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Convex geometry of max-stable distributions

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Abstract It is shown that max-stable random vectors in $[0,\infty)^d$ with unit Fréchet marginals are in one to one correspondence with convex sets K in $[0,\infty)^d$ called max-zonoids. The max-zonoids can be characterised as sets obtained as limits of Minkowski sums of cross-polytopes or, alternatively, as the selection expectation of a random cross-polytope whose distribution is controlled by the spectral measure of the max-stable random vector. Furthermore, the cumulative distribution function $\mathbf{P}\{\xi \le x\}$ of a max-stable random vector ξ with unit Fréchet marginals is determined by the norm of the inverse to x, where all possible norms are given by the support functions of (normalised) max-zonoids. As an application, geometrical interpretations of a number of well-known concepts from the theory of multivariate extreme values and copulas are provided.

Keywords Copula · Max-stable random vector · Norm · Cross-polytope · Spectral measure · Support function · Zonoid

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1 Introduction

A random vector ξ in $\mathbb{R}^d$ is said to have a *max-stable* distribution if, for every $n \geq 2$, the coordinatewise maximum of n i.i.d. copies of ξ coincides in distribution with an affine transform of ξ, i.e.

$$\xi^{(1)} \vee \cdots \vee \xi^{(n)} \overset{\mathrm{d}}{\sim} a_n \xi + b_n \tag{1}$$

for $a_n > 0$ and $b_n \in \mathbb{R}^d$. If Eq. 1 holds with $b_n = 0$ for all n, then ξ is called *strictly* max-stable, see, e.g., Beirlant et al. (2004), Kotz and Nadarajah (2000), Resnick (1987).

Since every max-stable random vector ξ is infinitely divisible with respect to coordinatewise maximum, its cumulative distribution function satisfies

$$F(x) = \mathbf{P}\{\xi \leq x\} = \begin{cases} \exp\{-\mu([-\infty, x]^{\mathbf{c}})\}, & x \geq a, \\ 0, & \text{otherwise}, \end{cases} \qquad x \in \mathbb{R}^d, \tag{2}$$

where $a \in [-\infty, \infty)^d$, the superscript $\mathbf{c}$ denotes the complement and μ is a measure on $[a, \infty] \setminus \{a\}$ called the *exponent* measure of ξ, see Resnick (1987, Proposition 5.8). Note that all inequalities and segments (intervals) for vectors are understood coordinatewise.

Representation Eq. 2 shows that the cumulative distribution function of ξ can be represented as the exponential $F(x) = e^{-\nu(x)}$ of another function ν. If ξ is strictly max-stable and $a = 0$, then ν is homogeneous, i.e. $\nu(sx) = s^{-\alpha}\nu(x)$ for all $s > 0$ and some $\alpha > 0$. This fact can be also derived from general results concerning semigroup-valued random elements (Davydov et al. 2005). If $\alpha = 1$, an example of such function $\nu(x)$ is provided by $\nu(x) = \|x^*\|$, i.e. a norm of $x^* = (x_1^{-1}, \ldots, x_d^{-1})$ for $x = (x_1, \ldots, x_d) \in [0, \infty)^d$. One of the main aims of this paper is to show that this is the *only* possibility and to characterise all norms that give rise to strictly max-stable distributions with $\alpha = 1$.

Every norm is homogeneous and sublinear. It is known (Schneider 1993, Th. 1.7.1) that each bounded homogeneous and sublinear function g on $\mathbb{R}^d$ can be described as the *support function* of a certain convex compact set K, i.e.

$$g(x) = h(K, x) = \sup\{\langle x, y \rangle : y \in K\},$$

where $\langle x, y \rangle$ is the scalar product of x and y. In Section 2 we show that every standardised strictly max-stable distribution with $\alpha = 1$ is associated with the unique compact convex set $K \subset [0, \infty)^d$ called the dependency set. The dependency sets are suitably rescaled sets from the family of sets called max-zonoids. While classical zonoids appear as limits for the sums of segments (Schneider 1993, Section 3.5), max-zonoids are limits of the sums of cross-polytopes. The contributions of particular cross-polytopes to this sum are controlled by the spectral measure of the max-stable random vector. It is shown that not every convex compact set for $d \geq 3$ corresponds to a strictly max-stable distribution, while if $d = 2$, then the family of dependency sets is the family of all “standardised” convex sets, see also Falk (2006) for the treatment of the bivariate case. This, in particular, shows a substantial difference between possible dependency structures for bivariate extremes on one hand and multivariate extremes in dimensions three and more on the other hand.

The geometrical interpretation of max-stable distributions opens a possibility to use tools from convex geometry in the framework of the theory of extreme values. For instance, the polar sets to the dependency set K appear as multivariate quantiles of the corresponding max-stable random vector, i.e. the level sets of its cumulative distribution function. In the other direction, some useful families of extreme values distributions may be used to construct new norms in $\mathbb{R}^d$ which acquire an explicit probabilistic interpretation. The norms corresponding to max-stable distributions are considered in Section 3.

Section 4 deals with relationships between spectral measures of max-stable laws and geometric properties of the corresponding dependency set. In Section 5 it is shown that a number of dependency concepts for max-stable random vectors can be expressed using geometric functionals of the dependency set and its polar. Here also relationships to copulas are considered. It is shown that max-zonoids are only those convex sets whose support functions generate multivariate extreme value copulas.

It is well known that Z is (classical) zonoid if and only if $e^{-h(Z,x)}$ is positive definite, see Schneider (1993, p. 194). In Section 6 we establish a similar result for the positive definiteness of the exponential with respect to the coordinatewise maximum operation in case Z is a max-zonoid.

Section 7 describes some relationships between operations with convex sets and operations with max-stable random vectors. Finally, Section 8 briefly mentions an infinite-dimensional extension for max-stable sample continuous random functions.

2 Dependency Sets and Max-zonoids

Let ξ be a max-stable random vector with non-degenerate marginals. By an affine transformation it is possible to standardise the marginals of ξ, so that ξ has Φ_α (Fréchet distributed) marginals, where

$$\Phi_\alpha(x) = \begin{cases} 0, & x < 0\,, \\ e^{-x^{-\alpha}}, & x \geq 0\,, \end{cases} \qquad \alpha > 0\,,$$

or Ψ_α (Weibull or negative exponential distributed) marginals, i.e.

$$\Psi_\alpha(x) = \begin{cases} e^{-(-x)^\alpha}, & x < 0\,, \\ 1, & x \geq 0\,, \end{cases} \qquad \alpha > 0\,,$$

or Λ (Gumbel or double exponentially distributed) marginals, i.e.

$$\Lambda(x) = \exp\{-e^{-x}\}\,, x \in \mathbb{R}\,.$$

By using (possibly non-linear) monotonic transformations applied to the individual coordinates it is possible to assume that all marginals are Φ_1, see Resnick (1987, Proposition 5.10) and Beirlant et al. (2004, Section 8.2.2).

In this case we say that ξ has *unit Fréchet marginals* or has a *simple* max-stable distribution, see also Einmahl et al. (1997). Sometimes we say that $\xi = (\xi_1, \dots, \xi_d)$ has a *semi-simple* max-stable distribution if its rescaled version $(c_1\xi_1, \dots, c_d\xi_d)$ has a simple max-stable distribution for some $c_1, \dots, c_d > 0$.

If ξ has a simple max-stable distribution, then Resnick (1987, Proposition 5.11) implies that the exponent in Eq. 2 has the following representation

$$\nu(x) = \mu([0, x]^{\mathrm{c}}) = \int_{\mathbb{S}_+} \max_{1 \le i \le d} \left(\frac{a_i}{x_i} \right) \sigma(da) , \quad x \in [0, \infty]^d \setminus \{0\} , \tag{3}$$

where

$$[0, x] = \times_{i=1}^d [0, x_i] , \quad x = (x_1, \dots, x_d) ,$$

$\mathbb{S}_+ = \{x \in \mathbb{E} : \ \|x\| = 1\}$ is a sphere in $\mathbb{E} = [0, \infty)^d$ with respect to any chosen norm (from now on called the *reference sphere* and the *reference norm*) and σ is a finite measure on $\mathbb{S}_+$ (called the *spectral measure* of ξ) such that

$$\int_{\mathbb{S}_+} a_i \sigma(da) = 1 , \quad i = 1, \dots, d . \tag{4}$$

A similar representation is described in Falk et al. (2004, Theorem 4.3.1) for the special case of $\mathbb{S}_+$ being the unit simplex.

We now aim to relate the function $\nu(x)$ from Eq. 3 to the support function of a certain compact convex set. Recall that the *support function* of a set $M \subset \mathbb{R}^d$ is defined as

$$h(M, x) = \sup\{\langle z, x \rangle : \ z \in M\} ,$$

where $\langle z, x \rangle$ is the scalar product in $\mathbb{R}^d$. Let $e_1, \dots, e_d$ be the standard orthonormal basis in $\mathbb{R}^d$. For every $a = (a_1, \dots, a_d) \in \mathbb{R}^d$ consider the *cross-polytope*

$$\Delta_a = \mathrm{conv}(\{0, a_1 e_1, \dots, a_d e_d\}) ,$$

where $\mathrm{conv}(\cdot)$ denotes the convex hull of the corresponding set. Note that $\mathrm{conv}(\{a_1 e_1, \dots, a_d e_d\})$ is a simplex. Then

$$h(\Delta_a, x) = h(\Delta_x, a) = \max_{1 \le i \le d} (a_i x_i)$$

for every $a \in \mathbb{S}_+$ and $x \in \mathbb{E}$. For $x = (x_1, \dots, x_d) \in \mathbb{E}$ write $x^* = (x_1^{-1}, \dots, x_d^{-1})$. Then Eq. 3 can be expressed as

$$\nu(x^*) = \int_{\mathbb{S}_+} h(\Delta_a, x) \sigma(da) , \quad x \in \mathbb{E} . \tag{5}$$

The function $l(x) = \nu(x^*)$ is called the *stable tail dependence function*, see Beirlant et al. (2004, p. 257).

It is well known that the arithmetic sum of support functions of two convex compact sets K and L is the support function of their Minkowski sum

$$K + L = \{x + y : \ x \in K, \ y \in L\} ,$$

i.e. $h(K + L, x)$ equals $h(K, x) + h(L, x)$. Extending this idea to integrals of support functions leads to the expectation concept for random compact sets, see Artstein and Vitale (1975) and Molchanov (2005, Section 2.1). If X is a random compact set (Molchanov 2005) such that $\|X\| = \sup\{\|x\| : x \in X\}$ is integrable, then the *selection expectation* (also called the Aumann expectation) of X is the set of expectations of $\mathbf{E}\,\xi$ for all random vectors ξ such that $\xi \in X$ a.s. If the underlying probability space is non-atomic, or X is a.s. convex, then $\mathbf{E}\,X$ is the unique compact convex set that satisfies

$$\mathbf{E}\,h(X, x) = h(\mathbf{E}\,X, x)$$

for all x, see Molchanov (2005, Theorem II.1.22).

Let σ_1 be the spectral measure σ normalised to have the total mass 1. If η is distributed on $\mathbb{S}_+$ according to σ_1, then Δ_η is a random convex compact set whose selection expectation satisfies

$$h(\mathbf{E}\,\Delta_\eta, x) = \frac{1}{\sigma(\mathbb{S}_+)} \int_{\mathbb{S}_+} h(\Delta_a, x)\sigma(da)\,. \qquad (6)$$

Condition Eq. 4 further implies that

$$\sigma(\mathbb{S}_+) h(\mathbf{E}\,\Delta_\eta, e_i) = 1\,, \quad i = 1, \ldots, d\,. \qquad (7)$$

Since $h(\mathbf{E}\,\Delta_\eta, e_i) = \mathbf{E}\,h(\Delta_\eta, e_i) = \mathbf{E}\,\eta_i$, we have $\sigma(\mathbb{S}_+)\,\mathbf{E}\,\eta_i = 1$ for $i = 1, \ldots, d$. Together with Eqs. 3 and 2 these reasons lead to the following result.

Theorem 1 *A random vector ξ is max-stable with unit Fréchet marginals if and only if its cumulative distribution function $F(x) = \mathbf{P}\{\xi \le x\}$ satisfies*

$$F(x) = \exp\{-ch(\mathbf{E}\,\Delta_\eta, x^*)\}\,, \quad x \in \mathbb{E}\,,$$

for a constant $c > 0$ and a random vector $\eta \in \mathbb{S}_+$ such that $c\,\mathbf{E}\,\eta = (1, \ldots, 1)$.

If $K = c\,\mathbf{E}\,\Delta_\eta$, then

$$F(x) = e^{-h(K,x^*)}\,, \quad x \in \mathbb{E}\,. \qquad (8)$$

Furthermore, note that $K = \mathbf{E}\,\Delta_{c\eta}$ with $\mathbf{E}(c\eta) = (1, \ldots, 1)$.

Definition 1 The set $K = c\,\mathbf{E}\,\Delta_\eta$ where $c > 0$ and η is a random vector on $\mathbb{S}_+$ is said to be a *max-zonoid*. If σ_1 is the distribution of η, then $\sigma = c\sigma_1$ is the *spectral measure* of K. If $c\,\mathbf{E}\,\eta = (1, \ldots, 1)$, then the max-zonoid K is called the *dependency set* associated with the spectral measure σ (or associated with the corresponding simple max-stable random vector).

Proposition 1 *A convex set K is a max-zonoid if and only if there exists a semi-simple max-stable vector ξ with cumulative distribution function $F(x) = e^{-h(K,x^*)}$ for all $x \in \mathbb{E}$.*

Proof Sufficiency. A semi-simple max-stable ξ can be obtained as $\xi = a\xi' = (a_1\xi'_1, \dots, a_d\xi'_d)$ for simple max-stable vector ξ' and $a = (a_1, \dots, a_d) \in (0, \infty)^d$. Let K' be the dependency set of ξ'. By Theorem 1,

$$\mathbf{P}\{\xi \le x\} = \mathbf{P}\{a\xi' \le x\} = e^{-h(K', ax^*)} = e^{-h(K, x^*)}\,, \quad x \in \mathbb{E}\,,$$

for $K = aK' = \{(a_1x_1, \dots, a_dx_d) : (x_1, \dots, x_d) \in K\}$.

Necessity. If K is a max-zonoid, then $K' = aK$ is a dependency set for some $a \in (0, \infty)^d$. If ξ' is max-stable with dependency set K', then it is easily seen that $a\xi'$ has the cumulative distribution function $e^{-h(K,x^*)}$. □

Proposition 1 means that each max-zonoid can be rescaled to become a dependency set.

Proposition 2 *A max-zonoid K always satisfies*

$$\Delta_z \subset K \subset [0, z] \tag{9}$$

for some $z \in \mathbb{E}$.

Proof The result follows from the following bound on the support function of $\mathbf{E}\,\Delta_\eta$

$$h(\Delta_y, x) = \max_{1 \le i \le d} \mathbf{E}(\eta_i x_i) \le \mathbf{E}\,h(\Delta_\eta, x) \le \mathbf{E}\sum_{i=1}^{d} \eta_i x_i = h([0, y], x)$$

where $y = \mathbf{E}\,\eta$, so that Eq. 9 holds with $z = cy$. □

The normalisation condition (7) and (9) imply that the dependency set of a simple max-stable distribution satisfies

$$\Delta_{(1,\dots,1)} = \mathrm{conv}\{0, e_1, \dots, e_d\} \subset K \subset [0, 1]^d\,, \tag{10}$$

where $\Delta_{(1,\dots,1)}$ is called the unit cross-polytope.

The selection expectation of Δ_η has the support function given by

$$h(\mathbf{E}\,\Delta_\eta, x) = \int_{\mathbb{S}_+} \|(a_1x_1, \dots, a_dx_d)\|_\infty \sigma(da)\,, \tag{11}$$

where $\|\cdot\|_\infty$ is the ℓ_∞-norm in $\mathbb{R}^d$. If the ℓ_∞-norm in Eq. 11 is replaced by the ℓ_1-norm, i.e. the absolute value of the sum of the coordinates and integration is carried over the whole sphere, then Eq. 11 yields the support function of a *zonoid*, see Schneider (1993, Section 3.5). This provides one of the reasons for calling $\mathbf{E}\,\Delta_\eta$ a max-zonoid. Note that max-zonoids form a sub-family of sets called d-zonoids in Ricker (1982).

It is possible to define a max-zonoid as the selection expectation of Δ_ζ, where ζ is any random vector in $\mathbb{E}$ (not necessarily on $\mathbb{S}_+$). The corresponding spectral measure σ on $\mathbb{S}_+$ can be found from

$$\int_{\mathbb{S}_+} g(a)\sigma(da) = \mathbf{E}\left[\|\zeta\| g\left(\frac{\zeta}{\|\zeta\|}\right)\right] \tag{12}$$

for all integrable functions g on $\mathbb{S}_+$. Indeed,

$$\int_{\mathbb{S}_+} h(\Delta_u, x)\sigma(du) = \mathbf{E}[\|\zeta\| h(\Delta_{\zeta/\|\zeta\|}, x)] = \mathbf{E}\, h(\Delta_\zeta, x)\,.$$

If all coordinates of ζ have the unit mean, then the selection expectation of Δ_ζ becomes a dependency set.

An alternative representation of max-stable laws (Resnick 1987, Proposition 5.11) yields that

$$F(x) = \exp\left\{-\int_0^1 \max\left(\frac{f_1(s)}{x_1}, \ldots, \frac{f_d(s)}{x_d}\right) ds\right\} \tag{13}$$

for non-negative integrable functions $f_1, \ldots, f_d$ satisfying

$$\int_0^1 f_i(s)ds = 1\,, \quad i = 1, \ldots, d\,.$$

Thus

$$h(K, x) = \int_0^1 \max(f_1(s)x_1, \ldots, f_d(s)x_d)\, ds\,,$$

i.e. the dependency set K is given by the selection expectation of the cross-polytope $\Delta_{f(\eta)}$, where $f(\eta) = (f_1(\eta), \ldots, f_d(\eta))$ and η is uniformly distributed on $[0, 1]$. The corresponding spectral measure can be found from Eq. 12 for $\zeta = f(\eta)$.

Theorem 2 *If $d = 2$, then each convex set K satisfying Eq. 10 is the dependency set of a simple max-stable distribution. If $d \geq 3$, then only those K that satisfy Eq. 10 and are max-zonoids correspond to simple max-stable distributions.*

Proof Consider a planar convex polygon K satisfying Eq. 10, so that its vertices are $a^0 = e_1, a^1, \ldots, a^m = e_2$ in the anticlockwise order. Then K equals the sum of triangles with vertices $(0, 0), (a_1^{i-1} - a_1^i, 0), (0, a_2^i - a_2^{i-1})$ for $i = 1, \ldots, m$, where $a^i = (a_1^i, a_2^i)$. Thus Eq. 5 holds with σ having atoms at $u_i/\|u_i\|$ with mass $\|u_i\|$ where $u_i = (a_1^{i-1} - a_1^i, a_2^i - a_2^{i-1})$ for $i = 1, \ldots, m$. The approximation by polytopes yields that a general convex K satisfying Eq. 10 can be represented as the expectation of a random cross-polytope and so corresponds to a simple max-stable distribution.

Theorem 1 implies that all max-zonoids satisfying Eq. 10 correspond to simple max-stable distributions. It remains to show that not every convex set K satisfying Eq. 10 is a dependency set in dimension $d \geq 3$. For instance, consider set L in $\mathbb{R}^3$ which is the convex hull of $0, e_1, e_2, e_3$ and $(2/3, 2/3, 2/3)$.

All its 2-dimensional faces are triangles, so that this set is indecomposable by Grünbaum (1967, Theorem 15.3). Since L is a polytope, but not a cross-polytope, it cannot be represented as a sum of cross-polytopes and so is not a max-zonoid. □

The support function of the dependency set K equals the tail dependence function Eq. 5. If an estimate $\hat{l}(\cdot)$ of the tail dependence function is given for a finite set of directions $u_1, \dots, u_m$, it is possible to estimate K, e.g. as the intersection of half-spaces $\{x \in \mathbb{E} : \langle x, a_i \rangle \leq \hat{l}(a_i)\}$. However, this estimate should be used very cautiously, since the obtained polytope K is not necessarily a max-zonoid in dimensions three and more. While this approach is justified in the bivariate case (see also Hall and Tajvidi 2004), in general, it is better to use an estimate $\hat{\sigma}$ of the spectral measure σ in order to come up with an estimator of K as

$$h(\hat{K}, x) = \int_{\mathbb{S}_+} h(\Delta_a, x)\hat{\sigma}(da)\,.$$

Being the expectation of a cross-polytope, the obtained set is necessarily a max-zonoid.

The set

$$K^o = \{x \in \mathbb{E} : \ h(K, x) \leq 1\}$$

is called the *polar* (or *dual*) set to K in $\mathbb{E}$, see Schneider (1993, Section 1.6) for the conventional definition where $\mathbb{E}$ is replaced by $\mathbb{R}^d$. If K is convex and satisfies Eq. 10, then its polar K^o is also convex and satisfies the same condition. Furthermore,

$$\begin{aligned}\{x \in \mathbb{E} : \ F(x) \geq \alpha\} &= \{x \in \mathbb{E} : \ e^{-h(K,x^*)} \geq \alpha\} \\ &= \{x^* : \ x \in \mathbb{E}, \ h(K, x) \leq -\log\alpha\} \\ &= (-\log\alpha)\{x^* : \ x \in K^o\}\,,\end{aligned}$$

i.e. multivariate quantiles of the cumulative distribution function of a simple max-stable random vector are inverted rescaled variants of the polar set to the dependency set K. The level sets of multivariate extreme values distributions have been studied in de Haan and de Ronde (1998). Note that the dimension effect described in Theorem 2 restricts the family of sets that might appear as multivariate quantiles in dimensions $d \geq 3$.

The ordering of dependency sets by inclusion corresponds to the stochastic ordering of simple max-stable random vectors, i.e. if ξ' and ξ'' have dependency sets K' and K'' with $K' \subset K''$, then $\mathbf{P}\{\xi' \leq x\} \geq \mathbf{P}\{\xi'' \leq x\}$ for all $x \in \mathbb{E}$.

A metric on the family of dependency sets may be used to measure the distance between random vectors ξ' and ξ' with simple max-stable distributions. Such distance can be defined as the Hausdorff distance between the dependency sets of ξ and ξ' or any other metric for convex sets (e.g. the Lebesgue measure of the symmetric difference or the L_p-distance between the support functions). In the spirit of the Banach-Mazur metric for convex sets (or linear spaces), a distance between two dependency sets K' and K'' can be defined as

$$m(K', K'') = \log\inf\left\{\prod_{i=1}^{d}\lambda_i :\ K' \subset \lambda K'',\ K'' \subset \lambda K', \lambda \in (0,\infty)^d\right\},$$

where $\lambda K = \{(\lambda_1 x_1, \dots, \lambda_d x_d) :\ (x_1, \dots, x_d) \in K\}$ with $\lambda = (\lambda_1, \dots, \lambda_d)$. If ξ' and ξ'' have dependency sets K' and K'' respectively, then $m(K', K'')$ is the logarithm of the smallest value of $(\lambda_1 \cdots \lambda_d)$ such that ξ' is stochastically smaller than $\lambda\xi''$ and ξ'' is stochastically smaller than $\lambda\xi'$. For instance, the distance between the unit cross-polytope and the unit square (for $d = 2$) is $\log 4$, which is the largest possible distance between two simple bivariate max-stable laws.

3 Norms Associated with Max-stable Distributions

Note that the support function of a compact set L is sublinear, i.e. it is homogeneous and subadditive. If L is convex symmetric and contains the origin in its interior, then its support function $h(L, x)$ defines a norm in $\mathbb{R}^d$. Conversely, every norm defines a symmetric convex compact set in $\mathbb{R}^d$ with the origin in its interior, see Rockafellar (1970, Theorem 15.2).

Let K be a convex set satisfying Eq. 9. The corresponding norm $\|\cdot\|_K$ can be defined as the support function of the set L obtained as the union of all symmetries of K with respect to coordinate planes, i.e.

$$\|x\|_K = h(L, x) = h(K, |x|), \quad x \in \mathbb{R}^d,$$

where $|x| = (|x_1|, \dots, |x_d|)$. The norm $\|x\|_K$ is said to be generated by the max-zonoid K. Note that the origin belongs to the interior of L and $\|x\|_K = h(K, x)$ for $x \in \mathbb{E}$. The following result shows that distributions of max-stable vectors correspond to norms generated by max-zonoids.

Theorem 3 *Let $\|\cdot\|$ be a norm on $\mathbb{R}^d$. The function*

$$F(x) = \exp\{-\|x^*\|\}, \quad x \in \mathbb{E}, \tag{14}$$

is the cumulative distribution function of a random vector ξ in $\mathbb{E}$ if and only if $\|x\| = h(K, |x|)$ is the norm generated by a max-zonoid K. In this case the random vector ξ is necessarily semi-simple max-stable.

Proof Sufficiency. If K is a max-zonoid, Proposition 1 implies that there exists a semi-simple max-stable vector ξ with cumulative distribution function $e^{-h(K,x^*)} = e^{-\|x^*\|}$.

Necessity. If Eq. 14 is the cumulative distribution function of a random vector ξ, then

$$\mathbf{P}\left\{\xi^{(1)} \vee \cdots \vee \xi^{(n)} \le x\right\} = e^{-n\|x^*\|} = e^{-\|(n^{-1}x)^*\|} = \mathbf{P}\left\{\xi \le n^{-1}x\right\}$$

for all $x \in \mathbb{E}$ and i.i.d. copies $\xi^{(1)}, \dots, \xi^{(n)}$ of ξ. Thus, ξ is necessarily semi-simple max-stable. Proposition 1 implies that Eq. 14 holds with the norm generated by a max-zonoid K. $\square$

The space $\mathbb{R}^d$ with the norm $\|\cdot\|_K$ becomes a finite-dimensional normed linear space, also called the *Minkowski space*, see Thompson (1988). If this is an inner product space, then the norm is necessarily Euclidean. Indeed, if K is the intersection of a centred ellipsoid with $\mathbb{E}$ and satisfies Eq. 10, then this ellipsoid is necessarily the unit ball.

Another common way to standardise the marginals of a multivariate extreme value distribution is to bring them to the reverse exponential distribution (or unit Weibull distribution), see Falk et al. (2004, Section 4.1). In this case, the cumulative distribution function turns out to be

$$F(x) = e^{-\|x\|_K} \quad x \in (-\infty, 0]^d .$$

The fact that every max-stable distribution with reverse exponential marginals gives rise to a norm has been noticed in Falk et al. (2004, p. 127), however without giving a characterisation of these norms.

Note that the norm of $\|x\|_K$ can be expressed as

$$\|x\|_K = \|x\| \; \|u_x\|_K ,$$

where $u_x = x/\|x\|$ belongs to the reference sphere $\mathbb{S}_+$. If the reference norm is ℓ_1, then $\mathbb{S}_+$ is the unit simplex and the norm $\|u_x\|_K$ of $u = (t_1, \dots, t_{d-1}, 1 - t_1 - \dots - t_{d-1}) \in \mathbb{S}_+$ can be represented as a function $A(t_1, \dots, t_{d-1})$ of $t_1, \dots, t_{d-1} \geq 0$ such that $t_1 + \dots + t_{d-1} \leq 1$. If $d = 2$, then $A(t)$, $0 \leq t \leq 1$, is called the *Pickands function*, see Kotz and Nadarajah (2000) and for the multivariate case also Falk and Reiss (2005), Kotz and Nadarajah (2000). In general, the norm $\|u\|$, $u \in \mathbb{S}_+$, is an analogue of the Pickands function.

Example 1 The dependency set K being the unit cube $[0, 1]^d$ (so that $\|x\|_K$ is the ℓ_1-norm) corresponds to the independence case, i.e. independent coordinates of $\xi = (\xi_1, \dots, \xi_d)$. The corresponding spectral measure allocates unit atoms to the points from the coordinate axes.

Furthermore, K being the unit cross-polytope (so that $\|x\|_K$ is the ℓ_∞-norm) gives rise to the random vector $\xi = (\xi_1, \dots, \xi_1)$ with all identical Φ_1-distributed coordinates, i.e. the completely dependent random vector. The corresponding spectral measure has its only atom at the point from $\mathbb{S}_+$ having all equal coordinates. Note that the unit cube is dual (or polar) set to the unit cross-polytope.

Example 2 The ℓ_p-norm $\|x\|_p$ with $p \geq 1$ generates the symmetric logistic distribution (Beirlant et al. 2004, (9.11)) with parameter $\alpha = 1/p$. The strength of dependency increases with p.

Example 3 A useful family of simple max-stable bivariate distribution appears if the functions f_1, f_2 in Eq. 13 are chosen to be the density functions of normal distributions, see Kotz and Nadarajah (2000, Section 3.4.5) and Beirlant et al. (2004, p. 309). It is shown in Hüsler and Reiss (1989) that these distributions appear as limiting distributions for maxima of bivariate i.i.d. Gaussian random vectors. The corresponding norm (which we call the Hüsler-Reiss norm) is given by

$$\|x\|_K = x_1 \Phi\left(\lambda + \frac{1}{2\lambda}\log\frac{x_1}{x_2}\right) + x_2 \Phi\left(\lambda - \frac{1}{2\lambda}\log\frac{x_1}{x_2}\right),$$

where $\lambda \in [0, \infty]$. The cases $\lambda = 0$ and $\lambda = \infty$ correspond to complete dependence and independence, respectively.

Example 4 The bivariate symmetric negative logistic distribution (Beirlant et al. 2004, p. 307) corresponds to the norm given by

$$\|x\|_K = \|x\|_1 - \lambda\|x\|_p ,$$

where $\lambda \in [0, 1]$ and $p \in [-\infty, 0]$.

4 Spectral Measures

Since the dependency set determines uniquely the distribution of a simple max-stable random vector, there is one to one correspondence between dependency sets and normalised spectral measures. It is possible to extend this correspondence to max-zonoids on one side and all finite measures on $\mathbb{S}_+$ on the other one, since both uniquely identify semi-simple max-stable distributions. While the spectral measure depends on the choice of the reference norm, the dependency set remains the same whatever the reference norm is.

It is shown in Coles and Tawn (1991) that the densities of the spectral measure on the reference simplex $\mathbb{S}_+ = \operatorname{conv}\{e_1, \ldots, e_d\}$ can be obtained by differentiating the function $\nu(x) = \mu([0, x]^{\mathbf{c}})$ for the exponent measure μ. This is possible if the spectral measure is absolutely continuous with respect to the surface area measures on relative interiors of all faces of the simplex and, possibly, has atoms at the vertices of $\mathbb{S}_+$. Following the proof of this fact in Beirlant et al. (2004, Section 8.6.1), we see that

$$\lim_{z_j \to 0,\ j \notin A} D_A \nu(z) = (-1)^{|A|-1} D_A \mu(\{x \in \mathbb{E} :\ x_j > z_j,\ j \in A;\ x_j = 0,\ j \notin A\}),$$

where $A \subset \{1, \ldots, d\}$, $|A|$ is the cardinality of A, and D_A denotes the mixed partial derivative with respect to the coordinates with numbers from A.

The derivatives of ν can be expressed by means of the derivatives of the stable tail dependence function $l(z) = \nu(z^*) = \|z\|_K$. Indeed,

$$D_A \nu(z) = D_A l(z^*)(-1)^{|A|} \prod_{j \in A} z_j^{-2} .$$

Thus, the densities of the exponent measure can be found from

$$D_A\mu(\{x \in \mathbb{E}:\ x_j \le z_j,\ j \in A;\ x_j = 0,\ j \notin A\})$$
$$= (-1)^{|A|-1} \lim_{z_j \to 0,\ j \notin A} D_A l(z^*) \prod_{j \in A} z_j^{-2}\,.$$

In particular, the density of μ in the interior of $\mathbb{E}$ can be found from the dth mixed partial derivative of the norm as

$$f(z) = (-1)^{d-1} \frac{\partial^d l}{\partial z_1 \cdots \partial z_d}(z^*) \prod_{i=1}^{d} z_i^{-2}\,, \qquad z \in (0,\infty)^d\,.$$

After decomposing these densities into the radial and directional parts, it is possible to obtain the spectral measure by

$$\sigma(G) = \mu(\{tu:\ u \in G,\ t \ge 1\}) = \int_{\{tu:\, u\in G,\, t\ge 1\}} f(z)dz$$

for every measurable G from the relative interior of $\mathbb{S}_+$. The relationship between spectral measures on two different reference spheres is given in Beirlant et al. (2004, p. 264).

Proposition 3 *A d-times continuously differentiable function $l(x)$, $x \in E$, is the tail dependency function of a simple max-stable distribution if and only if l is sublinear, takes value 1 on all basis vectors, and all its mixed derivatives of even orders are non-positive and of odd orders are non-negative.*

Proof The necessity follows from Theorem 1 and the non-negativity condition on the exponent measure μ. In the other direction, the sublinearity property implies that l is the support function of a certain convex set K, see Schneider (1993, Theorem 1.7.1). The condition on the sign of mixed derivatives yields that the corresponding densities of μ are non-negative, i.e. K is the max-zonoid corresponding to a certain spectral measure. □

In the planar case, Schneider (1993, Theorem 1.7.2) implies that the second mixed derivative of the (smooth) support function is always non-positive. Accordingly, all smooth planar convex sets satisfying Eq. 10 are dependency sets.

A number of interesting measures on the unit sphere appear as curvature measures of convex sets (Schneider 1993, Section 4.2). A complete interpretation of these curvature measures is possible in the planar case, where the curvature measure becomes the length measure. The *length measure* $S_1(L, A)$ generated by a smooth set L associates with every measurable $A \subset \mathbb{S}^1$ the 1-dimensional Hausdorff measure of the boundary of L with unit normals from A. The length measure for a general L is defined by approximation. Recall that $\mathbb{S}^1_+$ is the part of the unit circle lying in the first quadrant.

Theorem 4 *A measure σ on $\mathbb{S}^1_+$ is the spectral measure of a simple max-stable random vector ξ with dependency set K if and only if σ is the restriction on $\mathbb{S}^1_+$ of the length measure generated by $\check{K} = \{(x_1, x_2) : (x_2, x_1) \in K\}$ with K satisfying Eq. 10.*

Proof Sufficiency. Consider a planar convex set K satisfying Eq. 10. Let $\sigma(da)$ be the length measure of $L = \check{K}$, i.e. the first-order curvature measure $S_1(L, da)$. Then

$$\int_{\mathbb{S}^1} h(\Delta_a, x)\sigma(da) = \int_{\mathbb{S}^1} h(\Delta_x, a) S_1(L, da) = 2V(\Delta_x, L)\,,$$

where $V(\Delta_x, L)$ denotes the mixed volume (the mixed area in the planar case) of the sets Δ_x and L, i.e.

$$2V(\Delta_x, L) = V_2(L + \Delta_x) - V_2(L) - V_2(\Delta_x)\,,$$

see Schneider (1993, Section 5.1). Because of Eq. 10, the integral over the full circle $\mathbb{S}^1$ with respect to σ coincides with the integral over $\mathbb{S}^1 \cap [0, \infty)^2 = \mathbb{S}^1_+$. It remains to show that $2V(\Delta_x, L)$ equals $h(K, x)$. If $z = (z_1, z_2) \in L$ is any support point of L in direction $x = (x_1, x_2)$, i.e. $h(L, x) = \langle z, x\rangle$, then

$$2V(\Delta_x, L) = z_1 x_2 + z_2 x_1 = h(K, x)\,.$$

An alternative proof follows the construction from Theorem 2. Indeed, a polygonal K can be obtained as the sum of triangles. A triangle $\Delta_{(t,s)}$ with vertices $(0, 0)$, $(t, 0)$ and $(0, s)$ corresponds to the spectral measure having the atom at $(t, s)c^{-1}$ with mass $c = \sqrt{t^2 + s^2}$ and therefore coincides with the length measure of $\Delta_{(s,t)} = \check{\Delta}_{(t,s)}$ restricted onto $\mathbb{S}^1_+$. Since the spectral measure of K is the sum of spectral measures of these triangles, it can be alternatively represented as the sum of the length measures. A general K can be then approximated by polygons.

Necessity. Assume that a measure σ on $\mathbb{S}^1_+$ is the spectral measure of a simple max-stable law with dependency set K. If now σ' is chosen to be the length measure of $\check{K}$ restricted onto $\mathbb{S}^1_+$, then σ' generates the max-zonoid K. Finally, $\sigma = \sigma'$ by the uniqueness of the spectral measure. $\square$

Theorem 4 implies that the length of the boundary of K inside $(0, \infty)^2$ equals the total mass of the spectral measure on $\mathbb{S}^1_+$. Given Eq. 10, an obvious bound on this boundary length implies that this total mass lies between $\sqrt{2}$ and 2.

The total mass of the spectral measure on the reference *simplex* has a simple geometric interpretation. Assume that the reference norm is ℓ_1, i.e. $\|x\| = x_1 + \cdots + x_d$ for $x \in \mathbb{E}$. If $\eta \in \mathbb{S}_+$, then $\mathbf{E}\,\eta_1 + \cdots + \mathbf{E}\,\eta_d = 1$, so that the ℓ_1-norm of $\mathbf{E}\,\eta$ is 1. Since $c\,\mathbf{E}\,\eta = (1, \ldots, 1)$ in Theorem 1, we have $c = d$, i.e. the spectral measure has the total mass d.

The weak convergence of simple max-stable random vectors can be interpreted as convergence of the corresponding max-zonoids.

Theorem 5 *Let $\xi, \xi_1, \xi_2, \dots$ be a sequence of simple max-stable random vectors with spectral measures $\sigma, \sigma_1, \sigma_2, \dots$ and dependency sets $K, K_1, K_2, \dots$. Then the following statements are equivalent.*

(i) *ξ_n converges in distribution to ξ;*
(ii) *σ_n converges weakly to σ;*
(iii) *K_n converges in the Hausdorff metric to K.*

Proof The equivalence of (i) and (ii) is well known, see de Haan and Ferreira (2006, Corollary 6.1.15).

The weak convergence of σ_n, the continuity of $h(\Delta_a, x)$ for $a \in \mathbb{S}_+$ and Eq. 6 imply that the support function of K_n converges pointwisely to the support function of K. Because dependency sets are contained inside the unit cube and so are uniformly bounded, their convergence in the Hausdorff metric is equivalent to the pointwise convergence of their support functions.

The Hausdorff convergence of K_n to K implies the pointwise convergence of their support functions and so the pointwise convergence of the cumulative distribution functions given by Eq. 8. The latter entails that ξ_n converges in distribution to ξ. □

A random vector $\zeta \in \mathbb{E}$ belongs to the *domain of attraction* of a simple max-stable distribution if and only if the measure

$$\sigma_s(A) = s\mathbf{P}\{\frac{\zeta}{\|\zeta\|} \in A\,,\ \|\zeta\| \geq s\}\,, \quad A \subset \mathbb{S}_+\,, \tag{15}$$

converges weakly as $s \to \infty$ to a finite measure on $\mathbb{S}_+$, which then becomes the spectral measure of the limiting random vector, see Beirlant et al. (2004, (8.95)). The equivalence of (ii) and (iii) in Theorem 5 implies the following result.

Proposition 4 *A random vector $\zeta \in \mathbb{E}$ belongs to the domain of attraction of a simple max-stable random vector ξ with spectral measure σ if and only if the max-zonoids generated by σ_s from Eq. 15 converge in the Hausdorff metric as $s \to \infty$ to the max-zonoid generated by σ.*

5 Copulas and Association

The dependency structure of a distribution with fixed marginals can be explored using the copula function C defined on $\mathbb{I}^d = [0, 1]^d$ by the following equation

$$F(x) = F(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d))\,,$$

where $F_1, \dots, F_d$ are the marginals of F, see Nelsen (2006). In case of a simple max-stable distribution, we obtain

$$C(u_1, \dots, u_d) = \exp\{-\|(-\log u_1, \dots, -\log u_d)\|_K\}\,. \tag{16}$$

Theorem 6 *The function* (16) *is a copula function if and only if K is a max-zonoid.*

Proof The sufficiency is trivial, since the right-hand side of Eq. 16 can be used to construct a max-stable distribution. In the other direction, we can substitute into C the Φ_1-distribution functions, i.e. $u_i = e^{-1/x_i}$. This yields a multivariate cumulative distribution function given by $F(x) = e^{-\|x^*\|_K}$. The result then follows from Theorem 3. □

Note that Eq. 16 in the bivariate case appears in Falk (2006). A rich family of copulas consists of the *Archimedean copulas* that in the bivariate case satisfy $\varphi(C(x_1, x_2)) = \varphi(x_1) + \varphi(x_2)$ for a strictly decreasing continuous function φ and all $x_1, x_2 \in [0, 1]$, see Nelsen (1999, Ch. 4). Using Eq. 16, it is easy to see that in this case $\psi(\|(x_1, x_2)\|_K) = \psi(x_1) + \psi(x_2)$ for a monotone increasing continuous function ψ and all $x_1, x_2 \geq 0$. It is known (see Nelsen 2006, Theorem 4.5.2; Genest and Rivest 1989) that all Archimedean copulas that correspond to max-stable distributions are so-called Gumbel copulas, where $\psi(t) = t^p$. Thus, Eq. 16 is an Archimedean copula if and only if K is ℓ_p-ball with $p \in [1, \infty]$, see Example 2.

The bivariate copulas are closely related to several association concepts between random variables, see Nelsen (1991). The *Spearman correlation coefficient* is expressed as $\rho_S = 12J - 3$, where

$$\begin{aligned} J &= \int_0^1 \int_0^1 C(u_1, u_2) du_1 du_2 = \int_0^1 \int_0^1 e^{-\|(-\log u_1, -\log u_2)\|_K} du_1 du_2 \\ &= \int_0^\infty \int_0^\infty e^{-\|(x_1, x_2)\|_K} e^{-x_1 - x_2} dx_1 dx_2 \\ &= \frac{1}{4} \int_{(0,\infty)^2} e^{-\|x\|_L} dx\,, \end{aligned}$$

and

$$L = \frac{1}{2}\left(K + \mathbb{I}^2\right).$$

It is possible to calculate J by changing variables $x = r(t, 1-t)$ with $r \geq 0$ and $t \in [0, 1]$, which leads to the following known expression

$$J = \frac{1}{4} \int_0^1 \frac{1}{\|(t, 1-t)\|_L^2} dt = \int_0^1 \frac{1}{(1 + \|(t, 1-t)\|_K)^2} dt\,,$$

see Hürlimann (2003). The following proposition is useful to provide another geometric interpretation of ρ_S and also an alternative way to compute J.

Proposition 5 *If L is a convex set in $\mathbb{R}^d$, then*

$$\int_{[0,\infty)^d} e^{-h(L,x)} dx = \Gamma(d+1) V_d(L^o)\,,$$

where $V_d(\cdot)$ is the d-dimensional Lebesgue measure, L^o is the polar set to L and Γ is the Gamma function.

Proof The proof follows the argument mentioned in Vitale (1996, p. 2173). Let ζ be the exponentially distributed random variable of mean 1. Then

$$\begin{aligned}\int_{[0,\infty)^d} e^{-h(L,x)}dx &= \mathbf{E}\int_{[0,\infty)^d} \mathbb{I}_{\zeta \geq h(L,x)}\,dx\\ &= \mathbf{E}\,V_d(\{x\in\mathbb{E}:\ h(L,x)\leq\zeta\})\\ &= \mathbf{E}\,V_d(\zeta L^o) = V_d(L^o)\,\mathbf{E}\,\zeta^d\,.\end{aligned}$$

It remains to note that $\mathbf{E}\,\zeta^d = \Gamma(d+1)$. □

Thus, in the planar case

$$\rho_S = 3(2V_2(L^o)-1)\,.$$

As a multivariate extension, an affine function $\rho_S = c(V_d(L^o)-a)$ of the d-dimensional volume of L^o may be used to define the Spearman correlation coefficient for a d-dimensional max-stable random vector with unit Fréchet marginals. By considering the independent case $L = K = \mathbb{I}^d$ with $\rho_S = 0$ and using the fact that the volume of the unit cross-polytope L^o is $(d!)^{-1}$, we see that $\rho_S = c(d!V_d(L^o)-1)$ for some constant $c>0$. The choice $c = (d+1)/(2^d-d-1)$ ensures that $\rho_S = 1$ in the totally dependent case, where $V_d(L^o) = 2^d/(d+1)!$.

The *Kendall correlation coefficient* of a bivariate copula C is given by

$$\begin{aligned}\tau &= 4\int_0^1\int_0^1 C(z_1,z_2)dC(z_1,z_2)-1\\ &= 1-4\int_0^1\int_0^1 \frac{\partial}{\partial z_1}C(z_1,z_2)\frac{\partial}{\partial z_2}C(z_1,z_2)dz_1dz_2\,,\end{aligned}$$

see Nelsen (1991, (2.3)). By Eq. 16, the partial derivatives of C can be expressed using partial derivatives of the support function of K. The directional derivative of the support function $h(K,x)$ at point x in direction u is given by $h(F(K,x),u)$, where

$$F(K,x) = \{y\in K:\ \langle y,x\rangle = h(K,x)\}$$

is the *support set* of K in direction x, see Schneider (1993, Theorem 1.7.2). Thus the partial derivatives of $h(K,x)$ are given by

$$\frac{\partial h(K,x)}{\partial x_i} = h(F(K,x),(1,0)) = y_i(K,x)\,,\quad i=1,2\,,$$

where $y_1(K,x)$ and $y_2(K,x)$ are respectively the maximum first and second coordinates of the points from $F(K,x)$. If the dependency set K is strictly convex in $(0,\infty)^2$, i.e. the boundary of K inside $(0,\infty)^2$ does not contain any segment, then $F(K,x) = \{(y_1(K,x), y_2(K,x))\}$ is a singleton for all $x \in \mathbb{E}$. In this case denote

$$y(K,x) = y_1(K,x)y_2(K,x)\,.$$

By using Eq. 16 and changing variables we arrive at

$$\tau = 1 - 4\int_{[0,\infty)^2} e^{-2\|x\|_K} y(K,x)dx\,.$$

The fact that $y(K,tx) = y(K,x)$ and a similar argument to Proposition 5 yield that

$$\tau = 1 - 2\int_{K^o} y(K,x)dx\,. \tag{17}$$

For instance, $\tau = 1/2$ if ξ has the logistic distribution with parameter 1/2, i.e. $\|\cdot\|_K$ is the Euclidean norm. By changing variables $x = (t, 1-t)r$, we arrive at

$$\tau = 1 - \int_0^1 \frac{y_1(K,(t,1-t))y_2(K,(t,1-t))}{\|(t,1-t)\|_K^2}dt\,,$$

which also corresponds to Hürlimann (2003, Theorem 3.1).

The *Pearson correlation coefficient* for the components of a bivariate simple max-stable random vector is not defined, since the unit Fréchet marginals are not integrable. However it is possible to compute it for the inverted coordinates of ξ.

Proposition 6 *If ξ is a simple max-stable bivariate random vector, then $\mathbf{E}(\xi_1^{-1}\xi_2^{-1}) = 2V_2(K^o)$, and the covariance between ξ_1^{-1} and ξ_2^{-1} is $2V_2(K^o) - 1$.*

Proof Integrating by parts, it is easy to see that

$$\mathbf{E}\left(\xi_1^{-1}\xi_2^{-1}\right) = \int_0^\infty \int_0^\infty F(x^*)dx_1dx_2 = \int_{\mathbb{E}} e^{-h(K,x)}dx\,.$$

The result follows from Proposition 5 and the fact that $\mathbf{E}(\xi_1^{-1}) = \mathbf{E}(\xi_2^{-1}) = 1$. □

Proposition 6 corresponds to the formula

$$\rho = \int_0^1 \frac{1}{\|(t,1-t)\|_K^2}dt - 1\,.$$

for the covariance obtained in Tawn (1988) for the exponential marginals.

Extending this concept for the higher-dimensional case, we see that the covariance matrix of ξ^* is determined by the areas of polar sets to the 2-dimensional projections of K and

$$\rho = \frac{d!V_d(K^o) - 1}{d! - 1}$$

can be used to characterise the multivariate dependency of a simple d-dimensional max-stable random vector ξ, so that ρ varies between zero (complete independence) and 1 (complete dependence).

Example 5 Assume that $\|x\|_K = \|x\|_p$ is the ℓ_p-norm with $p \geq 1$, i.e. the corresponding ξ has the logistic distribution with parameter $\alpha = 1/p$, see Example 2. The volume of the ℓ_p-ball $\{x \in \mathbb{R}^d : \|x\|_p \leq 1\}$ equals

$$v_d(p) = \frac{(2\Gamma(1+1/p))^d}{\Gamma(1+d/p)},$$

see Pisier (1989, p. 11). Thus, the volume of K^o is $2^{-d} v_d(p)$ and the multivariate dependency of ξ can be described by

$$\rho = \frac{1}{d!-1}\left(d!\frac{(\Gamma(1+1/p))^d}{\Gamma(1+d/p)} - 1\right).$$

If $d = 2$, then $\rho = \alpha B(\alpha, \alpha) - 1$ with B being the Beta-function.

The *tail dependency index* for $\xi = (\xi_1, \xi_2)$ with identical marginal distributions supported by the whole positive half-line is defined as

$$\chi = \lim_{t\to\infty} \mathbf{P}\{\xi_2 > t | \xi_1 > t\}.$$

An easy argument shows that $\chi = 2 - \|(1,1)\|_K$ if ξ has a simple max-stable distribution with dependency set K, cf Coles et al. (1999).

It is easy to see that ξ has all independent coordinates if and only if $\|(1,\ldots,1)\|_K = d$ and the completely dependent coordinates if and only if $\|(1,\ldots,1)\|_K = 1$, cf Takahashi (1994) and Beirlant et al. (2004, p. 266). It is well known (Beirlant et al. 2004, p. 266) that the pairwise independence of the coordinates of ξ implies the joint independence. Indeed, the spectral measure of the set $u \in \mathbb{S}_+$ such that at least two coordinates of u are positive is less than the sum of $\sigma\{u \in \mathbb{S}_+ : u_i > 0, u_j > 0\}$ over all $i \neq j$. Each of these summands vanishes, since

$$x_i + x_j - \int_{\mathbb{S}_+} \max_{1\leq k\leq d}(u_k x_k)\sigma(du) = \int_{\mathbb{S}_+} ((u_i x_i + u_j x_j) - (u_i x_i \vee u_j x_j))\sigma(du) = 0$$

by the pairwise independence, where x has all vanishing coordinates apart from x_i and x_j. This leads to the following property of max-zonoids.

Proposition 7 *If K is a max-zonoid with all its two-dimensional projections being unit squares, then K is necessarily the unit cube.*

6 Complete Alternation and Extremal Coefficients

Consider a numerical function f defined on a semigroup S with a commutative binary operation $+$. For $n \geq 1$ and $x_1, \ldots, x_n \in S$ define the following successive differences

$$\Delta_{x_1} f(x) = f(x) - f(x + x_1),$$
$$\cdots$$
$$\Delta_{x_n} \cdots \Delta_{x_1} f(x) = \Delta_{x_{n-1}} \cdots \Delta_{x_1} f(x) - \Delta_{x_{n-1}} \cdots \Delta_{x_1} f(x + x_n).$$

The function f is said to be *completely alternating* (resp. *monotone*) if all these successive difference are non-positive (resp. non-negative), see Berg et al. (1984, Section 4.6) and Molchanov (2005, Section I.1.2). We will use these definitions in the following cases: S is the family of closed subsets of $\mathbb{R}^d$ with the union operation, S is $\mathbb{R}^d$ or $\mathbb{E}$ with coordinatewise minimum or coordinatewise maximum operation. Then we say shortly that the function is max-completely alternating or monotone (resp. min-completely or union-completely).

Every cumulative distribution function F is min-completely monotone. This is easily seen by considering the random set $X = \{\xi\}$ where ξ has the distribution F, then noticing that $F(x) = \mathbf{P}\{X \cap L_x = \emptyset\} = Q(L_x)$ with L_x being the complement to $x + (-\infty, 0)^d$ is the avoiding functional of X, and finally using the fact that

$$F(\min(x, y)) = \mathbf{P}\{X \cap (L_x \cup L_y) = \emptyset\} = Q(L_x \cup L_y)$$

together with the union-complete monotonicity of Q, see Molchanov (2005, Section 1.6).

Theorem 7 *A convex set $K \subset \mathbb{E}$ is a max-zonoid if and only if $h(K, x)$ is a max-completely alternating function of x.*

Proof It follows from Berg et al. (1984, Proposition 4.6.10) that a function $f(x)$ on a general semigroup is completely alternating if and only if $F(x) = e^{-tf(x)}$ is completely monotone for all $t > 0$. Since $(\min(x, y))^* = \max(x^*, y^*)$ for $x, y \in \mathbb{E}$, the function $x \mapsto f(x^*)$ is max-completely alternating on $\mathbb{E}$ if and only if $x \mapsto f(x)$ is min-completely alternating.

If K is the max-zonoid corresponding to a simple max-stable random vector with distribution function $F(x) = e^{-h(K,x^*)}$, then F^t is also a cumulative distribution function (and so is min-completely monotone) for each $t > 0$. Thus, $h(K, x^*)$ is min-completely alternating, whence $h(K, x)$ is max-completely alternating.

In the other direction, if h is max-completely alternating, then $F(x) = e^{-h(K,x^*)}$ is min-completely monotone, whence it is a cumulative distribution function. The corresponding law is necessarily semi-simple max-stable, so that K is indeed a max-zonoid. □

Theorem 7 can be compared with a similar characterisation of classical zonoids, where the complete alternation of $h(K, x)$ and monotonicity of $e^{-h(K,x)}$ are understood with respect to the vector addition on $\mathbb{R}^d$, see Schneider (1993, p. 194).

The remainder of this section concerns extensions for the support function defined on a finite subset of $\mathbb{E}$.

Theorem 8 *Let M be a finite set in $\mathbb{E}$, which is closed with respect to coordinatewise maxima, i.e. $u \vee v \in M$ for all $u, v \in M$. Assume that for each $u, v \in M$, we have $tu \le v$ if and only if $u \le v$ and $t \le 1$. Then a non-negative function h on M can be extended to the support function of a max-zonoid if and only if h is max-completely alternating on M.*

Proof The necessity trivially follows from Theorem 7. To prove the sufficiency we explicitly construct (following the ideas of Schlather and Tawn (2002)) a max-stable random vector ξ such that the corresponding norm coincides with the values of h on the points from M.

For any set $A \subset M$, let $\vee A$ denote the coordinatewise maximum of A. Furthermore, define $T(A) = h(\vee A)/h(\vee M)$. Since h is max-completely alternating, T is union-completely alternating on subsets of M. The Choquet theorem (Molchanov 2005, Theorem I.1.13) implies that a union-completely alternating function on a discrete set is the capacity functional $T(A) = \mathbf{P}\{X \cap A \neq \emptyset\}$ of a random closed set $X \subset M$. Define $c_u = h(\vee M)\mathbf{P}\{\vee X = u\}$ for $u \in M$.

Let ζ_u, $u \in M$, be the family of i.i.d. unit Fréchet random variables which are also chosen to be independent of X and let ξ be the coordinatewise maximum of $c_u u \zeta_u$ over all $u \in M$. It remains to show that ξ has the required distribution. Consider an arbitrary point $v \in M$. By the condition on M, $tu \le v$ is possible for some $t > 0$ if and only if $u \le v$ and $t \le 1$. Thus,

$$\mathbf{P}\{\xi \le v\} = \prod_{u\in M} \mathbf{P}\{c_u \zeta_u u \le v\} = \prod_{u\in M,\, u\le v} \mathbf{P}\{c_u \zeta_u \le 1\} = \exp\left\{-\sum_{u\in M,\, u\le v} c_u\right\}$$

$$= \exp\{-h(\vee M)\mathbf{P}\{X \cap \{u :\ u \le v\} \neq \emptyset\}\}$$

$$= \exp\{-h(\vee M)T(\{u :\ u \le v\})\} = \exp\{-h(v)\}\ . \qquad \square$$

A simple example of set M from Theorem 8 is the smallest set which contains all basis vectors in $\mathbb{R}^d$ and is closed with respect to coordinatewise maxima. Then M consists of the vertices of the unit cube $\mathbb{I}^d$ without the origin and the values of h on M become the extremal coefficients. The *extremal coefficients* θ_A of a simple max-stable random vector $\xi = (\xi_1, \dots, \xi_d)$ are defined from the equations

$$\mathbf{P}\Big\{\max_{j\in A} \xi_j \le z\Big\} = (\mathbf{P}\{\xi_1 \le z\})^{\theta_A}\,, \quad z > 0\,,\ A \subset \{1, \dots, d\}\,, \tag{18}$$

see Schlather and Tawn (2002, 2003). Since the marginals are unit Fréchet, it suffices to use Eq. 18 for $z = 1$ only. If $e_A = \sum_{i \in A} e_i$, then Eq. 14 implies

$$\theta_A = h(K, e_A) = \|e_A\|_K \, .$$

Every nonempty set $A \subset \{1, \ldots, d\}$ can be associated with the unique vertex of the unit cube $\mathbb{I}^d \setminus \{0\}$. The consistency condition for the extremal coefficients follows directly from Theorem 8 and can be formulated as follows.

Corollary 1 *A family of non-negative numbers θ_A, $A \subset \{1, \ldots, d\}$, is a set of extremal coefficients for a simple max-stable distribution if and only if $\theta_\emptyset = 0$ and θ_A is a union-completely alternating function of A.*

This consistency result for the extremal coefficients has been formulated in Schlather and Tawn (2002) as a set of inequalities that, in fact, mean the complete alternation property of θ_A.

7 Operations with Dependency Sets

Rescaling For a dependency set K and $\lambda_1, \ldots, \lambda_d > 0$ define

$$\lambda K = \{(\lambda_1 x_1, \ldots, \lambda_d x_d) : \ x = (x_1, \ldots, x_d) \in K\} \, . \tag{19}$$

Then $e^{-h(\lambda K, x^*)}$ is the cumulative distribution function of $\lambda^* \xi = (\xi_1/\lambda_1, \ldots, \xi_d/\lambda_d)$.

Projection If ξ' denotes the vector composed from the first k-coordinates of d-dimensional vector ξ with the dependency set K, then

$$\begin{aligned} \mathbf{P}\{\xi' \le (x_1, \ldots, x_k)\} &= \exp\{-\|(x_1, \ldots, x_k, \infty, \ldots, \infty)^*\|_K\} \\ &= \exp\{-\|(x_1, \ldots, x_k)^*\|_{K'}\} \, , \end{aligned}$$

where K' is the projection of K onto the subspace spanned by the first k coordinates in $\mathbb{R}^d$. Thus, taking a sub-vector of ξ corresponds to projecting of K onto the corresponding coordinate subset. Recall Proposition 7 which says that if all two-dimensional projections of K are squares, then K is necessarily the cube.

Proposition 8 *If $\mathbb{L}$ is the subspace spanned by some coordinate axes in $\mathbb{R}^d$, the projection of K onto $\mathbb{L}$ coincides with $K \cap \mathbb{L}$.*

Proof By definition, $K = c\,\mathbf{E}\,\Delta_\eta$. Then it suffices to note that the projection of Δ_η on $\mathbb{L}$ equals $\Delta_\eta \cap \mathbb{L}$. Indeed every selection of $\Delta_\eta \cap \mathbb{L}$ can be associated with the projection of a selection of Δ_η. □

An interesting open question concerns a reconstruction of K from its lower-dimensional projections. In various forms this question was discussed in Kotz and Nadarajah (2000, Section 3.5.6) and Joe (1997, Section 4.7).

Cartesian product If K' and K'' are two dependency sets of simple max-stable random vectors ξ' and ξ'' with dimensions d' and d'' respectively, then the Cartesian product $K' \times K''$ is the dependency set corresponding to the max-stable random vector ξ obtained by concatenating of independent copies of ξ' and ξ''. Indeed, if $x = (x', x'')$, then

$$\begin{aligned}\mathbf{P}\{\xi \le x\} &= \exp\left\{-h\left(K' \times K'', x\right)\right\} = \exp\left\{-h\left(K', x'\right) - h\left(K'', x''\right)\right\} \\ &= \mathbf{P}\left\{\xi' \le x'\right\}\mathbf{P}\left\{\xi'' \le x''\right\}.\end{aligned}$$

Minkowski sum If K' and K'' are dependency sets of two independent max-stable random vectors ξ' and ξ'' of dimension d, then the weighted Minkowski sum $K = \lambda K' + (1-\lambda)K''$ with $\lambda \in [0,1]$ is the dependency set of the max-stable random vector

$$\xi = \left(\lambda\xi'\right) \vee \left((1-\lambda)\xi''\right). \tag{20}$$

The cumulative distribution functions of ξ', ξ'' and ξ are related as

$$F_\xi(x) = F_{\xi'}(x)^\lambda F_{\xi''}(x)^{(1-\lambda)}.$$

It is possible to generalise the Minkowski summation scheme for multivariate weights. Consider $K = \lambda K' + (1-\lambda)K''$ for some $\lambda \in [0,1]^d$, where the products of vectors and sets are defined in Eq. 19. Then $\|x\|_K = \|\lambda^* x\|_{K'} + \|(1-\lambda)^* x\|_{K''}$, so that Eq. 20 also holds with the products defined coordinatewisely.

Example 6 If ξ_1 and ξ_2 are independent with unit Fréchet distributions and $\alpha_1, \alpha_2 \in [0,1]$, then setting $\lambda = (\alpha_1, 1-\alpha_2)$ we obtain the max-stable random vector

$$\xi = (\alpha_1\xi_1 \vee (1-\alpha_1)\xi_2, (1-\alpha_2)\xi_1 \vee \alpha_2\xi_2)$$

with the dependency set $K = \mathrm{conv}\{(0,0), (0,1), (1,0), (\alpha_1, 1), (1, \alpha_2)\}$. If $\alpha_1 = \alpha_2 = \alpha$, then ξ has the Marshall-Olkin distribution, cf Falk et al. (2004, Example 4.1.1).

Example 7 (Matrix weights) Let a_{ij}, $i = 1, \dots, m$, $j = 1, \dots, d$, be a matrix of positive numbers such that $\sum_{i=1}^m a_{ij} = 1$ for all j. Furthermore, let $\zeta_1, \dots, \zeta_m$ be i.i.d. random variables with Φ_1-distribution. Define $\xi = (\xi_1, \dots, \xi_d)$ by

$$\xi_j = \max_{1 \le i \le m} \zeta_i a_{ij}, \quad j = 1, \dots, d,$$

cf Falk et al. (2004, Lemma 4.1.2). Then ξ is simple max-stable with the corresponding norm

$$\|x\|_K = \sum_{i=1}^{m} \max_{1\le j\le d} a_{ij}x_j\,,$$

i.e. its dependency set is $K = \Delta_{(a_{11},\dots,a_{1d})} + \cdots + \Delta_{(a_{m1},\dots,a_{md})}$.

Power sums A *power-mean* of two convex compact sets K' and K'' containing the origin in their interior is defined to be a convex set K such that

$$h(K,x)^p = \lambda h(K',x)^p + (1-\lambda)h(K,x)^p\,, \tag{21}$$

where $\lambda \in [0,1]$ and $p \ge 1$, see Firey (1967). The power-mean definition is applicable also if K' and K'' satisfy Eq. 10, despite the fact that the origin is not their interior point. In the plane, the power sum is a dependency set if K' and K'' satisfy Eq. 10. Therefore, the power sum of dependency sets leads to a new operation with distributions of *bivariate* max-stable random vectors. For instance, if K' is the unit cross-polytope and K'' is the unit square, then, for $p = 2$,

$$\|x\|_K = ((x_1+x_2)^2 + (\max(x_1,x_2))^2)^{1/2}\,.$$

Minkowski difference Let K' and K'' be two dependency sets. For any $\lambda > 0$ define

$$L = K' - \lambda K'' = \{x:\ x + \lambda K'' \subset K'\}\,.$$

If the spectral measures σ' and σ'' of K' and K'' are such that $\sigma = \sigma' - \lambda\sigma''$ is a non-negative measure, then L is a max-zonoid regardless of the dimension of the space. The negative logistic distribution from Example 4 illustrates this construction.

Convex hull and intersection In the space of a dimension $d \ge 3$ the convex hull or intersection of dependency sets do not necessarily remain dependency sets. However, on the plane this is always the case.

Let K' and K'' be the dependency sets of bivariate simple max-stable random vectors ξ' and ξ''. Since $h(\mathrm{conv}(K'\cup K''),x) = h(K',x)\vee h(K'',x)$, the dependency set $K = \mathrm{conv}(K'\cup K'')$ corresponds to a max-stable random vector ξ such that

$$\mathbf{P}\{\xi \le x\} = \min(\mathbf{P}\{\xi' \le x\}, \mathbf{P}\{\xi'' \le x\})\,,\quad x \in [0,\infty)^2\,.$$

The intersection of two planar dependency sets also remains the dependency set and so yields another new operation with distributions of simple max-stable bivariate random vectors.

Duality If the polar to the dependency set K of ξ is a max-zonoid, then the corresponding simple max-stable random vector ξ^o is said to be the dual to ξ. In the plane, the polar to a max-zonoid is max-zonoid; it is not known when it holds in higher dimensions. This duality operation is a new operation with distributions of bivariate max-stable random vectors, see also Example 1.

8 Infinite Dimensional Case

It is possible to define the dependency set for max-stable stochastic processes studied in de Haan (1984), Falk et al. (2004), Giné et al. (1990). The spectral representation (Giné et al. 1990, Proposition 3.2) of a sample continuous max-stable process $\xi(t)$, $t \in S$, on a compact metric space S with unit Fréchet marginals yields that

$$-\log \mathbf{P}\{\xi < f\} = \int_{\mathbb{S}_+} \|g/f\|_\infty d\sigma(g)\,,$$

where $\mathbb{S}_+$ is the family of non-negative continuous functions g on S that their maximum value $\|g\|_\infty$ equals 1, and σ is a finite Borel measure on $\mathbb{S}_+$ such that $\int_{\mathbb{S}_+} g d\sigma(g)$ is the function identically equal to 1.

The corresponding dependency set is the set in the space of finite measures with the total variation distance, which is the dual space to the family of non-negative continuous functions. For a continuous function g, define $\varDelta_g$ to be the closed convex hull of the family of atomic measures $g(x)\delta_x$ for $x \in S$. Then the dependency set is the expectation of $c\varDelta_\eta$, where $c = \sigma(\mathbb{S}_+)$ and η is distributed according to the normalised σ.

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