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# CONVEX FOLIATED PROJECTIVE STRUCTURES AND THE HITCHIN COMPONENT FOR $\mathrm{PSL}_4(\mathbf{R})$

OLIVIER GUICHARD AND ANNA WIENHARD

ABSTRACT. In this article we give a geometric interpretation of the Hitchin component  $\mathcal{T}^4(\Sigma) \subset \mathrm{Rep}(\pi_1(\Sigma), \mathrm{PSL}_4(\mathbf{R}))$  of a closed oriented surface of genus  $g \geq 2$ . We show that representations in  $\mathcal{T}^4(\Sigma)$  are precisely the holonomy representations of properly convex foliated projective structures on the unit tangent bundle of  $\Sigma$ . From this we also deduce a geometric description of the Hitchin component  $\mathcal{T}(\Sigma, \mathrm{Sp}_4(\mathbf{R}))$  of representations into the symplectic group.

## INTRODUCTION

In his article [11] N. Hitchin discovered a special connected component, the ‘‘Teichmüller component’’, of the representation variety of the fundamental group of a closed Riemann surface  $\Sigma$  into a simple adjoint  $\mathbf{R}$ -split Lie group  $G$ . He showed that the Teichmüller component, now usually called ‘‘Hitchin component’’, is diffeomorphic to a ball of dimension  $|\chi(\Sigma)| \dim(G)$ . In this article we interpret the Hitchin component for  $G = \mathrm{PSL}_4(\mathbf{R})$  as moduli space of certain locally homogeneous geometric structures. Our main result is the following

**Theorem 1.** *The Hitchin component for  $\mathrm{PSL}_4(\mathbf{R})$  is naturally homeomorphic to the moduli space of (marked) properly convex foliated projective structures on the unit tangent bundle of  $\Sigma$ .*

Let us briefly describe these geometric structures (see Section 2.2.1 for a precise definition). Properly convex foliated projective structures are locally homogeneous  $(\mathrm{PGL}_4(\mathbf{R}), \mathbb{P}^3(\mathbf{R}))$ -structures on the unit tangent bundle  $M = S\Sigma$  of the surface  $\Sigma$  satisfying the following additional conditions:

- every orbit of the geodesic flow on  $M$  is locally a projective line,
- every (weakly) stable leaf of the geodesic flow is locally a projective plane and the projective structure on the leaf obtained by restriction is convex.

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There is a natural map from the moduli space of projective structures to the variety of representation  $\pi_1(M) \rightarrow \mathrm{PGL}_4(\mathbf{R})$ . We show that the restriction of this map to the moduli space of properly convex foliated projective structures is a homeomorphism onto the Hitchin component; in particular, the holonomy representation of a properly convex foliated projective structure factors through the projection  $\pi_1(M) \rightarrow \pi_1(\Sigma)$  and takes values in  $\mathrm{PSL}_4(\mathbf{R})$ .

Appealing to N. Hitchin's result, we conclude

**Corollary 2.** *The moduli space of (marked) properly convex foliated projective structures on the unit tangent bundle of  $\Sigma$  is a ball of dimension  $-15\chi(\Sigma)$ .*

We describe (see Section 3) several examples of projective structures on  $M$ , including families of projective structures with “quasi-Fuchsian” holonomy  $\pi_1(M) \rightarrow \pi_1(\Sigma) \rightarrow \mathrm{PSL}_2(\mathbf{C}) \rightarrow \mathrm{PSL}_4(\mathbf{R})$ . Those examples show that for the holonomy representation to lie in the Hitchin component the above additional conditions cannot be weakened.

Geometric interpretations of the Hitchin component were previously known when  $G$  is  $\mathrm{PSL}_2(\mathbf{R})$  or  $\mathrm{PSL}_3(\mathbf{R})$ . For  $\mathrm{PSL}_2(\mathbf{R})$  the Hitchin component is the image of the embedding of the Fricke-Teichmüller space of  $\Sigma$  into  $\mathrm{Rep}(\pi_1(\Sigma), \mathrm{PSL}_2(\mathbf{R}))$  obtained by associating to a (marked) hyperbolic structure its holonomy homomorphism. For  $\mathrm{PSL}_3(\mathbf{R})$ , S. Choi and W. Goldman showed in [4] that the Hitchin component is homeomorphic to the moduli space of (marked) convex real projective structures on  $\Sigma$ . In both cases it can be proved directly that these moduli spaces of geometric structures are balls of the expected dimension (see [10] for the  $\mathrm{PSL}_3$ -case).

As a consequence of Theorem 1 we obtain a similar description for the Hitchin component of the symplectic group. The symplectic form on  $\mathbf{R}^4$  induces a natural contact structure on  $\mathbb{P}^3(\mathbf{R})$  and  $\mathrm{PSp}_4(\mathbf{R})$  is the maximal subgroup of  $\mathrm{PSL}_4(\mathbf{R})$  preserving this contact structure. We call a locally homogeneous structure modeled on  $\mathbb{P}^3(\mathbf{R})$  with this contact structure a projective contact structure.

**Theorem 3.** *The Hitchin component for  $\mathrm{PSp}_4(\mathbf{R})$  is naturally homeomorphic to the moduli space of (marked) properly convex foliated projective contact structures on the unit tangent bundle  $M = S\Sigma$ .*

The symplectic form on  $\mathbf{R}^4$  gives rise to an involution on the representation variety  $\mathrm{Rep}(\pi_1(\Sigma), \mathrm{PSL}_4(\mathbf{R}))$  and on the Hitchin component for  $\mathrm{PSL}_4(\mathbf{R})$ . The set of fixed points of this involution is the Hitchin component for  $\mathrm{PSp}_4(\mathbf{R})$ .

As above we conclude

**Corollary 4.** *The moduli space of (marked) properly convex foliated projective contact structures on the unit tangent bundle  $M = S\Sigma$  is a ball of dimension  $-10\chi(\Sigma)$ .*

Our results rely on recent progress in understanding representations in the Hitchin component for  $\mathrm{PSL}_n(\mathbf{R})$  made by F. Labourie [19]. In particular, he proved that all representations in the Hitchin components are faithful, discrete and semisimple. V. Fock and A. Goncharov [7] showed that representations in the Hitchin component for a general simple  $\mathbf{R}$ -split Lie group have the same properties, using among other things the positivity theory for Lie groups developed by G. Lustzig. More important for our work is that F. Labourie (supplemented by the first author [12]) gives the following geometric characterization of representations inside the Hitchin component of  $\mathrm{PSL}_n(\mathbf{R})$ .

**Theorem 5** (Labourie [19], Guichard [12]). *A representation  $\rho : \pi_1(\Sigma) \rightarrow \mathrm{PSL}_n(\mathbf{R})$  lies in the Hitchin component if, and only if, there exists a continuous  $\rho$ -equivariant convex curve  $\xi^1 : \partial\pi_1(\Sigma) \rightarrow \mathbb{P}^{n-1}(\mathbf{R})$ .*

A curve  $\xi^1 : \partial\pi_1(\Sigma) \rightarrow \mathbb{P}^{n-1}(\mathbf{R})$  is said to be *convex* (see Definition 4.1) if for every  $n$ -tuple of pairwise distinct points in  $\partial\pi_1(\Sigma)$  the corresponding lines are in direct sum. Convex curves into  $\mathbb{P}^2(\mathbf{R})$  are exactly injective maps whose image bounds a strictly convex domain in  $\mathbb{P}^2(\mathbf{R})$ .

It is easy to prove that the existence of such a curve for  $\mathrm{PSL}_2(\mathbf{R})$  implies that the representation is in the Teichmüller space (see Lemma A.2).

Let us quickly indicate that Theorem 5 is equivalent to the result of S. Choi and W. Goldman [4] that the representations in the Hitchin component for  $\mathrm{PSL}_3(\mathbf{R})$  are precisely the holonomy representations of (marked) convex real projective structure on  $\Sigma$ .

A (marked) convex real projective structure on  $\Sigma$  is a pair  $(N, f)$ , where  $N$  is a convex real projective manifold, that is  $N$  is the quotient  $\Omega/\Gamma$  of a strictly convex domain  $\Omega$  in  $\mathbb{P}^2(\mathbf{R})$  by a discrete subgroup  $\Gamma$  of  $\mathrm{PSL}_3(\mathbf{R})$ , and  $f : \Sigma \rightarrow N$  is a diffeomorphism.

Given a representation  $\rho : \pi_1(\Sigma_g) \rightarrow \mathrm{PSL}_3(\mathbf{R})$  in the Hitchin component for  $\mathrm{PSL}_3(\mathbf{R})$ , let  $\Omega_\xi \subset \mathbb{P}^2(\mathbf{R})$  be the strictly convex domain bounded by the convex curve  $\xi^1(\partial\pi_1(\Sigma)) \subset \mathbb{P}^2(\mathbf{R})$ . Then  $\rho(\pi_1(\Sigma))$  is a discrete subgroup of the group of Hilbert isometries of  $\Omega_\xi$  and hence acts freely and properly discontinuously on  $\Omega_\xi$ . The quotient  $\Omega_\xi/\rho(\pi_1(\Sigma))$  is a real projective convex manifold, diffeomorphic to  $\Sigma$ . Conversely given a real projective structure on  $\Sigma$ , we can  $\rho$ -equivariantly identify  $\partial\pi_1(\Sigma)$  with the boundary of  $\Omega$  and get a convex curve  $\xi^1 : \partial\pi_1(\Sigma) \rightarrow \partial\Omega \subset \mathbb{P}^2(\mathbf{R})$ .

Our starting point to associate a geometric structure to a representation  $\rho : \pi_1(\Sigma) \rightarrow \mathrm{PSL}_4(\mathbf{R})$  in the Hitchin component was to look for domains of discontinuity for the action of  $\rho(\pi_1(\Sigma))$  in  $\mathbb{P}^{n-1}(\mathbf{R})$ , similar to  $\Omega_\xi$  for  $\mathrm{PSL}_3(\mathbf{R})$ . It turns out that for this it is useful to consider the convex curve  $\xi^{n-1} : \partial\pi_1(\Sigma) \rightarrow \mathrm{Gr}_{n-1}^n(\mathbf{R}) \simeq \mathbb{P}^{n-1}(\mathbf{R})^*$  associated to the contragredient representation of  $\rho$ . For example, the set  $\Lambda \subset \mathbb{P}^{n-1}(\mathbf{R})$  of lines which are contained in  $n-1$  distinct hyperplanes  $\xi^{n-1}(t)$  is invariant and the action of  $\Gamma$  on it is free and properly discontinuous when  $n \geq 4$ , but cocompact only if  $n = 4$ . The open set  $\Omega = \mathbb{P}^{n-1}(\mathbf{R}) - \overline{\Lambda}$ , which coincides with  $\Omega_\xi$  for

$n = 3$ , turns out to have the right topology and cohomological dimension in order to admit a cocompact action of  $\rho(\pi_1(\Sigma))$ . Unfortunately, the action on  $\Omega$  is proper only for  $n = 3$  and  $n = 4$  ( $\Omega$  is empty for  $n = 2$ ).

When  $n = 4$ , the case of our concern, the action of  $\rho(\pi_1(\Sigma))$  on  $\mathbb{P}^3(\mathbf{R})$  is proper (see Paragraph 4.4) precisely on the complement of the ruled surface given by the union of projective lines tangents to the curve  $\xi^1(\partial\pi_1(\Sigma))$ . This complement is an open set which has two connected components, namely  $\Omega$  and  $\Lambda$ . The quotient  $\Omega/\rho(\pi_1(\Sigma))$  is a projective manifold homeomorphic to the unit tangent bundle  $M$  of  $\Sigma$  and induces a properly convex foliated projective structure on  $M$ . The quotient  $\Lambda/\rho(\pi_1(\Sigma))$  is a projective manifold which is homeomorphic to a quotient of  $M$  by  $\mathbf{Z}/3\mathbf{Z}$  and induces a projective structure on  $M$  which is foliated, but not properly convex.

The construction of these domains of discontinuity gives a map from the Hitchin component to the moduli space of properly convex foliated structures on  $M$ . Conversely, starting with a properly convex foliated projective structure on  $M$  we construct an equivariant convex curve and show that the projective structure is obtained by the above construction. This gives a more precise description of the content of Theorem 1.

Let us conclude with some open questions. One might try to describe how properly convex foliated projective structures on the unit tangent bundle of  $\Sigma$  can be glued from or decomposed into simpler pieces similar to pair of pants decompositions of a hyperbolic surface. This would probably lead to some generalized Fenchel-Nielsen coordinates for the Hitchin component for  $\mathrm{PSL}_4(\mathbf{R})$  and hopefully to a geometric proof that the Hitchin component is a ball. Gluing convex projective manifolds (with boundary) is one of the tools in W. Goldman's work [10] and was also used in a recent work of M. Kapovich [16] to produce convex projective structures on the manifolds, constructed by M. Gromov and W. Thurston in [11], that admit Riemannian metrics of pinched negative sectional curvature but no metrics of constant sectional curvature. This approach might be extendable to the construction of convex foliated projective manifolds, where the first and subtle point is to find natural submanifolds at which one should cut and glue.

The interpretation of the Hitchin component for  $\mathrm{PSL}_3(\mathbf{R})$  as holonomy representations of convex real projective structures plays an important role in independent work of F. Labourie [18] and J. Loftin [20] who associate to a convex real projective structure on  $\Sigma$  a complex structure and a cubic differential on  $\Sigma$ . One might hope that our interpretation of the Hitchin component for  $\mathrm{PSL}_4(\mathbf{R})$  as holonomy representations of properly convex foliated projective structures on  $M$  could help to associate a complex structure, a cubic and a quartic differential on  $\Sigma$  to every such representation. This would answer, for  $\mathrm{PSL}_4(\mathbf{R})$ , a conjecture of F. Labourie describing the quotient of the Hitchin component by the modular group.

Turning to higher dimensions, one might suspect that there is a moduli space of suitable geometric structures associated to the Hitchin component

for  $\mathrm{PSL}_n(\mathbf{R})$ . As we alluded to above, using the convex curve, natural domains of discontinuity for a convex representation can be described, but in general none will admit a cocompact action of  $\pi_1(\Sigma)$ . To find the right geometric structures on a suitable object associated to  $\Sigma$  for general  $n$  seems to be a delicate and challenging problem.

In this paper, we concentrate on the projective manifold  $\Omega/\rho(\pi_1(\Sigma))$  obtained from one of the connected components of the domain of discontinuity for a representation in the Hitchin component for  $\mathrm{PSL}_4(\mathbf{R})$ . The projective manifolds  $\Lambda/\rho(\pi_1(\Sigma))$  obtained from the other connected component also satisfy some additional properties, and it should be possible to obtain a similar description for the Hitchin component from them.

In a subsequent paper, we will concentrate more specifically on the question of describing the Hitchin component for  $\mathrm{PSp}_4(\mathbf{R})$  by geometric structures modeled on the space of Lagrangians rather than on projective space. Using the isomorphism  $\mathrm{PSp}_4(\mathbf{R}) = \mathrm{SO}^\circ(2, 3)$  Hitchin representations give rise to flat conformal Lorentzian structures on  $M$ . Since the Hitchin component of  $\mathrm{SO}^\circ(2, 3)$  embeds into the Hitchin component of  $\mathrm{PSL}_5(\mathbf{R})$  this might also help to understand the geometric picture for  $\mathrm{PSL}_5(\mathbf{R})$ .

**Structure of the paper:** In Section 1 we review some classical facts about the unit tangent bundle  $M = S\Sigma$  and fix some conventions and notation which are used throughout the paper. Properly convex foliated projective structures are introduced in Section 2, where we also recall some basic facts about locally homogeneous geometric structures. In Section 3 we describe several examples of projective structures on  $M$  whose properties justify the definitions made in Section 2. After reviewing the example of a properly convex foliated projective structure given in Section 3 we construct in Section 4 a properly convex foliated projective structure on  $M$  starting from the convex curve associated to a representation in the Hitchin component. Conversely, in Section 5 we construct an equivariant convex curve starting from a properly convex foliated projective structure on  $M$ , and show that every properly convex foliated projective structure on  $M$  arises by the construction given in Section 4. The consequences for representations into the symplectic group are discussed in Section 6. In the Appendix A we collect some useful facts.

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## 1. CONVENTIONS

In this section we review basic facts about the geometry of surfaces and their unit tangent bundle and introduce some notation. Everything is classical except maybe Section 1.2.

**1.1. The Unit Tangent Bundle.** Let  $\Sigma$  be a connected oriented closed surface of genus  $g \geq 2$  and  $\Gamma = \pi_1(\Sigma, x)$  its fundamental group. We denote by  $\tilde{\Sigma}$  the universal covering of  $\Sigma$ .

**Notation 1.1.** We denote by  $M$  the circle bundle associated to the tangent bundle of  $\Sigma$ .  $M$  is homeomorphic to the unit tangent bundle of  $\Sigma$  with respect to any Riemannian metric on  $\Sigma$ .

The fundamental group  $\bar{\Gamma} = \pi_1(M, m)$  is a central extension of  $\Gamma$

$$1 \longrightarrow \mathbf{Z} \longrightarrow \bar{\Gamma} \xrightarrow{p} \Gamma \longrightarrow 1.$$

This central extension and the group  $\Gamma$  admit the classical presentations:

$$\begin{aligned} \Gamma &= \langle a_1, \dots, a_g, b_1, \dots, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle \\ \bar{\Gamma} &= \langle a_1, \dots, a_g, b_1, \dots, b_g, \tau \mid [a_1, b_1] \cdots [a_g, b_g] \tau^{2g}, [a_i, \tau], [b_i, \tau] \rangle, \end{aligned}$$

where  $[a, b] = aba^{-1}b^{-1}$  is the commutator of two elements  $a, b$ .

There is an important covering  $\bar{M} = S\tilde{\Sigma}$ , the unit tangent bundle of  $\tilde{\Sigma}$  which is a Galois covering of  $M$  with covering group  $\Gamma$ . Topologically  $S\tilde{\Sigma}$  is the product  $S^1 \times \mathbf{R}^2$  so the universal cover  $\tilde{\bar{M}} = \widetilde{\bar{M}}$  is a 3-dimensional ball and the covering  $\tilde{\bar{M}} \rightarrow \bar{M}$  is an abelian covering with covering group  $\mathbf{Z}$ .

**1.1.1. Canonical Foliations.** We fix for a moment a hyperbolic metric on the surface  $\Sigma$ , that is a Riemannian metric of constant sectional curvature  $-1$ . The induced geodesic flow  $g_t : M \rightarrow M$  on the unit tangent bundle of  $\Sigma$  is Anosov (see [17, § 17.4–6]). In particular,  $M$  is endowed with two codimension one foliations, namely the (weakly) stable foliation and the (weakly) unstable foliation of the geodesic flow, and a codimension two foliation given by the flow lines, which we call the geodesic foliation.

**Notation 1.2.** We denote by  $\mathcal{F}$  the set of leaves of the (weakly) stable foliation and by  $\mathcal{G}$  the set of leaves of the geodesic foliation.

The set  $\mathcal{G}$  coincides with the set of (unparametrized) oriented geodesics in  $\Sigma$ . Correspondingly the lifts of the geodesic flow to  $\bar{M}$  and  $\tilde{\bar{M}}$  induce foliations denoted  $\bar{\mathcal{F}}, \bar{\mathcal{G}}$  and respectively  $\tilde{\mathcal{F}}, \tilde{\mathcal{G}}$ .

**Notation 1.3.** A typical element of  $\mathcal{G}$  will be denoted by  $g$  and a typical element of  $\mathcal{F}$  by  $f$ , similarly for  $\bar{\mathcal{F}}, \bar{\mathcal{G}}$  and  $\tilde{\mathcal{F}}, \tilde{\mathcal{G}}$ .

The sets  $\bar{\mathcal{F}}, \bar{\mathcal{G}}$  and  $\tilde{\mathcal{F}}, \tilde{\mathcal{G}}$  carry a natural topology coming from the Hausdorff distance on subsets, and admit a natural action of the corresponding covering groups  $\Gamma$  and  $\bar{\Gamma}$  respectively.



1.1.2. *A Topological Description of the Foliations.* The geodesic flow on  $M$  depends on the choice of hyperbolic metric on  $\Sigma$ , but it is well known that the foliations  $\mathcal{F}$  and  $\mathcal{G}$  admit a description which shows that topologically they are indeed independent of the metric. We recall this description briefly.

The group  $\Gamma$  is a hyperbolic group and hence there is a canonical boundary at infinity of  $\Gamma$  (see [8] for definition and properties).

**Notation 1.4.** *The boundary at infinity of the group  $\Gamma$  is denoted by  $\partial\Gamma$ . It is a topological circle with a natural  $\Gamma$ -action.*

The dynamics of the action of any element  $\gamma \in \Gamma - \{1\}$  is well understood; the element  $\gamma$  has exactly two fixed points  $t_{+,\gamma}, t_{-,\gamma} \in \partial\Gamma$ . For any  $t \in \partial\Gamma$  distinct from  $t_{\mp,\gamma}$  we have  $\lim_{n \rightarrow \pm\infty} \gamma^n \cdot t = t_{\pm,\gamma}$ .

Let us define  $\partial\Gamma^{(2)} := \partial\Gamma^2 - \Delta$ , where  $\Delta = \{(t, t) \mid t \in \partial\Gamma\}$  is the diagonal in  $\partial\Gamma^2$ . Then we have the following classical facts.

**Lemma 1.5.** *The action of  $\Gamma$  on  $\partial\Gamma$  is minimal.*

*The subset of pairs of fixed points  $\{(t_{+,\gamma}, t_{-,\gamma}) \mid \gamma \neq 1\}$  is dense in  $\partial\Gamma^{(2)}$ .*

The hyperbolic metric on  $\Sigma$  isometrically identifies  $\tilde{\Sigma}$  with the hyperbolic plane  $\mathbb{H}^2$ . Let  $\iota : \Gamma \rightarrow \mathrm{PSL}_2(\mathbf{R}) = \mathrm{Isom}^+(\mathbb{H}^2)$  be the homomorphism which makes this identification equivariant. Such a homomorphism is faithful and discrete, and will be called a *Fuchsian* representation or a uniformization. Then any orbital application  $\Gamma \rightarrow \mathbb{H}^2$ ,  $\gamma \mapsto \iota(\gamma) \cdot x_0$  is a quasiisometry, and hence induces a  $\Gamma$ -equivariant homeomorphism  $\partial\Gamma \xrightarrow{\sim} \partial\mathbb{H}^2$ , which is easily seen to be independent from the base point  $x_0$ . Using the upper half space model for  $\mathbb{H}^2$  the boundary  $\partial\mathbb{H}^2$  is identified with the projective line  $\mathbb{P}^1(\mathbf{R})$  with the natural  $\mathrm{PSL}_2(\mathbf{R})$  action.

The orientation of  $\Sigma$  induces an orientation on  $\partial\Gamma \simeq \partial\tilde{\Sigma} \simeq S_x\tilde{\Sigma}$ , for any  $x$  in  $\tilde{\Sigma}$ . This enables us say when a triple of pairwise distinct points of the boundary is positively oriented.

**Notation 1.6.** *We denote by  $\partial\Gamma^{3+}$  the subset of  $\partial\Gamma^3$  consisting of pairwise distinct positively oriented triples.*

The unit tangent bundle  $\overline{M} = S\tilde{\Sigma}$  can be  $\Gamma$ -equivariantly identified with  $\partial\Gamma^{3+}$

$$\begin{aligned} \overline{M} &\longrightarrow \partial\Gamma^{3+} \\ v &\longmapsto (t_+, t_0, t_-), \end{aligned}$$

where  $t_+$  is the endpoint at  $+\infty$ ,  $t_-$  is the endpoint at  $-\infty$  of the geodesic  $g_v$  defined by  $v$ , and  $t_0$  is the unique endpoint in  $\partial\Gamma \cong S^1$  of the geodesic perpendicular to  $g_v$  at the foot point of  $v$  such that  $(t_+, t_0, t_-)$  is positively oriented (see Figure 1).

In this model the leaves of the geodesic and (weakly) stable foliations  $\overline{\mathcal{G}}$ ,  $\overline{\mathcal{F}}$  of  $\overline{M}$  through the point  $v = (t_+, t_0, t_-)$  are explicitly given by

$$\begin{aligned} \overline{g}_v &= \{(s_+, s_0, s_-) \in \partial\Gamma^{3+} \mid s_+ = t_+, s_- = t_-\} \\ \text{and } \overline{f}_v &= \{(s_+, s_0, s_-) \in \partial\Gamma^{3+} \mid s_+ = t_+\}. \end{aligned}$$

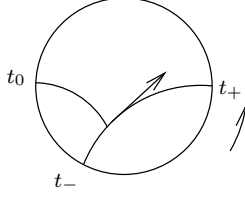


FIGURE 1. A positively oriented triple

In particular, the set of leaves of the geodesic foliation  $\overline{\mathcal{G}}$  on  $\overline{M}$  is  $\Gamma$ -equivariantly identified with  $\partial\Gamma^{(2)}$ , where the oriented geodesic  $\overline{g}_v$  through  $v = (t_+, t_0, t_-)$  is identified with  $(t_+, t_-) \in \partial\Gamma^{(2)}$ . The set of (weakly) stable leaves  $\overline{\mathcal{F}}$  is  $\Gamma$ -equivariantly identified with  $\partial\Gamma$  by mapping the (weakly) stable leaf  $\overline{f}_v$  to  $t_+ \in \partial\Gamma$ .

From now on, we will not distinguish anymore between a (weakly) stable leaf seen *as a subset of  $\overline{M}$*  or as *an element of  $\overline{\mathcal{F}}$*  or as *an element of  $\partial\Gamma$* . So for example  $m \in t \in \partial\Gamma$  will denote a point  $m$  of  $\overline{M}$  in the leaf of  $\overline{\mathcal{F}} \simeq \partial\Gamma$  corresponding to  $t$ . We will use a similar language for geodesics leaves.

1.1.3. *Identifying  $\overline{M}$  with  $\mathrm{PSL}_2(\mathbf{R})$ .* We will sometimes consider  $M$  as the quotient of  $\mathrm{PSL}_2(\mathbf{R})$  by the subgroup  $\iota(\Gamma) < \mathrm{PSL}_2(\mathbf{R})$  and identify leaves of the foliations  $\mathcal{G}$  and  $\mathcal{F}$  with orbits of subgroups of  $\mathrm{PSL}_2(\mathbf{R})$ .

The isometry  $\tilde{\Sigma} \simeq \mathbb{H}^2$ , which is  $\iota$ -equivariant for some  $\iota : \Gamma \rightarrow \mathrm{PSL}_2(\mathbf{R})$ , induces a diffeomorphism  $\overline{M} = S\tilde{\Sigma} \simeq S\mathbb{H}^2$  of the unit tangent bundles. Since the action of  $\mathrm{PSL}_2(\mathbf{R})$  on  $S\mathbb{H}^2$  is simply transitive, we can identify  $S\mathbb{H}^2$  with  $\mathrm{PSL}_2(\mathbf{R})$  and obtain a diffeomorphism:

$$M = \Gamma \backslash \overline{M} \xrightarrow{\sim} \iota(\Gamma) \backslash \mathrm{PSL}_2(\mathbf{R}).$$

Note that  $\mathrm{PSL}_2(\mathbf{R})$  acts by right multiplication on itself and hence also on  $\iota(\Gamma) \backslash \mathrm{PSL}_2(\mathbf{R})$ .

**Lemma 1.7.** *Under these identifications  $M \simeq \iota(\Gamma) \backslash \mathrm{PSL}_2(\mathbf{R})$  and  $\overline{M} \simeq \mathrm{PSL}_2(\mathbf{R})$*

- (i) *the leaves of the geodesic foliation are the (right) orbits of the Cartan subgroup:*

$$A = \left\{ \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \mid t \in \mathbf{R} \right\},$$

- (ii) *the leaves of the (weakly) stable foliation are the (right) orbits of the parabolic subgroup:*

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in \mathrm{PSL}_2(\mathbf{R}) \right\}.$$

**Remark 1.8.** – *The leaves of the (weakly) unstable foliation are the right orbits of  $P^{opp} = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in \mathrm{PSL}_2(\mathbf{R}) \right\}$ .*

- The (left) action of  $\mathrm{PSL}_2(\mathbf{R})$  on  $\overline{M}$  is transitive on the set of leaves  $\overline{\mathcal{F}}$  and  $\overline{\mathcal{G}}$ .
- The above identification  $M \simeq \iota(\Gamma) \backslash \mathrm{PSL}_2(\mathbf{R})$  endows  $M$  with a locally homogeneous  $(\mathrm{PSL}_2(\mathbf{R}), \mathrm{PSL}_2(\mathbf{R}))$ -structure with  $\mathrm{PSL}_2(\mathbf{R})$  acting by left multiplication on itself.

**1.2. The Action of  $\overline{\Gamma}$  on the Leaf Spaces.** In this paragraph we establish some facts about the action of  $\overline{\Gamma} = \pi_1(M)$  on the space of geodesics  $\widetilde{\mathcal{G}}$  of  $\widetilde{M}$  and on the space of (weakly) stable leaves  $\widetilde{\mathcal{F}}$ , which we will use frequently.

Note first that there is an identification of  $\widetilde{\mathcal{F}}$  with  $\widetilde{\partial\Gamma}$ , the universal cover of  $\partial\Gamma \simeq \mathcal{S}^1$ , lifting the natural isomorphism  $\overline{\mathcal{F}} \simeq \partial\Gamma$ . Two such identifications  $\widetilde{\mathcal{F}} \simeq \widetilde{\partial\Gamma}$  differ only by a translation by an element of the central subgroup  $\langle \tau \rangle < \overline{\Gamma}$ . In particular, there is a well defined action of  $\overline{\Gamma}$  on  $\widetilde{\partial\Gamma}$  making any of these isomorphisms  $\widetilde{\mathcal{F}} \simeq \widetilde{\partial\Gamma}$  equivariant.

The chosen orientation on  $\partial\Gamma$  induces an orientation on  $\widetilde{\partial\Gamma}$ . We choose the element  $\tau$  generating the center of  $\overline{\Gamma}$  so that  $(\tau^n \tilde{f}, \tau^m \tilde{f}, \tilde{f})$  is positively oriented precisely when  $n > m > 0$ .

1.2.1. *Minimality.*

**Lemma 1.9.** *The action of  $\overline{\Gamma}$  on  $\widetilde{\mathcal{F}} \simeq \widetilde{\partial\Gamma}$  is minimal.*

*Proof.* Recall that the action of  $\overline{\Gamma}$  on  $\widetilde{\partial\Gamma}$  is minimal if any closed  $\overline{\Gamma}$ -invariant subset of  $\widetilde{\partial\Gamma}$  is either empty or equal to  $\widetilde{\partial\Gamma}$ . Any  $\overline{\Gamma}$ -invariant subset  $\widetilde{A}$  of  $\widetilde{\partial\Gamma}$  is in particular  $\langle \tau \rangle$ -invariant. Hence it is of the form  $\widetilde{A} = \pi^{-1}(A)$  where  $\pi : \widetilde{\partial\Gamma} \rightarrow \partial\Gamma$  is the natural projection and  $A$  is a  $\Gamma$ -invariant subset of  $\partial\Gamma$ . The set  $\widetilde{A}$  is closed if and only if  $A$  is closed. Since  $\Gamma$  acts minimally on  $\partial\Gamma$ , any closed  $\overline{\Gamma}$ -invariant subset  $\widetilde{A} \subset \widetilde{\partial\Gamma}$  is either  $\pi^{-1}(\emptyset) = \emptyset$  or  $\pi^{-1}(\partial\Gamma) = \widetilde{\partial\Gamma}$ .  $\square$

The space of leaves of the (weakly) unstable foliation of the geodesic flow on  $\widetilde{M}$  can also be identified with  $\widetilde{\partial\Gamma}$ . This enables us to identify  $\widetilde{\mathcal{G}}$  with a set of pairs  $(\tilde{t}_+, \tilde{t}_-)$  in  $\widetilde{\partial\Gamma} \times \widetilde{\partial\Gamma}$ . Since the identification is a lift of the natural identification

$$\overline{\mathcal{G}} \simeq \partial\Gamma^{(2)} = \{(t_+, t_-) \in \partial\Gamma^2 \mid t_+ \neq t_-\},$$

the set  $\widetilde{\mathcal{G}}$  will be identified with a subset of:

$$\{(\tilde{t}_+, \tilde{t}_-) \in \widetilde{\partial\Gamma}^{(2)} \mid \pi(\tilde{t}_+) \neq \pi(\tilde{t}_-)\} = \bigcup_{n \in \mathbf{Z}} \widetilde{\partial\Gamma}_{[n]}^{(2)},$$

where

$$\widetilde{\partial\Gamma}_{[n]}^{(2)} := \{(\tilde{t}_+, \tilde{t}_-) \in \widetilde{\partial\Gamma}^{(2)} \mid (\tau^{n+1}\tilde{t}_-, \tilde{t}_+, \tau^n\tilde{t}_-) \text{ is oriented}\}.$$

This is in fact the decomposition of  $\{(\tilde{t}_+, \tilde{t}_-) \in \widetilde{\partial\Gamma}^{(2)} \mid \pi(\tilde{t}_+) \neq \pi(\tilde{t}_-)\}$  into connected components. As  $\widetilde{\mathcal{G}}$  is connected, there exist an  $n \in \mathbf{Z}$  such that  $\widetilde{\mathcal{G}}$  is equal to  $\widetilde{\partial\Gamma}_{[n]}^{(2)}$  and by changing the identification of the set of (weakly) unstable leaves with  $\widetilde{\partial\Gamma}$  we can suppose that  $n = 0$ .

**Remark 1.10.** *The manifold  $\widetilde{M}$  can be  $\overline{\Gamma}$ -equivariantly identified with the set of triples  $(\tilde{t}_+, \tilde{t}_0, \tilde{t}_-)$  of  $(\partial\widetilde{\Gamma})^3$  where  $(\tau\tilde{t}_-, \tilde{t}_+, \tilde{t}_0, \tilde{t}_-)$  is positively oriented.*

1.2.2. *Elements of Zero Translation.* Any element  $\overline{\gamma} \in \overline{\Gamma} - \langle \tau \rangle$  projects onto an element  $\gamma = p(\overline{\gamma}) \in \Gamma - \{1\}$ . Every element  $\gamma \in \Gamma - \{1\}$  has a unique attractive and repulsive fixed point  $t_{+, \gamma}$ , respectively  $t_{-, \gamma}$  in  $\partial\Gamma$ , which lift to two  $\langle \tau \rangle$ -orbits  $(\tau^n \tilde{t}_{+, \gamma})_{n \in \mathbf{Z}}$  and  $(\tau^n \tilde{t}_{-, \gamma})_{n \in \mathbf{Z}}$  in  $\partial\widetilde{\Gamma}$ . We choose a pair  $(\tilde{t}_{+, \gamma}, \tilde{t}_{-, \gamma})$  such that the triple  $(\tau \tilde{t}_{-, \gamma}, \tilde{t}_{+, \gamma}, \tilde{t}_{-, \gamma})$  is oriented. Two different choices of such a pair differ only by the action of a power of  $\tau$ .

Since the element  $\overline{\gamma}$  commutes with  $\tau$ , it leaves the two orbits  $(\tau^n \tilde{t}_{\pm, \gamma})_{n \in \mathbf{Z}}$  invariant and acts orientation preserving on  $\partial\widetilde{\Gamma}$ . Therefore there is a unique integer  $l$  such that

$$\overline{\gamma} \cdot \tilde{t}_{+, \gamma} = \tau^l \tilde{t}_{+, \gamma} \text{ and } \overline{\gamma} \cdot \tilde{t}_{-, \gamma} = \tau^l \tilde{t}_{-, \gamma}.$$

We will call  $l =: \mathbf{t}(\overline{\gamma})$  the *translation* of  $\overline{\gamma}$ . Obviously  $\mathbf{t}(\overline{\gamma}\tau^m) = \mathbf{t}(\overline{\gamma}) + m$ , hence in every orbit  $\{\overline{\gamma}\tau^m \mid m \in \mathbf{Z}\}$  there is a unique element of zero translation.

Elements in  $\overline{\Gamma}$  of zero translation can be characterized by considering the action on the space of geodesics. Given a non-trivial element  $\gamma \in \Gamma$ , it fixes exactly two geodesic leaves in  $\overline{\mathcal{G}} \simeq \partial\Gamma^{(2)}$ , namely  $\overline{g}_\gamma = (t_{+, \gamma}, t_{-, \gamma})$  and  $\overline{g}_{\gamma^{-1}} = (t_{-, \gamma}, t_{+, \gamma})$ , and its action on  $\overline{g}_\gamma = (t_{+, \gamma}, t_{-, \gamma})$  corresponds to a positive time map of the geodesic flow. The geodesic leave  $\overline{g}_\gamma$  lifts to one  $\langle \tau \rangle$ -orbit of geodesic leaves  $(g_n)_{n \in \mathbf{Z}}$  in  $\widetilde{\mathcal{G}}$ . Among the lifts of  $\gamma$  there is a unique element  $\overline{\gamma} \in \overline{\Gamma}$  fixing each of these geodesics. This element  $\overline{\gamma}$  is the unique element of translation zero in  $\{\overline{\gamma}\tau^m \mid m \in \mathbf{Z}\}$ .

**Lemma 1.11.** *The set of pairs of fixed points of zero translation elements*

$$\{(\tilde{t}_{+, \gamma}, \tilde{t}_{-, \gamma}) \mid \gamma \in \overline{\Gamma} - \{1\}\}$$

*is dense in  $\partial\widetilde{\Gamma}_{[0]}^{(2)}$ .*

*Proof.* The closure of this set is  $\tau$ -invariant (since we take all possible fixed pairs for a given element) so it is of the form  $\pi^{-1}(A)$  where  $A \subset \partial\Gamma^{(2)}$  is closed and contains all the pairs  $(t_{+, \gamma}, t_{-, \gamma})$  with  $\gamma \in \Gamma - \{1\}$ . We conclude by Lemma 1.5.  $\square$

1.3. **Projective Geometry.** Since we will consider manifolds which are locally modeled on the three-dimensional real projective space, we recall some basic notions from projective geometry.

Let  $E$  be a vector space, then  $\mathbb{P}(E)$  denotes the space of lines in  $E$ . We write  $\mathbb{P}^{n-1}(\mathbf{R})$  when  $E = \mathbf{R}^n$ . The Grassmanian of  $m$ -dimensional subspaces of  $E$  is denoted by  $\text{Gr}_m(E)$  or  $\text{Gr}_m^n(\mathbf{R})$  when  $E = \mathbf{R}^n$ . We denote by  $\mathbb{P}^{n-1}(\mathbf{R})^*$  the Grassmanian of hyperplanes in  $\mathbf{R}^n$ . The variety of full flags in  $E$  is denoted by  $\text{Flag}(E)$ .

If  $F \subset E$  is a subvector space,  $\mathbb{P}(F)$  is naturally a subspace of  $\mathbb{P}(E)$ , which is called a *projective line* when  $\dim F = 2$  and a *projective plane*

when  $\dim F = 3$ . We will regularly consider an element of  $\text{Gr}_2^4(\mathbf{R})$ , or of  $\mathbb{P}^3(\mathbf{R})^*$  as a projective line respectively a projective plane in  $\mathbb{P}^3(\mathbf{R})$  without introducing any additional notation. What is meant should always be clear from the context.

1.3.1. *Convexity.* A subset  $C$  of the projective space is said to be *convex* if its intersection with any projective line is connected. It is said to be *properly convex* if its closure does not contain any projective line.

A convex subset of the projective plane which is a connected component of the complement of two projective lines through some point  $x$  is called a *sector*. The point  $x$  will be called the *tip* of the sector.

## 2. GEOMETRIC STRUCTURES

In this section we introduce the notion of properly convex foliated projective structures and define the Hitchin component. For more background on geometric structures we refer the reader to [9].

### 2.1. Projective Structures.

**Definition 2.1.** A projective structure on an  $n$ -dimensional manifold  $M$  is a maximal atlas  $\{(U, \varphi_U)\}$  on  $M$  such that

- (i)  $\{U\}$  is an open cover of  $M$  and, for any  $U$ ,  $\varphi_U : U \rightarrow \varphi_U(U)$  is a homeomorphism onto an open subset of  $\mathbb{P}^n(\mathbf{R})$ .
- (ii) For each  $U, V$  the change of coordinates  $\varphi_V \circ \varphi_U^{-1} : \varphi_U(U \cap V) \rightarrow \varphi_V(U \cap V)$  is locally projective, i.e. it is (locally) the restriction of an element of  $\text{PGL}_{n+1}(\mathbf{R})$ .

A manifold endowed with a projective structure is called a  $\mathbb{P}^n(\mathbf{R})$ -manifold.

A projective structure is a locally homogeneous  $(\text{PGL}_{n+1}(\mathbf{R}), \mathbb{P}^n(\mathbf{R}))$ -structure.

Let  $M, N$  be two  $\mathbb{P}^n(\mathbf{R})$ -manifolds. A continuous map  $h : M \rightarrow N$  is *projective* if for every coordinate chart  $(U, \varphi_U)$  of  $M$  and every coordinate chart  $(V, \varphi_V)$  of  $N$  the composition

$$\varphi_V \circ h \circ \varphi_U^{-1} : \varphi_U(h^{-1}(V) \cap U) \longrightarrow \varphi_V(h(U) \cap V)$$

is locally projective. A projective map is always a local homeomorphism.

Two projective structures on  $M$  are equivalent if there is a homeomorphism  $h : M \rightarrow M$  isotopic to the identity which is a projective map with respect to the two projective structures on  $M$ . The set of equivalence classes of projective structures on  $M$  is denoted by  $\mathcal{P}(M)$ . We draw the reader's attention to the above restriction on the differentiability class for the coordinate changes; the projective structures constructed in Section 4 are only continuous.

Let  $\widetilde{M}$  be the universal covering of  $M$ . A projective structure on  $M$  defines a projective structure on  $\widetilde{M}$ . Since  $\widetilde{M}$  is simply connected, there

exists a global projective map  $\text{dev} : \widetilde{M} \rightarrow \mathbb{P}^n(\mathbf{R})$ . The action of  $\pi_1(M)$  on  $\widetilde{M}$  respects the projective structure on  $\widetilde{M}$ . More precisely, for every  $\gamma \in \pi_1(M)$ , there is a unique element  $\text{hol}(\gamma)$  in  $\text{PGL}_{n+1}(\mathbf{R})$  such that  $\text{dev}(\gamma \cdot m) = \text{hol}(\gamma) \cdot \text{dev}(m)$  for any  $m$  in  $\widetilde{M}$ . This defines a homomorphism  $\text{hol} : \pi_1(M) \rightarrow \text{PGL}_{n+1}(\mathbf{R})$  such that the map  $\text{dev}$  is hol-equivariant. The map  $\text{dev}$  is called the developing map and the homomorphism  $\text{hol}$  is called the holonomy homomorphism. Conversely the data of a developing pair  $(\text{dev}, \text{hol})$  defines a  $\pi_1(M)$ -invariant projective structure on  $\widetilde{M}$ , hence a projective structure on  $M$ :

**Proposition 2.2** ([9]). *A projective structure on  $M$  is equivalent to the data  $(\text{dev}, \text{hol})$  of a holonomy homomorphism  $\text{hol} : \pi_1(M) \rightarrow \text{PGL}_{n+1}(\mathbf{R})$  and a hol-equivariant local homeomorphism  $\text{dev} : \widetilde{M} \rightarrow \mathbb{P}^n(\mathbf{R})$ . Two pairs  $(\text{dev}_1, \text{hol}_1)$  and  $(\text{dev}_2, \text{hol}_2)$  define equivalent projective structures on  $M$  if and only if there exists a homeomorphism  $h : M \rightarrow M$  isotopic to the identity and an element  $g \in \text{PGL}_{n+1}(\mathbf{R})$  such that*

- $\text{hol}_2(\gamma) = g \text{hol}_1(\gamma) g^{-1}$  for all  $\gamma \in \pi_1(M)$ , and
- $\text{dev}_1 \circ \tilde{h} = g^{-1} \circ \text{dev}_2$ , where  $\tilde{h} : \widetilde{M} \rightarrow \widetilde{M}$  is the homeomorphism induced by  $h$ .

This defines an equivalence relation on the pairs  $(\text{dev}, \text{hol})$  such that  $\mathcal{P}(M)$  is identified with the set of equivalence classes of pairs  $(\text{dev}, \text{hol})$ . We endow the set of pairs  $(\text{dev}, \text{hol})$  with the topology coming from the compact-open topology on the spaces of maps  $\widetilde{M} \rightarrow \mathbb{P}^n(\mathbf{R})$  and  $\pi_1(M) \rightarrow \text{PGL}_{n+1}(\mathbf{R})$ , and consider  $\mathcal{P}(M)$  with the induced quotient topology.

The holonomy map

$$\text{hol} : \mathcal{P}(M) \longrightarrow \text{Hom}(\pi_1(M), \text{PGL}_{n+1}(\mathbf{R})) / \text{PGL}_{n+1}(\mathbf{R})$$

associates to a pair  $(\text{dev}, \text{hol})$  just the holonomy homomorphism.

**2.2. Foliated Projective Structures on  $S\Sigma$ .** We consider now the unit tangent bundle  $M = S\Sigma$  endowed with the (weakly) stable foliation  $\mathcal{F}$  and the geodesic foliation  $\mathcal{G}$ .

**Definition 2.3.** *A foliated projective structure on  $(M, \mathcal{F}, \mathcal{G})$  is a projective structure  $\{(U, \varphi_U)\}$  on  $M$  with the additional properties that*

- (i) *for every  $x \in U$  the image  $\varphi_U(U \cap g_x)$  of the geodesic leaf  $g_x$  through  $x$  is contained in a projective line, and*
- (ii) *for every  $x \in U$  the image  $\varphi_U(U \cap f_x)$  of the (weakly) stable leaf  $f_x$  through  $x$  is contained in a projective plane.*

Two foliated projective structures on  $(M, \mathcal{F}, \mathcal{G})$  are *equivalent* if there is a projective homeomorphism  $h : M \rightarrow M$  isotopic to the identity such that  $h^*\mathcal{G} = \mathcal{G}$  and  $h^*\mathcal{F} = \mathcal{F}$ . The space of equivalence classes of foliated projective structures on  $(M, \mathcal{F}, \mathcal{G})$  is denoted by  $\mathcal{P}_f(M)$ .

**Remark 2.4.** – Note that the natural map of  $\mathcal{P}_f(M)$  into  $\mathcal{P}(M)$  is not an inclusion since we do not only restrict to a subset of projective structures, but at the same time refine the equivalence relation. We do not know if this map is injective or not.

– It is a simple exercise to show that any homeomorphism  $h$  of  $M$  homotopic to the identity and respecting the foliation  $\mathcal{G}$  sends every geodesic to itself. So it is of the form  $m \mapsto \phi_{f(m)}(m)$  where  $f : M \rightarrow \mathbf{R}$  is a continuous function and  $(\phi_t)_{t \in \mathbf{R}}$  is the geodesic flow on  $M$ . Since  $m \mapsto \phi_{f(m)}(m)$  is a homeomorphism, we obtain that for any  $t > 0$  and  $m \in M$ , the inequality  $f(m) - f(\phi_t(m)) < t$  holds. Conversely for any continuous function  $f$  satisfying this inequality, the map  $m \mapsto \phi_{f(m)}(m)$  is a homeomorphism of  $M$  respecting both foliations leaf by leaf. So, using the family  $(\lambda f)_{\lambda \in [0,1]}$ ,  $h$  is isotopic to the identity through homeomorphisms respecting the foliations. We observe that the homeomorphisms homotopic to the identity and respecting the foliation  $\mathcal{G}$  are precisely the ones sending each geodesic leaf to itself.

A developing pair  $(\text{dev}, \text{hol})$  of a projective structure on  $M$  defines a foliated projective structure on  $(M, \mathcal{F}, \mathcal{G})$  if the following conditions are satisfied.

- (i) For every  $g \in \tilde{\mathcal{G}}$ , the image  $\text{dev}(g)$  is contained in a projective line, and
- (ii) for every  $f \in \tilde{\mathcal{F}}$ , the image  $\text{dev}(f)$  is contained in a projective plane.

### 2.2.1. Properly Convex Foliated Projective Structures.

**Definition 2.5.** A foliated projective structure on  $(M, \mathcal{F}, \mathcal{G})$  is said to be convex if for every  $f \in \tilde{\mathcal{F}}$  the image  $\text{dev}(f)$  is a convex set in the projective plane containing  $\text{dev}(f)$ . It is properly convex if for every  $f \in \tilde{\mathcal{F}}$  the image  $\text{dev}(f)$  is a properly convex set.

Note that we do not require that the restriction of  $\text{dev}$  to a (weakly) stable leaf is a homeomorphism onto its image but this will be a consequence of the other conditions. In fact, already for convex projective structures on closed surfaces one can show that such a condition on the image of the developing map suffices.

Let  $\mathcal{P}_{cf}(M)$  denote the set of equivalence classes of convex foliated projective structures on  $M$  and  $\mathcal{P}_{pcf}(M)$  the set of equivalence classes of properly convex foliated projective structures. They are naturally subsets of the moduli space  $\mathcal{P}_f(M)$  of foliated projective structures on  $M$ .

**2.3. The Hitchin Component.** Let  $\Gamma$  be as above the fundamental group of a closed Riemann surface  $\Sigma$ , and denote by  $\text{Rep}(\Gamma, \text{PGL}_n(\mathbf{R}))$  the set of conjugacy classes of representations, that is

$$\text{Rep}(\Gamma, \text{PGL}_n(\mathbf{R})) = \text{Hom}(\Gamma, \text{PGL}_n(\mathbf{R})) / \text{PGL}_n(\mathbf{R}).$$

Of course this space (with the quotient topology) is not Hausdorff, but we will not worry about this since we are just concerned with a component of it that has the topology of a ball.

Relying on the correspondence between stable Higgs bundle and irreducible representations of  $\pi_1(\Sigma)$  in  $\mathrm{PSL}_n(\mathbf{C})$  due to N. Hitchin [14], C. Simpson [22, 23], K. Corlette [5] and S. Donaldson [6], N. Hitchin proved in [15] that the connected component of  $\mathrm{Hom}(\Gamma, \mathrm{PSL}_n(\mathbf{R}))/\mathrm{PGL}_n(\mathbf{R})$  containing the representation

$$\rho_n \circ \iota : \Gamma \rightarrow \mathrm{PSL}_n(\mathbf{R}) \subset \mathrm{PGL}_n(\mathbf{R}),$$

is a ball of dimension  $(2g-2)(n^2-1)$ . Here  $\iota : \Gamma \rightarrow \mathrm{PSL}_2(\mathbf{R})$  is a uniformization and  $\rho_n$  is the  $n$ -dimensional irreducible representation of  $\mathrm{PSL}_2(\mathbf{R})$ .

**Notation 2.6.** *This component is called the Teichmüller component or Hitchin component and is denoted by  $\mathcal{T}^n(\Gamma)$  or  $\mathcal{T}^n(\Sigma)$ .*

The Hitchin component naturally embeds

$$\mathcal{T}^n(\Gamma) \subset \mathrm{Rep}(\Gamma, \mathrm{PGL}_n(\mathbf{R})) \subset \mathrm{Rep}(\bar{\Gamma}, \mathrm{PGL}_n(\mathbf{R}))$$

where the last inclusion comes from the projection  $\bar{\Gamma} \rightarrow \Gamma$ . Even though we do not use it, the Hitchin component can be seen as a connected component of  $\mathrm{Rep}(\bar{\Gamma}, \mathrm{PGL}_n(\mathbf{R}))$ :

**Lemma 2.7.** *The Hitchin component  $\mathcal{T}^n(\Gamma)$  is a connected component of  $\mathrm{Rep}(\bar{\Gamma}, \mathrm{PGL}_n(\mathbf{R}))$ .*

*Proof.* Clearly  $\mathcal{T}^n(\Gamma)$  is closed in  $\mathrm{Rep}(\bar{\Gamma}, \mathrm{PGL}_n(\mathbf{R}))$ . By [15] any representation  $\rho \in \mathcal{T}^n(\Gamma)$  is strongly irreducible (e.g. [19, Lemma 10.1] explains this fact). Therefore  $\mathcal{T}^n(\Gamma)$  is contained in the open subset of irreducible representations  $\mathrm{Rep}_{irr}(\bar{\Gamma}, \mathrm{PGL}_n(\mathbf{R}))$ . Since any irreducible representation  $\bar{\rho} : \bar{\Gamma} \rightarrow \mathrm{PGL}_n(\mathbf{R})$  necessarily factors through a representation  $\rho : \Gamma \rightarrow \mathrm{PGL}_n(\mathbf{R})$ , this shows that  $\mathcal{T}^n(\Gamma)$  is open in  $\mathrm{Rep}_{irr}(\bar{\Gamma}, \mathrm{PGL}_n(\mathbf{R}))$  and hence open in  $\mathrm{Rep}(\bar{\Gamma}, \mathrm{PGL}_n(\mathbf{R}))$ .  $\square$

With this we are now able to restate our main result:

**Theorem 2.8.** *The holonomy map*

$$\mathrm{hol} : \mathcal{P}_{pcf}(M) \longrightarrow \mathrm{Rep}(\bar{\Gamma}, \mathrm{PGL}_4(\mathbf{R}))$$

*is a homeomorphism onto the Hitchin component  $\mathcal{T}^4(\Gamma)$ .*

### 3. EXAMPLES

In this section we define several families of projective structures on  $M$ . These families will provide some justification for the definitions given in the previous section and make the reader acquainted with some geometric constructions which will be used in the following sections.

We can summarize this section in the following



**Proposition 3.1.** *All the inclusions*

$$\mathcal{P}_{pcf}(M) \subset \mathcal{P}_{cf}(M) \subset \mathcal{P}_f(M)$$

are strict and the projection from  $\mathcal{P}_f(M)$  to  $\mathcal{P}(M)$ , defined by forgetting the foliations, is not onto.

In order to define a  $(\mathrm{PGL}_4(\mathbf{R}), \mathbb{P}^3(\mathbf{R}))$ -structure, it is sufficient to give a pair  $(\mathrm{dev}, \mathrm{hol})$  consisting of the holonomy representation  $\mathrm{hol} : \bar{\Gamma} \rightarrow \mathrm{PGL}_4(\mathbf{R})$  and the hol-equivariant developing map  $\mathrm{dev} : \widetilde{M} \rightarrow \mathbb{P}^3(\mathbf{R})$ . In all examples given below the holonomy factors through  $\Gamma = \bar{\Gamma}/\langle \tau \rangle$  and the developing map factors through the quotient  $\overline{M} = \widetilde{M}/\langle \tau \rangle = S\widetilde{\Sigma}$ . In particular we will specify developing pairs  $(\mathrm{dev}, \mathrm{hol})$  with  $\mathrm{hol} : \Gamma \rightarrow \mathrm{PSL}_4(\mathbf{R})$  and  $\mathrm{dev} : \overline{M} \rightarrow \mathbb{P}^3(\mathbf{R})$ .

**3.1. Homogeneous Examples.** We first construct families of projective structures on  $M$  which are induced from homogeneous projective structures on  $\mathrm{PSL}_2(\mathbf{R})$  when we realize  $M$  as a quotient of  $\mathrm{PSL}_2(\mathbf{R})$  (see Paragraph 1.1.3). Since this procedure will be used several times we state it in the following lemma.

**Lemma 3.2.** *Let  $\iota : \Gamma \rightarrow \mathrm{PSL}_2(\mathbf{R})$  be a uniformization,  $\rho : \mathrm{PSL}_2(\mathbf{R}) \rightarrow \mathrm{PSL}_4(\mathbf{R})$  a homomorphism and  $x$  a point in  $\mathbb{P}^3(\mathbf{R})$  with*

$$\mathrm{Stab}_{\mathrm{PSL}_2(\mathbf{R})}(x) = \{g \in \mathrm{PSL}_2(\mathbf{R}) \mid \rho(g) \cdot x = x\} \text{ being finite.}$$

Then the following assignment

$$\begin{aligned} \mathrm{dev} : \overline{M} &= \mathrm{PSL}_2(\mathbf{R}) \rightarrow \mathbb{P}^3(\mathbf{R}), \quad g \mapsto \rho(g) \cdot x \\ \mathrm{hol} &= \rho \circ \iota \end{aligned}$$

defines a projective structure on  $M$ . Furthermore

- The images of geodesics are contained in projective lines if and only if  $\mathrm{dev}(A)$  is.
- The images of (weakly) stable leaves are contained in projective planes if and only if  $\mathrm{dev}(P)$  is.
- The restriction of  $\mathrm{dev}$  to every (weakly) stable leaf is a homeomorphism onto a (properly) convex set if and only if the restriction to  $P$  is so.

*Proof.* The fact that  $\mathrm{dev}$  is a hol-equivariant local homeomorphism is clear. The listed properties follow from the  $\mathrm{PSL}_2(\mathbf{R})$ -equivariance and the description of the leaves (Paragraph 1.1.3).  $\square$

**Remark 3.3.** *This lemma could also be stated by saying that given a locally homogeneous  $(\mathrm{PSL}_2(\mathbf{R}), \mathrm{PSL}_2(\mathbf{R}))$ -structure on  $M$  and a  $(\mathrm{PSL}_2(\mathbf{R}), U)$  geometry with  $U$  an open in  $\mathbb{P}^3(\mathbf{R})$  we automatically obtain a  $(\mathrm{PSL}_2(\mathbf{R}), U)$ -structure and hence a projective structure on  $M$ .*

3.1.1. *The Diagonal Embedding.* We keep the notation of Lemma 3.2. If we choose

$$\begin{aligned} \rho : \mathrm{PSL}_2(\mathbf{R}) &\longrightarrow \mathrm{PSL}_4(\mathbf{R}) \\ g &\longmapsto \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \text{ and } x = [1, 0, 0, 1], \end{aligned}$$

we obtain a convex foliated projective structure on  $M$  with non properly convex (weakly) stable leaves. Indeed  $\mathrm{Stab}_{\mathrm{PSL}_2(\mathbf{R})}(x) = \{\mathrm{Id}\}$  shows that  $\mathrm{dev}$  is a homeomorphism onto its image and:

$$\begin{aligned} \mathrm{dev}(A) &= \{[(e^t, 0, 0, 1)] \mid t \in \mathbf{R}\}, \\ \mathrm{dev}(P) &= \{[(1, 0, u, v)] \mid u \in \mathbf{R}, v > 0\}. \end{aligned}$$

So  $\mathrm{dev}(A)$  is contained in a projective line,  $\mathrm{dev}(P)$  is convex in a projective plane but its closure contains a projective line.

In Section 3.2 and Section 3.3 we generalize this example in two different ways.

3.1.2. *The Irreducible Example.* Let  $\rho_4 : \mathrm{PSL}_2(\mathbf{R}) \rightarrow \mathrm{PSL}_4(\mathbf{R})$  be the 4-dimensional irreducible representation. In other words this is the representation on the 3-fold symmetric power  $\mathbf{R}^4 \simeq \mathrm{Sym}^3 \mathbf{R}^2$ . Hence we will consider elements of  $\mathbf{R}^4$  as homogeneous polynomials of degree three in two variables  $X$  and  $Y$ , so it will make sense to speak of the roots of an element of  $\mathbf{R}^4$ . We choose  $x = [R] \in \mathbb{P}^3(\mathbf{R})$  such that  $R$  has only one real root counted with multiplicity.

A direct calculation shows that for the projective structure defined by Lemma 3.2  $\mathrm{dev}(A)$  and  $\mathrm{dev}(P)$  are contained respectively in a projective line and in a projective plane. Actually, if  $R = X(X^2 + Y^2) = (1, 0, 1, 0)$ ,

$$\mathrm{dev}(P) = \{[a^4 + a^2b^2, 2ab, 1, 0] \in \mathbb{P}^3(\mathbf{R}) \mid a, b \in \mathbf{R}\}$$

is the projection of the properly convex cone of  $\mathbf{R}^4 - \{0\}$ :

$$\{(\alpha, \beta, \gamma, 0) \in \mathbf{R}^4 \mid \beta^2 - 4\alpha\gamma < 0\}.$$

So this defines a properly convex foliated projective structure on  $M$ . In Section 4.1 we will give a more geometric description of this example.

**Remark 3.4.** *Note that the choice of  $R$  is related to the previous choice of a Cartan subgroup  $A$ . Any other polynomial  $g \cdot R$  in the orbit of  $R$  defines the same structure when parametrizing the geodesic flow by  $gAg^{-1}$  instead of  $A$ .*

3.1.3. *The Irreducible Example Revisited.* Taking the same representation  $\rho_4 : \mathrm{PSL}_2(\mathbf{R}) \rightarrow \mathrm{PSL}_4(\mathbf{R})$  but choosing  $x = [Q]$  where  $Q$  is a polynomial having three distinct real roots, the projective structure on  $M$  defined by Lemma 3.2 is foliated but not convex. In this case, the image of  $P$  is the complementary subset (in the projective plane containing it) of the closure of convex set described above and cannot be convex.

**3.2. Nonfoliated Structures with Quasi-Fuchsian Holonomy.** In this paragraph we generalize the example of Section 3.1.1 for any quasi-Fuchsian representation  $q : \Gamma \rightarrow \mathrm{PSL}_2(\mathbf{C})$ . We thank Bill Goldman for explaining us this construction.

Recall that any quasi-Fuchsian representation  $q$  is a deformation of a Fuchsian representation  $\Gamma \rightarrow \mathrm{PSL}_2(\mathbf{R})$ , and there exists a  $q$ -equivariant local orientation preserving homeomorphism  $u : \tilde{\Sigma} \rightarrow \mathbb{P}^1(\mathbf{C})$ .

Fixing an identification  $\mathbf{C} \simeq \mathbf{R}^2$  we have an embedding  $\mathrm{PSL}_2(\mathbf{C}) \subset \mathrm{PSL}_4(\mathbf{R})$  such that the Hopf fibration  $\mathbb{P}^3(\mathbf{R}) \rightarrow \mathbb{P}^1(\mathbf{C})$  is a  $\mathrm{PSL}_2(\mathbf{C})$ -equivariant fibration by circles.

**Proposition 3.5.** *Let  $q : \Gamma \rightarrow \mathrm{PSL}_2(\mathbf{C})$  be a quasi-Fuchsian representation and  $u : \tilde{\Sigma} \rightarrow \mathbb{P}^1(\mathbf{C})$  a  $q$ -equivariant local orientation preserving homeomorphism.*

- (i) *Then the pull back  $u^*\mathbb{P}^3(\mathbf{R})$  of the Hopf fibration  $\mathbb{P}^3(\mathbf{R}) \rightarrow \mathbb{P}^1(\mathbf{C})$  admits a  $\Gamma$ -invariant projective structure and the quotient of  $u^*\mathbb{P}^3(\mathbf{R})$  by  $\Gamma$  is homeomorphic to  $M$ .*
- (ii) *The induced projective structure on  $M$  is foliated if and only if the representation  $q$  is (conjugate to) a Fuchsian representation  $\Gamma \rightarrow \mathrm{PSL}_2(\mathbf{R})$ .*

*Proof.* For (i), the projective structure on the pull back  $u^*\mathbb{P}^3(\mathbf{R})$  is tautological since  $u^*\mathbb{P}^3(\mathbf{R}) \rightarrow \mathbb{P}^3(\mathbf{R})$  is a local homeomorphism. From the homotopy invariance of fiber bundles (see [24, p.53]) it is enough to show that the quotient of  $u^*\mathbb{P}^3(\mathbf{R})$  by  $\Gamma$  is homeomorphic to  $M$  when the representation  $q : \Gamma \rightarrow \mathrm{PSL}_2(\mathbf{R})$  is Fuchsian. In this case  $u$  is the composition  $\tilde{\Sigma} \simeq \mathbb{H}^2 \hookrightarrow \mathbb{P}^1(\mathbf{C})$ . Also the embedding  $\mathrm{PSL}_2(\mathbf{R}) \subset \mathrm{PSL}_2(\mathbf{C}) \subset \mathrm{PSL}_4(\mathbf{R})$  is the diagonal embedding of Section 3.1.1 and the map  $\mathrm{dev} : \overline{M} \simeq \mathrm{PSL}_2(\mathbf{R}) \rightarrow \mathbb{P}^3(\mathbf{R})$  defined in Section 3.1.1 fits into the commutative diagram:

$$\begin{array}{ccccc} \mathrm{dev} : \overline{M} & \xrightarrow{\sim} & \mathrm{PSL}_2(\mathbf{R}) & \xrightarrow{\sim} & \mathbb{P}^3(\mathbf{R}) \\ & & \downarrow & & \downarrow \\ u : \tilde{\Sigma} & \xrightarrow{\sim} & \mathbb{H}^2 & \xrightarrow{\sim} & \mathbb{P}^1(\mathbf{C}). \end{array}$$

So  $\overline{M}$  is homeomorphic to  $u^*\mathbb{P}^3(\mathbf{R})$  as  $\Gamma$ -space.

(ii) If the structure is foliated there exists a  $q$ -equivariant map  $\xi^3 : \partial\Gamma \rightarrow \mathbb{P}^3(\mathbf{R})^*$  (see Proposition 5.2). In particular any element  $\gamma$  in  $\Gamma - \{1\}$  will have an eigenline  $\xi^3(t)$  in  $\mathbf{R}^{4*}$ , with  $t$  a fixed point of  $\gamma$  in  $\partial\Gamma$ , and hence a real eigenvalue for its action on  $\mathbf{R}^4$ . Since the eigenvalues of an element of  $\mathrm{PSL}_2(\mathbf{C}) \subset \mathrm{PSL}_4(\mathbf{R})$  are  $\{\lambda, \lambda, \lambda^{-1}, \lambda^{-1}\}$  we deduce that  $q(\gamma)$  has only real eigenvalues. In particular  $\mathrm{tr}(q(\gamma)) = \mathrm{tr}\left(\overline{q(\gamma)}\right)$ . So the representation  $(q, \overline{q}) : \Gamma \rightarrow \mathrm{PSL}_2(\mathbf{C}) \times \mathrm{PSL}_2(\mathbf{C})$  cannot have Zariski-dense image. Therefore, by Lemma A.5, there exists  $g$  in  $GL_2(\mathbf{C})$  such that,  $q(\gamma) = g\overline{q(\gamma)}g^{-1}$  for all  $\gamma \in \Gamma$ . Since  $q(\Gamma)$  is Zariski dense the element  $g\overline{g}$  is central. Up to multiplying  $g$  by a scalar we have  $g\overline{g} = \pm \mathrm{Id}$ .

If we have  $g\bar{g} = \text{Id}$  then, for some  $\beta$  in  $\mathbf{C}$ ,  $h = \beta g + \bar{\beta}\text{Id}$  belongs to  $\text{GL}_2(\mathbf{C})$  and satisfies  $g\bar{h} = h$ . The discrete and faithful representation  $\gamma \rightarrow hq(\gamma)h^{-1}$  takes values in  $\text{PSL}_2(\mathbf{R})$  so  $q$  is conjugate to a Fuchsian representation.

If  $g\bar{g} = -\text{Id}$ , setting  $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  then, for  $\beta \in \mathbf{C}$ ,  $h = \beta g + \bar{\beta}T$  is invertible. The representation  $hqh^{-1}$  has values in  $\text{PSU}_2(\mathbf{R})$  but this is impossible since  $q$  is faithful and discrete.  $\square$

**3.3. Geometric Description of the Diagonal Embedding.** Recall that in this example the holonomy is

$$\begin{aligned} \text{hol} : \Gamma &\longrightarrow \text{PSL}_4(\mathbf{R}) \\ \gamma &\longmapsto \rho(\iota(\gamma)) = \begin{pmatrix} \iota(\gamma) & 0 \\ 0 & \iota(\gamma) \end{pmatrix} \end{aligned}$$

and the developing map is

$$\begin{aligned} \text{dev} : \text{PSL}_2(\mathbf{R}) \simeq \overline{M} &\longrightarrow \mathbb{P}^3(\mathbf{R}) \\ g &\longmapsto \rho(g) \cdot x \end{aligned}$$

where  $x = [(1, 0, 0, 1)]$ . We wish to describe  $\text{dev}$  as a map  $\partial\Gamma^{3+} \simeq \overline{M} \rightarrow \mathbb{P}^3(\mathbf{R})$ .

It will be useful to have a lift of  $\text{hol}$  to  $\text{SL}_4(\mathbf{R})$

$$\begin{aligned} \widehat{\text{hol}} : \Gamma &\longrightarrow \text{SL}_4(\mathbf{R}) \\ \gamma &\longmapsto \begin{pmatrix} \hat{\iota}(\gamma) & 0 \\ 0 & \hat{\iota}(\gamma) \end{pmatrix} \end{aligned}$$

where  $\hat{\iota} : \Gamma \rightarrow \text{SL}_2(\mathbf{R})$  is one of the  $2^g$  lifts of  $\iota$ . We also remind the reader that  $\partial\Gamma$  is being equivariantly identified with  $\mathbb{P}^1(\mathbf{R})$ . The sphere  $\mathbb{S}(\mathbf{R}^2) = \mathbf{R}^2 - \{0\}/\{x \sim \lambda^2 x\}$  is a  $\Gamma$ -space that projects onto  $\partial\Gamma = \mathbb{P}^1(\mathbf{R}) = \mathbb{S}(\mathbf{R}^2)/\{\pm 1\}$ , we denote it by  $\widehat{\partial\Gamma}$ . It has the following property:

for any  $(t_+, t_-)$  in  $\partial\Gamma^{(2)}$ , there are exactly two lifts  $(\hat{t}_+, \hat{t}_-)$  in  $\widehat{\partial\Gamma}^2$  of this pair so that  $(-\hat{t}_-, \hat{t}_+, \hat{t}_-)$  is oriented, these two lifts are exchanged by the action of  $-1$ .

A straightforward calculation shows that the image  $\text{dev}(t_+, t_-)$  of any geodesic is a segment in  $\mathbb{P}^3(\mathbf{R})$  with endpoints  $\xi^+(t_+)$  and  $\xi^-(t_-)$  where  $\xi^\pm : \partial\Gamma \rightarrow \mathbb{P}^3(\mathbf{R})$  are the two  $\rho$ -equivariant maps  $\mathbb{P}^1(\mathbf{R}) \hookrightarrow \mathbb{P}^3(\mathbf{R})$  coming from the decomposition  $\mathbf{R}^4 = \mathbf{R}^2 \oplus \mathbf{R}^2$ . To describe  $\text{dev}$  we need lifts of  $\xi^\pm$  to  $\mathbf{R}^4$ . No continuous lift exists, so we rather choose  $\eta^\pm : \widehat{\partial\Gamma} \rightarrow \mathbf{R}^4$  two continuous maps, equivariant by  $-1$ , lifting  $\hat{\xi}^\pm : \widehat{\partial\Gamma} \rightarrow \partial\Gamma \rightarrow \mathbb{P}^3(\mathbf{R})$ . Then there exists a continuous function  $\varphi : \partial\Gamma^{3+} \rightarrow \mathbf{R}$  such that, for all  $(t_+, t_0, t_-) \in \partial\Gamma^{3+}$

$$(1) \quad \text{dev}(t_+, t_0, t_-) = [\eta^+(\hat{t}_+) + \varphi(t_+, t_0, t_-)\eta^-(\hat{t}_-)] \in \mathbb{P}^3(\mathbf{R}),$$

where  $(\hat{t}_+, \hat{t}_-)$  is one of the two lifts of  $(t_+, t_-)$  with  $(-\hat{t}_-, \hat{t}_+, \hat{t}_-)$  oriented. Observing that  $\varphi$  never vanishes, we can suppose  $\varphi > 0$  up to changing

$\eta^+$  in  $-\eta^+$ . Since  $\text{dev}$  is a local homeomorphism,  $\varphi$  must be monotonely decreasing along geodesics.

To state the condition on  $\varphi$  coming from the equivariance of  $\text{dev}$  we consider the two maps  $f^\pm : \Gamma \times \partial\Gamma \rightarrow \mathbf{R}^*$  measuring the lack of equivariance of  $\eta^\pm$ : for all  $\hat{t} \in \widehat{\partial\Gamma}$  projecting on  $t \in \partial\Gamma$

$$(2) \quad \eta^\pm(\gamma \cdot \hat{t}) = f^\pm(\gamma, t) \widehat{\text{hol}}(\gamma) \cdot \eta^\pm(\hat{t}).$$

With this, for all  $(t_+, t_0, t_-) \in \partial\Gamma^{3+}$  and  $\gamma \in \Gamma$ , the developing map  $\text{dev}$  is equivariant if and only if

$$(3) \quad \varphi(\gamma \cdot (t_+, t_0, t_-)) = \frac{f^-(\gamma, t_-)}{f^+(\gamma, t_+)} \varphi(t_+, t_0, t_-).$$

Given maps with those conditions, we can construct a foliated projective structure on  $M$ .

**Proposition 3.6.** *Let  $\text{hol} : \Gamma \rightarrow \text{PSL}_4(\mathbf{R})$  be a representation and  $\widehat{\text{hol}} : \Gamma \rightarrow \text{SL}_4(\mathbf{R})$  a lift of  $\text{hol}$ . Suppose that*

- (i) *there exist two continuous hol-equivariant maps  $\xi^\pm : \partial\Gamma \rightarrow \mathbb{P}^3(\mathbf{R})$ , and*
  - (ii) *the image of  $\xi^-$  is contained in a projective line,*
  - (iii) *there exist two lifts  $\eta^\pm : \widehat{\partial\Gamma} \rightarrow \mathbf{R}^4$  of  $\xi^\pm : \widehat{\partial\Gamma} \rightarrow \partial\Gamma \rightarrow \mathbb{P}^3(\mathbf{R})$  and functions  $f^\pm : \Gamma \times \partial\Gamma \rightarrow \mathbf{R}^*$  satisfying (2),*
  - (iv) *there exists a continuous function  $\varphi : \partial\Gamma^{3+} \rightarrow \mathbf{R}_{>0}$  satisfying the identity (3)*
  - (v) *and  $\varphi$  satisfies the limit condition: for all  $(t_+, t_-)$  in  $\partial\Gamma^{(2)}$*
- $$(4) \quad \lim_{t_0 \rightarrow t_+} \varphi(t_+, t_0, t_-) = 0 \text{ and } \lim_{t_0 \rightarrow t_-} \varphi(t_+, t_0, t_-) = \infty.$$

Moreover, suppose that the map  $\text{dev} : \partial\Gamma^{3+} \rightarrow \mathbb{P}^3(\mathbf{R})$  defined by (1) is a local homeomorphism.

Then the pair  $(\text{dev}, \text{hol})$  defines on  $M$  a convex foliated projective structure which is not properly convex.

*Proof.* By construction  $\text{dev}$  is a hol-equivariant local homeomorphism and images of geodesics are contained in projective lines. The limit condition on  $\varphi$  and the fact that  $\xi^-$  is contained in a projective line  $L$  imply that the image of the (weakly) stable leaf  $t_+$  is a sector whose tip is  $\xi^-(t_+)$  and which is bounded by the projective lines  $L$  and  $\overline{\xi^+(t_+)\xi^-(t_+)}$ .  $\square$

A few remarks need to be made about this construction.

Instead of imposing that  $\text{dev}$  is a local homeomorphism, we could have stated conditions on  $\xi^\pm$  and  $\varphi$  which imply it. For example a condition of the type:

- The function  $\varphi$  is monotonely decreasing along geodesics,  $\xi^\pm$  are homeomorphisms onto  $\mathcal{C}^1$ -submanifolds of  $\mathbb{P}^3(\mathbf{R})$  satisfying the condition that if  $L^\pm$  are projective lines tangent to  $\xi^\pm(\partial\Gamma)$  at  $\xi^\pm(t_\pm)$  with  $t_+ \neq t_-$ , then  $L^+$  and  $L^-$  do not intersect.

would suffice. It is easy to see that  $\xi^\pm$  cannot be locally constant.

In a decomposition  $\mathbf{R}^4 = \mathbf{R}^2 \oplus \mathbf{R}^2$  adapted to the image of  $\xi^-$ ,  $\text{hol}$  (and  $\widehat{\text{hol}}$ ) has the form

$$\gamma \longmapsto \begin{pmatrix} \rho_Q(\gamma) & 0 \\ c(\gamma) & \rho_L(\gamma) \end{pmatrix}.$$

It can be shown that  $\rho_Q$  and  $\rho_L$  are Fuchsian (see Lemma A.2) and that  $\xi^\pm$  are uniquely determined by  $\text{hol}$ . In fact  $\xi^-$  is a homeomorphism onto its image; this was implicitly used in the above proof.

Our analysis shows that the holonomy of non-properly convex foliated projective structures have to be of the above form with  $\rho_Q$  and  $\rho_L$  Fuchsian. Unfortunately we do not have a construction of a nontrivial example besides the diagonal example.

Condition (4) does indeed depend only on  $\text{hol}$  and not on  $\varphi$ . If  $\varphi_1, \varphi_2$  satisfy this condition then their ratio  $\varphi_1/\varphi_2$  descends to a continuous function on  $M$  and hence is bounded. If we change the lift  $\eta^+$  then the function  $\varphi$  will change to  $(t_+, t_0, t_-) \mapsto \lambda(t_+)\varphi(t_+, t_0, t_-)$  for some continuous function  $\lambda : \partial\Gamma \rightarrow \mathbf{R}_{>0}$ . So the behavior at infinity of the continuous functions satisfying equality (3) depends only on  $\text{hol}$  and  $\xi^\pm$ , and the curves  $\xi^\pm$  are uniquely determined by  $\text{hol}$ .

The above description of the diagonal embedding and the following lemma will be used in the proof of Lemma 5.24.

**Lemma 3.7.** *Suppose that  $\text{hol}$  is the holonomy representation of a projective structure on  $M$  such that Conditions (i)-(iv) of Proposition 3.6 are satisfied. Assume that its semisimplification  $\text{hol}_0$  satisfies Conditions (i)-(v). Then all (weakly) stable leaves of the projective structure associated to  $\text{hol}$  are developed into sectors.*

*Proof.* First note that the images of all (weakly) stable leaves are sectors if and only if Condition (4) is satisfied. We write  $\text{hol}$  as above with respect to a decomposition  $\mathbf{R}^4 = \mathbf{R}^2 \oplus \mathbf{R}^2$  adapted to  $\xi^-$ . Let  $\eta^\pm : \widehat{\partial\Gamma} \rightarrow \mathbf{R}^4$  be the lifted curves for  $\text{hol}$ . We decompose

$$\eta^+(t) = \eta_Q^+(t) + \eta_L^+(t) \in \mathbf{R}^2 \oplus \mathbf{R}^2.$$

Then the curves for

$$\begin{aligned} \text{hol}_0 : \Gamma &\longrightarrow \text{PSL}_4(\mathbf{R}) \\ \gamma &\longmapsto \begin{pmatrix} \rho_Q(\gamma) & 0 \\ 0 & \rho_L(\gamma) \end{pmatrix} \end{aligned}$$

are  $\eta_Q^+$  and  $\eta_L^-$ . The functions  $f^\pm$  (Equation (2)) are the same for  $(\eta^+, \eta^-)$  and  $(\eta_Q^+, \eta_L^-)$ . Therefore the equivariance condition for  $\varphi$  is the same for both representations  $\text{hol}$  and  $\text{hol}_0$ . So by the above remark their behavior at infinity is the same. In particular Condition (4) holds for  $\text{hol}$ .  $\square$

## 4. FROM CONVEX REPRESENTATIONS TO PROPERLY CONVEX FOLIATED PROJECTIVE STRUCTURES

In this section we first describe the properly convex foliated projective structure on  $M = S\Sigma$  defined in Section 3.1.2 geometrically. This geometric description of the developing map enables us to construct properly convex foliated projective structure with  $\text{hol} = \rho$  for any representation  $\rho$  in the Hitchin component.

**4.1. A Different Description of the Homogeneous Example.** Our aim is now to describe the developing map of Example 3.1.2 as a map  $\partial\Gamma^{3+} \rightarrow \mathbb{P}^3(\mathbf{R})$ . Recall that the holonomy was  $\text{hol} = \rho_4 \circ \iota$  with  $\iota$  a Fuchsian representation and  $\rho_4$  the 4-dimensional irreducible representation of  $\text{PSL}_2(\mathbf{R})$ , that is the representation on  $\mathbf{R}^4 = \text{Sym}^3\mathbf{R}^2$  the space of homogeneous polynomials of degree three in two variables  $X$  and  $Y$ . The developing map was

$$(5) \quad \begin{aligned} \text{dev} : \overline{M} \simeq \text{PSL}_2(\mathbf{R}) &\longrightarrow \mathbb{P}^3(\mathbf{R}) \\ g &\longmapsto \rho_4(g) \cdot [R], \end{aligned}$$

where  $R$  is  $X(X^2 + Y^2)$ . In fact the description for the nonconvex example given in Section 3.1.3, where the holonomy is the same but the developing map is

$$\begin{aligned} \text{dev}' : \overline{M} \simeq \text{PSL}_2(\mathbf{R}) &\longrightarrow \mathbb{P}^3(\mathbf{R}) \\ g &\longmapsto \rho_4(g) \cdot [Q], \end{aligned}$$

with  $Q = XY(X + Y)$ , is easier to obtain.

It is convenient to consider  $\mathbf{R}^2 = \text{Sym}^1\mathbf{R}^2$  as the space of homogeneous polynomials of degree one in  $X$  and  $Y$ . The Veronese embedding

$$\begin{aligned} \xi^1 : \partial\Gamma \simeq \mathbb{P}^1(\mathbf{R}) &\longrightarrow \mathbb{P}^3(\mathbf{R}) \\ t = [S] &\longmapsto [S^3] \end{aligned}$$

is a  $\rho_4$ -equivariant map, so certainly hol-equivariant, and extends to an equivariant map into the flag variety

$$\xi = (\xi^1, \xi^2, \xi^3) : \partial\Gamma \longrightarrow \mathcal{F}lag(\mathbf{R}^4),$$

which is also  $\rho$ -equivariant.

The maps  $\xi^i$  can be described as follows:

- $\xi^1([S])$  is the line of polynomials divisible by  $S^3$ ,
- $\xi^2([S])$  is the plane of polynomials divisible by  $S^2$ ,
- $\xi^3([S])$  is the hyperplane of polynomials divisible by  $S$ .

The four orbits of  $\text{PSL}_2(\mathbf{R})$  in  $\mathbb{P}^3(\mathbf{R})$  can be described in term of  $\xi$ :

- one open orbit  $\Lambda_\xi$  which is the set of polynomials having three distinct real roots, *i.e.* the points in  $\mathbb{P}^3(\mathbf{R})$  that are contained in (exactly) three pairwise distinct  $\xi^3(t)$ ,

- the other open orbit  $\Omega_\xi$  which is the set of polynomials having a pair of complex conjugate roots, *i.e.* points of  $\mathbb{P}^3(\mathbf{R})$  contained in exactly one  $\xi^3(t)$ ,
- the relatively closed orbit is the surface  $\bigcup_{t \in \partial\Gamma} \xi^2(t) \setminus \xi^1(\partial\Gamma)$ ,
- and the closed orbit is the curve  $\xi^1(\partial\Gamma)$ .

Remark that the two open orbits are the connected components of the complementary of the surface, called *discriminant*,  $\xi^2(\partial\Gamma) = \bigcup_{t \in \partial\Gamma} \xi^2(t) \subset \mathbb{P}^3(\mathbf{R})$ .

4.1.1. *The Nonconvex Example.* Let us first describe the developing map of the nonconvex foliated projective structure  $(\text{dev}', \text{hol})$ . The open orbit  $\Lambda_\xi$  coincides with the image of  $\text{dev}'$ . The developing map  $\text{dev}'$  can be described as

$$(6) \quad \begin{aligned} \text{dev}' : \partial\Gamma^{3+} \simeq \overline{M} &\longrightarrow \mathbb{P}^3(\mathbf{R}) \\ (t_+, t_0, t_-) &\longmapsto \xi^3(t_+) \cap \xi^3(t_0) \cap \xi^3(t_-). \end{aligned}$$

The image of the geodesic  $(t_+, t_-) \in \partial\Gamma^{(2)}$  is an open segment in the projective line  $\xi^3(t_+) \cap \xi^3(t_-)$ . The image of the (weakly) stable leaf  $t_+$  is contained in the projective plane  $\xi^3(t_+)$ .

Note that the only property of the hol-equivariant map  $\xi : \partial\Gamma \rightarrow \text{Flag}(\mathbf{R}^4)$  needed to define the developing map is that, for any pairwise distinct  $t_1, t_2, t_3 \in \partial\Gamma$ , the three projective planes  $\xi^3(t_1)$ ,  $\xi^3(t_2)$  and  $\xi^3(t_3)$  intersect in a unique point in  $\mathbb{P}^3(\mathbf{R})$ . To ensure that the developing map is a local homeomorphism we need that for  $t_1, t_2, t_3, t_4$  pairwise distinct the intersection  $\bigcap_{i=1}^4 \xi^3(t_i) = \emptyset$ .

4.1.2. *The Properly Convex Example.* Let us now describe the developing map of the properly convex foliated structure  $(\text{dev}, \text{hol})$  defined above in terms of the hol-equivariant map  $\xi : \partial\Gamma \rightarrow \text{Flag}(\mathbf{R}^4)$ . First note that the image of  $\text{dev}$  is the open orbit  $\Omega_\xi$ .

The map  $\xi : \partial\Gamma \rightarrow \text{Flag}(\mathbf{R}^4)$  has the property that, for every distinct  $t, t' \in \partial\Gamma$ , the intersection  $\xi^3(t) \cap \xi^2(t')$  is a point. We can therefore define an equivariant map, or rather an equivariant family of maps  $(\xi_t)_{t \in \partial\Gamma}$

$$\begin{aligned} \partial\Gamma \times \partial\Gamma &\longrightarrow \mathbb{P}^3(\mathbf{R}) \\ (t, t') &\longmapsto \xi_t(t') = \begin{cases} \xi^3(t) \cap \xi^2(t') & \text{if } t \neq t' \\ \xi^1(t) & \text{otherwise} \end{cases} . \end{aligned}$$

Note that for every  $t$  the image of  $\xi_t^1$  in  $\xi^3(t)$  bounds the properly convex domain  $C_t = \text{dev}(t)$ .

Having this family of maps, we see that the image of the geodesic leaf  $g = (t_+, t_-) \in \partial\Gamma^{(2)}$  under the developing map  $\text{dev}$  of (5) is the intersection of the projective line  $\overline{\xi^1(t_+) \xi_{t_+}^1(t_-)} = \mathbb{P}(\xi^1(t_+) \oplus \xi_{t_+}^1(t_-))$  with the convex  $C_{t_+}$ . The endpoint at  $+\infty$  of the open segment  $\text{dev}(g)$  is  $\xi^1(t_+)$  and  $\xi_{t_+}^1(t_-)$  is the endpoint at  $-\infty$ . Moreover the projective line  $\xi^2(t_+)$  is tangent to the convex  $C_{t_+}$  at  $\xi^1(t_+)$ . The tangent line at the point  $\xi_{t_+}^1(t_-)$  is the



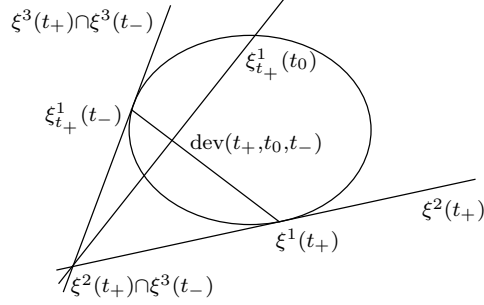


FIGURE 2. The image of the developing map

projective line  $\xi^3(t_+) \cap \xi^3(t_-)$ . These two projective lines intersect in the point  $\xi^2(t_+) \cap \xi^3(t_-) = \xi^1_{t_+}(t_+)$ .

Note that given  $t_0 \in \partial\Gamma$  distinct from  $t_+$  and  $t_-$  the two projective lines  $\overline{\xi^1(t_+)\xi^1_{t_+}(t_-)}$  and  $\overline{\xi^1_{t_+}(t_+)\xi^1_{t_+}(t_0)}$  intersect in a unique point that belongs to  $C_{t_+}$ . With this we can now give an explicit formula for the developing map

$$(7) \quad \begin{aligned} \text{dev} : \partial\Gamma^{3+} \simeq \overline{M} &\longrightarrow \mathbb{P}^3(\mathbf{R}) \\ (t_+, t_0, t_-) &\longmapsto \overline{\xi^1(t_+)\xi^1_{t_+}(t_-)} \cap \overline{\xi^1_{t_+}(t_+)\xi^1_{t_+}(t_0)}. \end{aligned}$$

This gives a description of the developing map of the homogeneous properly convex foliated structure in geometric terms using only ‘‘convexity’’ properties of the curve  $\xi : \mathbb{P}^1(\mathbf{R}) \rightarrow \mathcal{F}lag(\mathbf{R}^4)$ .

#### 4.2. Convex Curves and Convex Representations.

**Definition 4.1.** *A curve*

$$\xi^1 : S^1 \longrightarrow \mathbb{P}^{n-1}(\mathbf{R})$$

*is said to be convex if for, every  $n$ -tuple  $(t_1, \dots, t_n)$  of pairwise distinct points  $t_i \in \mathbb{P}^1(\mathbf{R})$ , we have*

$$\bigoplus_{i=1}^n \xi^1(t_i) = \mathbf{R}^n.$$

In [19, 13, 12] convex curves are called hyperconvex. They were previously known and studied under the name of convex curves (see e.g. [21]) and we stick to this terminology. A convex curve  $\xi^1 : S^1 \rightarrow \mathbb{P}^2(\mathbf{R})$  is precisely an injective curve which parametrizes the boundary of a strictly convex domain in  $\mathbb{P}^2(\mathbf{R})$ .

We are interested in convex curves  $\partial\Gamma \rightarrow \mathbb{P}^{n-1}(\mathbf{R})$  which are equivariant with respect to some representation  $\rho : \Gamma \rightarrow \text{PSL}_n(\mathbf{R})$ .

**Definition 4.2.** *A representation  $\rho : \Gamma \rightarrow \text{PSL}_n(\mathbf{R})$  is said to be convex if there exists a  $\rho$ -equivariant (continuous) convex curve  $\xi^1 : \partial\Gamma \rightarrow \mathbb{P}^{n-1}(\mathbf{R})$ .*

Convex representation are deeply related with the Hitchin component:

**Theorem 4.3** (Labourie [19], Guichard [12]). *The Hitchin component  $\mathcal{T}^n(\Gamma)$  is the set of (conjugacy classes of) convex representations  $\rho : \Gamma \rightarrow \mathrm{PSL}_n(\mathbf{R})$ .*

Let us recall some facts and properties about  $\rho$ -equivariant convex curves.

**Definition 4.4.** *A curve  $\xi = (\xi^1, \dots, \xi^{n-1}) : S^1 \rightarrow \mathcal{F}lag(\mathbf{R}^n)$  is Frenet if*

- (i) *For every  $(n_1, \dots, n_k)$  with  $\sum_{i=1}^k n_i = n$  and every  $x_1, \dots, x_k \in S^1$  pairwise distinct, the following sum is direct:*

$$\bigoplus_{i=1}^k \xi^{n_i}(x_i) = \mathbf{R}^n.$$

- (ii) *For every  $(m_1, \dots, m_k)$  with  $\sum_{i=1}^k m_i = m \leq n$  and for every  $x \in S^1$*

$$\lim_{(x_i) \rightarrow x} \bigoplus_{i=1}^k \xi^{m_i}(x_i) = \xi^m(x),$$

*the limit is taken over  $k$ -tuples  $(x_1, \dots, x_k)$  of pairwise distinct  $x_i$ .*

If  $\xi$  is Frenet, the curve  $\xi^1$  is convex and entirely determines  $\xi$  by the limit condition.

**Theorem 4.5** (Labourie [19]). *If a representation  $\rho$  is a convex, then there exists a (unique)  $\rho$ -equivariant Frenet curve  $\xi = (\xi^1, \dots, \xi^{n-1}) : \partial\Gamma \rightarrow \mathcal{F}lag(\mathbf{R}^n)$ .*

Frenet curves satisfy a duality property

**Theorem 4.6** ([13] Théorème 5). *Let  $\xi = (\xi^1, \dots, \xi^{n-1}) : S^1 \rightarrow \mathcal{F}lag(V)$  be a Frenet curve. Then the dual curve  $\xi^\perp : S^1 \rightarrow \mathcal{F}lag(V^*)$ ,  $t \mapsto (\xi^{n-1, \perp}(t), \dots, \xi^{1, \perp}(t))$  is also Frenet.*

**Remark 4.7.** *This duality of Frenet curves is more natural in the context of positive curves into flag variety for which we refer the reader to [7] where the connection between convex curves into  $\mathbb{P}^n(\mathbf{R})$  and positive curves into  $\mathcal{F}lag(\mathbf{R}^n)$  is discussed.*

This means that we can check if a curve is Frenet indifferently by investigating sums or intersections of vectors spaces. From this we can deduce

**Proposition 4.8** ([2]). *Let  $\xi = (\xi^1, \xi^2, \xi^3) : S^1 \rightarrow \mathcal{F}lag(\mathbf{R}^4)$  be a Frenet curve. For  $t \in S^1$ , let  $\xi_t : S^1 \rightarrow \mathcal{F}lag(\xi^3(t))$  be the curve defined by*

$$\begin{aligned} \xi_t : S^1 &\longrightarrow \mathcal{F}lag(\xi^3(t)) \\ t' &\longmapsto \begin{cases} (\xi^3(t) \cap \xi^2(t'), \xi^3(t) \cap \xi^3(t')) & \text{if } t' \neq t \\ (\xi^1(t), \xi^2(t)) & \text{otherwise.} \end{cases} \end{aligned}$$

*Then  $\xi_t$  is a Frenet curve.*

*Proof.* By the duality property, it suffices to check:

- for all  $t'_1 \neq t'_2$ ,  $\xi_{t'_1}^1(t'_1) \cap \xi_{t'_2}^2(t'_2) = \{0\}$ ,

- for all pairwise distinct  $t'_1, t'_2$  and  $t'_3$ ,  $\xi_t^2(t'_1) \cap \xi_t^2(t'_2) \cap \xi_t^2(t'_3) = \{0\}$ ,
- and for all  $t'$ ,  $\lim_{t'_1 \neq t'_2 \rightarrow t'} \xi_t^2(t'_1) \cap \xi_t^2(t'_2) = \xi_t^1(t')$ .

These three properties follow from the corresponding properties for  $\xi$ .  $\square$

### 4.3. The Properly Convex Foliated Structure of a Convex Representation.

**Theorem 4.9.** *Let  $\rho : \Gamma \rightarrow \mathrm{PSL}_4(\mathbf{R})$  be a convex representation. Then there exists a developing pair  $(\mathrm{dev}, \mathrm{hol})$ , with holonomy homomorphism  $\mathrm{hol} = \rho \circ p : \overline{\Gamma} \rightarrow \mathrm{PSL}_4(\mathbf{R})$  defining a properly convex foliated projective structure on  $M$ .*

In fact we construct a section  $s : \mathcal{T}^4(\Gamma) \rightarrow \mathcal{P}_{pcf}(M)$  of the holonomy map. This section will be automatically continuous, injective with image a connected component of  $\mathcal{P}_{pcf}(M)$ .

*Proof.* We want to use formula (7). Since the holonomy homomorphism  $\mathrm{hol}$  factors through  $\Gamma$ , the developing map  $\mathrm{dev}$  will be defined by a  $\rho$ -equivariant map  $\mathrm{dev} : \overline{M} \rightarrow \mathbb{P}^3(\mathbf{R})$ . Let

$$\xi = (\xi^1, \xi^2, \xi^3) : \partial\Gamma \rightarrow \mathcal{F}lag(\mathbf{R}^4)$$

be the  $\rho$ -equivariant Frenet curve. Let  $(\xi_t^1)_{t \in \partial\Gamma}$  be the  $\rho$ -equivariant family of continuous curves

$$\begin{aligned} \xi_t^1 : \partial\Gamma &\longrightarrow \xi^3(t) \subset \mathbb{P}^3(\mathbf{R}) \\ t' &\longmapsto \xi^3(t) \cap \xi^2(t') \quad \text{if } t' \neq t \\ t &\longmapsto \xi^1(t). \end{aligned}$$

By Proposition 4.8, for every  $t$ , the curve  $\xi_t^1$  is convex and hence bounds a properly convex domain  $C_t \subset \xi^3(t)$ . Note that, as in the homogeneous example above, the tangent line to  $C_t$  at  $\xi^1(t)$  is  $\xi^2(t)$  and the tangent line to  $C_t$  at  $\xi_t^1(t')$ ,  $t' \neq t$ , is  $\xi^3(t') \cap \xi^3(t)$  (see Figure 2). In particular, the point

$$\xi_{t'}^1(t) = \xi^2(t) \cap \xi^3(t') = \xi^2(t) \cap (\xi^3(t') \cap \xi^3(t))$$

is the intersection of the two tangent lines.

We define the developing map by

$$\begin{aligned} \mathrm{dev} : \partial\Gamma^{3+} \simeq \overline{M} &\longrightarrow \mathbb{P}^3(\mathbf{R}) \\ (t_+, t_0, t_-) &\longmapsto \overline{\xi^1(t_+) \xi_{t_+}^1(t_-)} \cap \overline{\xi_{t_-}^1(t_+) \xi_{t_+}^1(t_0)} \end{aligned}$$

Then  $\mathrm{dev}$  is a  $\rho$ -equivariant, it is continuous and injective so it is a homeomorphism onto its image. The image of the (weakly) stable leaf  $t$  is the proper convex set  $C_t$  and the image of the geodesic  $(t_+, t_-)$  is contained in the projective line  $\overline{\xi^1(t_+) \xi_{t_+}^1(t_-)}$ . Therefore, the pair  $(\mathrm{dev}, \mathrm{hol})$  defines a properly convex foliated projective structure on  $M$ .  $\square$

Similar to the foliated structure with non-convex leaves in the homogeneous example above we also get the following:

**Theorem 4.10.** *Let  $\rho : \Gamma \rightarrow \mathrm{PSL}_4(\mathbf{R})$  be a convex representation. Then there exists a developing pair  $(\mathrm{dev}', \mathrm{hol})$ , with holonomy homomorphism  $\mathrm{hol} = \rho \circ p : \overline{\Gamma} \rightarrow \mathrm{PSL}_4(\mathbf{R})$  defining a foliated projective structure on  $M$  which is not convex.*

As above this could be stated as the existence of a section of  $\mathrm{hol}$  from  $\mathcal{T}(\Gamma)$  to  $\mathcal{P}_f(M)$ .

*Proof.* Let  $\xi = (\xi^1, \xi^2, \xi^3) : \partial\Gamma \rightarrow \mathcal{F}lag(\mathbf{R}^4)$  be the  $\rho$ -equivariant Frenet curve. We define the  $\rho$ -equivariant developing map

$$\begin{aligned} \mathrm{dev}' : \partial\Gamma^{3+} \simeq \overline{M} &\longrightarrow \mathbb{P}^3(\mathbf{R}) \\ (t_+, t_0, t_-) &\longmapsto \xi^3(t_+) \cap \xi^3(t_0) \cap \xi^3(t_-). \end{aligned}$$

By Proposition 4.8  $\mathrm{dev}'$  is well defined. It is also continuous and locally injective, hence a local homeomorphism. The image of a geodesic leaf  $(t_+, t_-) \in \partial\Gamma^{(2)}$  is contained in the projective line  $\xi^3(t_+) \cap \xi^3(t_-)$ . The image of the (weakly) stable leaf  $t_+$  is contained in the projective plane  $\xi^3(t_+)$ , but it cannot be convex since it is the complementary subset in  $\xi^3(t_+)$  of the closure of the convex set  $\mathrm{dev}(t_+)$  given in the preceding Theorem.  $\square$

Note that  $\mathrm{dev}$  is a global homeomorphism whereas  $\mathrm{dev}'$  is only a local homeomorphism, two points of  $\partial\Gamma^{3+}$  have the same images under  $\mathrm{dev}'$  if, and only if, they differ by a permutation.

**4.4. Domains of Discontinuity.** The above foliated projective structures  $(\mathrm{dev}, \mathrm{hol})$  and  $(\mathrm{dev}', \mathrm{hol})$  appear naturally when we consider domains of discontinuity for the action of  $\rho(\Gamma)$  on  $\mathbb{P}^3(\mathbf{R})$ . The action of  $\rho(\Gamma)$  on  $\mathbb{P}^3(\mathbf{R})$  is not free or proper, since  $\rho(\gamma)$  has fixed points for every  $\gamma \in \Gamma$ .

But if we remove the ruled surface (discriminant)  $\xi^2(\partial\Gamma) \subset \mathbb{P}^3(\mathbf{R})$ , the complement  $\mathbb{P}^3(\mathbf{R}) \setminus \xi^2(\partial\Gamma)$  has two connected components  $\Lambda_\xi = \mathrm{dev}'(\overline{M})$  and  $\Omega_\xi = \mathrm{dev}(\overline{M})$ . Namely, the image  $\mathrm{dev}'(\overline{M})$  is contained in  $\mathbb{P}^3(\mathbf{R}) \setminus \xi^2(\partial\Gamma)$  and, using the Frenet property of  $\xi$ , the boundary of  $\mathrm{dev}'(\overline{M})$  is  $\xi^2(\partial\Gamma)$ . This implies that  $\mathrm{dev}'(\overline{M})$  is one connected component of  $\mathbb{P}^3(\mathbf{R}) \setminus \xi^2(\partial\Gamma)$ . Furthermore, by Proposition 4.8 and Figure 2,  $\mathrm{dev}(\overline{M})$  is the complementary of the closure of  $\mathrm{dev}'(\overline{M})$ , hence the other connected component of  $\mathbb{P}^3(\mathbf{R}) \setminus \xi^2(\partial\Gamma)$ .

In particular,  $\rho(\Gamma)$  acts freely and properly discontinuously on  $\Lambda_\xi$  and on  $\Omega_\xi$ . If  $t$  is a (weakly) stable leaf, then  $\mathrm{dev}(t) = \Omega_\xi \cap \xi^3(t) = C_t$  and  $\mathrm{dev}'(t) = \Lambda_\xi \cap \xi^3(t) = \xi^3(t) \setminus \overline{C}_t$ .

## 5. FROM PROPERLY CONVEX FOLIATED STRUCTURES TO CONVEX REPRESENTATIONS

In this section we will prove the following

**Theorem 5.1.** *The holonomy representation  $\mathrm{hol} : \overline{\Gamma} \rightarrow \mathrm{PGL}_4(\mathbf{R})$  of a properly convex foliated projective structure factors through a convex representation  $\rho : \Gamma \rightarrow \mathrm{PSL}_4(\mathbf{R})$  and the foliated projective structure on  $M$  is equivalent to the one associated to  $\rho$  in Theorem 4.9.*

Basically, we will construct an equivariant continuous curve  $\partial\Gamma \rightarrow \mathbb{P}^3(\mathbf{R})^*$  and show that it is convex.

### 5.1. Maps Associated to Foliated Projective Structures.

**Proposition 5.2.** *Let  $(\text{dev}, \text{hol})$  be the developing pair of a foliated projective structure on  $M$ . Then the two maps*

$$\begin{aligned} \xi^3 : \widetilde{\mathcal{F}} \simeq \widetilde{\partial\Gamma} &\longrightarrow \mathbb{P}^3(\mathbf{R})^* \\ \mathcal{D} : \widetilde{\mathcal{G}} \simeq \widetilde{\partial\Gamma}_{[0]}^{(2)} &\longrightarrow \text{Gr}_2^4(\mathbf{R}), \end{aligned}$$

defined by

$$\begin{aligned} \xi^3(f) &= \xi^3(t_+) \text{ is the projective plane containing } \text{dev}(f), \text{ and} \\ \mathcal{D}(g) &= \mathcal{D}(t_+, t_-) \text{ is the projective line containing } \text{dev}(g) \end{aligned}$$

are continuous and hol-equivariant. Moreover

- (i) the map  $\xi^3$  is locally injective.
- (ii) for all  $(t_+, t_-)$  in  $\widetilde{\mathcal{G}}$ ,  $\mathcal{D}(t_+, t_-) \subset \xi^3(t_+)$ .
- (iii) for all  $t_+$ , the map  $t_- \mapsto \mathcal{D}(t_+, t_-)$  is not locally constant.

*Proof.* The developing pair  $(\text{dev}, \text{hol})$  defining a foliated projective structure on  $M$  consists of the homomorphism

$$\text{hol} : \overline{\Gamma} = \pi_1(M) \longrightarrow \text{PGL}_4(\mathbf{R})$$

and the hol-equivariant local homeomorphism

$$\text{dev} : \widetilde{M} \longrightarrow \mathbb{P}^3(\mathbf{R})$$

satisfying the following properties

- (i) for any geodesic  $g$  in  $\widetilde{\mathcal{G}}$ ,  $\text{dev}(g)$  is contained in a projective line, and
- (ii) for every (weakly) stable leaf  $f$  in  $\widetilde{\mathcal{F}}$ ,  $\text{dev}(f)$  is contained in a projective plane.

This shows that the above definitions of  $\xi^3$  and  $\mathcal{D}$  are meaningful. The properties of  $\xi^3$  and  $\mathcal{D}$  are direct consequences of the fact that  $\text{dev}$  is a hol-equivariant local homeomorphism.  $\square$

In fact we can be a little more precise about the injectivity of the map  $\xi^3$ .

**Lemma 5.3.** *For every  $(t_+, t_-) \in \widetilde{\partial\Gamma}_{[0]}^{(2)}$  we have  $\xi^3(t_+) \neq \xi^3(t_-)$ .*

*Proof.* Take  $(t_+, t_-) \in \widetilde{\partial\Gamma}_{[0]}^{(2)}$ . By Lemma 1.11 there exists an element  $\gamma \in \overline{\Gamma}$  of zero translation such that  $(\gamma^n t_+), (\gamma^n t_-)$  converge to  $\tilde{t}_{+, \gamma}$ . By the above local injectivity, for big enough  $n$

$$\rho(\gamma)^n \xi^3(t_+) = \xi^3(\gamma^n t_+) \neq \xi^3(\gamma^n t_-) = \rho(\gamma)^n \xi^3(t_-).$$

$\square$

## 5.2. The Holonomy Action on Convex Sets.

**Proposition 5.4.** *Suppose that  $(\text{dev}, \text{hol})$  is a developing pair defining a properly convex foliated projective structure on  $M$ . Then for every element  $\bar{\gamma} \in \bar{\Gamma} - \langle \tau \rangle$  of zero translation  $\text{hol}(\bar{\gamma})$  is diagonalizable over  $\mathbf{R}$  with all eigenvalues being of the same sign. Moreover if  $t_+ \in \widetilde{\partial\Gamma}$  is an attractive fixed point of  $\bar{\gamma}$ , the eigenvalues of the action of  $\text{hol}(\bar{\gamma})$  restricted to  $\xi^3(t_+)$  satisfy  $|\lambda_+| \geq |\lambda_0| > |\lambda_-|$ .*

**Remark 5.5.** *Eigenvalues of an element of  $\text{hol}(\bar{\gamma}) \in \text{PGL}_4(\mathbf{R})$  are of course only defined up to a common multiple. Our statements about eigenvalues will clearly be invariant by scalar multiplication.*

*From Proposition 5.4 one could already conclude some properties of the action of the central element  $\tau \in \bar{\Gamma}$  as it commutes with an  $\mathbf{R}$ -diagonalizable element with at least two distinct eigenvalues. We will not state any of these properties until we are indeed able to prove that  $\text{hol}(\tau)$  is trivial.*

5.2.1. *Some Observations.* Since the developing pair  $(\text{dev}, \text{hol})$  defines a properly convex foliated projective structure we have that the image of each (weakly) stable leaf is convex.

**Notation 5.6.** *We denote by  $C_{t_+}$  the properly convex subset in  $\xi^3(t_+)$  equal to  $\text{dev}(t_+)$ .*

Let  $\bar{\gamma} \in \bar{\Gamma} - \langle \tau \rangle$  be an element of zero translation and let  $(t_+, t_-) \in \widetilde{\partial\Gamma}_{[0]}^{(2)}$  be a pair of an attractive and a repulsive fixed point of  $\bar{\gamma}$ . Then  $\text{hol}(\bar{\gamma}) \in \text{PGL}_4(\mathbf{R})$  stabilizes the projective plane  $\xi^3(t_+)$  and also the open properly convex set  $C_{t_+} = \text{dev}(t_+) \subset \xi^3(t_+)$ . Moreover  $\text{hol}(\bar{\gamma})$  stabilizes the projective line  $\mathcal{D}(t_+, t_-)$  containing the image  $\text{dev}(t_+, t_-)$  of the geodesic  $(t_+, t_-) \in \widetilde{\mathcal{G}}$ .

**Lemma 5.7.** *The action of  $\text{hol}(\bar{\gamma})$  on  $\mathcal{D}(t_+, t_-) \subset \xi^3(t_+)$  has two eigenlines  $x_+, x_-$  with eigenvalues  $\lambda_+, \lambda_-$  satisfying:*

$$\lambda_+ \lambda_- \geq 0 \text{ and } |\lambda_+| > |\lambda_-|.$$

*Proof.* Since  $C_{t_+}$  is properly convex,  $\overline{C}_{t_+}$  does not contain any projective line; hence the intersection

$$\mathcal{D}(t_+, t_-) \cap \partial C_{t_+}$$

consists of two points  $x_+ = x_+(\bar{\gamma})$  and  $x_- = x_-(\bar{\gamma})$  which are fixed by  $\text{hol}(\bar{\gamma})$ . The points  $x_+$  and  $x_-$  are the endpoints at  $+\infty$  and at  $-\infty$  of the segment  $\text{dev}(t_+, t_-)$ . So  $x_+$  and  $x_-$  are eigenlines for  $\text{hol}(\bar{\gamma})$  corresponding to real eigenvalues  $\lambda_+ = \lambda_+(\bar{\gamma})$  and  $\lambda_- = \lambda_-(\bar{\gamma})$ . The segment  $\text{dev}(t_+, t_-)$  with endpoints  $x_+$  and  $x_-$  is stable by  $\text{hol}(\bar{\gamma})$ . This implies that the two eigenvalues  $\lambda_+$  and  $\lambda_-$  are of the same sign,

$$\lambda_+ \lambda_- > 0.$$

The action of  $\bar{\gamma}$  on the geodesic  $(t_+, t_-) \subset \widetilde{M}$  corresponds to a positive time map of the geodesic flow. This implies that for every point  $x$  in  $\text{dev}(t_+, t_-)$ ,

the limit of the sequence  $(\text{hol}(\bar{\gamma})^n x)_{n \in \mathbf{N}}$  is equal to  $x_+$ . This gives the inequality  $|\lambda_+| > |\lambda_-|$ .  $\square$

**Lemma 5.8.** *The action of  $\text{hol}(\bar{\gamma})$  does not interchange the two components of  $C_{t_+} - \mathcal{D}(t_+, t_-)$ .*

*Proof.* By proper convexity, the set  $C_{t_+} - \mathcal{D}(t_+, t_-)$  has indeed two connected components,  $C_1$  and  $C_2$  and  $\text{hol}(\bar{\gamma})$  either exchanges them or send them into themselves.

Since the image of a geodesic  $(t_+, t_-)$  is an open segment, the restriction of  $\text{dev}$  to this geodesic is necessarily a homeomorphism onto its image. Fix a point  $x$  in this geodesic  $(t_+, t_-) \subset t_+$ . In the leaf  $t_+$  there is a neighborhood  $U$  of the geodesic segment  $[x, \bar{\gamma} \cdot x]$  such that  $\text{dev}|_U$  is a homeomorphism onto its image.

The complementary of  $(t_+, t_-)$  in the leaf  $t_+$  has also two connected components,  $D_1$  and  $D_2$  and we have (up to reindexing)  $\text{dev}(U \cap D_1) = \text{dev}(U) \cap C_1$  and  $\text{dev}(U \cap D_2) = \text{dev}(U) \cap C_2$ . Moreover, as the action of  $\bar{\gamma}$  does not exchange  $D_1$  and  $D_2$ , there are points  $y$  (close to  $x$ ) in  $U \cap D_1$  such that  $\bar{\gamma} \cdot y$  is in  $U \cap D_1$ . Hence the point  $m = \text{dev}(y)$  is in  $C_1$  and its image  $\text{hol}(\bar{\gamma}) \cdot m = \text{dev}(\bar{\gamma} \cdot y)$  is also in  $C_1$ . This implies that  $\text{hol}(\bar{\gamma}) \cdot C_1 = C_1$  and the same for  $C_2$ .  $\square$

**Lemma 5.9.** *The action of  $\text{hol}(\bar{\gamma})$  on  $\xi^3(t_+)$  has a third eigenline  $x_0$  with eigenvalue  $\lambda_0$  being of the same sign as  $\lambda_+$  and  $\lambda_-$ .*

*Proof.* By Lemma 5.8, the left tangent line to  $C_{t_+}$  at the point  $x_+$  and the right tangent line to  $C_{t_+}$  at  $x_-$  are preserved by  $\text{hol}(\bar{\gamma})$ . Their intersection is a third eigenline  $x_0 = x_0(\bar{\gamma})$  in  $\xi^3(t_+)$ , fixed by  $\text{hol}(\bar{\gamma})$ . The point  $x_0$  does not lie on the projective line  $\mathcal{D}(t_+, t_-)$ . Therefore the restriction of  $\text{hol}(\bar{\gamma})$  to  $\xi^3(t_+)$  is diagonalizable over  $\mathbf{R}$ . Moreover, since  $\text{hol}(\bar{\gamma})$  does not interchange the two connected components of  $C_{t_+} - \mathcal{D}(t_+, t_-)$ , the third eigenvalue  $\lambda_0 = \lambda_0(\bar{\gamma})$  is of the same sign as  $\lambda_+$  and  $\lambda_-$ .  $\square$

Summarizing we have the following

**Lemma 5.10.** *For all  $\bar{\gamma} \in \bar{\Gamma}$  of zero translation, the element  $\text{hol}(\bar{\gamma})$  is diagonalizable over  $\mathbf{R}$  with all eigenvalues being of the same sign. In particular  $\text{hol}(\bar{\gamma}) \in \text{PSL}_4(\mathbf{R})$ .*

*Proof.* Let  $(t_+, t_-) \in \widetilde{\partial\Gamma}_{[0]}^{(2)}$  be the chosen pair of an attractive and a repulsive fixed point for  $\bar{\gamma}$ . Then the restriction of  $\text{hol}(\bar{\gamma})$  to  $\xi^3(t_+)$  is diagonalizable over  $\mathbf{R}$  with three eigenvalues of the same sign. Applying the above arguments to  $\bar{\gamma}^{-1}$  the same holds for the restriction of  $\text{hol}(\bar{\gamma})$  to  $\xi^3(t_-)$ . Note that the inclusion  $\xi^3(t_+) \cap \xi^3(t_-) \subset \xi^3(t_-)$  is strict by Lemma 5.3, so the fourth eigenvalue of  $\text{hol}(\bar{\gamma})$  is the one corresponding to the eigenline of  $\text{hol}(\bar{\gamma})$  in  $\xi^3(t_-)$  which is not contained in the intersection  $\xi^3(t_+) \cap \xi^3(t_-)$ .  $\square$

Keeping the same notation as above we get the following

**Lemma 5.11.** *For every nontrivial element  $\bar{\gamma} \in \bar{\Gamma}$  of zero translation, the inequalities  $|\lambda_-| < |\lambda_0| \leq |\lambda_+|$  hold.*

*Proof.* Suppose on the contrary that  $|\lambda_0| > |\lambda_+|$  and let  $x_0 \in \xi^3(t_+)$  be the corresponding eigenline for  $\lambda_0$ . The convex  $C_{t_+}$  contains a neighborhood of the segment  $\text{dev}(t_+, t_-)$ . The image of this neighborhood under  $\text{hol}(\bar{\gamma}^n)_{n \in \mathbf{Z}}$  will be a sector bounded by the two projective lines  $\overline{x_0 x_+}$  and  $\overline{x_0 x_-}$ . Therefore the  $\text{hol}(\bar{\gamma})$ -invariant convex  $C_{t_+}$  has to be this sector, contradicting the hypothesis that  $C_{t_+}$  is a properly convex set.

For any geodesic  $(t_+, t)$  in the (weakly) stable leaf  $t_+$  consider the intersection of  $\mathcal{D}(t_+, t)$  with a tangent line  $L$  to  $C_{t_+}$  in  $x_-$ . By continuity and equivariance of  $\mathcal{D}$

$$\lim_{n \rightarrow +\infty} \text{hol}(\bar{\gamma})^{-n}(\mathcal{D}(t_+, t) \cap L) = \mathcal{D}(t_+, t_-) \cap L = x_-.$$

The restriction of  $\text{hol}(\bar{\gamma})$  to  $L$  is diagonalizable with two real eigenvalues  $\lambda_-$  and  $\lambda_0$ , which satisfy hence the inequality  $|\lambda_-| < |\lambda_0|$   $\square$

With this Proposition 5.4 is proved.

5.2.2. *Two Cases for the Action of  $\text{hol}(\bar{\gamma})$ .* The two possible cases  $\lambda_+ = \lambda_0$  and  $|\lambda_+| > |\lambda_0|$  for the eigenvalues of  $\text{hol}(\bar{\gamma})$  of a nontrivial element  $\bar{\gamma} \in \bar{\Gamma}$  of zero translation have different consequences for the convex sets. We continue to use the notation from the previous paragraph.

**Case (T):**  $\lambda_+ = \lambda_0$ .

There is a unique  $\text{hol}(\bar{\gamma})$ -invariant projective line  $T = \overline{x_0 x_+}$  through the point  $x_+$  and distinct from  $\mathcal{D}(t_+, t_-)$ , which is the unique tangent line to  $C_{t_+}$  at  $x_+$ . Any projective line  $L$  containing the point  $x_-$  is invariant by  $\text{hol}(\bar{\gamma})$ . Since  $\text{hol}(\bar{\gamma})$  fixes  $T$  pointwise, by Lemma 5.11 it acts on  $L$  with one repulsive fixed point  $x_-$  and the attractive fixed point being  $L \cap T$ . Therefore the only proper invariant convex sets in  $L$  with non empty interior relative to  $L$  are the two segments with endpoints  $x_-$  and  $L \cap T$ . In particular  $L \cap C_{t_+}$  is such a segment for any  $L$ , and  $C_{t_+}$  is a union of such segments. This means that  $C_{t_+}$  is a triangle with one side supported by  $T$  and the third vertex being  $x_-$  (Figure 3).

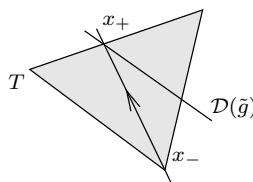


FIGURE 3. A triangle

We will finally show that case (T) will not occur.

**Case (C):**  $|\lambda_0| < |\lambda_+|$ .

In this case the third eigenline  $x_0$  in the projective plane  $\xi^3(t_+)$  is the



eigenspace corresponding to  $\lambda_0$ . The projective line  $\overline{x_0x_+}$  is the unique  $\text{hol}(\overline{\gamma})$ -invariant projective line through  $x_+$  which is contained in  $\xi^3(t_+)$  and different from  $\mathcal{D}(t_+, t_-)$ . Similarly  $\overline{x_0x_-}$  is the unique  $\text{hol}(\overline{\gamma})$ -invariant projective line through  $x_-$  which is contained in  $\xi^3(t_+)$  and different from  $\mathcal{D}(t_+, t_-)$ . In particular the tangent lines to  $C_{t_+}$  at  $x_+$  and  $x_-$  are unique. We might have degenerate cases as in Figure 4. But as we will see the

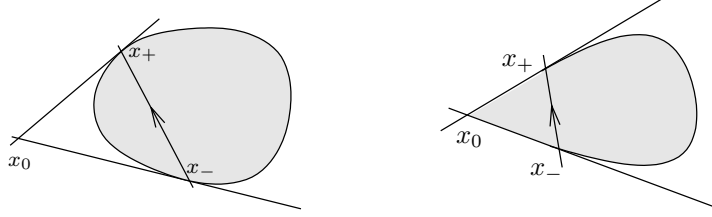


FIGURE 4. A “nice” convex and a degenerate proper convex

degenerate case actually never occurs.

### 5.3. Defining the Map $\xi^1$ .

**Proposition 5.12.** *Let  $(\text{dev}, \text{hol})$  be a developing pair defining a properly convex foliated projective structure on  $M$ . Then there exists a continuous  $\text{hol}$ -equivariant map*

$$\xi^1 : \widetilde{\partial\Gamma} \rightarrow \mathbb{P}^3(\mathbf{R})$$

such that  $\xi^1(t_+) \in \mathcal{D}(g)$  for all  $t_+ \in \widetilde{\partial\Gamma}$  and all  $g \subset t_+$ .

We continue to use the notations from the previous section.

**Lemma 5.13.** *We have the following alternative:*

– for all  $t_+ \in \widetilde{\partial\Gamma}$  the intersection

$$\bigcap_{g \subset t_+} \mathcal{D}(g) = \bigcap_{\substack{t \in \widetilde{\partial\Gamma} \\ (\tau t, t_+, t) \text{ oriented}}} \mathcal{D}(t_+, t) = \emptyset$$

is empty, or

– for all  $t_+ \in \widetilde{\partial\Gamma}$  the intersection  $\bigcap_{g \subset t_+} \mathcal{D}(g)$  is a point in  $\mathbb{P}^3(\mathbf{R})$ .

*Proof.* The injectivity of the map  $\mathcal{D}$  (Proposition 5.2) implies that the intersection  $\bigcap_{g \subset t_+} \mathcal{D}(g)$  is either empty or a point. The continuity of  $\mathcal{D}$  implies that the  $\overline{\Gamma}$ -invariant set

$$\{t_+ \in \widetilde{\partial\Gamma} \mid \bigcap_{g \subset t_+} \mathcal{D}(g) = \emptyset\}$$

is open. Since the action of  $\overline{\Gamma}$  on  $\widetilde{\partial\Gamma}$  is minimal (Lemma 1.9) this set is either empty or equal to  $\widetilde{\partial\Gamma}$  this proves the claim.  $\square$

**Lemma 5.14.** *Let  $\bar{\gamma}$  be of zero translation and  $t_+$  an attractive fixed point. For all geodesics  $g_t = (t_+, t)$  in the (weakly) stable leaf  $t_+$ , the projective line  $\mathcal{D}(g_t)$  contains the point  $x_+$ .*

*Proof.* As observed in the proof of Lemma 5.11 the sequence  $(\text{hol}(\bar{\gamma})^{-n}\mathcal{D}(g_t))_{n \in \mathbf{N}}$  converges to  $\overline{x_+x_-}$ . Let  $L$  be a  $\text{hol}(\bar{\gamma})$ -invariant projective line tangent to  $C_{t_+}$  at  $x_+$ , then

$$\lim_{n \rightarrow +\infty} \text{hol}(\bar{\gamma})^{-n}(\mathcal{D}(g_t) \cap L) = L \cap \overline{x_+x_-} = x_+$$

Since the restriction of  $\text{hol}(\bar{\gamma})$  to  $L$  is diagonalizable with two real eigenvalues  $\lambda_+$  and  $\lambda_0$  satisfying the inequality  $|\lambda_0| \leq |\lambda_+|$  and with  $x_+$  corresponding to  $\lambda_+$ , this implies

$$\mathcal{D}(g_t) \cap L = x_+,$$

and in particular,  $x_+ \in \mathcal{D}(g_t)$ .  $\square$

Lemma 5.14 implies that we are in the second case of Lemma 5.13.

**Definition 5.15.** *For all  $t$  in  $\widetilde{\partial\Gamma}$  we define  $\xi^1(t)$  to be the common intersection of the projective lines  $\mathcal{D}(g)$  for  $g$  in the leaf  $t$ :*

$$\xi^1(t) := \bigcap_{g \subset t} \mathcal{D}(g).$$

*Proof of Proposition 5.12.* Continuity and equivariance of  $\xi^1$  follow from the corresponding properties of  $\mathcal{D}$ . From the very definition, for all  $t \in \widetilde{\partial\Gamma}$  and all  $g \subset t$ ,  $\xi^1(t) \subset \mathcal{D}(g) \subset \xi^3(t)$ .  $\square$

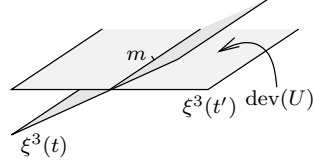
**5.4. Intersections of  $\xi^3$ .** As we already indicated, our ultimate goal is to show that the representation  $\text{hol} : \bar{\Gamma} \rightarrow \text{PGL}_4(\mathbf{R})$  factors through a representation of  $\Gamma$  and that the induced curve  $\xi^3 : \partial\Gamma \rightarrow \mathbb{P}^3(\mathbf{R})^*$  is convex. In this paragraph we establish some facts about the possible intersections of  $\xi^3$ . Since  $\xi^3$  is not constant, the intersection

$$\bigcap_{t \in \widetilde{\partial\Gamma}} \xi^3(t) \subset \mathbb{P}^3(\mathbf{R})$$

is either empty, a point or a projective line.

**Lemma 5.16.** *If the intersection  $\bigcap_{t \in \widetilde{\partial\Gamma}} \xi^3(t)$  is a projective line, this line does not meet the image  $\text{dev}(\widetilde{M})$ .*

*Proof.* Suppose that the intersection  $L = \bigcap_{t \in \widetilde{\partial\Gamma}} \xi^3(t)$  is a projective line and that there is a point  $m$  in  $\widetilde{M}$  such that  $\text{dev}(m)$  belongs to  $L$ . Then, for a small enough neighborhood  $U$  of  $m$ ,  $\text{dev}(U)$  will, in some affine chart, be contained in one of the sectors bounded by the two planes  $\xi^3(t)$ ,  $\xi^3(t')$  (Figure 5), contradicting that  $\text{dev}$  is a local homeomorphism.  $\square$


 FIGURE 5.  $\text{dev}(U)$ .

**Lemma 5.17.** *Suppose that there is a non-empty open subset  $U$  in  $\widetilde{\partial\Gamma}$  such that  $L = \bigcap_{t \in U} \xi^3(t)$  is a projective line, then*

$$\bigcap_{t \in \widetilde{\partial\Gamma}} \xi^3(t) = L.$$

*Proof.* The subset

$$\{t \in \widetilde{\partial\Gamma} \mid \exists \text{ open } U_t \ni t \text{ such that } \bigcap_{s \in U_t} \xi^3(s) \text{ is a projective line}\}$$

is a non-empty, open and  $\bar{\Gamma}$ -invariant subset of  $\widetilde{\partial\Gamma}$ , so by minimality it equals  $\widetilde{\partial\Gamma}$ . Note that by the local injectivity of  $\xi^3$  the intersection  $\bigcap_{s \in U_t} \xi^3(s)$  is independent of the choice of the open  $U_t$ . Therefore we have a locally constant continuous map

$$\widetilde{\partial\Gamma} \longrightarrow \text{Gr}_2^4(\mathbf{R}), \quad t \longmapsto \bigcap_{s \in U_t} \xi^3(s).$$

This map is constant equal to  $L$ . □

**Lemma 5.18.** *If there exists a non-empty open subset  $U$  of  $\widetilde{\partial\Gamma}$ , such that*

$$x = \bigcap_{t \in U} \xi^3(t)$$

*is a point in  $\mathbb{P}^3(\mathbf{R})$ , then*

$$\bigcap_{t \in \widetilde{\partial\Gamma}} \xi^3(t) \text{ is equal to } x.$$

*Proof.* We argue along the lines of the proof of the preceding lemma. The set

$$\{t \in \widetilde{\partial\Gamma} \mid \exists \text{ open } U_t \ni t \text{ such that } \bigcap_{s \in U_t} \xi^3(s) \text{ is a point}\}$$

equals  $\widetilde{\partial\Gamma}$ . Since by the previous lemma the intersection  $\bigcap_{t \in V} \xi^3(t)$  is never a projective line for  $V$  a non-empty open subset of  $\widetilde{\partial\Gamma}$  we get a well defined locally constant map

$$\widetilde{\partial\Gamma} \longrightarrow \mathbb{P}^3(\mathbf{R}), \quad t \longmapsto \bigcap_{s \in U_t} \xi^3(s).$$

This map is constant equal to  $x$ . □

**5.5. Semi-Continuity of  $C_t$ .** The map that sends  $t$  in  $\widetilde{\partial\Gamma}$  to the closure  $\overline{C}_t$  of the convex  $C_t = \text{dev}(t)$  has some semi-continuity properties.

Fixing any continuous distance on  $\mathbb{P}^3(\mathbf{R})$  the space of compact subsets of  $\mathbb{P}^3(\mathbf{R})$  is endowed with the Hausdorff distance.

**Lemma 5.19.** *Let  $(t_n)_{n \in \mathbf{N}}$  be a sequence in  $\widetilde{\partial\Gamma}$  converging to  $t \in \widetilde{\partial\Gamma}$  and such that the sequence of convex sets  $(\overline{C}_{t_n})_{n \in \mathbf{N}}$  has a limit, then*

$$\lim_{n \rightarrow +\infty} \overline{C}_{t_n} \supset \overline{C}_t.$$

*Proof.* It is enough to show that  $C_t$  is contained in the limit  $\lim_{n \rightarrow +\infty} \overline{C}_{t_n}$ . For this, it is sufficient to show that, for all  $x$  in  $C_t$ , there is a sequence  $(x_n)_{n \in \mathbf{N}}$  of  $\mathbb{P}^3(\mathbf{R})$  converging to  $x$  and such that  $x_n$  belongs to  $\overline{C}_{t_n}$  for all  $n$ .

Choose a point  $m$  in  $\widetilde{M}$  with  $\text{dev}(m) = x$ . Since  $(t_n)_{n \in \mathbf{N}}$  converges to  $t$ , there is a sequence  $(m_n)_{n \in \mathbf{N}}$  converging to  $m$  such that  $m_n$  is contained in the leaf  $t_n$  for all  $n$ . The sequence  $x_n = \text{dev}(m_n)$  satisfies the above conditions.  $\square$

Lemma 5.19 has the following refinement:

**Lemma 5.20.** *Under the same hypothesis as in the preceding lemma, suppose that  $P$  is a projective line or a projective plane transversal to  $\xi^3(t)$  (i.e. the intersection  $P \cap \xi^3(t)$  is of the smallest possible dimension) and intersecting  $C_t$ , then*

$$\lim_{n \rightarrow +\infty} \overline{C}_{t_n} \cap P \supset \overline{C}_t \cap P.$$

**Remark 5.21.** *Instead of taking a fixed  $P$  we could also work with a sequence  $(P_n)_{n \in \mathbf{N}}$  converging to  $P$ .*

*Proof.* Let  $x = \text{dev}(m)$  be in  $C_t \cap P$  and let  $U, V$  be neighborhoods of  $m \in \widetilde{M}$  and  $x \in \mathbb{P}^3(\mathbf{R})$  respectively, such that the restriction of  $\text{dev}$  is a homeomorphism from  $U$  onto  $V$ . The transversality condition implies that there exists a sequence  $(m_n)_{n \geq N}$ , defined only for large enough  $N$ , such that  $m_n \in t_n \cap \text{dev}^{-1}(P \cap V)$ . Now one can conclude as in the proof of Lemma 5.19.  $\square$

## 5.6. Nontriviality of $\xi^1$ .

**Proposition 5.22.** *The map  $\xi^1 : \widetilde{\partial\Gamma} \rightarrow \mathbb{P}^3(\mathbf{R})$  is not constant.*

*Proof.* We argue by contradiction. Assume that  $\xi^1$  is constant equal to  $x \in \mathbb{P}^3(\mathbf{R})$ .

Before we give the formal argument which leads to a contradiction, let us summarize the idea of the proof. We will show that all the convex sets  $C_t$  have to be triangles and that these triangles all share a common edge. Then we show that for every triangle the vertex opposite to this edge is contained in a fixed projective line. This forces the image of the developing map to be contained in a two dimensional subspace, which gives the desired contradiction.

Let  $\bar{\gamma} \in \bar{\Gamma} - \langle \tau \rangle$  be an element of zero translation and  $(\tilde{t}_{+, \gamma}, \tilde{t}_{-, \gamma}) \in \widetilde{\partial\Gamma}_{[0]}^{(2)}$  a pair of fixed points. Since  $x = \xi^1(\tilde{t}_{+, \gamma})$  is an eigenline for  $\text{hol}(\bar{\gamma})$  corresponding to the largest eigenvalue of  $\text{hol}(\bar{\gamma})|_{\xi^3(\tilde{t}_{+, \gamma})}$  and  $x = \xi^1(\tilde{t}_{-, \gamma})$  is an eigenline corresponding to the largest eigenvalue of  $\text{hol}(\bar{\gamma})|_{\xi^3(\tilde{t}_{-, \gamma})}^{-1}$ , we are necessarily in Case (T), so both convex sets  $C_{\tilde{t}_{+, \gamma}}$  and  $C_{\tilde{t}_{-, \gamma}}$  are triangles. The intersection  $L = \xi^3(\tilde{t}_{+, \gamma}) \cap \xi^3(\tilde{t}_{-, \gamma})$  is an eigenspace for  $\text{hol}(\bar{\gamma})$  (corresponding to the eigenvalue  $\lambda_+(\bar{\gamma}) = \lambda_+(\bar{\gamma}^{-1})^{-1}$ ). Moreover  $L$  is the tangent line to  $C_{\tilde{t}_{+, \gamma}}$  at  $x$  which is the same as the tangent line to  $C_{\tilde{t}_{-, \gamma}}$  at  $x$ .

Therefore for all  $(\tilde{t}_{+, \gamma}, \tilde{t}_{-, \gamma})$  we have  $C_{\tilde{t}_{+, \gamma}} \cap \xi^3(\tilde{t}_{-, \gamma}) = \emptyset$ . Since the set  $\{(t_+, t_-) \in \widetilde{\partial\Gamma}_{[0]}^{(2)} \mid \xi^3(t_+) \cap C_{t_-} = \emptyset\}$  is closed, we deduce from Lemma 1.11 that for all  $t$ , with  $(\tilde{t}_{+, \gamma}, t) \in \widetilde{\partial\Gamma}_{[0]}^{(2)}$  the intersection  $C_{\tilde{t}_{+, \gamma}} \cap \xi^3(t)$  is empty. In particular, the projective line  $\xi^3(\tilde{t}_{+, \gamma}) \cap \xi^3(t)$  is the line tangent to  $C_{\tilde{t}_{+, \gamma}}$  at  $x$  and hence equals  $L$ . So there exists a non-empty open subset  $U$  of  $\partial\Gamma$  such that  $L = \bigcap_{t \in U} \xi^3(t)$ . By Lemma 5.17 this implies  $L = \bigcap_{t \in \widetilde{\partial\Gamma}} \xi^3(t)$ . **Claim 1:** The segment  $I \subset L$  corresponding to the side of the triangle  $C_{\tilde{t}_{+, \gamma}}$  is independent of  $\bar{\gamma}$ .

Note that, since  $\tau^{-2g} = \prod_{i=1}^g [a_i, b_i]$  and the restriction of  $\text{hol}(a_i)$  and  $\text{hol}(b_i)$  to  $L$  are trivial, the element  $\text{hol}(\tau)|_L^{-2g}$  is the identity. Therefore  $\text{hol}(\tau)|_L = 1$  since  $x$  is an eigenline for  $\tau$ . This implies that the sides of the triangles  $C_{\tilde{t}_{+, \gamma}}$  and  $C_{\tilde{t}_{-, \gamma}}$  in  $L$  do not depend on the pair  $(\tilde{t}_{+, \gamma}, \tilde{t}_{-, \gamma})$ . We denote this segments by  $I_+(\bar{\gamma})$  and  $I_-(\bar{\gamma})$ . We now want to show that they are independent of  $\bar{\gamma}$ .

For this consider  $t \in \widetilde{\partial\Gamma}$  such that  $(\tilde{t}_{+, \gamma}, t) \in \widetilde{\partial\Gamma}_{[0]}^{(2)}$ . Let  $w \in L$  be a point and  $P$  the projective plane spanned by  $w$ ,  $x_-(\bar{\gamma})$  and  $x_-(\bar{\gamma}^{-1})$  (using the notation from Section 5.2). The plane  $P$  is  $\text{hol}(\bar{\gamma})$ -invariant. If  $w$  is in  $I_+$ , then Lemma 5.20 implies that

$$\lim_{n \rightarrow +\infty} \text{hol}(\bar{\gamma})^n \bar{C}_t \cap P \supset \bar{C}_{\tilde{t}_{+, \gamma}} \cap P.$$

This is possible if and only if  $w$  belongs to  $\bar{C}_t \cap P$ . Applying this to all points  $w \in I_+(\bar{\gamma})$  implies that if  $t = t_{+, \bar{\gamma}'}$  is the attractive fixed point for another element  $\bar{\gamma}' \in \bar{\Gamma} - \langle \tau \rangle$  of zero translation, then  $I_+(\bar{\gamma}') \supset I_+(\bar{\gamma})$ . Interchanging the roles of  $\bar{\gamma}$  and  $\bar{\gamma}'$  we get that  $I = I_+(\bar{\gamma}) = I_+(\bar{\gamma}') = I_-(\bar{\gamma})$  is independent of  $\bar{\gamma}$ .

**Claim 2:** There exists a projective line  $Q$  such that for every element  $\bar{\gamma} \in \bar{\Gamma} - \langle \tau \rangle$  of zero translation we have  $x_-(\bar{\gamma}) \subset Q$ .

Let  $y$  be the intersection point of  $\xi^3(t_{+, \bar{\gamma}'})$  with the  $\text{hol}(\bar{\gamma})$ -invariant projective line  $Q$  spanned by  $x_-(\bar{\gamma})$  and  $x_-(\bar{\gamma}^{-1})$ . Then for any projective line  $D$  through  $y$  which is contained in  $\xi^3(t_{+, \bar{\gamma}'})$ , the intersection  $\bar{C}_{t_{+, \bar{\gamma}'}} \cap D$  is either empty or contains the intersection point  $L \cap D$ . This shows that  $y$

is contained in the sector with tip  $x_-(\bar{\gamma}')$ , which is bounded by the projective lines supporting the triangle  $C_{t_+, \bar{\gamma}'}$  and which contains  $C_{t_+, \bar{\gamma}'}$  (Figure 6 shows what cannot happen). Similarly  $y$  cannot be contained in the open

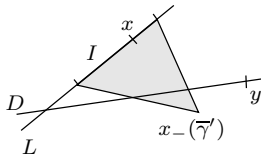


FIGURE 6. The point  $y$  cannot be in the other sector

triangle  $C_{t_+, \bar{\gamma}'}$ .

A short calculation, using this last condition for  $\bar{\gamma}^{\pm 1}$  and  $\bar{\gamma}'^{\pm 1}$ , gives

$$\overline{x_-(\bar{\gamma})x_-(\bar{\gamma}^{-1})} = \overline{x_-(\bar{\gamma}')x_-(\bar{\gamma}'^{-1})} = Q,$$

and this holds for any  $\bar{\gamma}$  and  $\bar{\gamma}'$ .

Hence the projective line  $Q$  is invariant by every element of zero translation, and also by  $\tau$  since  $x_-(\bar{\gamma}^{\pm 1})$  are eigenlines for  $\text{hol}(\tau)$ . Thus  $Q$  is  $\text{hol}(\bar{\Gamma})$ -invariant. We can define a  $\text{hol}$ -equivariant continuous map

$$\begin{aligned} \xi_- : \widetilde{\partial\Gamma} &\longrightarrow Q \\ t &\longmapsto \xi_-(t) = \xi^3(t) \cap Q. \end{aligned}$$

Then the closed  $\bar{\Gamma}$ -invariant set

$$\{(t, t') \in \widetilde{\partial\Gamma}_{[0]}^{(2)} \mid \mathcal{D}(t, t') = x \oplus \xi_-(t)\}$$

contains the pairs  $(\tilde{t}_{+, \gamma}, \tilde{t}_{-, \gamma})$  for all elements  $\gamma \in \bar{\Gamma} - \{1\}$  of zero translation. Hence Lemma 1.11 implies that for any  $(t, t') \in \widetilde{\partial\Gamma}_{[0]}^{(2)}$

$$\mathcal{D}(t, t') = x \oplus \xi_-(t)$$

which contradicts the local injectivity of the map  $t' \mapsto \mathcal{D}(t, t')$  (see Proposition 5.2).  $\square$

**5.7. The Holonomy Factors.** In this paragraph we study the possible intersections of  $\xi^3$  and show that

**Proposition 5.23.** *The intersection  $\bigcap_{t \in \widetilde{\partial\Gamma}} \xi^3(t)$  is empty and  $\text{hol}(\tau) = \text{Id}$ .*

**Lemma 5.24.** *The intersection  $\bigcap_{t \in \widetilde{\partial\Gamma}} \xi^3(t)$  cannot be a projective line.*

*Proof.* Suppose  $L = \bigcap_{t \in \widetilde{\partial\Gamma}} \xi^3(t)$  is a projective line. Then we have two representations

$$\begin{aligned} \rho_L : \bar{\Gamma} &\longrightarrow \text{PGL}(L) \\ \rho_Q : \bar{\Gamma} &\longrightarrow \text{PGL}(\mathbf{R}^4/L). \end{aligned}$$

Consider the set

$$C = \{t \in \widetilde{\partial\Gamma} \mid \xi^1(t) \subset L\}$$

This set is a  $\overline{\Gamma}$ -invariant closed set, which by minimality (Lemma 1.9) is either empty or equal to  $\widetilde{\partial\Gamma}$ . So we have to consider two cases

- **(Case 1):**  $\forall t \in \widetilde{\partial\Gamma} \quad \xi^1(t) \subset L$ ,
- **(Case 2):**  $\forall t \in \widetilde{\partial\Gamma} \quad \xi^1(t) \oplus L = \xi^3(t)$ .

Let us first assume that we are in **(Case 1)**. We show that in this case the two representations  $\rho_L$  and  $\rho_Q$  are Fuchsian with length functions (see Appendix A)  $\ell_{\rho_L} < \ell_{\rho_Q}$ . This will contradict Fact A.4

Since we are in (Case 1) the curve  $\xi^1$  is a continuous  $\rho_L$ -equivariant curve

$$\xi^1 : \widetilde{\partial\Gamma} \rightarrow \mathbb{P}(L) \subset \mathbb{P}^3(\mathbf{R})$$

and  $\xi^3$  is a continuous  $\rho_Q$ -equivariant curve

$$\xi^3 : \widetilde{\partial\Gamma} \rightarrow \mathbb{P}(\mathbf{R}^4/L) \subset \mathbb{P}^3(\mathbf{R})^*.$$

Let  $\overline{\gamma} \in \overline{\Gamma} - \langle \tau \rangle$  be an element of zero translation with  $(\tilde{t}_{+, \gamma}, \tilde{t}_{-, \gamma}) \in \widetilde{\partial\Gamma}_{[0]}^{(2)}$  a pair of an attractive and a repulsive fixed point. Then  $\xi^1(\tilde{t}_{+, \gamma})$  is the eigenspace corresponding to the largest eigenvalue (in modulus) of  $\text{hol}(\overline{\gamma})|_L$ . At least one  $\rho_L(\overline{\gamma})$  is nontrivial since otherwise  $\xi^1$  would be constant contradicting Proposition 5.22. For this particular element  $\xi^1(\tilde{t}_{+, \gamma})$  is independent of the choice of  $\tilde{t}_{+, \gamma}$  in its  $\langle \tau \rangle$ -orbit. In other words  $\xi^1(\tau\tilde{t}_{+, \gamma}) = \xi^1(\tilde{t}_{+, \gamma})$ . Thus  $\xi^1$  is  $\tau$ -invariant, since the set

$$\{t \in \widetilde{\partial\Gamma} \mid \xi^1(\tau t) = \xi^1(t)\}$$

is closed,  $\overline{\Gamma}$ -invariant and non-empty. Consequently  $\rho_L(\tau) = \text{Id}_L$ . An analogous argument shows that  $\rho_Q(\tau) = \text{Id}_{\mathbf{R}^4/L}$ .

In particular, the two representations  $\rho_L$  and  $\rho_Q$  factor as

$$\begin{aligned} \rho_L : \Gamma &\longrightarrow \text{PGL}(L) \\ \rho_Q : \Gamma &\longrightarrow \text{PGL}(\mathbf{R}^4/L). \end{aligned}$$

Since for any element  $\overline{\gamma}$  of zero translation the eigenvalues of  $\text{hol}(\overline{\gamma})$  are of the same sign (by Lemma 5.10), these two representations have image in  $\text{PSL}_2(\mathbf{R})$ . Moreover, they satisfy the hypothesis of Lemma A.2 so  $\rho_L$  and  $\rho_Q$  are Fuchsian.

Consider now the length functions of  $\rho_L$  and  $\rho_Q$  (see Appendix A).

Let  $\gamma \in \Gamma - \{1\}$  and  $\overline{\gamma}$  its lift in  $\overline{\Gamma}$  of zero translation with a pair of fixed points  $(\tilde{t}_{+, \gamma}, \tilde{t}_{-, \gamma}) \in \widetilde{\partial\Gamma}_{[0]}^{(2)}$ . By Lemma 5.16 the projective line  $L$  does not meet  $\text{dev}(\widetilde{M})$ . Therefore  $L$  is tangent to  $C_{\tilde{t}_{+, \gamma}}$  at the point  $x_+(\overline{\gamma}) = \xi^1(\tilde{t}_{+, \gamma})$  and tangent to  $C_{\tilde{t}_{-, \gamma}}$  at the point  $x_+(\overline{\gamma}^{-1}) = \xi^1(\tilde{t}_{-, \gamma})$ . The eigenvalues of the restriction  $\text{hol}(\overline{\gamma})|_{\xi^3(\tilde{t}_{+, \gamma})}$  are (we choose a lift of  $\text{hol}(\overline{\gamma})$  with positive eigenvalues)

$$\lambda_+(\overline{\gamma}) > \lambda_0(\overline{\gamma}) > \lambda_-(\overline{\gamma}) > 0,$$

where  $\lambda_+$  and  $\lambda_0$  are the eigenvalues corresponding to  $L$  which are distinct since  $\rho_L$  is faithful. Similarly the eigenvalues of  $\text{hol}(\bar{\gamma}^{-1})|_{\xi^3(\tilde{t}_{-, \gamma})}$  are:

$$\lambda_+(\bar{\gamma}^{-1}) > \lambda_0(\bar{\gamma}^{-1}) > \lambda_-(\bar{\gamma}^{-1}) > 0,$$

with relations  $\lambda_+(\bar{\gamma})^{-1} = \lambda_0(\bar{\gamma}^{-1})$  and  $\lambda_0(\bar{\gamma})^{-1} = \lambda_+(\bar{\gamma}^{-1})$ . Therefore

$$\ell_{\rho_L}(\gamma) = \ln \left( \frac{\lambda_+(\bar{\gamma})}{\lambda_0(\bar{\gamma})} \right) < \ln \left( \frac{\lambda_-(\bar{\gamma}^{-1})^{-1}}{\lambda_-(\bar{\gamma})} \right) = \ell_{\rho_Q}(\gamma),$$

contradicting Fact A.4. Therefore (Case 1) cannot occur.

If we are in **(Case 2)**, that is for all  $t \in \widetilde{\partial\Gamma}$

$$\xi^1(t) \oplus L = \xi^3(t),$$

we show that  $\rho_L$  and  $\rho_Q$  have to be conjugate Fuchsian representations. Then we apply the analysis of Section 3.3 to get a contradiction.

As above the  $\rho_Q$ -equivariant map  $\xi^3 : \widetilde{\partial\Gamma} \rightarrow \mathbb{P}(\mathbf{R}^4/L)$  is  $\tau$ -invariant. The representation  $\rho_Q$  factors as  $\rho_Q : \Gamma \rightarrow \text{PGL}(\mathbf{R}^4/L)$  satisfying the hypothesis of Lemma A.2, so  $\rho_Q$  is Fuchsian.

Let  $\bar{\gamma} \in \bar{\Gamma}$  be an element of zero translation with fixed points  $(\tilde{t}_{+, \gamma}, \tilde{t}_{-, \gamma})$  as before. In  $\xi^3(\tilde{t}_{+, \gamma})$  the line  $L$  is spanned by  $x_-(\gamma)$  and another invariant point  $x_0(\gamma)$  distinct from  $x_+(\gamma) = \xi^1(\tilde{t}_{+, \gamma})$ . In particular  $\rho_L(\bar{\gamma})$  is split over  $\mathbf{R}$  with distinct eigenvalues of the same sign. This implies that  $\rho_L(\tau)$  is also split over  $\mathbf{R}$  in the same basis. By Lemma A.6 we get that  $\rho_L(\tau) = \text{Id}$ .

Therefore  $\rho_L$  factors through  $\Gamma$  and again we have  $\rho_L : \Gamma \rightarrow \text{PSL}(L)$ . Furthermore, for any nonzero  $\gamma$ ,  $\rho_L(\gamma)$  is split with two distinct real eigenvalues. Hence  $\rho_L$  is faithful and discrete by Lemma A.1, so  $\rho_L$  is Fuchsian.

We already observed that  $\xi^1$  factors through a curve  $\xi^1 : \partial\Gamma \rightarrow \mathbb{P}^3(\mathbf{R})$ . For any  $t \in \partial\Gamma$  the line  $\xi^1(t)$  is an eigenline for  $\text{hol}(\tau)$ . In particular  $\text{hol}(\tau)$  is split over  $\mathbf{R}$  and since  $\text{hol}(\tau)$  acts trivially on  $L$  and on  $\mathbf{R}^4/L$ , there are only two possibilities

$$\text{hol}(\tau) = \text{Id} \text{ or } \text{hol}(\tau) = \begin{pmatrix} \text{Id} & 0 \\ 0 & -\text{Id} \end{pmatrix}.$$

In the first case the holonomy factors as  $\text{hol} : \Gamma \rightarrow \text{PSL}_4(\mathbf{R})$ , and in the second case we get  $\text{hol} : \widehat{\Gamma} = \bar{\Gamma}/\langle 2\tau \rangle \rightarrow \text{PSL}_4(\mathbf{R})$ . In both case what we already know about the eigenvalues implies that the holonomy lifts (non uniquely and up to taking a subgroup of index two) to  $\widehat{\text{hol}} : \Gamma \rightarrow \text{SL}_4(\mathbf{R})$  or  $\widehat{\text{hol}} : \widehat{\Gamma} \rightarrow \text{SL}_4(\mathbf{R})$ .

Suppose that  $\text{hol}(\tau) = \text{Id}$ . In a basis adapted to  $L$ , we have:

$$\begin{aligned} \text{hol} : \Gamma &\longrightarrow \text{SL}_4(\mathbf{R}) \\ \gamma &\longmapsto \text{hol}(\gamma) = \begin{pmatrix} \chi(\gamma)^{-1}\rho_Q(\gamma) & 0 \\ * & \chi(\gamma)\rho_L(\gamma) \end{pmatrix}, \end{aligned}$$



with  $\chi : \Gamma \rightarrow \mathbf{R}_{>0}$  a character and  $\rho_L, \rho_Q : \Gamma \rightarrow \mathrm{SL}_2(\mathbf{R})$  Fuchsian. For all  $\gamma$  in  $\Gamma - \{1\}$

$$\begin{aligned} |\lambda_+(\gamma)| &= \chi(\gamma)^{-1} \exp(\ell_{\rho_Q}(\gamma)/2) \\ |\lambda_0(\gamma)| &= |\lambda_-(\gamma^{-1})^{-1}| = \chi(\gamma) \exp(\ell_{\rho_L}(\gamma)/2) \\ |\lambda_-(\gamma)| &= |\lambda_0(\gamma^{-1})^{-1}| = \chi(\gamma) \exp(-\ell_{\rho_L}(\gamma)/2) \\ |\lambda_+(\gamma^{-1})^{-1}| &= \chi(\gamma)^{-1} \exp(-\ell_{\rho_Q}(\gamma)/2). \end{aligned}$$

From the inequalities:

$$|\lambda_+(\gamma)| \geq |\lambda_0(\gamma)| > |\lambda_-(\gamma)| \geq |\lambda_+(\gamma^{-1})^{-1}|,$$

we obtain  $\ell_{\rho_L}(\gamma) \leq \ell_{\rho_Q}(\gamma)$  hence  $\rho_L$  is conjugate to  $\rho_Q$  by Fact A.4. So  $\ell_{\rho_L}(\gamma) = \ell_{\rho_Q}(\gamma)$ , this implies that  $\chi^2(\gamma) \geq 1$  for all  $\gamma$  and  $\chi$  must be the trivial character. In particular the holonomy homomorphism  $\mathrm{hol}$  and its semisimplification  $\widehat{\mathrm{hol}}$  satisfy the conditions of Lemma 3.7. Thus the convex sets  $C_t$  must all be sectors, which is a contradiction.

The other case  $\mathrm{hol}(\tau) = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -\mathrm{Id} \end{pmatrix}$  cannot happen. Using the same argument as above, we can show that  $\widehat{\mathrm{hol}}$  has to be of the form

$$\begin{aligned} \widehat{\mathrm{hol}} : \Gamma \times \mathbf{Z}/2\mathbf{Z} &\longrightarrow \mathrm{SL}_4(\mathbf{R}) \\ \gamma &\longmapsto \begin{pmatrix} \rho(\gamma) & 0 \\ 0 & \rho(\gamma) \end{pmatrix} \\ -1 &\longmapsto \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -\mathrm{Id} \end{pmatrix}. \end{aligned}$$

The developing map  $\mathrm{dev}$  factors through the quotient  $\widehat{M} = \widetilde{M}/\langle 2\tau \rangle$  which can be identified with the set of triples  $(\hat{t}_+, \hat{t}_0, \hat{t}_-)$  in  $\widehat{\partial\Gamma} = \widetilde{\partial\Gamma}/\langle 2\tau \rangle$  of pairwise distinct elements such that  $(-\hat{t}_-, \hat{t}_+, \hat{t}_0, \hat{t}_-)$  is oriented. Following the argument and the notation of Section 3.3, one can write the developing map as

$$\mathrm{dev}(\hat{t}_+, \hat{t}_0, \hat{t}_-) = [\eta_+(\hat{t}_+) + \varphi(\hat{t}_+, \hat{t}_0, \hat{t}_-)\eta_-(\hat{t}_-)],$$

where  $\varphi : \widehat{M} \rightarrow \mathbf{R}$  is continuous, non-zero and satisfies  $\varphi(-\hat{t}_+, -\hat{t}_0, -\hat{t}_-) = -\varphi(\hat{t}_+, \hat{t}_0, \hat{t}_-)$ . This is a contradiction.  $\square$

**Lemma 5.25.** *Let  $\gamma \in \overline{\Gamma} - \{1\}$  be an element of zero translation. with fixed points  $(\tilde{t}_{+, \gamma}, \tilde{t}_{-, \gamma}) \in \widetilde{\partial\Gamma}_{[0]}^{(2)}$  Then*

- $\xi^3(\tilde{t}_{+, \gamma}) \cap \xi^3(\tilde{t}_{-, \gamma})$  contains  $x_-(\gamma)$  and  $x_-(\gamma^{-1})$ .
- $\xi^1(\tilde{t}_{+, \gamma}) \oplus \xi^3(\tilde{t}_{-, \gamma}) = \mathbf{R}^4 = \xi^3(\tilde{t}_{+, \gamma}) \oplus \xi^1(\tilde{t}_{-, \gamma})$ .
- $\xi^3(\tilde{t}_{+, \gamma}) \cap \xi^3(\tilde{t}_{-, \gamma}) = \overline{x_0(\gamma)x_-(\gamma)}$ .

Moreover  $x_0(\gamma) = x_-(\gamma^{-1})$ .

*Proof.* Assume that  $x_-(\gamma) \notin L := \xi^3(\tilde{t}_{+, \gamma}) \cap \xi^3(\tilde{t}_{-, \gamma})$ . Then  $L = \overline{x_+(\gamma)x_0(\gamma)}$ . Let  $t \in \widetilde{\partial\Gamma}$  be close to  $\tilde{t}_{-, \gamma}$  and consider the line  $L_t = \xi^3(t) \cap \xi^3(\tilde{t}_{+, \gamma})$ . Then

$\lim_{n \rightarrow \infty} \text{hol}(\gamma^{-n})L_t = L$ . On the other hand the point  $q = L_t \cap \overline{x_-(\gamma)x_+(\gamma)}$  satisfies  $\lim_{n \rightarrow \infty} \text{hol}(\gamma^{-n})q \in L$  if and only if  $q = x_+(\gamma)$ , and similarly we get  $p = L_t \cap \overline{x_-(\gamma)x_0(\gamma)} = x_0(\gamma)$ . Thus  $L_t = \overline{x_+(\gamma)x_0(\gamma)} = L$  for every  $t$  close enough to  $\tilde{t}_{-, \gamma}$ . But then Lemma 5.17 implies  $\bigcap_{t \in \widetilde{\partial\Gamma}} \xi^3(t) = L$ , which contradicts Lemma 5.24. This concludes the first claim.

Assume that  $\xi^1(\tilde{t}_{+, \gamma}) \subset \xi^3(\tilde{t}_{-, \gamma})$ , then  $\xi^3(\tilde{t}_{+, \gamma}) \cap \xi^3(\tilde{t}_{-, \gamma}) = \overline{x_+(\gamma)x_-(\gamma)}$ . Let  $t \in \widetilde{\partial\Gamma}$  be close to  $\tilde{t}_{-, \gamma}$  and consider the line  $L_t = \xi^3(t) \cap \xi^3(\tilde{t}_{+, \gamma})$ . Then  $\lim_{n \rightarrow \infty} \text{hol}(\gamma^{-n})L_t = \overline{x_+(\gamma)x_-(\gamma)}$ . But the intersection  $p = L_t \cap \overline{x_+(\gamma)x_0(\gamma)}$  converges to  $x_+(\gamma)$  only if  $p = x_+(\gamma)$ . Hence Lemma 5.17, Lemma 5.16 and Lemma 5.18 imply that  $\bigcap_{t \in \widetilde{\partial\Gamma}} \xi^3(t) = x_+(\gamma) = \xi^1(\tilde{t}_{+, \gamma})$ . Since  $\overline{\Gamma}$  acts minimally on  $\widetilde{\partial\Gamma}$ ,  $\xi^1$  is constant. This is a contradiction.

From the above, the equality

$$\xi^3(\tilde{t}_{+, \gamma}) \cap \xi^3(\tilde{t}_{-, \gamma}) = \overline{x_0(\gamma)x_-(\gamma)} = \overline{x_0(\gamma^{-1})x_-(\gamma^{-1})}$$

follows. Since  $\lambda_0(\gamma) > \lambda_-(\gamma)$  and  $\lambda_0(\gamma^{-1}) > \lambda_-(\gamma^{-1})$ , we necessarily have  $x_0(\gamma) = x_-(\gamma^{-1})$ .  $\square$

**Lemma 5.26.** *For every  $(t_+, t_-) \in \widetilde{\partial\Gamma}_{[0]}^{(2)}$  we have*

$$\xi^3(t_+) \cap C_{t_-} = \emptyset.$$

*Proof.* Let  $\gamma \in \overline{\Gamma}$  be of zero translation, then  $\xi^3(\tilde{t}_{-, \gamma}) \cap C_{\tilde{t}_{+, \gamma}} = \emptyset$ , since otherwise  $\xi^3(\tilde{t}_{-, \gamma}) \cap C_{\tilde{t}_{+, \gamma}} = \mathcal{D}(\tilde{t}_{+, \gamma}, \tilde{t}_{-, \gamma})$ , which is spanned by  $x_+(\gamma)$  and  $x_-(\gamma)$ , contradicting Lemma 5.25.

Since the set  $\{(t_+, t_-) \in \widetilde{\partial\Gamma}_{[0]}^{(2)} \mid \xi^3(t_+) \cap C_{t_-} = \emptyset\}$  is closed, we conclude with Lemma 1.11.  $\square$

**Remark 5.27.** *The order of  $t_+$  and  $t_-$  in Lemma 5.26 is of no importance.*

**Lemma 5.28.** *The intersection  $\bigcap_{t \in \widetilde{\partial\Gamma}} \xi^3(t)$  is not a point.*

*Proof.* Suppose that  $x = \bigcap_{t \in \widetilde{\partial\Gamma}} \xi^3(t)$  is a point. Then, for any  $\gamma$ ,  $x = x_-(\gamma)$  or  $x = x_-(\gamma^{-1})$ .

In particular we have

$$x \in \mathcal{D}(\tilde{t}_{+, \gamma}, \tilde{t}_{-, \gamma}) \cup \mathcal{D}(\tilde{t}_{+, \gamma^{-1}}, \tilde{t}_{-, \gamma^{-1}}) = \mathcal{D}(\tilde{t}_{+, \gamma}, \tilde{t}_{-, \gamma}) \cup \mathcal{D}(\tilde{t}_{-, \gamma}, \tau^{-1}\tilde{t}_{+, \gamma}).$$

Thus the set

$$C = \{(t_+, t_-) \in \widetilde{\partial\Gamma}_{[0]}^{(2)} \mid x \in \mathcal{D}(t_+, t_-) \cup \mathcal{D}(t_-, \tau^{-1}t_+)\}$$

is closed and contains  $(\tilde{t}_{+, \gamma}, \tilde{t}_{-, \gamma})$  for every  $\gamma \in \overline{\Gamma} - \{1\}$  of zero translation, hence, by Lemma 1.11,  $C = \widetilde{\partial\Gamma}_{[0]}^{(2)}$ .

Consider  $\gamma \in \overline{\Gamma} - \{1\}$  an element of zero translation with  $(\tilde{t}_{+, \gamma}, \tilde{t}_{-, \gamma})$ . Then, by Lemma 5.25

$$\mathcal{D}(\tilde{t}_{+, \gamma}, \tilde{t}_{-, \gamma}) \cap \mathcal{D}(\tilde{t}_{+, \gamma^{-1}}, \tilde{t}_{-, \gamma^{-1}}) = \emptyset$$

and assume that  $x \in \mathcal{D}(\tilde{t}_{+, \gamma}, \tilde{t}_{-, \gamma})$ , so  $x \notin \mathcal{D}(\tilde{t}_{+, \gamma^{-1}}, \tilde{t}_{-, \gamma^{-1}})$ . Then by continuity of  $\mathcal{D}$  there exists a neighborhood  $U \subset \partial\Gamma$  of  $\tilde{t}_{+, \gamma^{-1}}$  such that  $x \notin \mathcal{D}(t, \tilde{t}_{-, \gamma^{-1}}) = \mathcal{D}(t, \tau^{-1}\tilde{t}_{+, \gamma})$  for all  $t \in U$ . Thus for all  $t \in U$  we have that  $x \in \mathcal{D}(\tilde{t}_{+, \gamma}, t)$ , hence  $x = \bigcap_{t \in U} \mathcal{D}(\tilde{t}_{+, \gamma}, t) = \xi^1(\tilde{t}_{+, \gamma})$  by local injectivity of  $\mathcal{D}$ . This again implies that  $\xi^1$  is constant and gives a contradiction.  $\square$

*Proof of Proposition 5.23.* The first statement has been proved by elimination. Now as we already observed,  $\xi^3$  is  $\tau$ -invariant. This means that for all  $t$ ,  $\xi^3(t)$ , as a line in the dual space of  $\mathbf{R}^4$ , is invariant by  $\text{hol}(\tau)$ . Now an element of  $\text{PGL}(V)$  having a continuous family of invariant lines generating the vector space  $V$  is necessarily trivial, so  $\text{hol}(\tau) = \text{Id}$ .  $\square$

The main consequence of Proposition 5.23 is the following

**Corollary 5.29.** *Let  $(\text{dev}, \text{hol})$  be the developing pair defining a properly convex foliated projective structure on  $M$ , then the holonomy factors through a representation*

$$\text{hol} : \overline{\Gamma} \rightarrow \Gamma \longrightarrow \text{PSL}_4(\mathbf{R}) \subset \text{PGL}_4(\mathbf{R}).$$

We will consider now the holonomy as a homomorphism

$$\text{hol} : \Gamma \longrightarrow \text{PSL}_4(\mathbf{R})$$

and the developing map

$$\text{dev} : \overline{M} = S\tilde{\Sigma} \longrightarrow \mathbb{P}^3(\mathbf{R}).$$

Moreover, we get that the maps  $\xi^3$ ,  $\xi^1$  and  $\mathcal{D}$  also factor through the corresponding quotient, which is  $\partial\Gamma$  and  $\partial\Gamma^{(2)}$  respectively.

### 5.7.1. Some Other Consequences.

**Lemma 5.30. Case (T)** *never occurs, that is  $|\lambda_+(\gamma)| > |\lambda_0(\gamma)|$  for all  $\gamma$  in  $\Gamma - \{1\}$*

*Proof.* If  $\lambda_+ = \lambda_0$  for some  $\gamma \in \Gamma$  with fixed points  $(t_+, t_-)$ , then the line  $L$  spanned by  $x_+(\gamma) = \xi^1(t_+)$  and  $x_0(\gamma)$  is pointwise fixed by  $\text{hol}(\gamma)$ . Here, by Lemma 5.25,  $x_0(\gamma)$  can be chosen to lie in the projective plane  $\xi^3(t_-)$ .

For  $t$  in  $\partial\Gamma - \{t_+\}$ , the sequence  $(\gamma^{-n}t)_{n \in \mathbf{N}}$  converges to  $t_-$ , hence

$$\lim_{n \rightarrow +\infty} \text{hol}(\gamma)^{-n} \xi^3(t) = \xi^3(t_-).$$

In particular, applying negative powers of  $\text{hol}(\gamma)$  to the intersection  $L \cap \xi^3(t)$ , this point must converge to  $x_0(\gamma)$ . This implies that  $x_0 = L \cap \xi^3(t)$  and in particular  $x_0 \in \bigcap_{t \in \partial\Gamma} \xi^3(t)$  contradicting Proposition 5.23.  $\square$

**Lemma 5.31.** *Let  $(t_+, t_-) \in \partial\Gamma^{(2)}$ . Assume that  $t_- = t_{-, \gamma}$  for some  $\gamma \in \Gamma - \{1\}$ . Then  $\xi^1(t_+) \oplus \xi^3(t_-) = \mathbf{R}^4 = \xi^1(t_-) \oplus \xi^3(t_+)$ .*

*Proof.* Assume that  $\xi^1(t_+) \subset \xi^3(t_{-, \gamma})$ . Then

$$\xi^1(t_{+, \gamma}) = \lim_{n \rightarrow \infty} \text{hol}(\gamma^n) \xi^1(t_+) \subset \xi^3(t_{-, \gamma}),$$

which contradicts Lemma 5.25. The other statement follows by a similar argument.  $\square$

**5.8. Convexity of  $\xi^3$ .** In this section we will show that the curve

$$\xi^3 : \partial\Gamma \longrightarrow \mathbb{P}^3(\mathbf{R})^*$$

is convex. For this we will define for every  $t \in \partial\Gamma$  an auxiliary convex set  $D_t \subset \xi^3(t)$  containing  $C_t$ .

Define for all  $t \in \partial\Gamma$

$$D_t := \xi^3(t) - \bigcup_{t' \neq t} \xi^3(t) \cap \xi^3(t').$$

Then, by Lemma A.7,  $D_t$  is a convex subset in the projective plane  $\xi^3(t)$ , which, by Proposition 5.23, is properly convex. By Lemma 5.26,  $D_t$  contains the properly convex set  $C_t$ . In particular  $D_t$  has nonempty interior in  $\xi^3(t)$  and by properties of convex sets we have

$$\overset{\circ}{D}_t \subset D_t \subset \overline{D}_t.$$

**Remark 5.32.** *Of course the two convex sets  $C_t$  and  $\overset{\circ}{D}_t$  should be equal (see Section 5.10). But for the moment we work with  $D_t$  because of the following semi-continuity property, which the reader might compare with Lemma 5.19.*

**Lemma 5.33.** *Suppose that  $(t_n)_{n \in \mathbf{N}} \subset \partial\Gamma$  is a sequence converging to  $t \in \partial\Gamma$  such that the sequence of convex sets  $(\overline{D_{t_n}})_{n \in \mathbf{N}}$  converges in the Hausdorff topology for compact subsets of  $\mathbb{P}^3(\mathbf{R})$ . Then*

$$\lim_{n \rightarrow \infty} \overline{D_{t_n}} \subset \overline{D_t}.$$

*Proof.* Let  $D_\infty := \lim_{n \rightarrow \infty} \overline{D_{t_n}}$ . Then  $D_\infty$  is a convex set in  $\xi^3(t)$  containing  $C_t$  (Lemma 5.19). If  $D_\infty - \overline{D_t} \neq \emptyset$  then, by properties of convex sets,  $D_\infty - \overline{D_t}$  contains an open set  $U$ .

By definition of  $D_t$  there exists then  $t' \in \partial\Gamma$ ,  $t' \neq t$  such that

$$\xi^3(t') \cap U \neq \emptyset.$$

Fix some  $t_0 \in \partial\Gamma$ ,  $t_0 \neq t, t'$  such that for  $n$  big enough  $D_{t_n}$  is a convex set in the affine chart  $\mathbb{P}^3(\mathbf{R}) - \xi^3(t_0)$ . Choose coordinates  $(x, y, z)$  in this affine chart such that  $\xi^3(t) = \{(x, y, z) \mid x = 0\}$  and  $\xi^3(t') = \{(x, y, z) \mid z = 0\}$ .

Since  $U$  intersects  $\xi^3(t')$  nontrivially there are points  $p, q, r \in U \subset D_\infty$  such that

$$p = (x_p, y_p, z_p), \quad q = (x_q, y_q, z_q) \quad \text{and} \quad r \notin \overline{pq},$$

with  $z_p > 0$  and  $z_q < 0$ . By hypothesis there exists points  $p_n, q_n, r_n \in \overline{D_{t_n}}$  such that  $\lim_{n \rightarrow \infty} p_n = p$ ,  $\lim_{n \rightarrow \infty} q_n = q$  and  $\lim_{n \rightarrow \infty} r_n = r$ . Hence for  $n$  big enough  $p_n, q_n, r_n \in \mathbb{P}^3(\mathbf{R}) - \xi^3(t_0)$  and

$$p_n = (x_{p_n}, y_{p_n}, z_{p_n}) \text{ and } q_n = (x_{q_n}, y_{q_n}, z_{q_n})$$

with  $z_{p_n} > 0$  and  $z_{q_n} < 0$ . In particular, the open triangle with vertices  $p_n, q_n, r_n$  intersects the hyperplane  $\xi^3(t') = \{(x, y, z) \mid z = 0\}$  nontrivially. But this is a contradiction since by convexity this open triangle is contained in  $D_{t_n}$  and  $D_{t_n} \cap \xi^3(t') = \emptyset$  for all  $t_n \neq t'$ .  $\square$

Using Lemma 5.33 we will now be able to show that  $\xi^3(t) \cap \xi^3(t')$  is tangent to the convex set  $D_t$ , and this will suffice to show that the curve  $\xi^3 : \partial\Gamma \rightarrow \mathbb{P}^3(\mathbf{R})^*$  is convex.

**Lemma 5.34.** *For all  $(t_+, t_-) \in \partial\Gamma^{(2)}$ , the point*

$$\xi^3(t_+) \cap \xi^3(t_-) \cap \mathcal{D}(t_+, t_-)$$

*is contained in  $\overline{D_{t_+}}$ .*

*Proof.* Since  $\mathcal{D}(t_+, t_-) \cap C_{t_+} = \text{dev}((t_+, t_-))$  Lemma 5.26 implies that the intersection  $\xi^3(t_-) \cap \mathcal{D}(t_+, t_-) = \xi^3(t_+) \cap \xi^3(t_-) \cap \mathcal{D}(t_+, t_-)$  is a point.

Note that it follows from Section 5.2 and Lemma 5.25 that the statement is true for the pairs of fixed points  $(t_{+, \gamma}, t_{-, \gamma})$  for any  $\gamma \in \Gamma - \{1\}$ . So, by Lemma 1.11 we only have to show that

$$C = \{(t_+, t_-) \mid \xi^3(t_+) \cap \xi^3(t_-) \cap \mathcal{D}(t_+, t_-) \in \overline{D_{t_+}}\}$$

is closed.

Let  $((t_{+, n}, t_{-, n}))_{n \in \mathbf{N}} \subset C$  and  $(t_+, t_-) = \lim_{n \rightarrow \infty} (t_{+, n}, t_{-, n})$ . Then Lemma 5.33 implies that

$$\begin{aligned} \xi^3(t_+) \cap \xi^3(t_-) \cap \mathcal{D}(t_+, t_-) &= \\ \lim_{n \rightarrow \infty} \xi^3(t_{+, n}) \cap \xi^3(t_{-, n}) \cap \mathcal{D}(t_{+, n}, t_{-, n}) &\in \lim_{n \rightarrow \infty} \overline{D_{t_{+, n}}} \subset \overline{D_{t_+}}. \end{aligned}$$

Hence  $(t_+, t_-) \in C$  and  $C$  is closed.  $\square$

The intersection  $\xi^3(t_+) \cap \xi^3(t_-) \cap \mathcal{D}(t_+, t_-)$  is one of the points of intersection

$$\mathcal{D}(t_+, t_-) \cap \partial D_{t_+} = \{e_+(t_+, t_-), e_-(t_+, t_-)\},$$

where  $e_-(t_+, t_-)$  is the point such that the open segment  $]\xi^1(t_+), e_-(t_+, t_-)[$  in  $\overline{D_{t_+}}$  intersects  $C_{t_+}$ .

For all  $t_+ \in \partial\Gamma$  the map

$$\begin{aligned} e_- : \partial\Gamma - \{t_+\} &\rightarrow \xi^3(t_+) \\ t_- &\mapsto e_-(t_+, t_-) \end{aligned}$$

is continuous.

**Lemma 5.35.** *For all  $(t_+, t_-) \in \partial\Gamma^{(2)}$  we have*

$$\xi^3(t_+) \cap \xi^3(t_-) \cap \mathcal{D}(t_+, t_-) = e_-(t_+, t_-).$$

*Proof.* We first prove that, for all  $(t_+, t_-) \in \partial\Gamma^{(2)}$ , we have

$$\xi^3(t_+) \cap \xi^3(t_-) \cap \mathcal{D}(t_+, t_-) \neq \xi^1(t_+).$$

Assume the contrary. Then there exists  $(t_+, t_-) \in \partial\Gamma^{(2)}$  such that  $\xi^1(t_+) \in \xi^3(t_+) \cap \xi^3(t_-)$  and hence  $\xi^1(t_+) \notin D_{t_+}$ . But since  $\xi^1(t_+) \in \overline{D_{t_+}}$  (Lemma 5.34) we have that  $\xi^1(t_+) = e_+(t_+, t_-)$ .

By Lemma 5.31 we have that for all  $t_{-, \gamma}, \gamma \in \Gamma - \{1\}$

$$\xi^3(t_+) \cap \xi^3(t_{-, \gamma}) \cap \mathcal{D}(t_+, t_{-, \gamma}) = e_-(t_+, t_{-, \gamma}) \neq \xi^1(t_+).$$

Let  $(t_{-, \gamma_n})_{n \in \mathbf{N}} \subset \partial\Gamma$  be a sequence of fixed points of elements  $\gamma_n \in \Gamma - \{1\}$  with  $\lim_{n \rightarrow \infty} t_{-, \gamma_n} = t_-$ . Then, by continuity of  $\xi^3$  and  $\mathcal{D}$  the point of intersection

$$\xi^3(t_+) \cap \xi^3(t_{-, \gamma_n}) \cap \mathcal{D}(t_+, t_{-, \gamma_n})$$

converges to  $\xi^1(t_+)$ . But on the other hand by continuity of  $e_-(t_+, \cdot)$

$$\xi^3(t_+) \cap \xi^3(t_{-, \gamma_n}) \cap \mathcal{D}(t_+, t_{-, \gamma_n}) = e_-(t_+, t_{-, \gamma_n})$$

should converge to  $e_-(t_+, t_-)$ . This is a contradiction.

Note that the set

$$C = \{(t_+, t_-) \mid \xi^3(t_+) \cap \xi^3(t_-) \cap \mathcal{D}(t_+, t_-) = e_-(t_+, t_-)\}$$

contains the set of pairs  $(t_+, t_{-, \gamma})$  of fixed points for any  $\gamma \in \Gamma - \{1\}$ . So by Lemma 1.11 we only have to show that  $C$  is closed. Let  $((t_+, t_{-, n}))_{n \in \mathbf{N}} \subset C$  be a sequence converging to  $(t_+, t_-)$ . Then Lemma 5.33 and the definition of  $e_-$  imply that

$$\begin{aligned} \xi^3(t_+) \cap \xi^3(t_-) \cap \mathcal{D}(t_+, t_-) &= \lim_{n \rightarrow \infty} \xi^3(t_+, n) \cap \xi^3(t_{-, n}) \cap \mathcal{D}(t_+, n, t_{-, n}) \\ &= \lim_{n \rightarrow \infty} e_-(t_+, n, t_{-, n}) \in [\xi^1(t_+), e_-(t_+, t_-)] \subset \overline{D_{t_+}}. \end{aligned}$$

By the above  $\xi^3(t_+) \cap \xi^3(t_-) \cap \mathcal{D}(t_+, t_-) \neq \xi^1(t_+)$ , hence, since  $\xi^3(t_+) \cap \xi^3(t_-) \cap \mathcal{D}(t_+, t_-)$  is not in  $D_{t_+}$ , we necessarily have  $\xi^3(t_+) \cap \xi^3(t_-) \cap \mathcal{D}(t_+, t_-) = e_-(t_+, t_-)$  and  $C$  is closed.  $\square$

**Lemma 5.36.** *For all  $t_+ \in \partial\Gamma$  the map*

$$\begin{aligned} f_{t_+} : \partial\Gamma - \{t_+\} &\rightarrow \mathbb{P}(\xi^3(t_+)/\xi^1(t_+)) \\ t_- &\rightarrow \mathcal{D}(t_+, t_-) \end{aligned}$$

*is injective.*

*Proof.* By definition of  $\xi^1$  we have  $\xi^1(t_+) \subset \mathcal{D}(t_+, t_-)$  for all  $t_- \neq t_+$ , so  $f_{t_+}$  is well defined. Since  $C_{t_+}$  is a convex set,  $f_{t_+}$  is not surjective and we can think of  $f_{t_+}$  as a map from  $\mathbf{R}$  to  $\mathbf{R}$ .

Assume that  $f_{t_+}$  is not injective. Then there exist a point  $t_- \in \partial\Gamma$ , a neighborhood  $U_- \subset \partial\Gamma$  of  $t_-$  and an interval

$$[l_-, l_0[ \subset \mathbb{P}(\xi^3(t_+)/\xi^1(t_+))$$

such that

$$f_{t_+}(t_-) = l_- \quad \text{and} \quad f_{t_+}(U_-) \subset [l_-, l_0[.$$

This implies that there exist a point  $(t_+, t_0, t_-)$  in the leaf  $t_+ \subset \overline{M}$  and an open neighborhood  $V$  of  $(t_+, t_0, t_-)$

$$\text{dev}(V) \subset U = \bigcup_{l \in [l_-, l_0[} l \subset \xi^3(t_+),$$

but  $U$  is not a neighborhood of  $\text{dev}(t_+, t_0, t_-) \in l_-$ . This contradicts the fact that  $\text{dev}$  is a local homeomorphism.  $\square$

**Lemma 5.37.** *For all  $t_+ \in \partial\Gamma$  the map*

$$\begin{aligned} e_- : \partial\Gamma - \{t_+\} &\longrightarrow \xi^3(t_+) \\ t_- &\longmapsto e_-(t_+, t_-) \end{aligned}$$

*is injective. Moreover, for all  $(t_+, t_-) \in \partial\Gamma^{(2)}$  there is no open segment in  $\partial\overline{D}_{t_+}$  containing  $e_-(t_+, t_-)$ .*

*Proof.* Injectivity follows from the injectivity of  $\mathcal{D}(t_+, \cdot)$  (Lemma 5.36).

Assume that  $\partial\overline{D}_{t_+}$  contains a segment containing  $e_-(t_+, t_-)$ . Let  $L$  be the projective line supporting this segment. Then for  $t'$  close to  $t_-$  the point  $e_-(t_+, t') \in \partial\overline{D}_{t_+}$  will be an interior point of this segment. Since  $\xi^3(t_+) \cap \xi^3(t')$  is a line tangent to  $D_{t_+}$  at  $e_-(t_+, t')$  this implies  $\xi^3(t_+) \cap \xi^3(t') = L$ . By Lemma 5.17 this implies  $\bigcap_{t \in \partial\Gamma} \xi^3(t) = L$ , which contradicts Lemma 5.24.  $\square$

**Proposition 5.38.** *The hol-equivariant curve*

$$\xi^3 : \partial\Gamma \longrightarrow \mathbb{P}^3(\mathbf{R})^*$$

*is convex, that is*

$$\text{for all } (t_1, t_2, t_3, t_4) \text{ in } \partial\Gamma^4 \text{ pairwise distinct } \bigcap_{i=1}^4 \xi^3(t_i) = \emptyset.$$

*Proof.* Let us rename the four points  $t_1, t_2, t_3, t_4 \in \partial\Gamma$  by  $t_{-,1}, t_{-,2}, t_{-,3}, t_+$  such that  $(t_+, t_{-,3}, t_{-,2}, t_{-,1})$  is positively oriented. By Lemma 5.37 the three lines  $\xi^3(t_{-,i}) \cap \xi^3(t_+)$ ,  $i = 1, 2, 3$  are tangent to the convex  $D_{t_+}$  at the three distinct points  $e_-(t_+, t_{-,1})$ ,  $e_-(t_+, t_{-,2})$  and  $e_-(t_+, t_{-,3})$ . By Lemma 5.37 they cannot intersect. In particular, the intersection

$$\bigcap_{i=1}^4 \xi^3(t_i) = (\xi^3(t_{-,1}) \cap \xi^3(t_+)) \cap (\xi^3(t_{-,2}) \cap \xi^3(t_+)) \cap (\xi^3(t_{-,3}) \cap \xi^3(t_+))$$

is empty.  $\square$

This shows that the holonomy representation  $\rho = \text{hol}$  of a properly convex foliated projective structure factors through a convex representation and hence lies in the Hitchin component in view of Theorem 4.3. In particular Theorem 4.5 states that there exists a unique hol-equivariant curve

$$\xi = (\xi^1, \xi^2, \xi^3) : \partial\Gamma \longrightarrow \text{Flag}(\mathbf{R}^4).$$

In the following section we want to describe a specific construction of the missing

$$\xi^2 : \partial\Gamma \longrightarrow \text{Gr}_2^4.$$

The following figure shows the configuration of the images by the developing map of two (weakly) stable leaves and serves as motivation for the geometric construction of  $\xi^2$  which is already pictured.

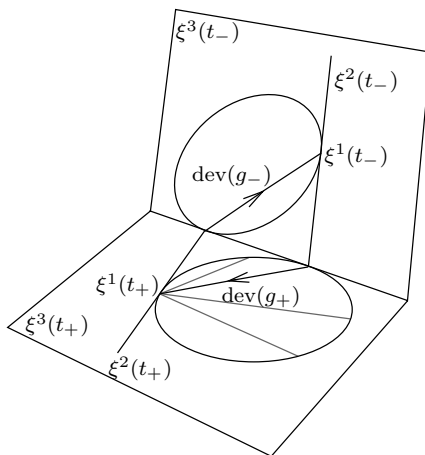


FIGURE 7. Two (weakly) stable leaves

In Section 5.10 we will compare the properly convex foliated structure  $(\text{dev}, \text{hol})$  and the properly convex foliated structure associated to  $\rho$  in Section 4, this will complete the proof of Theorem 5.1.

**5.9. Definition of  $\xi^2$ .** The following is a consequence of Lemma 5.35, which also follows from the Frenet property of  $\xi = (\xi^1, \xi^2, \xi^3)$ : for all  $(t_+, t_-) \in \partial\Gamma^{(2)}$  we have

$$\xi^1(t_+) \oplus \xi^3(t_-) = \mathbf{R}^4 = \xi^1(t_-) \oplus \xi^3(t_+).$$

This implies that the continuous hol-equivariant map:

$$\begin{aligned} \mathcal{P} : \partial\Gamma^{(2)} &\longrightarrow \text{Gr}_2^4(\mathbf{R}) \\ (t_+, t_-) &\longmapsto (\mathcal{D}(t_+, t_-) \cap \xi^3(t_-)) \oplus \xi^1(t_-) \end{aligned}$$

is well defined.

**Lemma 5.39.** For all  $(t_+, t_-) \in \partial\Gamma^{(2)}$

$$\mathcal{P}(t_+, t_-) \cap C_{t_-} = \emptyset.$$

For all  $t_-$  in  $\partial\Gamma$ , the function

$$\begin{aligned} \mathcal{P}_{t_-} : \partial\Gamma - \{t_-\} &\longrightarrow \text{Gr}_2^4(\mathbf{R}) \\ t_+ &\longmapsto \mathcal{P}(t_+, t_-) \end{aligned}$$

is constant.



*Proof.* The subset

$$\{(t_+, t_-) \in \partial\Gamma^{(2)} \mid \mathcal{P}(t_+, t_-) \cap C_{t_-} = \emptyset\}$$

is closed. Since the pairs consisting of fixed points  $(t_{+, \gamma}, t_{-, \gamma})$  of a nontrivial element  $\gamma \in \Gamma$  are contained in this set (see Figure 7 and Lemma 5.25), it equals  $\partial\Gamma^{(2)}$ .

The set

$$\{t_- \in \partial\Gamma \mid \mathcal{P}_{t_-} \text{ is constant}\}$$

is  $\Gamma$ -invariant and closed by the continuity of  $\mathcal{P}$ . This set does contain points  $t_{-, \gamma}$  which are the fix points of a non-trivial element  $\gamma \in \Gamma$ , since in that case the projective line  $\mathcal{P}(t, t_-)$  is necessarily the unique tangent line to  $C_{t_-}$  at  $\xi^1(t_-)$ . Hence, this set equals  $\partial\Gamma$ .  $\square$

**Definition 5.40.** *Define the map*

$$\xi^2 : \partial\Gamma \longrightarrow \text{Gr}_2^4(\mathbf{R}),$$

by setting  $\xi^2(t_-)$  to be the value of the constant function  $\mathcal{P}_{t_-}$ .

The map  $\xi^2 : \partial\Gamma \longrightarrow \text{Gr}_2^4(\mathbf{R})$  is a continuous, hol-equivariant injective map.

**5.10. Proof of Theorem 2.8.** In Section 5.8 we showed that the image of the holonomy map

$$\text{hol} : \mathcal{P}_{pcf}(M) \longrightarrow \text{Rep}(\bar{\Gamma}, \text{PGL}_4(\mathbf{R}))$$

is contained in the Hitchin component  $\mathcal{T}^4(\Gamma) \subset \text{Rep}(\bar{\Gamma}, \text{PGL}_4(\mathbf{R}))$ . In Section 4 we defined a section

$$s : \mathcal{T}^4(\Gamma) \longrightarrow \mathcal{P}_{pcf}(M)$$

by constructing a developing pair  $(\text{dev}_\xi, \text{hol})$  of a properly convex foliated projective structure on  $M$  starting with a hol-equivariant Frenet curve  $\xi = (\xi^1, \xi^2, \xi^3)$ .

To finish the proof of Theorem 2.8 we have to show that the maps  $\text{hol}$  and  $s$  are inverse to each other. That  $\text{hol} \circ s = \text{Id}$  is immediate. To show that  $s \circ \text{hol} = \text{Id}$  we have to show that the developing pairs  $(\text{dev}, \text{hol})$  and  $(\text{dev}_\xi, \text{hol})$  are equivalent (see Section 2.2).

**Lemma 5.41.** *The developing map  $\text{dev}$  of a properly convex foliated projective structure is a homeomorphism onto its image*

$$\text{dev} : \bar{M} \longrightarrow \bigcup_{t \in \partial\Gamma} C_t \subset \mathbb{P}^3(\mathbf{R}).$$

Furthermore the image of  $\text{dev}$  equals the image of  $\text{dev}_\xi$ .

*Proof.* The two convex sets  $C_{t_+}$  and  $C_{s_+}$  intersect only when  $t_+ = s_+$ , and by Lemma 5.36 the two projective lines  $\mathcal{D}(t_+, t_-)$  and  $\mathcal{D}(t_+, s_-)$  intersect in the convex  $C_{t_+}$  if and only if  $t_- = s_-$ . Therefore, if two points  $(t_+, t_0, t_-)$  and  $(s_+, s_0, s_-)$  have the same image under  $\text{dev}$  then necessarily  $t_+ = s_+$  and

$t_- = s_-$ , but we already observed that the restriction of  $\text{dev}$  to the geodesic  $(t_+, t_-)$  is a homeomorphism onto its image (see proof of Lemma 5.8), so  $(t_+, t_0, t_-) = (s_+, s_0, s_-)$ . Thus  $\text{dev}$  is a bijective local homeomorphism, hence a homeomorphism.

The image  $\text{dev}(\overline{M})$  is a  $\text{hol}(\Gamma)$ -invariant open subset of  $\text{dev}_\xi(\overline{M}) = \Omega_\xi$ . The group  $\text{hol}(\Gamma)$  acts properly discontinuous with compact quotient on both sets  $\text{dev}(\overline{M})$  and  $\text{dev}_\xi(\overline{M})$ , so  $\text{dev}(\overline{M})/\text{hol}(\Gamma)$  is a compact and open subset of  $\text{dev}_\xi(\overline{M})/\text{hol}(\Gamma)$ , which is connected, hence  $\text{dev}(\overline{M})/\text{hol}(\Gamma) = \text{dev}_\xi(\overline{M})/\text{hol}(\Gamma)$ .  $\square$

A direct corollary is that, for all  $t \in \partial\Gamma$ , we have

$$C_t = \overset{\circ}{D}_t.$$

So  $\text{dev}^{-1} \circ \text{dev}_\xi : \overline{M} \rightarrow \overline{M}$  is a  $\Gamma$ -equivariant homeomorphism, hence it descends to a homeomorphism  $h : M \rightarrow M$ . Since  $\text{dev}$  and  $\text{dev}_\xi$  send the geodesic  $(t_+, t_-)$  onto the same segment of  $C_{t_+}$  (namely the segment with endpoints  $\xi^1(t_+)$  and  $\xi^3(t_+) \cap \xi^2(t_-) = e_-(t_+, t_-)$ , see Lemma 5.35), the map  $h$  sends each geodesic leaf of  $M$  into itself. As noted in Remark 2.4 this is enough to ensure that the two pairs are equivalent and define the same element in  $\mathcal{P}_{pcf}(M)$ . This proves Theorem 2.8.

## 6. PROJECTIVE CONTACT STRUCTURES

In this section we conclude the description of the Hitchin component for the symplectic group  $\text{PSp}_4(\mathbf{R})$  from Theorem 2.8.

The 4-dimensional irreducible representation

$$\rho_4 : \text{PSL}_2(\mathbf{R}) \longrightarrow \text{PSL}_4(\mathbf{R})$$

preserves a symplectic form  $\omega$  on  $\mathbf{R}^4 = \text{Sym}^3\mathbf{R}^2$ , and so takes values in  $\text{PSp}_4(\mathbf{R}) = \text{PSp}(\mathbf{R}^4, \omega) \subset \text{PSL}_4(\mathbf{R})$

The symplectic form  $\omega$  on  $\mathbf{R}^4$  defines a canonical contact structure on  $\mathbb{P}^3(\mathbf{R})$ , that is a non integrable distribution. Let  $[l]$  be the line spanned by a vector  $l \in \mathbf{R}^4$ . The tangent space  $T_{[l]}\mathbb{P}^3(\mathbf{R})$  is naturally identified with the space of homomorphisms  $\text{Hom}([l], \mathbf{R}^4/[l])$ . Let  $H_l = [l]^\perp$ . Then the contact distribution is

$$\mathcal{H}_l = \text{Hom}([l], H_l/[l]) \subset \text{Hom}([l], \mathbf{R}^4/[l]) \simeq T_{[l]}\mathbb{P}^3(\mathbf{R}).$$

We consider  $\mathbb{P}^3(\mathbf{R})$  with this contact structure. The maximal subgroup of  $\text{PSL}_4(\mathbf{R})$  preserving this contact structure is  $\text{PSp}_4(\mathbf{R})$ .

**Definition 6.1.** *A projective contact structure on  $M$  is a projective structure  $(U, \varphi_U)$  on  $M$ , such that the coordinates changes  $\varphi_V \circ \varphi_U^{-1}$  are locally in  $\text{PSp}_4(\mathbf{R})$ .*

Two projective contact structures are equivalent if they are equivalent as projective structure by a homeomorphism  $h$  which preserves the contact structure.

A projective contact structure on  $M$  is given by a developing pair  $(\text{dev}, \text{hol})$ , where  $\text{dev} : \widetilde{M} \rightarrow \mathbb{P}^3(\mathbf{R})$  is a local homeomorphism which is equivariant with respect to the holonomy homomorphism  $\text{hol} : \overline{\Gamma} \rightarrow \text{PSp}_4(\mathbf{R})$  with values in the symplectic group. A projective contact structure which is properly convex foliated as projective structure will be called a properly convex foliated projective contact structure, and we denote by  $\mathcal{PC}_{pcf}(M)$  be the equivalence classes of properly convex foliated projective contact structures.

**Theorem 6.2.** *The holonomy map is a homeomorphism between  $\mathcal{PC}_{pcf}(M)$  and the Hitchin component  $\mathcal{T}(\Gamma, \text{PSp}_4(\mathbf{R})) \subset \text{Rep}(\Gamma, \text{PSp}_4(\mathbf{R}))$ .*

This statement is indeed a direct Corollary of Theorem 2.8 since the Hitchin component  $\mathcal{T}^4(\Gamma, \text{PSp}_4(\mathbf{R}))$  is canonically identified with the subset of representations in  $\mathcal{T}^4(\Gamma)$  preserving the symplectic form  $\omega$  on  $\mathbf{R}^4$ .

Using the existence of a hol-equivariant Frenet curve  $\xi = (\xi^1, \xi^2, \xi^3) : \partial\Gamma \rightarrow \mathcal{Flag}(\mathbf{R}^4)$  for a properly convex foliated projective structure  $(\text{dev}, \text{hol})$  on  $M$  given in Section 5 we get that properly convex foliated projective contact structures carry indeed a much richer structure.

**Proposition 6.3.** *The contact distribution of a properly convex foliated projective structure carries a contact vector field and a natural Hermitian structure.*

*Proof.* Note that a Frenet curve equivariant with respect to a representation  $\rho : \Gamma \rightarrow \text{PSp}(4, \mathbf{R})$  is of the form  $\xi = (\xi^1, \xi^2 = \xi^{2\perp\omega}, \xi^3 = \xi^{1\perp\omega})$ . In particular for every  $t \in \partial\Gamma$ , the plane  $\xi^2(t)$  is a Lagrangian subspace of  $\mathbf{R}^4$ . Moreover, for  $(t_+, t_0, t_-)$  the three Lagrangians  $\xi^2(t_+)$ ,  $\xi^2(t_0)$ ,  $\xi^2(t_-)$  are pairwise transverse and define a complex structure  $J_{(t_+, t_0, t_-)}$  on  $\mathbf{R}^4 = \xi^2(t_-) \oplus \xi^2(t_+)$  such that the symmetric bilinear form  $\omega(\cdot, J_{(t_+, t_0, t_-)}\cdot)$  is positive definite (this is true in greater generality for maximal representations [3], here this statement can be proved by deformation starting from Fuchsian representations).

Let us recall this construction for the reader's convenience: If  $(L_+, L_0, L_-)$  is a triple of pairwise transverse Lagrangians, then  $L_0$  can be realized as the graph of  $F_+ \in \text{Hom}(L_+, L_-)$  and as graph of  $F_- \in \text{Hom}(L_-, L_+)$ . Then  $F_+ \circ F_- = \text{Id}_{L_-}$  and  $F_- \circ F_+ = \text{Id}_{L_+}$ , and the assignment

$$l_+ + l_- \in L_+ \oplus L_- \mapsto -F_-(l_-) + F_+(l_+)$$

defines a complex structure  $J_{L_+, L_0, L_-}$ . We write  $J_{(t_+, t_-, t_0)}$  for  $J_{\xi^2(t_+), \xi^2(t_0), \xi^2(t_-)}$ . We can use the complex structure  $J$  to define a hol-equivariant vector field:  $V_J(t_+, t_0, t_-) \in T_{\text{dev}(t_+, t_0, t_-)}\mathbb{P}^3(\mathbf{R})$  is the linear map  $l \mapsto J_{(t_+, t_0, t_-)} \cdot l$ . The definiteness of the quadratic form implies that  $V_J$  is orthogonal to the distribution  $\mathcal{H}_l$  and descends to a contact vector field on  $M$ . Moreover  $J_{(t_+, t_0, t_-)}$  induces a Hermitian structure on  $\mathcal{H}_l$ .  $\square$

## APPENDIX A. SOME USEFUL FACTS

Let  $\Gamma = \pi_1(\Sigma)$  be the fundamental group of a closed oriented surface of genus  $g \geq 2$ . We recall some facts about representations of  $\Gamma$  and its central extension  $\bar{\Gamma}$  into  $\mathrm{PSL}_2(\mathbf{R})$  and  $\mathrm{PSL}_2(\mathbf{C})$ .

**A.1. Equivariant curves.** Fuchsian representations are the representations coming from a uniformization of the surface  $\Sigma$ . They are precisely the faithful and discrete representations:

**Lemma A.1.** *If  $\iota : \Gamma \rightarrow \mathrm{PSL}_2(\mathbf{R})$  is faithful and discrete, then it is Fuchsian. This will be for example the case if for all  $\gamma$  in  $\Gamma - \{1\}$ ,  $\rho(\gamma)$  is a nontrivial hyperbolic element of  $\mathrm{PSL}_2(\mathbf{R})$ .*

*Proof.* The hypothesis imply that the action of  $\Gamma$  on  $\mathbb{H}^2$  is proper and free. The quotient surface  $\iota(\Gamma) \backslash \mathbb{H}^2$  is a surface whose fundamental group is  $\Gamma = \pi_1(\Sigma)$  hence is diffeomorphic to  $\Sigma$ . So we get a uniformization  $\Sigma \simeq \iota(\Gamma) \backslash \mathbb{H}^2$  and its holonomy is  $\iota$ .

For the second statement, the hypothesis already implies that  $\rho$  is faithful. To show that it is discret we need to show that the neutral component  $\overline{\rho(\Gamma)}^\circ$  is trivial. This subgroup is normalized by  $\rho(\Gamma)$  and by  $\mathrm{PSL}_2(\mathbf{R})$  since  $\rho$  is Zariski dense (the proper Lie subgroups of  $\mathrm{PSL}_2(\mathbf{R})$  are virtually solvable, so a representation having value in one of this subgroup has a nontrivial kernel). Since the elliptic elements in  $\mathrm{PSL}_2(\mathbf{R})$  form an open subset,  $\rho(\Gamma)$  cannot be dense. This implies that  $\overline{\rho(\Gamma)}^\circ$  is trivial.  $\square$

**Lemma A.2.** *Let  $\rho : \Gamma \rightarrow \mathrm{PSL}_2(\mathbf{R})$  be a representation. Suppose that there exists a continuous non-constant,  $\rho$ -equivariant curve  $\xi : \partial\Gamma \rightarrow \mathbb{P}^1(\mathbf{R})$  then  $\rho$  is a discrete and faithful representation, hence Fuchsian.*

*Proof.* Since  $\Gamma$  is torsion-free both faithfulness and discreteness will follow from the property:

$$\begin{aligned} &\text{there is no sequence } (\gamma_n)_{n \in \mathbf{N}} \text{ in } \Gamma \text{ such that } \lim \gamma_n = \infty \\ &\text{and } \lim \rho(\gamma_n) = \mathrm{Id}. \end{aligned}$$

Suppose that such a sequence exists. Applying the Theorem of Abels, Margulis and Soifer (see next Lemma) there exists an element  $f$  in  $\Gamma$  such that, up to extracting a subsequence, the sequence  $\delta_n = \gamma_n f$  satisfies that

$$\lim t_{+, \delta_n} = t_+ \text{ and } \lim t_{-, \delta_n} = t_- \text{ exist with } t_+ \neq t_-.$$

The sequence  $(\delta_n)_{n \in \mathbf{N}}$  has the following property:

$$\lim \delta_n = \infty \text{ and } \lim \rho(\delta_n) = \rho(f).$$

For any  $t \neq t_+$  the limit of  $(\delta_n^{-1} \cdot t)$  equals  $t_-$  so that

$$\rho(f)^{-1} \xi(t) = \lim \rho(\delta_n)^{-1} \xi(t) = \xi(t_-).$$

Hence  $\xi$  is constant on  $\partial\Gamma - \{t_+\}$  and therefore constant by continuity, contradicting the hypothesis.  $\square$

To be complete we explain now how the statement we used follows easily from Abels, Margulis and Soifer's Theorem.

**Lemma A.3.** *There exists a finite subset  $F$  in  $\Gamma$  such that for any sequence  $(\gamma_n)_{n \in \mathbf{N}}$  in  $\Gamma$  converging to infinity there exists some  $f$  in  $F$  such that (up to extracting a subsequence) the sequence  $\delta_n = \gamma_n f$  satisfies:*

$$t_+ = \lim t_{+, \delta_n} \neq t_- = \lim t_{-, \delta_n},$$

with  $t_{\pm, \delta}$  being the fixed points of  $\delta$  in  $\partial\Gamma$ .

Also for  $(\delta_n)_{n \in \mathbf{N}}$  one has the following property:

$$\text{for all } t \neq t_+, \lim \delta_n^{-1} \cdot t = t_-.$$

*Proof.* First we realize  $\Gamma$  as a cocompact lattice in  $\mathrm{SL}_2(\mathbf{R})$  so that the boundary at infinity is identified with  $\mathbb{P}^1(\mathbf{R})$ .

Recall that an  $\mathbf{R}$ -split element  $A$  of  $\mathrm{SL}_2(\mathbf{R})$  is called  $(r, \varepsilon)$ -proximal if, with  $t_{\pm, A} \in \mathbb{P}^1(\mathbf{R})$  being its eigenlines, we have

- $d_{\mathbb{P}^1(\mathbf{R})}(t_{+, A}, t_{-, A}) \geq 2r$ ,
- if  $x_1, x_2$  in  $\mathbb{P}^1(\mathbf{R})$  are two points satisfying  $d_{\mathbb{P}^1(\mathbf{R})}(x_i, t_{-, A}) \geq \varepsilon$  then

$$\begin{aligned} d_{\mathbb{P}^1(\mathbf{R})}(A \cdot x_i, t_{+, A}) &\leq \varepsilon \\ d_{\mathbb{P}^1(\mathbf{R})}(A \cdot x_1, A \cdot x_2) &\leq \varepsilon d_{\mathbb{P}^1(\mathbf{R})}(x_1, x_2). \end{aligned}$$

Here  $d_{\mathbb{P}^1(\mathbf{R})}$  is a distance on  $\mathbb{P}^1(\mathbf{R})$  coming from a norm on  $\mathbf{R}^2$ .

The Theorem of Abels, Margulis and Soifer (see [1]) states that there exist  $r > \varepsilon > 0$  and a finite subset  $F \subset \Gamma$  such that

for any  $\gamma$  in  $\Gamma$  there exists  $f$  in  $F$  such that  $\gamma f$  is  $(r, \varepsilon)$ -proximal.

In particular for  $\gamma_n$  we find  $f_n \in F$  such that  $\delta_n = \gamma_n f_n$  is  $(r, \varepsilon)$ -proximal hence  $d_{\mathbb{P}^1(\mathbf{R})}(t_{+, \delta_n}, t_{-, \delta_n}) \geq 2r$ . By compactness we can extract a subsequence such that the limits  $t_+ = \lim t_{+, \delta_n}$  and  $t_- = \lim t_{-, \delta_n}$  exist and such that  $(f_n)$  is constant equal to  $f \in F$ . Then  $d_{\mathbb{P}^1(\mathbf{R})}(t_+, t_-) \geq 2r$  and  $t_+ \neq t_-$ .

Denote by  $\lambda_n > 1$  the maximum absolute value of the eigenvalues of  $\delta_n$ . Then  $\lim \lambda_n = +\infty$ , because if not there exists a subsequence such that  $\lim \lambda_n = \lambda$  and  $(\delta_n)$  converges to an element of  $\mathrm{SL}_2(\mathbf{R})$  with eigenlines  $t_{\pm}$  and eigenvalues  $\lambda^{\pm 1}$  (or  $-\lambda^{\pm 1}$ ), contradicting  $\lim \delta_n = \infty$ .

Also for any  $t \neq t_+$ , if  $n$  is big enough, then the absolute value of the crossratio

$$|[t, \delta_n^{-1} \cdot t, t_{+, \delta_n}, t_{-, \delta_n}]| = \frac{1}{\lambda_n},$$

if  $t \neq t_{-, \delta_n}$ . Hence for any accumulation point  $s$  of the sequence  $(\delta_n^{-1} \cdot t)_{n \in \mathbf{N}}$  the equality

$$[t, s, t_+, t_-] = 0$$

holds, which implies that  $s = t_-$  and  $\lim \delta_n^{-1} \cdot t = t_-$ . □

**A.2. The Length Function.** Let  $\rho : \Gamma \rightarrow \mathrm{PSL}_2(\mathbf{R})$  be a Fuchsian representation. The length function for  $\rho$  is defined by:

$$\text{for any } \gamma \text{ in } \Gamma - \{1\}, \ell_\rho(\gamma) = t \Leftrightarrow \rho(\gamma) \text{ is conjugate to } \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}.$$

We recall the well known

**Fact A.4.** *Let  $\rho, \rho' : \Gamma \rightarrow \mathrm{PSL}_2(\mathbf{R})$  be two discrete and faithful representations, such that for all  $\gamma \in \Gamma$  the inequality  $\ell_\rho(\gamma) \leq \ell_{\rho'}(\gamma)$  holds, then  $\rho$  and  $\rho'$  are conjugate by an element of  $\mathrm{SL}_2^\pm(\mathbf{R})$ . In particular,  $\ell_\rho = \ell_{\rho'}$ .*

*Proof.* Denote by  $K(\rho, \rho')$  the supremum

$$K(\rho, \rho') = \sup_\gamma \log \frac{\ell_\rho(\gamma)}{\ell_{\rho'}(\gamma)}.$$

Then [25, Theorem 3.1] states that, if  $\rho$  and  $\rho'$  are not conjugate, then  $K(\rho, \rho') > 0$ .  $\square$

### A.3. Conjugate Representations.

**Lemma A.5.** *Let  $\rho_1, \rho_2 : \Gamma \rightarrow \mathrm{PSL}_2(\mathbf{C})$  be two representations with Zariski dense image (this will be automatically the case when  $\rho_i$  is faithful). We have the following alternative:*

- either  $\rho_1$  and  $\rho_2$  are conjugate (by an element of  $\mathrm{SL}_2(\mathbf{C})$ ),
- or the representation  $\rho = (\rho_1, \rho_2)$  of  $\Gamma$  into  $\mathrm{PSL}_2(\mathbf{C}) \times \mathrm{PSL}_2(\mathbf{C})$  has Zariski-dense image.

*Proof.* Let  $G$  denote the Zariski closure of  $\rho(\Gamma)$  and  $p_1, p_2 : \mathrm{PSL}_2(\mathbf{C}) \times \mathrm{PSL}_2(\mathbf{C}) \rightarrow \mathrm{PSL}_2(\mathbf{C})$  the two projections. By hypothesis the restrictions  $p_{i|G}$  are onto, in particular the two kernels  $\ker p_{i|G}$  have the same dimension.

Suppose that this dimension is not zero then  $p_2(\ker p_1 \cap G)$  has positive dimension and is normalized by  $\rho_2(\Gamma)$ , hence  $p_2(\ker p_1 \cap G) = \mathrm{PSL}_2(\mathbf{C})$ . Since this holds also for the first projection, we have  $G = \mathrm{PSL}_2(\mathbf{C}) \times \mathrm{PSL}_2(\mathbf{C})$ .

If the above dimension is zero, the group  $p_2(\ker p_1 \cap G)$  is finite and normal hence trivial. This means that  $G \simeq \mathrm{PSL}_2(\mathbf{C})$  and  $\rho_1, \rho_2$  are conjugate.  $\square$

### A.4. Representations of $\bar{\Gamma}$ .

**Lemma A.6.** *Let  $\rho : \bar{\Gamma} \rightarrow \mathrm{PSL}_2(\mathbf{R})$  a representation, then  $\rho(\tau)$  cannot be nontrivial  $\mathbf{R}$ -split.*

*Proof.* Up to conjugation, we can suppose  $\rho(\tau) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$  with  $\lambda > 1$  then  $\rho(\bar{\Gamma})$  is contained in the centralizer of this element hence in the subgroup of diagonal matrices. Hence  $\rho$  is trivial on the commutator subgroup  $[\bar{\Gamma}, \bar{\Gamma}]$ , but  $\tau^{2g}$  is in this group with  $\rho(\tau)^{2g} \neq \mathrm{Id}$ .  $\square$

### A.5. Convex sets.

**Lemma A.7.** *Let  $I$  be an interval and  $\mathcal{L} : I \rightarrow \mathbb{P}^2(\mathbf{R})^*$  a continuous map. Then the subset*

$$D = \mathbb{P}^2(\mathbf{R}) - \bigcup_{t \in I} \mathcal{L}(t)$$

*is a convex set in  $\mathbb{P}^2(\mathbf{R})$ .*

*Proof.* Suppose that  $D$  is not empty. Assume that  $D$  is not convex. Then there exists  $p, q \in D$ ,  $p \neq q$  such that the intersection of the line  $L$  spanned by  $p$  and  $q$  in  $\mathbb{P}^2(\mathbf{R})$  with  $D$  is not connected. Therefore there are  $t_1 < t_2 \in I$  such that  $\mathcal{L}(t_1)$  and  $\mathcal{L}(t_2)$  intersect the two different connected components of  $L - \{p, q\}$ .

We choose coordinates  $(x, y)$  in the affine chart  $\mathbb{P}^2(\mathbf{R}) - L$  such that  $\mathcal{L}(t_1)$  is the  $y$ -axis  $\{(x, y) \mid x = 0\}$  and  $\mathcal{L}(t_2)$  is the  $x$ -axis  $\{(x, y) \mid y = 0\}$ . Since  $p, q \in D$  we have for all  $t \in I$  that  $\mathcal{L}(t) \neq L$ , and there exist continuous functions  $a, b, c : I \rightarrow \mathbf{R}$  such that

$$\mathcal{L}(t) = \{(x, y) \mid a(t)x + b(t)y + c(t) = 0\}.$$

In particular  $a(t_1) \neq 0$ ,  $b(t_1) = c(t_1) = 0$  and  $a(t_2) = c(t_2) = 0$ ,  $b(t_2) \neq 0$ .

The points  $p, q$  in  $L = \mathbb{P}(\{(x, y)\})$  have homogeneous coordinates  $[x_p, y_p]$  and  $[x_q, y_q]$ , and since  $p$  and  $q$  do not belong to  $\mathcal{L}(t_1)$  or  $\mathcal{L}(t_2)$  the products  $x_p y_p \neq 0$  and  $x_q y_q \neq 0$  are nonzero. Moreover since  $\mathcal{L}(t_1)$  and  $\mathcal{L}(t_2)$  intersect  $L - \{p, q\}$  in different connected components we have

$$(8) \quad \text{sign}(x_p y_p) \neq \text{sign}(x_q y_q).$$

Furthermore, since  $\mathcal{L}(t)$  does not contain neither  $p$  nor  $q$ ,

$$(9) \quad a(t)x_p + b(t)y_p \neq 0 \quad \text{and} \quad a(t)x_q + b(t)y_q \neq 0.$$

In particular the quantities in Equation (9) do not change their sign for all  $t \in I$ . Therefore

$$\text{sign}(a(t_1)x_p) = \text{sign}(b(t_2)y_p) \quad \text{and} \quad \text{sign}(a(t_1)x_q) = \text{sign}(b(t_2)y_q),$$

which implies

$$\text{sign}(x_p y_p) = \text{sign}(a(t_1)b(t_2)) = \text{sign}(x_q y_q),$$

contradicting Equation (8). □

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