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CONVEX HULLS OF SOME CLASSICAL FAMILIES OF UNIVALENT FUNCTIONS[1]

BY
L. BRICKMAN, T. H. MACGREGOR AND D. R. WILKEN

Abstract. Let S denote the functions that are analytic and univalent in the open unit disk and satisfy $f(0)=0$ and $f'(0)=1$. Also, let K, St, S_R, and C be the subfamilies of S consisting of convex, starlike, real, and close-to-convex mappings, respectively. The closed convex hull of each of these four families is determined as well as the extreme points for each. Moreover, integral formulas are obtained for each hull in terms of the probability measures over suitable sets. The extreme points for each family are particularly simple; for example, the Koebe functions $f(z)=z/(1-xz)^2$, $|x|=1$, are the extreme points of cl co St. These results are applied to discuss linear extremal problems over each of the four families. A typical result is the following: Let J be a "nontrivial" continuous linear functional on the functions analytic in the unit disk. The only functions in St that satisfy $\operatorname{Re} J(f)=\max\{\operatorname{Re} J(g) : g \in St\}$ are Koebe functions and there are only a finite number of them.

Introduction. We shall be concerned with the closed convex hulls of various families of functions that are analytic and univalent in the open unit disk $\Delta=\{z \in C : |z|<1\}$. For each family considered we obtain integral representations for the closed convex hull, and we determine all the extreme points. In each case the extreme points are strikingly simple and familiar functions. Thus we obtain a powerful tool for solving linear extremum problems over such families.

Let us establish some notation and outline our main results. We shall let A denote the set of all functions analytic in Δ. With the natural topology of uniform convergence on compact subsets of Δ, A is a locally convex linear topological space [15, p. 150]. Let S be the subset of A consisting of functions f that are univalent (one-to-one) in Δ and satisfy $f(0)=0, f'(0)=1$. It is well known [7, p. 217] that S is compact in A, or, equivalently, that S is closed and locally uniformly bounded. (On each compact subset of Δ there is a common bound for all the functions in S.) We shall be particularly interested in the following subfamilies of S.

$K=\{f\in S : f(\Delta) \text{ is convex}\}$,

$St=\{f\in S : f(\Delta) \text{ is starlike with respect to } 0\}$,

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$S_R=\{f\in S : f \text{ is real on } (-1, 1)\}$,

$C=\{f\in S : f \text{ is "close-to-convex"}\}$.

(A function f is close-to-convex if $\operatorname{Re}\{zf'(z)/g(z)\}>0$, $z\in\Delta$, for some function g that is analytic and univalent in Δ, and such that $g(\Delta)$ is starlike with respect to 0. For any such g it is clear that $g(0)=0$ and $\operatorname{Re} g'(0)>0$. Functions in the class C can be characterized by a geometric mapping property [4].) Since these four families are locally uniformly bounded (in fact, they are compact) so are their convex hulls. (The reader is referred to [2, Chapter 5] for the definition of convex hull and other related terms as well as for some results we shall use.) Hence the closure of these hulls, denoted cl co K, cl co St, cl co S_R, cl co C, are compact. Therefore we know a priori [2, p. 440] that the extreme points of cl co K, cl co St, cl co S_R, cl co C belong to K, St, S_R, C, respectively; in symbols $\mathscr{E}(\text{cl co } K)\subset K$, $\mathscr{E}(\text{cl co } St)\subset St$, $\mathscr{E}(\text{cl co } S_R)\subset S_R$, $\mathscr{E}(\text{cl co } C)\subset C$. (An extreme point of a set F is a point of F that cannot be written as a proper convex combination of two other points of F.) Indeed, we find that $\mathscr{E}(\text{cl co } K)$ is the set of Möbius functions $z/(1-xz)$, $|x|=1$; $\mathscr{E}(\text{cl co } St)$ is the set of Koebe functions $z/(1-xz)^2$, $|x|=1$; $\mathscr{E}(\text{cl co } S_R)$ is the set of functions $z/(1-2az+z^2)$, $-1\leqq a\leqq 1$; and $\mathscr{E}(\text{cl co } C)$ is the set of functions $[z-\frac{1}{2}(x+y)z^2]/[1-yz]^2$, $|x|=|y|=1$, $x\neq y$. (The functions in $\mathscr{E}(\text{cl co } C)$ are precisely those functions in S that map Δ onto the complement of a half-line.) Moreover we obtain the integral formulas

$$\int_{|x|=1}\frac{z}{1-xz}\,d\mu(x),\qquad \int_{|x|=1}\frac{z}{(1-xz)^2}\,d\mu(x),$$

$$\int_0^{\pi}\frac{z}{1-2z\cos t+z^2}\,d\mu(t),\qquad \int_{|x|=1}\int_{|y|=1}\frac{z-((x+y)/2)z^2}{(1-yz)^2}\,d\mu(x,y)$$

for the functions in cl co K, cl co St, cl co S_R, and cl co C, respectively. In each formula μ is a probability measure on the indicated set (that is, a positive measure of total mass 1). These formulas illustrate the Krein-Milman theorem for compact convex subsets of a locally convex linear topological space. Namely, each point of such a set is the limit of convex combinations of extreme points [2, p. 440]. We also note that the integral formula for cl co S_R is also the formula for functions belonging to T, the set of typically real functions, defined by $T=\{f\in A : f(0)=0, f'(0)=1$, $f(z)$ is real if and only if z is real$\}$ (see [12]). Thus cl co $S_R=T$. This and other related facts about S_R were noted independently by W. E. Kirwan and one of the present authors using a method different from that employed here. Various inclusion relations among the families considered in this paper are discussed in §4.

Perhaps the fundamental interest in the results outlined above arises from the fact that the maximum (or minimum) real part on K, St, S_R, or C of a continuous linear functional on A occurs on $\mathscr{E}(\text{cl co } K)$, $\mathscr{E}(\text{cl co } St)$, $\mathscr{E}(\text{cl co } S_R)$, or $\mathscr{E}(\text{cl co } C)$, respectively. For instance if J is a continuous linear functional on A, then

$$\max\{\operatorname{Re} J(f) : f\in St\} = \max\left\{\operatorname{Re} J\left(\frac{z}{(1-xz)^2}\right) : |x| = 1\right\},$$

a highly useful result. We note that this relation also follows from our integral formula for cl co *St*. We also obtain the further refinement that if J is "nontrivial", then $\operatorname{Re} J(f)$ is maximized over *St* only by finitely many Koebe functions. Analogous results are proved for K and S_R.

Our viewpoint in the paper, that of considering convex hulls and extreme points in connection with univalent functions, is rather new. (Consideration of extreme points is of course routine for convex sets such as analytic functions with positive real part and the unit balls of the H^p spaces.) We can mention only the following papers. In [13] some results are obtained concerning meromorphic, univalent functions and extreme points; a necessary condition on an extreme point of S is proved in [1]. The present authors intend to write a second paper in the same vein on related topics.

1. A general theorem on integral representations and applications to K, *St*, and S_R.

In this section we present a general theorem establishing the basic properties of integral representations of the type we shall encounter. Following this we determine the closed convex hulls of K, *St*, and S_R.

THEOREM 1. *Let Δ be the open unit disk in the complex plane $\mathbf{C}$, and let X be any compact Hausdorff space. Let $k\colon \Delta\times X\to \mathbf{C}$ have the following three properties*:

(1) *For each x in X the map $z\to k(z,x)$ is analytic in Δ.*

(2) *For each z in Δ the map $x\to k(z,x)$ is continuous on X.*

(3) *For each r, $0<r<1$, there exists $M_r>0$ such that $|k(z,x)|\leqq M_r$ for $|z|\leqq r$ and for x in X.*

Let $\mathscr{P}$ denote the set of probability measures on the Borel subsets of X. For μ in $\mathscr{P}$, let

$$f_\mu(z) = \int_X k(z,x)\,d\mu(x) \qquad (z\in\Delta). \tag{4}$$

Finally, let $\mathscr{F}=\{f_\mu : \mu\in\mathscr{P}\}$. Then

(a) *Each function in $\mathscr{F}$ is analytic in Δ.*

(b) *The map $\mu\to f_\mu$ is continuous (with the relative weak-star topology on $\mathscr{P}$, regarded as a subset of $C(X)^*$, and the topology of uniform convergence on compacta on $\mathscr{F}$).*

(c) *$\mathscr{F}$ is compact and is the closed convex hull of the set of functions $\{z\to k(z,x) : x\in X\}$.*

(d) *The only possible extreme points of $\mathscr{F}$ are the functions $z\to k(z,x)$, $x\in X$. If $x_0\in X$ and $k(z,x_0)=\int_X k(z,x)\,d\mu(x)$ $(z\in\Delta)$ holds only for $\mu=\delta_{x_0}$ (unit point mass at x_0), then the function $z\to k(z,x_0)$ is an extreme point of $\mathscr{F}$. In particular, if the map $\mu\to f_\mu$ is one-to-one, then each function $z\to k(z,x)$, $x\in X$, is an extreme point of $\mathscr{F}$.*

Proof. (a) Suppose that $\mu\in\mathscr{P}$. Then there is a net $(\mu_\alpha)\subset\mathscr{P}$ converging to μ such that each μ_α consists of a finite set of "point masses." Hence (2) implies

$$f_{\mu_\alpha}(z) = \int_X k(z,x)\,d\mu_\alpha(x) \to \int_X k(z,x)\,d\mu(x) = f_\mu(z)$$

for each z in Δ. By (1) each f_{μ_α} is analytic in Δ. Hence (a) will follow if the convergence $f_{\mu_\alpha}(z) \to f_\mu(z)$ is uniform on compact subsets of Δ. But by (3) and (4), (f_{μ_α}) is locally uniformly bounded on Δ. Hence (f_{μ_α}) has a convergent subnet. But from what was shown above it follows that any convergent subnet converges pointwise on Δ to f_μ. Thus (f_{μ_α}) has f_μ for its only cluster point, and hence $f_{\mu_\alpha} \to f_\mu$.

(b) Weak-star convergence of any net $(\mu_\alpha) \subset \mathscr{P}$ to μ in $\mathscr{P}$ implies pointwise convergence $f_{\mu_\alpha}(z) \to f_\mu(z)$, and this in turn implies that (f_{μ_α}) converges to f_μ in the topology of uniform convergence on compacta. The required arguments are as in (a) above.

(c) $\mathscr{F}$ is compact since $\mathscr{P}$ is compact and because of (b). As shown in the proof of (a), any f in $\mathscr{F}$ is the limit of a net of convex combinations of functions $z \to k(z, x)$. Hence $\mathscr{F} \subset \operatorname{cl} \operatorname{co} \{z \to k(z, x) : x \in X\}$. Conversely

$$\{z \to k(z, x) : x \in X\} \subset \mathscr{F},$$

and since $\mathscr{F}$ is convex and closed, $\operatorname{cl} \operatorname{co} \{z \to k(z, x) : x \in X\} \subset \mathscr{F}$.

(d) Let f be any extreme point of $\mathscr{F}$. As a result of (b), $\{\mu : \mu \in \mathscr{P}, f_\mu = f\}$ is compact, and therefore has an extreme point ν. But since f is extreme, ν must actually be an extreme point of $\mathscr{P}$. Hence ν is a unit point mass and $f(z) = f_\nu(z) = k(z, x)$ for some x in X. Finally, suppose that the function $z \to k(z, x_0)$ is not extreme. Then $k(z, x_0) = tf_\mu(z) + (1-t)f_\nu(z)$, where $0 < t < 1$, $\mu \in \mathscr{P}$, $\nu \in \mathscr{P}$, $\mu \neq \nu$. But $tf_\mu + (1-t)f_\nu = f_\lambda$, where $\lambda = t\mu + (1-t)\nu$. It follows that λ is not a unit point mass while $k(z, x_0) = f_\lambda(z) = \int_X k(z, x)\, d\lambda(x)$ as required.

REMARKS. 1. In all the applications of Theorem 1 throughout the paper (2) and (3) will hold by virtue of the continuity of k on $\Delta \times X$.

2. Probably the best known integral representation of the above type is the Herglotz formula for analytic functions having a positive real part in Δ. In this case it is known that the map $\mu \to f_\mu$ is one-to-one [8, p. 30], and we shall use this fact in Theorems 2 and 4 below.

THEOREM 2. *Let X be the unit circle $\{z : |z| = 1\}$, $\mathscr{P}$ the set of probability measures on X and $\mathscr{F}$ the set of functions f_μ on Δ defined by $f_\mu(z) = \int_X [z/(1-xz)]\, d\mu(x)$, $\mu \in \mathscr{P}$. Let K be the set of convex, univalent, normalized functions on Δ. Then $\mathscr{F} = \operatorname{cl} \operatorname{co} K$. Also the map $\mu \to f_\mu$ is one-to-one, and the extreme points of* cl co *K are precisely the functions $z \to z/(1-xz)$, $x \in X$.*

Proof. Since each function $z \to z/(1-xz)$, $x \in X$, belongs to K it is clear (Theorem 1, part (c)) that $\mathscr{F} \subset \operatorname{cl} \operatorname{co} K$. Next, let f be in K. It is known ([6], [14]) that $\operatorname{Re} \{f(z)/z\} > \frac{1}{2}$ for all z in Δ. Thus $2f(z)/z - 1$ has a positive real part for z in Δ. This function also has the value 1 at $z = 0$. Hence, according to Herglotz's formula, there exists μ in $\mathscr{P}$ such that $2f(z)/z - 1 = \int_X [(x+z)/(x-z)]\, d\mu(x)$. Therefore $f(z) = \int_X [z/(1-z/x)]\, d\mu(x)$, or replacing μ by $\bar{\mu}$ ($\bar{\mu}(E) = \mu(\bar{E})$ for $E \subset X$) we conclude that $f \in \mathscr{F}$. Therefore $K \subset \mathscr{F}$. Since $\mathscr{F}$ is convex and closed (Theorem 1, part (c)) cl co $K \subset \mathscr{F}$. Thus $\mathscr{F} = \text{cl co } K$ as stated. If $f_{\mu_1} = f_{\mu_2}$, then (for all z in Δ)

$$\int_X \frac{z}{1-z/x}\, d\mu_1(x) = \int_X \frac{z}{1-z/x}\, d\mu_2(x), \qquad \int_X \frac{x+z}{x-z}\, d\mu_1(x) = \int_X \frac{x+z}{x-z}\, d\mu_2(x),$$

and $\mu_1 = \mu_2$ by the one-to-one property of the Herglotz representation. The assertion concerning extreme points of cl co K follows from Theorem 1 (d).

THEOREM 3. *Let X be the unit circle $\{z : |z| = 1\}$, $\mathscr{P}$ be the set of probability measures on X, and $\mathscr{F}$ be the set of functions f_μ on Δ defined by*

$$f_\mu(z) = \int_X \frac{z}{(1-xz)^2}\, d\mu(x), \qquad \mu \in \mathscr{P}.$$

Let St be the set of starlike, univalent, normalized functions on Δ. Then $\mathscr{F} = \text{cl co } St$, the map $\mu \to f_\mu$ is one-to-one, and the extreme points of cl co St *are precisely the Koebe functions $z \to z/(1-xz)^2$, $x \in X$.*

Proof. The operator L defined by $Lf(z) = zf'(z)$ is a linear homeomorphism of the space of analytic functions on Δ that vanish at 0, and $L(K) = St$. This and Theorem 2 imply Theorem 3.

THEOREM 4. *Let $\mathscr{P}$ be the set of probability measures on $[0, \pi]$, $\mathscr{F}$ be the set of functions f_μ on Δ defined by*

$$f_\mu(z) = \int_0^\pi \frac{z}{1-2z\cos t + z^2}\, d\mu(t), \qquad \mu \in \mathscr{P}.$$

Let S_R be the set of analytic, univalent, normalized functions on Δ that are real on $(-1, 1)$, and let T be the set of typically real functions on Δ. Then cl co $S_R = T = \mathscr{F}$, *the map $\mu \to f_\mu$ is one-to-one, and each function $z \to z/(1-2z\cos t+z^2)$ is an extreme point of* cl co S_R.

Proof. It is known [10] that $T = \mathscr{F}$. Further, cl co $S_R \subset T$, for $S_R \subset T$ and T is a closed convex family. Also, since each function $z \to z/(1-2z\cos t + z^2)$ belongs to S_R, $\mathscr{F} \subset \text{cl co } S_R$. Thus cl co $S_R \subset T = \mathscr{F} \subset \text{cl co } S_R$. It remains to prove that $\mu \to f_\mu$ is one-to-one. If μ is in $\mathscr{P}$ define the probability measure μ^* on $[-\pi, \pi]$ by $\mu^*(E) = \frac{1}{2}\mu(E \cap [0, \pi]) + \frac{1}{2}\mu(-E \cap [0, \pi])$, where $E \subset [-\pi, \pi]$. Then

$$f_\mu(z) = \int_{-\pi}^{\pi} \frac{z}{1-2z\cos t + z^2}\, d\mu^*(t),$$

and

$$\frac{1-z^2}{z} f_\mu(z) = \int_{-\pi}^{\pi} \frac{1-z^2-2iz\sin t}{1-2z\cos t+z^2}\, d\mu^*(t) = \int_{-\pi}^{\pi} \frac{1+e^{-it}z}{1-e^{-it}z}\, d\mu^*(t).$$

Thus $f_{\mu_1} = f_{\mu_2}$ implies $\mu_1^* = \mu_2^*$ which implies $\mu_1 = \mu_2$.

2. **The determination of** cl co C **and** $\mathscr{E}$(cl co C). This section is devoted to the study of cl co C, the closed convex hull of the set of close-to-convex functions. We begin by discussing a sequence of families important for our investigations.

THEOREM 5. *Let X be the unit circle $\{z : |z|=1\}$, $\mathscr{P}$ be the set of probability measures on X, and $\mathscr{F}_k$ $(k=1, 2, \ldots)$ be the set of functions f on Δ given by*

$$f(z) = \int_X \frac{d\mu(x)}{(1-xz)^k}, \qquad \mu \in \mathscr{P}.$$

Then $\mathscr{F}_m\mathscr{F}_n \subset \mathscr{F}_{m+n}$; that is, for any positive integers m and n, the product of a function in $\mathscr{F}_m$ and a function in $\mathscr{F}_n$ belongs to $\mathscr{F}_{m+n}$.

Proof. The product of a function in $\mathscr{F}_m$ and a function in $\mathscr{F}_n$ can be written as an integral with respect to a probability measure on $X \times X$, the integrand being of the form $(1-xz)^{-m}(1-yz)^{-n}$, where $|x|=|y|=1$. Since $\mathscr{F}_{m+n}$ is a closed, convex family, it is sufficient to prove that this integrand belongs to $\mathscr{F}_{m+n}$. This is trivially true if $x=y$ so we let $x \neq y$. Let $c(t)=tx+(1-t)y$, and $f_1(z)=\int_0^1 dt/(1-c(t)z)$, where $z \in \Delta$. Since $|c(t)|<1$ it follows that $\operatorname{Re} f_1(z)>\frac{1}{2}$. This and $f_1(0)=1$ imply that $f_1 \in \mathscr{F}_1$ as a consequence of the Herglotz formula. Now, in general, if $f_k \in \mathscr{F}_k$, then the function $\{d/dz[z^k f_k(z)]\}/kz^{k-1}$ belongs to $\mathscr{F}_{k+1}$. Therefore $d/dz\,[zf_1(z)]$ belongs to $\mathscr{F}_2$. But

$$\frac{d}{dz}[zf_1(z)] = \int_0^1 \frac{dt}{[1-c(t)z]^2} = \frac{1}{(x-y)z}\left(\frac{1}{1-xz}-\frac{1}{1-yz}\right) = \frac{1}{1-xz}\cdot\frac{1}{1-yz}.$$

Thus the theorem is true for $m=n=1$.

Next we differentiate the identity $\int_0^1 dt/[1-c(t)z]^2=[1/(1-xz)]\cdot[1/(1-yz)]$ $m-1$ times with respect to x and $n-1$ times with respect to y. This yields

$$A_{mn}\int_0^1 \frac{t^{m-1}(1-t)^{n-1}}{[1-c(t)z]^{m+n}}\,dt = \frac{1}{(1-xz)^m}\cdot\frac{1}{(1-yz)^n},$$

where $A_{mn}=(m+n-1)!/[(m-1)!(n-1)!]$. Let

$$B_k(z) = A_{mn}\int_0^1 \frac{t^{m-1}(1-t)^{n-1}}{[1-c(t)z]^k}\,dt.$$

Then $B_1 \in \mathscr{F}_1$ because

$$\operatorname{Re} B_1(z) \geqq A_{mn}\int_0^1 \tfrac{1}{2}t^{m-1}(1-t)^{n-1}\,dt = \tfrac{1}{2}B_{m+n}(0) = \tfrac{1}{2},$$

and $B_1(0)=B_{m+n}(0)=1$. Finally, the rule mentioned above for passing from $\mathscr{F}_k$ to $\mathscr{F}_{k+1}$ leads to the conclusion $B_k \in \mathscr{F}_k$ for $k=1, 2, \ldots$. In particular $B_{m+n} \in \mathscr{F}_{m+n}$ as required.

REMARKS. 1. Since the function that is identically 1 belongs to $\mathscr{F}_1$, Theorem 5 implies that $\mathscr{F}_1 \subset \mathscr{F}_2 \subset \mathscr{F}_3 \subset \cdots$.

2. Theorem 5 provides a means of giving alternative proofs to some of the results in [11] (see in particular pp. 4–8) as well as proving some generalizations of these results.

THEOREM 6. *Let X be the torus $\{(x,y) : |x|=|y|=1\}$, $\mathscr{P}$ be the set of probability measures on X, $k(z,x,y)=[z-(x+y)z^2/2]/[1-yz]^2$, where $z\in\Delta$, $|x|=|y|=1$, and let $\mathscr{F}$ be the set of functions f_μ on Δ defined by*

$$f_\mu(z) = \int_X k(z,x,y)\,d\mu(x,y), \qquad \text{where } \mu\in\mathscr{P}.$$

If C is the set of normalized, close-to-convex functions on Δ then $\mathscr{F}=\operatorname{cl}\operatorname{co} C$, and the extreme points of $\operatorname{cl}\operatorname{co} C$ are precisely the functions $z\to k(z,x,y)$, $x\neq y$. Each extreme point maps Δ conformally onto the complement of a half-line.

Proof. We first show that $\operatorname{cl}\operatorname{co} C\subset\mathscr{F}$, or equivalently that $C\subset\mathscr{F}$. If $f\in C$ there is a starlike, univalent function g such that $p(z)=zf'(z)/g(z)$ has a positive real part in Δ. Let Γ denote the unit circle. Then because of the Herglotz formula, there is a probability measure α on Γ so that

$$p(z) = \int_\Gamma \frac{p(0)u+\overline{p(0)}z}{u-z}\,d\alpha(u).$$

Also, by Theorem 3 there is a probability measure β on Γ such that $g(z)/g'(0)=\int_\Gamma [z/(1-vz)^2]\,d\beta(v)$. Hence

$$f'(z) = g'(0)\int_\Gamma \frac{1}{(1-vz)^2}\,d\beta(v)\cdot\int_\Gamma \frac{p(0)u+\overline{p(0)}z}{u-z}\,d\alpha(u),$$

and since $g'(0)p(0)=1$ we obtain

$$f'(z) = \int_X \frac{1+\varepsilon\bar{u}z}{(1-\bar{u}z)(1-vz)^2}\,d\alpha(u)\,d\beta(v), \qquad \text{where } |\varepsilon| = 1.$$

To show that $f\in\mathscr{F}$ it is sufficient to show that $f'\in\mathscr{F}'$, the set of deviatives of functions belonging to $\mathscr{F}$. (Note that both f and the functions belonging to $\mathscr{F}$ vanish at 0.) For this it is enough to prove that $(1+\varepsilon\bar{u}z)/(1-\bar{u}z)(1-vz)^2$ belongs to $\mathscr{F}'$ for arbitrary u and v with $|u|=|v|=1$. Given u and v, Theorem 5 implies there is a probability measures γ on Γ such that

$$\frac{1+\varepsilon\bar{u}z}{(1-\bar{u}z)(1-vz)^2} = \int_\Gamma \frac{1+\varepsilon\bar{u}z}{(1-wz)^3}\,d\gamma(w).$$

It is now clear that we need only prove that the function $(1+\varepsilon\bar{u}z)/(1-wz)^3$ belongs to $\mathscr{F}'$ for arbitrary w with $|w|=1$. However, $d/dz\, k(z,x,y)=(1-xz)/(1-yz)^3$, and therefore we can choose a unit point mass at $(-\varepsilon\bar{u}, w)$ to obtain this function from $\mathscr{F}'$. Thus $\operatorname{cl}\operatorname{co} C\subset\mathscr{F}$.

To prove that $\mathscr{F}\subset\operatorname{cl}\operatorname{co} C$ it is enough to observe that the functions $z\to k(z,x,y)$ belong to C. For this, let $|x|=|y|=1$. Then there exists δ such that

$$\operatorname{Re}\{\delta(1-xz)/(1-yz)\} > 0,$$

$g(z)=z/\delta(1-yz)^2$ is starlike and univalent, and

$$\operatorname{Re}\{z\, d/dz\, k(z, x, y)/g(z)\} = \operatorname{Re}\{\delta(1-xz)/(1-yz)\} > 0$$

as required.

We now prove the assertion concerning extreme points of cl co C. By Theorem 1(d), the only possible extreme points are the functions $z \to k(z, x, y)$. In the case $x=y$, $k(z, x, y)=z/(1-yz)$. Since this function belongs to cl co St but is not an extreme point of cl co St (by Theorem 3), it cannot be an extreme point of the larger set cl co C. Next we suppose that $|x_0|=|y_0|=1$, $x_0 \neq y_0$, and

$$k(z, x_0, y_0) = \int_X k(z, x, y)\, d\mu(x, y), \qquad z \in \Delta.$$

If we can show that μ is a unit point mass at (x_0, y_0), then by Theorem 1(d) the function $z \to k(z, x_0, y_0)$ is an extreme point of cl co C. Differentiating the above formula for $k(z, x_0, y_0)$ we obtain

$$\frac{1-x_0z}{(1-y_0z)^3} = \int_{\Gamma\times\{y_0\}} \frac{1-xz}{(1-yz)^3}\, d\mu(x, y) + \int_{\Gamma\times\Gamma\setminus\{y_0\}} \frac{1-xz}{(1-yz)^3}\, d\mu(x, y),$$

$$\left\{1-x_0z-\int_{\Gamma\times\{y_0\}} (1-xz)\, d\mu(x, y)\right\}\Big/(1-y_0z)^3 = \int_{\Gamma\times\Gamma\setminus\{y_0\}} \frac{1-xz}{(1-yz)^3}\, d\mu(x, y).$$

Let $a=\mu(\Gamma\times\{y_0\})$. Then

$$1-x_0\bar{y}_0-a+\bar{y}_0\int_{\Gamma\times\{y_0\}} x\, d\mu(x, y) = \lim_{z\to\bar{y}_0}\left[(1-y_0z)^3\int_{\Gamma\times\Gamma\setminus\{y_0\}} \frac{1-xz}{(1-yz)^3}\, d\mu(x, y)\right].$$

We shall prove below that this limit is zero, and therefore $\bar{y}_0\int_{\Gamma\times\{y_0\}} x\, d\mu(x, y) = x_0\bar{y}_0-1+a$. Now $|x_0\bar{y}_0-(1-a)| \geqq 1-(1-a)=a$, while $|\bar{y}_0\int_{\Gamma\times\{y_0\}} x\, d\mu(x,y)| \leqq a$. Therefore $|x_0\bar{y}_0-(1-a)|=1-(1-a)$, and this implies that either $x_0\bar{y}_0=1$ or $a=1$. Since $x_0\neq y_0$ we must have $a=1$. Hence $\int_{\Gamma\times\{y_0\}} x\, d\mu(x, y)=x_0$, and this is possible only if $\mu\{(x_0, y_0)\}=1$.

To show that the above limit equals zero, we shall let z approach $\bar{y}_0$ radially from Δ. Let ε be an arbitrary positive number and let N be a corresponding neighborhood of y_0 in Γ such that $\mu(\Gamma\times N\setminus\{y_0\})<\varepsilon$. Then

$$\begin{aligned}
&\lim_{z\to\bar{y}_0}\left[(1-y_0z)^3\int_{\Gamma\times\Gamma\setminus\{y_0\}} \frac{1-xz}{(1-yz)^3}\, d\mu(x, y)\right] \\
&= \lim_{z\to\bar{y}_0}\left[(1-y_0z)^3\int_{\Gamma\times N\setminus\{y_0\}} \frac{1-xz}{(1-yz)^3}\, d\mu(x, y)+(1-y_0z)^3\int_{\Gamma\times\Gamma\setminus N} \frac{1-xz}{(1-yz)^3}\, d\mu(x,y)\right] \\
&= \lim_{z\to\bar{y}_0}\left[(1-y_0z)^3\int_{\Gamma\times N\setminus\{y_0\}} \frac{1-xz}{(1-yz)^3}\, d\mu(x, y)\right].
\end{aligned}$$

Now $|(1-y_0z)^3/(1-yz)^3| \leqq 1$ for every y with $|y|=1$ and every z of the form $r\bar{y}_0$, $0<r<1$. Therefore the above limit has absolute value less than 2ε, and so must be zero as asserted.

The last assertion of the theorem concerning the image of Δ under the mapping $z \to k(z, x, y)$ is well known [5].

REMARK. We note that the set of extreme points for $\operatorname{cl}\operatorname{co} K$, $\operatorname{cl}\operatorname{co} St$, and $\operatorname{cl}\operatorname{co} S_R$ is closed. The set of extreme points of $\operatorname{cl}\operatorname{co} C$ is not closed since the functions $z \to z/(1-yz)$, $|y|=1$, associated with the case $x=y$ of Theorem 6, are not in $\mathscr{E}(\operatorname{cl}\operatorname{co} C)$. It would be interesting to determine whether the set of extreme points of $\operatorname{cl}\operatorname{co} S$ is closed.

3. **Linear extremal problems.** We now discuss the problem of maximizing (or minimizing) a linear functional over K, St, S_R, or C. More precisely, if J is a continuous, complex-valued, linear functional on A, we wish to determine $\max \operatorname{Re} J(f)$, where f ranges over any of these four families. Easy arguments based on linearity, continuity, and [2, Lemma 2, p. 439 and Lemma 3, p. 440] show that if F is any of these families, then

$$\begin{aligned}\max\{\operatorname{Re} J(f) : f \in F\} &= \max\{\operatorname{Re} J(f) : f \in \operatorname{cl}\operatorname{co} F\}\\ &= \sup\{\operatorname{Re} J(f) : f \in \mathscr{E}(\operatorname{cl}\operatorname{co} F)\}.\end{aligned}$$

Since we have completely determined $\mathscr{E}(\operatorname{cl}\operatorname{co} F)$ in all four cases, our problem reduces to that of maximizing a real-valued function of the parameters defining the functions in $\mathscr{E}(\operatorname{cl}\operatorname{co} F)$. To illustrate this we consider the example $F=C$. Then

$$\max\{\operatorname{Re} J(f) : f \in C\} = \max\left\{\operatorname{Re} J\left[\frac{z-(x+y)z^2/2}{(1-yz)^2}\right] : |x| = |y| = 1\right\}.$$

For instance, if for some positive integer n, $J(f)$ is the nth coefficient a_n of the Taylor expansion of f in Δ, then

$$\max\{\operatorname{Re} a_n : f \in C\} = \max\{\operatorname{Re} \tfrac{1}{2}[(n+1)y^{n-1}-(n-1)xy^{n-2}] : |x| = |y| = 1\} = n.$$

(See [9] for the original proof.) Other classical results concerning K, St, T, and C are also readily obtainable by these methods. Additional extremal problems over subfamilies $\mathscr{F}$ of S, such as $\max\{|a_n| : f \in C\}$, can often be reduced to a problem of the form $\max\{\operatorname{Re} J(f) : f \in \mathscr{F}\}$ since a number of families are rotationally invariant; that is, if $f \in \mathscr{F}$ then the function $z \to f(xz)/x$ is in $\mathscr{F}$ for all x, $|x|=1$. However, rather than pursuing this we shall show that if $F=K$, St, or S_R, and if J is "nontrivial," then $\max\{\operatorname{Re} J(f) : f \in F\}$ is attained on F only by extreme points of $\operatorname{cl}\operatorname{co} F$ and there are only finitely many of them. We begin by recalling a concrete representation for the continuous linear functionals on A [16]; the analogous result for entire functions can be found in [3, p. 222].

THEOREM 7. *Let $\{b_n\}$ be a sequence of complex numbers such that* $\limsup_{n\to\infty} |b_n|^{1/n} < 1$, *and set $J(f)=\sum_{n=0}^{\infty} a_n b_n$ where $f(z)=\sum_{n=0}^{\infty} a_n z^n$ and $f \in A$. Then J is a continuous linear functional on A. Conversely, any continuous linear functional on A is given by such a sequence $\{b_n\}$.*

THEOREM 8. *Let J be a continuous linear functional on A not of the form $J(f)=af(0)+bf'(0)$. The only functions f in St that satisfy*

$$\operatorname{Re} J(f) = \max\{\operatorname{Re} J(g) : g \in St\}$$

are the Koebe functions $z \to z/(1-xz)^2$, $|x|=1$, and there are only finitely many of them.

Proof. According to Theorem 7 J is given by a suitable sequence $\{b_n\}$. If f is a Koebe function $f(z)=z/(1-xz)^2$, $|x|=1$, then its nth Taylor coefficient is $a_n=nx^{n-1}$, and therefore $J(f)=\sum_{n=1}^{\infty} nb_nx^{n-1}$. This defines an analytic function $F(x)$ for $|x|\leqq 1$ since $\limsup_{n\to\infty} |nb_n|^{1/n}=\limsup_{n\to\infty} |b_n|^{1/n}<1$. The image of the circle $|x|=1$ under F can intersect a line for only a finite number of values of x, unless F is constant. (A proof of this familiar fact is given in the Appendix.) But F is not a constant because J does not have the form $J(f)=af(0)+bf'(0)$. Therefore there are only finitely many numbers x such that $|x|=1$ and $\operatorname{Re} F(x)=\max\{\operatorname{Re} F(y) : |y|=1\}$. Equivalently, if $S^*=\{f : f(z)=z/(1-xz)^2, |x|=1\}$ then there are only finitely many functions f in S^* such that $\operatorname{Re} J(f)=\max\{\operatorname{Re} J(g) : g \in S^*\}$.

Let $G=\{f : f\in \operatorname{cl}\operatorname{co} St, \operatorname{Re} J(f)=\max \operatorname{Re} J(g), g\in \operatorname{cl}\operatorname{co} St\}$. G is compact, convex, and nonvoid and therefore has extreme points [2, p. 439, Lemma 2]. G also is an extremal subset of $\operatorname{cl}\operatorname{co} St$; that is, if $tf+(1-t)g\in G$, $0<t<1$, $f\in \operatorname{cl}\operatorname{co} St$, and $g\in \operatorname{cl}\operatorname{co} St$, then f and g are in G. Thus the extreme points of G are extreme points of $\operatorname{cl}\operatorname{co} St$; that is, they are Koebe functions. Because of what was first proved there can be only a finite number of such functions, say $f_1, f_2, \ldots, f_m$, where $f_k(z)=z/(1-x_kz)^2$, $|x_k|=1$ $(k=1,\ldots,m)$, $x_j\neq x_k$ for $j\neq k$. Since $\mathscr{E}(G)=\{f_1, f_2, \ldots, f_m\}$ it follows that

$$G = \left\{f : f = \sum_{k=1}^{m} \lambda_k f_k, \lambda_k \geqq 0, \sum_{k=1}^{m} \lambda_k = 1\right\}.$$

The function f_k has a pole of order two at $z=\bar{x}_k$, and therefore if $\lambda_k\neq 0$ for at least two values of k, the function $\sum_{k=1}^{m} \lambda_k f_k$ has two poles on the unit circle each of order two. Such a function is not univalent in Δ since its local behavior at the poles shows that it takes on most large values at least twice in Δ. Hence the only functions in G which are in St are the functions $f_1, f_2, \ldots, f_m$. Because

$$\max\{\operatorname{Re} J(g) : g\in St\}=\max\{\operatorname{Re} J(g) : g \in \operatorname{cl}\operatorname{co} St\}$$

this completes the proof that $f_1,\ldots,f_m$ are the only functions in St satisfying $\operatorname{Re} J(f)=\max\{\operatorname{Re} J(g) : g\in St\}$.

REMARKS. A simple application of Theorem 8 is the known fact: if f is in St and $f(z)=z+\sum_{n=2}^{\infty} a_nz^n$ then $\operatorname{Re} a_n\leqq n$, and equality occurs only for the functions $f(z)=z/(1-xz)^2$, $x^{n-1}=1$. Indeed, Theorem 8 indicates that this problem is equivalent to $\max\{\operatorname{Re} nx^{n-1} : |x|=1\}$.

With regard to the actual number of solutions to a linear extremal problem over St the last example is a particular case of the following situation. Let J be defined by $J(f)=\sum_{k=0}^{n} a_k b_k$, where $n \geqq 2, f \in A, f(z)=\sum_{k=0}^{\infty} a_k z^k$, and $b_0, b_1, \ldots, b_n$ are given complex numbers, $b_n \neq 0$. If $f \in St$ and $\operatorname{Re} J(f)=\max \{\operatorname{Re} J(g) : g \in St\}$, then f is a Koebe function $f(z)=z/(1-xz)^2$ and x is any solution to

$$\operatorname{Re}\left\{\sum_{k=1}^{n} kb_k x^{k-1}\right\} = \max_{|y|=1}\left\{\operatorname{Re}\sum_{k=1}^{n} kb_k y^{k-1}\right\}.$$

This equation, which determines x, is equivalent to finding the values of θ in $[0, 2\pi)$ that maximize a trigonometric polynomial $T(\theta)$ of degree $n-1$. Between each pair of numbers giving relative maxima of $T(\theta)$ is a point at which $T(\theta)$ has a relative minima. Likewise there is a point at which $T(\theta)$ has a relative maxima between any two points producing relative minima of $T(\theta)$. Therefore $T(\theta)$ has the same number of relative maxima and minima on $[0, 2\pi)$. As $T(\theta)$ has degree $n-1$ it takes on a value at most $2(n-1)$ times on $[0, 2\pi)$. Thus there are at most $n-1$ numbers in $[0, 2\pi)$ at which $T(\theta)$ achieves its (absolute) maxima. Our conclusion is that the functionals $\operatorname{Re} J$ of the above kind are maximized over St for at most $n-1$ (Koebe) functions.

Next we shall show that any finite collection of Koebe functions is the solution set to $\max \{\operatorname{Re} J(g) : g \in St\}$ for a suitable continuous linear functional J. Because of our previous arguments it is clear that such an assertion is equivalent to the existence of a function F analytic for $|z| \leqq 1$ such that $\operatorname{Re} F(z)$ is maximal at prescribed points on $|z|=1$. We now prove such a result. Another proof of this can be given containing the additional information that there is such a function F which is also univalent for $|z| \leqq 1$.

THEOREM 9. *Let B be any finite set of points on the unit circle. There is a function F analytic for $|z| \leqq 1$ such that* $\operatorname{Re} F(z) \geqq 0$ *and* $\operatorname{Re} F(z)=0$ *only for z in B.*

Proof. Let $x_1, x_2, \ldots, x_m$ be m distinct points on the unit circle. Define the complex numbers c_j by

$$\prod_{n=1}^{m} (e^{i\theta}-x_n)(e^{-i\theta}-\bar{x}_n) = \sum_{j=-m}^{m} c_j e^{ij\theta}$$

and note that $c_{-j}=\bar{c}_j$ (c_0 is real). Let $F(z)=c_0+2\sum_{j=1}^{m} c_j z^j$. Then

$$\operatorname{Re} F(e^{i\theta}) = \tfrac{1}{2}[F(e^{i\theta})+\overline{F(e^{i\theta})}] = c_0+\sum_{j=1}^{m} c_j e^{ij\theta}+\sum_{j=1}^{m} \bar{c}_j e^{-ij\theta}$$

$$= \sum_{j=-m}^{m} c_j e^{ij\theta} = \prod_{n=1}^{m} (e^{i\theta}-x_n)(e^{-i\theta}-\bar{x}_n) = \prod_{n=1}^{m} |e^{i\theta}-x_n|^2.$$

This shows that $\operatorname{Re} F(e^{i\theta}) \geqq 0$ and thus $\operatorname{Re} F(z) \geqq 0$ for $|z| \leqq 1$. Also, since $\operatorname{Re} F(z)$ is not constant it can be zero only on $|z|=1$ and therefore only at $e^{i\theta}=x_n$ $(n=1, 2, \ldots, m)$.

THEOREM 10. *Let $f_1, f_2, \ldots, f_m$ be m distinct Koebe functions. There is a continuous linear functional J on A such that the functions f in St which satisfy* $\operatorname{Re} J(f) = \max \{\operatorname{Re} J(g) : g \in St\}$ *are precisely the functions $f_1, f_2, \ldots, f_m$.*

Proof. Let f_n be given by $f_n(z)=z/(1-x_nz)^2$ $(n=1, 2, \ldots, m)$, $|x_n|=1$, where the numbers $x_1, x_2, \ldots, x_m$ are distinct. Let F be the polynomial defined in the proof of Theorem 9 and define G by $G(z)=-F(z)=\sum_{n=0}^{m} d_nz^n$. Then $\operatorname{Re} G(z) \leqq 0$ for $|z| \leqq 1$ and $\operatorname{Re} G(z)=0$ only for $z=x_n$, $n=1, 2, \ldots, m$. Set $b_0=0$ and $b_n=d_{n-1}/n$ for $n=1, 2, \ldots, m$. The continuous linear functional J is defined on A by $J(f) =\sum_{n=0}^{m} a_nb_n$ where $f(z)=\sum_{n=0}^{\infty} a_nz^n$. According to Theorem 8 the only functions f in St satisfying $\operatorname{Re} J(f)=\max\{\operatorname{Re} J(g) : g \in St\}$ are Koebe functions $f(z)=z/(1-xz)^2$, $|x|=1$. For these functions $\operatorname{Re} J(f)=\operatorname{Re} G(x)$ and this has its maximum value of zero only for $x=x_n$, $n=1, 2, \ldots, m$.

We next prove two theorems concerning linear problems over K and S_R analogous to the result for St given in Theorem 8. The first theorem follows by appealing to Theorem 8 and the fact that $f(z) \in K$ if and only if $zf'(z) \in St$. (See the remarks in the proof of Theorem 3.)

THEOREM 11. *Let J be a continuous linear functional on A not of the form $J(f) =af(0)+bf'(0)$. The only functions f in K satisfying* $\operatorname{Re} J(f)=\max\{\operatorname{Re} J(g) : g \in K\}$ *are the functions $f(z)=z/(1-xz)$, $|x|=1$, and there are only finitely many of them.*

THEOREM 12. *Let J be a continuous linear functional on A such that* $\operatorname{Re} J(f)$ *is not constant on S_R. The only functions f in S_R satisfying*

$$\operatorname{Re} J(f) = \max\{\operatorname{Re} J(g) : g \in S_R\}$$

are the functions $f(z)=z/(1-2az+z^2)$, $-1 \leqq a \leqq 1$, and there are only a finite number of them.

Proof. Let

$$\begin{aligned} F(z, t) &= \frac{z}{1-2z\cos t+z^2} \\ &= \frac{z}{(1-e^{it}z)(1-e^{-it}z)} = z+\sum_{n=2}^{\infty} a_n(x)z^n, \qquad x = e^{it}. \end{aligned}$$

Computing $a_n(x)$ in terms of a polynomial in x and $1/x$ we find that $|a_n(x)| \leqq nr^{n-1}$ $(n=2, 3, \ldots)$ if $1/r \leqq |x| \leqq r$, $r>1$. The functional J is given by a suitable sequence $\{b_n\}$ according to Theorem 7. If J is applied to the above functions we obtain $b_1+\sum_{n=2}^{\infty} a_n(x)b_n$. Since $\limsup_{n\to\infty} |b_n|^{1/n}<1$ the above estimates on $a_n(x)$ show that this series is majorized by a convergent geometric series for r sufficiently close to 1. It follows that when J is applied to the functions $F(z, t)$ a function $F(t)$ is defined for real values of t which can be analytically extended into some strip $\{t : |\operatorname{Im} t|<\rho, \rho>0\}$. Likewise $\operatorname{Re} F(t)$ can be analytically extended into such a strip.

Now, $\operatorname{Re} F(t)$ cannot be constant for all real values of t. Indeed, if $\operatorname{Re} F(t)=A$ for all real numbers t it follows that $\operatorname{Re} J(f)=A$ for all f in T since the functions $F(z, t)$ are the extreme points of T. In particular, this implies that $\operatorname{Re} J(f)=A$ for each f in S_R contrary to our assumption. Because of the identity theorem for analytic functions, we now conclude that Re $F(t)$ takes on any value for at most a finite number of real numbers t in $[0, 2\pi)$. This shows that there is only a finite number of functions $f_1, f_2, \ldots, f_m$ in $\mathscr{E}(T)$ that maximize Re $J(f)$.

Let $G = \{f : f \in T, \text{Re } J(f) = \max \text{Re } J(g), g \in T\}$. G is compact, convex, and nonvoid and therefore has extreme points. Each such point is also an extreme point of T because G is an extremal subset of T. But as we showed above there is only a finite number of such functions, say $f_1, f_2, \ldots, f_m$, where

$$f_k(z) = z/(1-2z\cos t_k + z^2), \qquad 0 \leqq t_k \leqq \pi \qquad (k = 1, \ldots, m),$$

and the numbers t_k are distinct. Therefore

$$G = \left\{ f : f = \sum_{k=1}^{m} \lambda_k f_k, \lambda_k \geqq 0, \sum_{k=1}^{m} \lambda_k = 1 \right\}.$$

The function $F(z, t)$ either has two simple poles or one double pole on the unit circle. Therefore a function of the form $f = \sum_{k=1}^{m} \lambda_k f_k$ where $\lambda_k \neq 0$ for at least two values of k has poles on the unit circle with a total order of at least four. Such a function is not univalent in Δ as it takes on most large values at least twice in Δ. This implies that the only functions in G which are also in S_R are the functions $f_1, f_2, \ldots, f_m$. Because $\max\{\text{Re } J(g) : g \in S_R\} = \max\{\text{Re } J(g) : g \in T\}$ we conclude that $f_1, f_2, \ldots, f_m$ are the only functions f in S_R satisfying

$$\text{Re } J(f) = \max\{\text{Re } J(g) : g \in S_R\}.$$

This completes the proof. We add the observation that Re $J(f)$ is constant on S_R only for a functional J such that $b_2, b_3, \ldots$ are purely imaginary.

4. **Inclusion relations.** In this section we describe some additional properties of the closed convex hulls treated earlier, focusing attention on the inclusion relations between them as subsets of A. One of the first questions one would like to answer is whether every normalized univalent function can be obtained as a limit (uniform on compact subsets of Δ) of convex combinations of close-to-convex functions. The implications of a positive answer to this question are enormous, including a positive conclusion to the Bieberbach conjecture. We show, however, that cl co $C \not\supset S$ and we decide this by showing that some of the "half-lines" which are extreme points of cl co C are not extreme in cl co S. To see this we appeal to the result in [1] that a necessary condition for f to be extreme in cl co S is that the complement of the range of f be a continuous arc tending to infinity with increasing modulus.

The general extreme point of cl co C has the form

$$f(z) = [z-(x+y)z^2/2]/[1-yz]^2$$

with $|x| = |y| = 1$, $x \neq y$. By an appropriate rotation it is sufficient to consider those of the form $f(z) = (z - az^2)/(1-z)^2$ where $a = \frac{1}{2}(1+e^{i\theta})$, $0 < |\theta| \leqq \pi$.

THEOREM 13. *$f(z) = (z-az^2)/(1-z)^2$ is not extreme in* cl co S *if* $|\theta| < \pi/2$.

Proof. Since f omits a half-line containing the values $f(\pm i) = -\frac{1}{2} \pm \frac{1}{2}ia$, f also omits $-\frac{1}{2}$, the average of these values. The base of the half-line occurs at $f(b)$ where $f'(b)=0$. Note that $b=1/(2a-1)$ and $f(b)=\frac{1}{4}(a-1)^{-1}=\frac{1}{2}(e^{i\theta}-1)^{-1}=-\frac{1}{4}$ $-(i\sin\theta)/4(1-\cos\theta)$. Therefore the half-line contains two points of equal modulus iff $|(\sin\theta)/4(1-\cos\theta)|>\frac{1}{4}$, i.e. $|\theta|<\pi/2$.

COROLLARY. *$S \not\subset$ cl co C and cl co $S \neq$ cl co C.*

REMARK. We do not know whether $\frac{1}{2}\pi \leq |\theta| < \pi \Rightarrow f$ is extreme in cl co S. Indeed we are unable at present to exhibit any extreme points of cl co S aside from rotations of the Koebe function.

The starlike functions lie in a distinguished subclass C_0 of C where $C_0 = \{f \in S : \text{Re}\{zf'(z)/g(z)\} > 0$ for some $g \in St\}$.

We are able to show that some functions in C_0 which are not starlike can nevertheless be approximated by convex combinations of starlike functions—so that cl co $St \cap S \neq St$. Although it may seem plausible that cl co St contains all of C_0, an elementary computation with coefficient restrictions shows this not to be the case. This information is contained in the following two theorems.

THEOREM 14. *Let $f'(z)=[(1+\bar{u}z)/(1-\bar{u}z)][1/(1-uz)^2]$ where $u=e^{i\theta}$. If $5/6<\cos^2\theta<1$, then $f \notin$ cl co St.*

THEOREM 15. *Let*

$$f'(z) = [(1+\bar{u}z)/(1-\bar{u}z)][1/(1-uz)^2]$$

where $u=e^{i\theta}$. There exists a positive number η such that if $\theta=\pm\pi/2+\rho$ and $|\rho|\leq\eta$, then $f \in$ cl co St; that is, there exists a probability measure μ on the unit circle X such that $f(z)=\int_X z/(1-xz)^2\,d\mu(x)$.

REMARKS. 1. The generic form for f', if $f \in C_0$, is given by

$$f'(z) = [(1+\bar{u}z)/(1-\bar{u}z)][1/(1-vz)^2].$$

By the rotation $z \to u^{1/2}\bar{v}^{1/2}z$ we obtain the form discussed in the theorem.

2. We do not know if there exists a unique dividing value for θ to distinguish the two cases in the theorems.

3. A direct computation shows that if u is near i or $-i$ $(u \neq \pm i)$ on the unit circle, then there exist complex numbers with $|z|<1$ such that $\text{Re}\{zf'(z)/f(z)\}<0$. Hence the functions described in Theorem 15 are close-to-convex, *not* starlike, but approximable by convex combinations of starlike functions.

4. It is *always* the case that for f in C_0 there exists a finite *real* measure ν on the unit circle X such that $f(z)=\int_X z/(1-xz)^2\,d\nu(x)$. The proof of this follows from a description of the necessary conditions imposed on the Fourier-Stieltjes coefficients of the measure ν if such an integral representation is to exist.

Proof of Theorem 14. If $f \in$ cl co St and we let $g(z)=\int_0^z f(w)/w\,dw$, then $\text{Re}\,g(z)/z>\frac{1}{2}$, i.e. $\text{Re}\,p(z)>0$ where $p(z)=2(g(z)/z)-1$. If we write $p(z)$

$=1+p_1z+p_2z^2+\cdots$, it follows from known results that $|2p_2-p_1^2|\leqq 4-|p_1|^2$. For instance, we can apply the inequality $|c_1|\leqq 1-|c_0|^2$ (see [7, p. 168, (28)]) to $(1/z)(p(z)-1)/(p(z)+1)=c_0+c_1z+\cdots=p_1/2+(2p_2-p_1^2)z/4+\cdots$. If we use $f'(z)=[(1+\bar{u}z)/(1-\bar{u}z)][1/(1-uz)^2]$ and expand in a power series we obtain the coefficients of $p(z)$ expressed as $p_1=u+\bar{u}$, $p_2=(2/9)(3u^2+4+2\bar{u}^2)$. If $u=e^{i\theta}$, then

$$|2p_2-p_1^2|^2 = (16/81)(1+14\cos^2\theta-15\cos^4\theta)$$

and

$$(4-|p_1|^2)^2 = 16(1-2\cos^2\theta+\cos^4\theta).$$

Thus if θ is chosen so that

$$16(1-2\cos^2\theta+\cos^4\theta) < (16/81)(1+14\cos^2\theta-15\cos^4\theta),$$

then $f\notin \text{cl co } St$. This inequality reduces to $(1-\cos^2\theta)(5-6\cos^2\theta)<0$. Hence if $5/6<\cos^2\theta<1$, then $f\notin \text{cl co } St$.

Proof of Theorem 15. $f\in \text{cl co } St$ provided $f(z)=\int_X [z/(1-vz)^2]\,d\mu(v)$ for a positive measure μ on the unit circle X. If $f'(z)=[(1+\bar{u}z)/(1-\bar{u}z)][1/(1-uz)^2]$, then $f\in \text{cl co } St$ provided $\operatorname{Re} F(z)>\frac{1}{2}$ where

$$F(z) = \frac{1}{z}\int_0^z \frac{1}{w}\int_0^w \frac{1+\bar{u}\zeta}{1-\bar{u}\zeta}\frac{1}{(1-u\zeta)^2}\,d\zeta\,dw.$$

Direct integration shows that

$$F(z) = \frac{1}{z}\int_0^z \left[\frac{1}{w}\frac{2u}{(1-u^2)^2}\int_{\bar{u}w}^{uw}\frac{-1}{1-\zeta}\,d\zeta-\frac{1+u^2}{1-u^2}\frac{1}{1-uw}\right]dw.$$

If we let $\lambda(t)=tu+(1-t)\bar{u}$, $\zeta=\lambda(t)w$, then

$$F(z) = \frac{1}{z}\int_0^z \frac{1}{\bar{u}-u}\left[\int_0^1 \frac{2\bar{u}}{1-\lambda(t)w}\,dt-\frac{u+\bar{u}}{(1-uw)}\right]dw.$$

Now let $u=ie^{i\alpha}=-\sin\alpha+i\cos\alpha$, so that

$$F(z) = \frac{1}{z}\int_0^z\int_0^1 \frac{1}{1-\lambda(t)w}\,dt\,dw-\frac{\sin\alpha}{\cos\alpha}\,i\,\frac{1}{z}\int_0^z\int_0^1\left[\frac{1}{1-\lambda(t)w}-\frac{1}{1-\lambda(1)w}\right]dt\,dw.$$

A similar computation shows that we obtain the same expression if $\delta=-ie^{i\alpha}$ and $\lambda(t)$ is changed accordingly. To prove the theorem we must show that $\operatorname{Re} F(z)>\frac{1}{2}$ for all z in Δ provided α is near zero; i.e. there exists $\eta>0$ so that $\operatorname{Re} F(z)>\frac{1}{2}$ if $|z|<1$ and $|\alpha|\leqq\eta$. We need to use the following Lemma.

Lemma. *If* $h(z)=1/z\int_0^z g(w)\,dw$ *where* $g(0)=1$, $\operatorname{Re} g(w)>\frac{1}{2}$, $|w|<1$, *then* $\operatorname{Re} h(z)>\log 2>\frac{1}{2}$, $|z|<1$.

Proof. It follows from known facts about functions with positive real part that $\operatorname{Re} g(w)\geqq 1/(1+|w|)$. If we write $h(z)=\int_0^1 g(zu)\,du$, then

$$\operatorname{Re} h(z) = \int_0^1 \operatorname{Re} g(zu)\,du \geqq \int_0^1 \frac{1}{1+|z|u}\,du = \frac{1}{|z|}\log(1+|z|) > \log 2.$$

(The case $z=0$ is clear.)

To apply this to $F(z)$ note that $\text{Re}\{1/(1-\lambda(t)w)\} > \frac{1}{2}$ so that $\text{Re}\{\int_0^1 1/(1-\lambda(t)w)\,dt\} > \frac{1}{2}$ also. By the Lemma

$$\text{Re}\left\{\frac{1}{z}\int_0^z\int_0^1 \frac{1}{1-\lambda(t)w}\,dt\,dw\right\} > \log 2.$$

Thus to show $\text{Re}\,F(z) > \frac{1}{2}$ it suffices to show

$$\text{Re}\left\{-\frac{\sin\alpha}{\cos\alpha}\,i\,\frac{1}{z}\int_0^z\int_0^1\left[\frac{1}{1-\lambda(t)w}-\frac{1}{1-\lambda(1)w}\right]dt\,dw\right\}$$

goes uniformly to 0 in z as $\alpha \to 0$. This occurs if $\text{Im}\,H(z)$ is uniformly bounded in z, where

$$H(z) = \frac{1}{z}\int_0^z\int_0^1\left[\frac{1}{1-\lambda(t)w}-\frac{1}{1-\lambda(1)w}\right]dt\,dw.$$

Now

$$H(z) = \int_0^1\left[\frac{1}{-\lambda(t)z}\log(1-\lambda(t)z)-\frac{1}{-\lambda(1)z}\log(1-\lambda(1)z)\right]dt$$

and the integrand is uniformly bounded ($\text{Im}\,1/w\log(1+w)$ is uniformly bounded for $|w|<1$). Hence $H(z)$ is uniformly bounded in z for $|z|<1$ and we can choose $\eta>0$ such that $\text{Re}\,F(z)>\frac{1}{2}$ for $|z|<1$ and $|\alpha|\leqq\eta$. This completes the proof of Theorem 15.

Some additional inclusion relations are the following: $\text{cl co}\,K \subsetneqq \text{cl co}\,St$, $\text{cl co}\,S_R \subsetneqq \text{cl co}\,St$, $\text{cl co}\,S_R \not\subset \text{cl co}\,K$, $\text{cl co}\,K \not\subset \text{cl co}\,S_R$. The relation $\text{cl co}\,K \subset \text{cl co}\,St$ is immediate from $K \subset St$. That they are distinct follows from $f \in \text{cl co}\,K$ if and only if $\text{Re}\{f(z)/z\} > \frac{1}{2}$ ($|z|<1$) and $\text{Re}\{1/(1-z)^2\} < \frac{1}{2}$ for z near 1. The inequality $\text{cl co}\,S_R \neq \text{cl co}\,St$ is clear, while $\text{cl co}\,S_R \subset \text{cl co}\,St$ is a consequence of Theorems 3, 4, and 5. Indeed, by Theorem 4 it is sufficient to show that the function $z/(1-2z\cos t+z^2)$ belongs to $\text{cl co}\,St$ for each real t. But $1/(1-2z\cos t+z^2) = 1/(1-e^{it}z)[1/(1-e^{-it}z)] \in \mathscr{F}_2$ by Theorem 5 and the conclusion then follows from Theorem 3. The relations $\text{cl co}\,K \not\subset \text{cl co}\,S_R$ and $\text{cl co}\,S_R \not\subset \text{cl co}\,K$ are obvious.

5. **Appendix.** In this section we prove the result that was used earlier in the proof of Theorem 8.

THEOREM 16. *Let f be an analytic function in the closed unit disk. If $f(e^{i\theta})$ lies on a line for infinitely many values of θ in* $[0, 2\pi)$, *then f is constant.*

Proof. Without loss of generality, the given line is the imaginary axis. Let

$$g(z) = (f(z)+\overline{f(1/\bar{z})})/2.$$

Then g is analytic in a neighborhood of the unit circle, and $g(e^{i\theta}) = \text{Re}\,f(e^{i\theta})$ for θ in $[0, 2\pi)$. Hence g vanishes on an infinite set with a limit point in the domain of analyticity. Therefore g vanishes identically, and in particular $\text{Re}\,f(e^{i\theta})=0$ for all

θ in $[0, 2\pi)$. But then $\operatorname{Re} f(z)=0$ for $|z| \leqq 1$ by the maximum and minimum principles for harmonic functions. Hence f is constant.

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State University of New York,
Albany, New York 12203