with DP is the "curse of dimensionality," an expression coined by Bellman, the inventor of this method. Since the cost for a family of trajectories is computed (rather than a single trajectory as in the Pontryagin maximum principle), the problem grows exponentially in the number of states.

The advantage with this method, however, is its applicability and ease of use for low-dimension systems. The discretization method presented in this note allows problems with up to three continuous states on a 336-MHz Ultra Sparc II.

A set of MATLAB commands has been compiled by the authors to make it easy to test the aforementioned methods and implement the examples. The LP solver that is used is "PCx," developed by the Optimization Technology Center, Illinois. The MATLAB commands and a manual of usage are available free of charge upon request from the authors.

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Output Violation Compensation for Systems With Output Constraints

Matthew C. Turner and Ian Postlethwaite

***Abstract*—The problem of output constraints in linear systems is considered, and a new methodology which helps the closed loop respect these limits is described. The new methodology invokes ideas from the antiwindup literature in order to address the problem from a practical point of view. This leads to a design procedure very much like that found in antiwindup design. First, a linear controller ignoring output constraints is designed. Then, an additional compensation network which ensures that the output limits are, as far as possible, respected is added. As the constraints occur at the output, global results can be obtained for both stable and unstable plants.**

***Index Terms*—Linear systems, output constraints, saturation.**

I. Introduction

The literature reveals a vast and varied treatment of linear systems subject to input, or saturation, constraints. This problem has been tackled from many different perspectives and its study has formed one of the most important topics in the control community over several decades. To avoid repeating prior work, we do not describe this work in detail; it suffices to mention that there are now several mature techniques available to cope with input constraints [1]. The amount of attention devoted to this problem is perhaps not surprising when one considers the virtual omnipresence of control constraints in real engineering systems.

Control constraints are not the only time-domain constraint present in control systems, however. In addition to constraints on the transient response of various closed-loop signals (e.g., rise time, settling time), there are sometimes "hard" or "soft" limits imposed on the magnitude of certain plant outputs, or states. These limits reflect issues such as safety requirements or are there to prevent excessive maintenance to system components. For example, in certain aircraft, during the approach to land, there is a limit on the angle of attack to prevent accidents caused by stall or pilot error, etc. Alternatively, if a certain value

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The authors are with the Department of Engineering, University of Leicester, LE1 7RH Leicester, U.K. (e-mail: mct6@le.ac.uk, ixp@le.ac.uk).

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