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ABSTRACT

In this paper certain convex averaging procedures are investigated with the purpose of:

(a) approximating fixed points of nonlinear nonexpansive mappings in uniformly convex spaces,

(b) approximating solutions of certain linear operator equations in reflexive Banach spaces, and

(c) summing divergent sequences and series in Banach spaces.

Chapter I consists of preliminary material concerning mappings in Banach spaces, abstract ergodic theory, summability theory and miscellaneous results.

In Chapter II the iterative process

$$v_{n+1} = (1-\lambda_n)v_n + \lambda_n Tv_n \, , \quad 0 \le \lambda_n \le 1$$

is studied. It is shown that $\Sigma \lambda_n(1-\lambda_n) = \infty$ is a necessary and sufficient condition for the convergence of the process to a fixed point for every member of a certain extensive class of nonlinear nonexpansive mappings in uniformly convex spaces. Convergence is considered in both the strong and the weak topology. As an application it is shown that solutions of variational inequalities involving nonlinear operators satisfying a monotonicity condition of F. E. Browder may be approximated by the above procedure.

In Chapter III results from abstract ergodic theory
are used to show that solutions of linear operator equa-
tions of the type $u - Tu = f$ may be approximated by
the iterative sequence

$$v_{n+1} = (1-\lambda_n)v_n + \lambda_n(Tv_n + f) \; .$$

Theorems are proved for operators T which are asympto-
tically bounded on reflexive Banach spaces. Approximation
theorems for various types of linear operator equations
in L^p spaces and Hilbert spaces are proved as applications.

In the final chapter it is shown that for affine
mappings the iterative sequence defined above is the
transform of the sequence of ordinary iterates by an
infinite matrix which is permanent if and only if
$\Sigma\lambda_n = \infty$. This gives rise to a large class of summability
methods which contains the Euler-Knopp methods as a small
subclass. This class is studied in detail. It is shown
that this new class of methods contains methods which are
strictly stronger than any of the Euler-Knopp methods.
Summability of series is studied; it is shown that each
of the summability methods is absolutely permanent and an
analogue of the Abel limit theorem is proved. Applications
to iterative solution of linear operator equations are
given and, in the final section, some unsolved problems
are mentioned.

INTRODUCTION

Much of applied mathematics is concerned with the problem of finding solutions of linear and nonlinear operator equations (e.g. integral equations, boundary value problems, etc.) in infinite dimensional function spaces. The most practical method of solution for many of these problems is to show that the solution is a fixed point of a certain operator and to approximate this fixed point by iteration. The best known general iteration theorem is the classical Picard-Banach theorem for strictly contractive mappings in complete metric spaces. It is easy to see, however, that the classical iteration process does not serve to approximate fixed points of mappings which are nonexpansive (i.e., mappings T which satisfy $d(Tx,Ty) \leq d(x,y)$ for all x and y).

In 1955 M. A. Krasnoselskiĭ [20] showed that fixed points of compact nonexpansive mappings defined on closed convex subsets of uniformly convex spaces can be approximated by the process

$$v_{n+1} = \frac{1}{2}(v_n + Tv_n) \ .$$

Recently this process was studied further by C. L. Outlaw [26]. H. H. Schaefer [28] showed in 1957 that under the same conditions as in the Krasnoselskiĭ theorem, the more

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general process

$$v_{n+1} = (1-\alpha)v_n + \alpha Tv_n \ , \ 0 < \alpha < 1$$

converges to a fixed point of T . Schaefer also showed that this sequence converges weakly to a fixed point of T (providing, of course, that T has a fixed point) if T is a weakly continuous nonexpansive mapping on a real Hilbert space. Browder and Petryshyn [5] and Opial [25] extended these results by assuming less stringent conditions on the mapping and on the space.

In this paper we undertake a theoretical investigation of the general convex iteration procedure

$$v_{n+1} = (1-\lambda_n)v_n + \lambda_n Tv_n \ , \ 0 \leq \lambda_n \leq 1$$

and certain associated summability methods. Recently, Dotson [11] showed that the arguments of Browder-Petryshyn and Opial go through for this process under the assumption that λ_n is bounded away from 0 and 1 . It should be mentioned that all of the methods considered above are special cases of a very general iteration procedure introduced by W. R. Mann [22].

In Chapter II we find a necessary and sufficient condition on the weights $\{\lambda_n\}$ in order that the general convex iteration procedure converge for every member of a large class of nonexpansive nonlinear mappings on

uniformly convex spaces. We thus obtain generalizations of all of the results mentioned above. As an application we show that solutions of certain variational inequalities in Hilbert space may be approximated by the above procedure.

Dotson [12] first considered the connection between iteration theory for affine mappings and abstract ergodic theory. In Chapter III we employ Eberlein's abstract ergodic theory and a specialization of an argument of Dotson's to approximate solutions of the equation $u - Tu = f$ for asymptotically bounded linear operators T on reflexive Banach spaces. We also obtain theorems on the approximate solution of equations of the type $Au = f$ for linear operators A defined on a Hilbert space and a theorem on the approximation in L^p norm of solutions of a certain linear functional equation in L^∞ spaces.

In the final chapter it is shown that for affine mappings T, the convex iteration procedure defined above gives a sequence which is the transform of the ordinary sequence of iterates by an infinite matrix which is permanent if and only if $\Sigma \lambda_n = \infty$. Chapter IV is devoted to a detailed study of these associated summability methods. For the special case $\lambda_n = \frac{1}{2}$ for all n this gives what is perhaps the oldest known summability method, that studied by Euler in 1755. Some

of the results obtained by studying these summability methods are used to give theorems on the iterative solution of linear operator equations (see 4.4).

Finally, we mention that all theorems, definitions and other such items are numbered consecutively, with a.b.c denoting item number c in section b of chapter a.

CHAPTER I

PRELIMINARIES

In this chapter we state the fundamental definitions
and theorems which will be used in the sequel. Our
standard references are the comprehensive treatises of
Dunford-Schwartz [14] and Zeller [32].

Section 1. <u>Mappings in Banach Spaces</u>

We shall consider mappings on spaces which satisfy
the following classical definition of Clarkson [8].

1.1.1. <u>Definition</u>. A norm $\|\cdot\|$ on a linear space X
is called <u>uniformly convex</u> if given $2 \geq \epsilon > 0$ there
exists a $\delta(\epsilon) > 0$ such that if $x, y \in X$ with
$\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x-y\| \geq \epsilon$, then
$\|\frac{1}{2}(x+y)\| \leq 1-\delta(\epsilon)$. A linear space with a uniformly
convex norm is called a <u>uniformly convex space</u>.

The class of uniformly convex spaces is extensive,
in fact, an easy application of the parallelogram law
shows that every Hilbert space is uniformly convex. The
spaces L^p with $1 < p < \infty$ are also uniformly convex.

1.1.2. <u>Definition</u>. A norm $\|\cdot\|$ on a linear space X is called <u>strictly convex</u> if x,y ϵ X with $\|x\| \leq 1$ and $\|y\| \leq 1$ and $x \neq y$, then $\|\frac{1}{2}(x+y)\| < 1$. A linear space endowed with a strictly convex norm is called a <u>strictly convex space</u>.

Of course, every uniformly convex space is also strictly convex. Conversely, a simple compactness argument shows that every finite dimensional strictly convex space is also uniformly convex. The next theorem serves to clarify the position held by uniformly convex spaces in the class of Banach spaces.

1.1.3. <u>Theorem</u> (Pettis-Milman). Every uniformly convex Banach space is reflexive.

<u>Proof</u>. The reader is referred to [19, p. 354].

Reflexive spaces enjoy the following compactness property.

1.1.4. <u>Theorem</u> (Eberlein-Šmulian). In a reflexive Banach space bounded sets are weakly sequencially compact.

<u>Proof</u>. The reader is referred to [14, p. 430].

1.1.5. <u>Theorem</u> (Šmulian). A convex subset K of a Banach space is weakly compact if and only if every decreasing sequence of nonempty closed convex subsets of K has a nonempty intersection.

<u>Proof</u>. The reader is referred to [14, p. 433].

We now consider some classes of mappings which will be studied in the next chapter.

1.1.6. <u>Definition</u>. A mapping T defined on a subset D(T) of a normed linear space is called <u>nonexpansive</u> if $\|Tx-Ty\| \leq \|x-y\|$ for x,y in D(T). T is called <u>quasi-nonexpansive</u> if T has a nonempty set of fixed points and $\|Tx-Tp\| \leq \|x-p\|$ for $x \in D(T)$ and p a fixed point of T.

The concept of mappings T which map all points closer to the fixed point set of T was appearently originated by Diaz and Metcalf [10]; the term quasi-nonexpansive is due to Dotson [11].

1.1.7. <u>Theorem</u> (Schauder-Tichonov). If T is a continuous mapping of a convex subset E of a locally convex linear topological space into itself with T(E)

relatively compact, then T has a fixed point in E .

Proof. The reader is referred to [14, p. 456].

1.1.8. Theorem (Browder). If T is a nonexpansive mapping of a closed bounded convex subset E of a uniformly convex Banach space into itself, then T has a fixed point in E .

Proof. The reader is referred to [2].

1.1.9. Definition. Let X be a Banach space, X* its dual space and (u,x) the value of the linear functional u ∈ X* at the element x of X . Let μ be a continuous strictly increasing real valued function on R^+ with μ(0) = 0 . A mapping $J:X \to X^*$ is called a duality mapping of X into X* with gauge function μ if: (a) For every x in X , $(Jx,x) = \|Jx\| \cdot \|x\|$ and (b) For every x in X , $\|Jx\| = \mu(\|x\|)$.

We now define two classes of mappings of the monotone type which have been studied extensively by F.E. Browder (see e.g. [7]) .

1.1.10. Definition. Let X be a Banach space, A

mapping $A:X \to X^*$ is called <u>strictly</u> <u>monotone</u> if there
is a $c > 0$ such that $(Ax-Ay, x-y) \geq c\|x-y\|^2$ for
$x,y \in D(A)$. A is called <u>β-monotone</u> if $(Ax-Ay, x-y)$
$\geq c\|Ax-Ay\|^2$ for some $c > 0$ and all $x,y \in D(A)$.

It is easy to see that every strictly monotone
Lipshitz mapping is β-monotone.

1.1.11. <u>Definition</u>. A linear operator T on a Banach
space is called <u>asymptotically</u> <u>bounded</u> if there is a
constant M such that $\|T^n\| \leq M$ for $n = 1,2,\ldots$.

For a given $M > 0$ one can define the operator T
on ℓ^2 by

$$T(x_1,x_2,x_3,\ldots) = (Mx_2,x_3,\ldots) .$$

This shows that there exist asymptotically bounded linear
operators of arbitrarily large norm. Of course, every
asymptotically bounded normal operator defined on a
Hilbert space is nonexpansive by virtue of the spectral
radius formula [14, p. 567].

If T is a bounded linear operator on a Hilbert
space H we denote the null space and range of T
by $N(T)$ and $R(T)$ respectively. The adjoint of T
is denoted by T^* and the orthogonal complement of a
subset C of H is denoted by $C^\perp$, i.e.,

$C^{\perp} = \{x \epsilon H: (x,y) = 0 \text{ for all } y \epsilon C \}$. Finally, we recall the elementary fact that $N(T*)^{\perp} = \overline{R(T)}$ and $N(T*) = \overline{R(T)}^{\perp}$, where the bar denotes the closure operation.

Section 2. Abstract Ergodic Theory.

In this section we state some fundamental results of Eberlein on abstract ergodic theory. The proofs of the theorems may be found in the paper of Eberlein [15].

Let X be a locally convex linear topological space and let G be a semi-group of linear transformations of X into itself. Denote by G* the family of transformations consisting of all finite convex combinations of elements of G , i.e. $G* = \{\Sigma a_j T_j : a_j \geq 0 , \Sigma a_j = 1 , T_j \epsilon G\}$. Let $O(x) = \{T*x: T* \epsilon G*\}$ denote the orbit of an element x in X under G* .

1.2.1. Definition. The semi-group G is called ergodic if there exists a net of linear transformations $\{T_{\alpha}\}_{\alpha \epsilon D}$ with the following properties:

(a) For every x and all α , $T_{\alpha}x \epsilon \overline{O}(x)$.

(b) $\{T_{\alpha}\}$ is an equicontinuous family.

(c) $\lim_{\alpha} (TT_{\alpha}x - T_{\alpha}x) = \lim_{\alpha} (T_{\alpha}Tx - T_{\alpha}x) = 0$

for every x in X and all T in G . Such a net

$\{T_\alpha\}$ is called a system of <u>almost invariant integrals</u> for G .

1.2.2. <u>Theorem</u> (Eberlein). If G is ergodic, x an element of X , and $\{T_\alpha\}$ any system of almost invariant integrals, then the following conditions on an element y in X are equivalent:

(1) $y \in \overline{O}(x)$, and $Ty = y$ for all $T \in G$;

(2) $y = \lim_\alpha T_\alpha x$;

(3) $y = \lim_\alpha T_\alpha x$ weakly;

(4) y is a weak cluster point of $\{T_\alpha x\}$.

1.2.3. <u>Definition</u>. If G is ergodic, an element x in X is called <u>ergodic</u> with (unique) limit fixed point y if and only if $y = \lim_\alpha T_\alpha x$ for some system of almost invariant integrals $\{T_\alpha\}$ (It follows easily from 1.2.2 that this does not depend on the particular system of almost invariant integrals used). The set of all ergodic elements is called the <u>ergodic subspace</u> of G . If the ergodic subspace is all of X , G is called <u>asymptotically convergent</u>.

1.2.4. <u>Theorem</u>. Let G be ergodic. Then the transformation $T_\infty x = \lim_\alpha T_\alpha x$ is a bounded linear trans-

formation on the ergodic subspace of G and

$$T_\infty = T_\infty^2 = T_\infty U = UT_\infty \quad \text{if} \quad U \in \overline{G}^* \quad \text{or} \quad U = T_\alpha \quad \text{for some} \quad \alpha \ .$$

Section 3. <u>Summability</u>.

Let there be given a sequence $\{s_n\}_0^\infty$ in a normed linear space. Employing an infinite real matrix $A = (a_{nk})$, a sequence $\{t_n\}_0^\infty$ is formally constructed where $t_n = \sum_{k=0}^\infty a_{nk} s_k$. We say that the sequence $\{s_n\}$ is transformed into the sequence $\{t_n\}$ by the ''method'' A .

1.3.1. <u>Definition</u>. The method A is called <u>permanent</u> if for every convergent sequence $\{s_n\}$ the sequence $\{t_n\}$ exists and converges to the same limit as $\{s_n\}$.

1.3.2. <u>Theorem</u> (Toeplitz). A method $A = (a_{nk})$ is permanent if and only if the following three conditions are satisfied:

(i) $\lim\limits_{n \to \infty} a_{nk} = 0$ for $k = 1, 2, \ldots$.

(ii) $\sum\limits_{k=0}^\infty |a_{nk}| \leq M$ for each n , where M is independent of n .

(iii) $\lim\limits_{n \to \infty} \sum\limits_{k=0}^\infty a_{nk} = 1$.

Proof. The reader is referred to [31, p. 367].

The importance of permanent methods resides in the fact that a divergent sequence $\{s_n\}$ may be transformed into a convergent sequence $\{t_n\}$ by the method A. In such a case we say that $\{s_n\}$ is A-summable to the limit of the sequence $\{t_n\}$.

1.3.3. Definition. The convergence field of a summability method A is denoted by $A^{\#}$ and is defined to be the class of all sequences which are summable by A. Two methods A and B are said to be consistent if given any sequence $s \in A^{\#} \cap B^{\#}$, s is summable by A and B to the same value.

1.3.4. Definition. A method A is said to be stronger than a method B (denoted $A \geq B$) if every B-summable sequence is A-summable to the same value (i.e., if $B^{\#} \subset A^{\#}$ and A and B are consistent). A method A is said to be strictly stronger than a method B (denoted $A > B$) if A is stronger than B and $A^{\#} \neq B^{\#}$.

A series is said to be A-summable if its sequence of associated partial sums is A-summable.

1.3.5. <u>Definition</u>. A sequence $\{s_n\}$ of numbers is called <u>absolutely convergent</u> if $\sum_{n=1}^{\infty} |s_n - s_{n-1}| < \infty$. If a summability method A transforms every absolutely convergent sequence $\{s_n\}$ into an abolutely convergent sequence $\{t_n\}$ and if $s_n \to s$ implies $t_n \to t$, then A is said to be <u>absolutely permanent</u>.

1.3.6. <u>Theorem</u> (Knopp-Lorentz). Suppose that the summability method A transforms a series $\sum_{n=0}^{\infty} a_n$ into a series $\sum_{n=0}^{\infty} \alpha_n$, where $\alpha_n = \sum_{k=0}^{\infty} b_{nk} a_k$. The method A is absolutely permanent if and only if:

(i) $\sum_{n=0}^{\infty} |b_{nk}| < M$ for $k = 0,1,2,\ldots$

and (ii) $\sum_{n=0}^{\infty} b_{nk} = 1$ for $k = 0,1,2,\ldots$.

<u>Proof</u>. The reader is referred to [18].

Section 4. <u>Miscellaneous Results</u>.

In this section we list three theorems which will be useful in the later chapters.

1.4.1. _Theorem_. Let a_n be a sequence of numbers with $0 \le a_n < 1$ for $n = 1, 2, \ldots$, then $\prod_{n=1}^{\infty} (1 - a_n) = 0$ if and only if $\sum_{n=1}^{\infty} a_n = \infty$.

An amusing (and appearently new) proof of this theorem may be obtained as an application of 4.3.5.

The next result is a familiar theorem of Riemann.

1.4.2. _Theorem_. If $\{a_n\}$ is any null sequence of real numbers, then there exists a divergent sequence $\{\epsilon_n\}$ with $\epsilon_n = \pm 1$ for $n = 1, 2, \ldots$ such that

$\sum_{n=1}^{\infty} \epsilon_n a_n$ converges.

Finally, we mention some concepts from probability theory which will be useful in the final chapter. Consider a sequence of independent trials in which the probability of a success on the ith trial is p_i (i.e., a Poisson scheme of trials). Let X_i be the random variable indicating the outcome of the ith trial, that is

$$P\{X_i = 1\} = p_i \quad \text{and} \quad P\{X_i = 0\} = 1 - p_i \, ,$$

where $P(E)$ denotes the probability of the event E . Let $S_n = X_1 + \ldots + X_n$ and let μ_n and σ_n^2 denote mean

and variance of S_n ,

$$\mu_n = \sum_{i=1}^{n} p_i \quad \text{and} \quad \sigma_n^2 = \sum_{i=1}^{n} p_i(1 - p_i) .$$

1.4.3. <u>Theorem</u> (DeMoivre-LaPlace). Using the notations introduced above,

$$\lim_{n \to \infty} P\{t_1 < \frac{S_n - \mu_n}{\sigma_n} < t_2\} = \frac{1}{\sqrt{2\pi}} \int_{t_1}^{t_2} \exp(-u^2/2)du .$$

The convergence is uniform in t_1 and t_2 .

<u>Proof</u>. The reader is referred to [30, p. 294].

CHAPTER II

APPROXIMATION OF FIXED POINTS
OF NONLINEAR NONEXPANSIVE MAPS

In this chapter we show that under certain conditions a fixed point of a (nonlinear) nonexpansive mapping T on a uniformly convex Banach space may be approximated by the iterative sequence

2.0.1. $\quad v_{n+1} = (1-\lambda_n)v_n + \lambda_n T v_n$, $n = 1,2,\ldots$

where $0 \leq \lambda_n \leq 1$ and v_1 is arbitrary.

Section 1. <u>Certain Sequences in Uniformly Convex Spaces</u>.

H. H. Schaeffer [27] showed that there exists a function $\delta : (0,1) \times (0,2) \to (0,1)$ such that if X is uniformly convex and $x,y \in X$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x-y\| \geq \epsilon$, then $\|(1-\lambda)x+\lambda y\| \leq 1-\delta(\lambda,\epsilon)$ for $0 < \lambda < 1$. Schaeffer gave an indirect argument which did not determine the form of the function δ . The next lemma has a very simple direct proof which gives a form for δ .

2.1.1. <u>Lemma</u>. Let X be a uniformly convex space

with modulus of convexity δ. If $x,y \in X$, $\|x\| \leq 1$,
$\|y\| \leq 1$ and $\|x-y\| \geq \epsilon$, then
$\|(1-\lambda)x+\lambda y\| \leq 1-2\lambda(1-\lambda)\delta(\epsilon)$, for $0 \leq \lambda \leq 1$.

Proof. Clearly, we may assume that $\lambda \leq \frac{1}{2}$. We then have

$$
\begin{aligned}
\|(1-\lambda)x+\lambda y\| &= \|(1-2\lambda)x+2\lambda(x+y)/2\| \\
&\leq 1-2\lambda+2\lambda(1-\delta(\epsilon)) \\
&= 1-2\lambda\delta(\epsilon) \leq 1-2\lambda(1-\lambda)\delta(\epsilon) .
\end{aligned}
$$

2.1.2. **Definition.** A sequence of numbers $\{\lambda_n\}$ with $0 \leq \lambda_n \leq 1$ and $\Sigma\lambda_n(1-\lambda_n) = \infty$ will be called a D-sequence.

2.1.3. **Theorem.** Let X be a uniformly convex space. Suppose $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\|y_n\| \leq \|x_n\|$ for all n and $x_{n+1} = (1-\lambda_n)x_n+\lambda_n y_n$ where $\{\lambda_n\}$ is a D-sequence, then 0 is a cluster point of the sequence $\{x_n-y_n\}$.

Proof. First note that $\|x_{n+1}\| \leq \|x_n\|$ for all n, therefore $\lim \|x_n\| = d$ exists. If $d = 0$, then $\lim x_n = \lim y_n = 0$ and the result is clear. Hence we may suppose that $d > 0$. Suppose there is an $\epsilon > 0$ such that $\|x_n-y_n\| \geq \epsilon$ for all n. We then have

$$\frac{\|x_n - y_n\|}{\|x_n\|} \geq \frac{\epsilon}{\|x_n\|} \geq \frac{\epsilon}{\|x_1\|} \ .$$

Let $b = 2\delta\left(\frac{\epsilon}{\|x_1\|}\right)$; lemma 2.1.1 then gives

$$\|x_{n+1}\| = \|(1-\lambda_n)x_n + \lambda_n y_n\|$$

$$\leq \|x_n\|(1-\lambda_n(1-\lambda_n)b) \ .$$

Inductively we have

$$\|x_n\| \leq \|x_1\| \prod_{i=1}^{n-1} (1-\lambda_i(1-\lambda_i)b) \ , \text{ for } n > 1 \ .$$

Since $\{\lambda_n\}$ is a D-sequence, by 1.4.1 we then have $d = \lim \|x_n\| = 0$, a contradiction.

Section 2. Approximation in the Strong Topology.

2.2.1. **Theorem**. Suppose T is a quasi-nonexpansive mapping of a convex subset E of a uniformly convex space into itself; then 0 is a cluster point of the sequence $\{v_n - Tv_n\}$ for each $v_1 \in E$, where $\{v_n\}$ is defined by 2.0.1.

Proof. Let p be a fixed point of T . In theorem 2.1.3 set $x_n = v_n - p$ and $y_n = Tv_n - p$, it then follows that $v_n - Tv_n = x_n - y_n$ clusters at 0 .

If T is nonexpansive and has a fixed point, we get the following stronger result.

2.2.2. __Theorem.__ If T is a nonexpansive mapping of a convex subset E of a uniformly convex space into itself with at least one fixed point, then $\{v_n - Tv_n\}$ converges strongly to 0 for each $v_1 \in E$, where $\{v_n\}$ is defined by 2.0.1.

__Proof.__ By 2.2.1, 0 is a cluster point of $\{v_n - Tv_n\}$. Since T is nonexpansive we have

$$\|v_{n+1} - Tv_{n+1}\| = \|(1-\lambda_n)v_n - (1-\lambda_n)Tv_n + Tv_n - Tv_{n+1}\|$$

$$\leq (1-\lambda_n)\|v_n - Tv_n\| + \|v_n - v_{n+1}\| .$$

But $\|v_n - v_{n+1}\| = \|v_n - (1-\lambda_n)v_n - \lambda_n Tv_n\|$

$$= \lambda_n \|v_n - Tv_n\| .$$

Therefore

$$\|v_{n+1} - Tv_{n+1}\| \leq (1-\lambda_n)\|v_n - Tv_n\| + \lambda_n \|v_n - Tv_n\|$$

$$= \|v_n - Tv_n\| .$$

Hence $\lim \|v_n - Tv_n\| = 0$.

2.2.3. __Lemma.__ If T is quasi-nonexpansive then $\|v_{n+1} - p\| \leq \|v_n - p\|$ for any fixed point p of T .

Proof. We have

$$\|v_{n+1} - p\| = \|(1-\lambda_n)(v_n - p) + \lambda_n(Tv_n - p)\|$$

$$\leq (1-\lambda_n)\|v_n - p\| + \lambda_n\|Tv_n - p\|$$

$$\leq \|v_n - p\| .$$

2.2.4. Theorem. If T is a quasi-nonexpansive mapping of a closed convex subset E of a uniformly convex space into itself with at least one fixed point and I-T maps closed bounded subsets of E into closed subsets of E , then $\{v_n\}$ converges strongly to a fixed point of T for arbitrary v_1 in E .

Proof. Let V be the strong closure of the set $\{v_n\}$. Then V is closed and also bounded since $\|v_n - p\| \leq \|v_1 - p\|$ for all n , by 2.2.3. By 2.2.1, 0 is in the closure of (I-T)(V) . Since this set is closed by hypothesis, there is a subsequence $\{v_{n_k}\}$ such that $v_{n_k} \to q$ where (I-T)q = 0 . Lemma 2.2.3 then shows that $\lim v_n = q$.

2.2.5. Corollary. If T is a nonexpansive mapping of a closed convex subset E of a uniformly convex space into itself such that T(E) is relatively compact, then $\{v_n\}$ converges strongly to a fixed point of T

for arbitrary v_1 in E .

Proof. Schauder's theorem 1.1.7 guarantees the existence of a fixed point of T . Since T(E) is relatively compact, I-T maps closed bounded sets into closed sets. The proof is finished by applying 2.2.4.

Petryshyn [27] has called a mapping T demi-compact if given a bounded sequence $\{u_n\}$ such that $\{(I-T)(u_n)\}$ converges then the sequence $\{Tu_n\}$ contains a convergent subsequence. It is clear that every compact mapping is demi-compact; it is also easy to see that if T is demi-compact then I-T maps closed bounded sets into closed sets. The identity mapping on an infinite dimensional space has the latter property but is not demi-compact. Theorem 2.2.4 was proved by Browder and Petryshyn [5] under the assumptions that T is demi-compact and $\lambda_n = \lambda$, with $0 < \lambda < 1$, for all n . Dotson[11] showed that the same proof carries through under the assumption that λ_n is bounded away from 0 and 1 . Petryshyn has shown that a mapping T in Hilbert space is demi-compact if it satisfies any one of the following three conditions:

(i) $Re(Tx-Ty, x-y) \leq a\|x-y\|^2$ with $a < \frac{1}{2}$.

(ii) $Re(Tx-Ty, x-y) \leq a\|Tx-Ty\|^2$ with $a \leq \frac{1}{2}$.

(iii) $(I-T)^{-1}$ exists and is continuous on its range.

The next two propositions give another class of demi-compact mappings.

2.2.6. **Proposition.** If T is demi-compact and K is compact then T + K is demi-compact.

Proof. The definition of demi-compactness may be verified in a straightforward manner.

2.2.7. **Proposition.** Every strict contraction on a Hilbert space is demi-compact.

Proof. Suppose $\|Tx-Ty\| \leq \alpha\|x-y\|$ for all $x,y \in D(T)$ with $\alpha < 1$. Then

$$(1-\alpha)\|x-y\| = \|x-y\| - \alpha\|x-y\|$$
$$\leq \|x-y\| - \|Tx-Ty\|$$
$$\leq \|(I-T)x - (I-T)y\| .$$

Hence T satisfies condition (iii).

If T has a unique fixed point, then the next corollary gives extra information about the convergence of the iterative sequence.

2.2.8. **Corollary.** Suppose T is a quasi-nonexpansive

mapping of a closed convex subset E of a uniformly
convex space into itself with a unique fixed point p .
Let $0 < \lambda < 1$ and define $\{v_n(\lambda)\}$ by
$v_{n+1}(\lambda) = (1-\lambda)v_n(\lambda) + \lambda Tv_n(\lambda)$, where $v_1 \in E$ is
arbitrary.   Then $\lim v_n(\lambda) = p$ and the convergence
is uniform in λ on compact subsets of $(0,1)$.

    <u>Proof</u>. Setting $\lambda_n = \lambda$ for all n in 2.2.4 we
see that $v_n(\lambda) \to p$ as $n \to \infty$ for each $\lambda \in (0,1)$.
The functions $f_n(\lambda) = \|v_n(\lambda)-p\|$ are continuous and
converge pointwise and monotonically to 0 . The
theorem then follows by an application of Dini's theorem.

    We have shown that the condition $\Sigma\lambda_n(1-\lambda_n) = \infty$ is
sufficient for the convergence of the sequence 2.0.1
for a wide class of mappings. We now show that a kind
of converse holds.

    2.2.9. <u>Theorem</u>. A sequence $\{\lambda_n\}$ with $0 \leq \lambda_n \leq 1$
has the property that the sequence 2.0.1 converges
strongly to a fixed point irrespective of the choice
of X, E and T satisfying the hypotheses of 2.2.4
if and only if $\{\lambda_n\}$ is a D-sequence.

    <u>Proof</u>. We need only to show the necessity. For

this it suffices to produce an example of X, E, v_1, and T satisfying 2.2.4 such that the convergence of $\{v_n\}$ implies $\Sigma \lambda_n (1-\lambda_n) = \infty$. Let X be the complex plane, E the closed unit disc and $v_1 = 1$. Choose θ such that $0 < \theta < \pi$ and $4\lambda_n(1-\lambda_n) \sin^2 (\frac{\theta}{2}) \neq 1$ for all n. Let T be the rotation of E about the origin by θ radians. For any two complex numbers u and w with $|u|^2 = |w|^2 = r$ and for $0 \leq t \leq 1$ we have $|(1-t)u+tw|^2 = r-t(1-t)|u-w|^2$. Using this we see that

$$
\begin{aligned}
|v_{n+1}|^2 &= |(1-\lambda_n)v_n - \lambda_n Tv_n|^2 \\
&= |v_n|^2 - \lambda_n(1-\lambda_n)|v_n-Tv_n|^2 \\
&= |v_n|^2 \{1-4\lambda_n(1-\lambda_n) \sin^2 (\frac{\theta}{2}) \} .
\end{aligned}
$$

Hence,

$$
|v_n|^2 = \prod_{k=1}^{n-1} \{1-4\lambda_k(1-\lambda_k) \sin^2 (\frac{\theta}{2}) \}
$$

for $n > 1$. Since 0 is the unique fixed point of T and v_n converges to a fixed point by assumption,

$$
\prod_{k=1}^{\infty} \{1-4\lambda_k(1-\lambda_k) \sin^2 (\frac{\theta}{2})\} = 0 .
$$

But this is equivalent to $\Sigma \lambda_n(1-\lambda_n) = \infty$, by 1.4.1.

Section 3.   <u>Approximation in the Weak Topology</u>.

We now generalize a result of Opial and show that fixed
points of nonexpansive maps in certain uniformly convex
spaces may be weakly approximated by the sequence 2.0.1
without assuming any ''compactness'' condition on the
mapping.

2.3.1.   <u>Lemma</u>.   The fixed point set of a quasi-nonexpansive
mapping in a strictly convex space is convex.

   <u>Proof</u>.   Let   T   be quasi-nonexpansive with fixed
points   x   and   y .   If $0 \leq t \leq 1$, then

$$\|x-y\| \leq \|x-T((1-t)x+ty)\| + \|T((1-t)x+ty)-y\|$$

$$\leq t\|x-y\| + (1-t)\|x-y\| = \|x-y\|$$

since   T   is quasi-nonexpansive.   Hence $T((1-t)x+ty)$
lies on the segment   [x,y] .   Since   x   and   y   are
fixed points and   T   maps no point away from a fixed
point it follows that   (1-t)x+ty   is a fixed point.

   The next two lemmas are due to Opial [25].

2.3.2.   <u>Lemma</u>.   Let   X   be a uniformly convex space with
weakly continuous duality mapping   J   (see 1.1.9).   If

the sequence $\{x_n\}$ is weakly convergent to x_0, then for any $x \neq x_0$,

$$\underline{\lim} \, \|x_n - x\| > \underline{\lim} \, \|x_n - x_0\| \, .$$

<u>Proof</u>. In the equation

$$(J(x_n - x_0), x_n - x_0) = (J(x_n - x_0), x_n - x) + (J(x_n - x_0), x - x_0)$$

the last term goes to 0 as $n \to \infty$ since J is weakly continuous. We then have

$$\underline{\lim} \, \mu(\|x_n - x_0\|)\|x_n - x_0\| \leq \underline{\lim} \, |(J(x_n - x_0), x_n - x)|$$

$$\leq \underline{\lim} \, \|J(x_n - x_0)\| \cdot \|x_n - x\|$$

$$= \underline{\lim} \, \mu(\|x_n - x_0\|)\|x_n - x\| \, .$$

Hence $\underline{\lim} \, \|x_n - x_0\| \leq \underline{\lim} \, \|x_n - x\|$. The equality cannot hold since if this were the case and y was the midpoint of x_0 and x, then taking a subsequence x_{n_k} such that

$$\|x_{n_k} - x_0\| \to \underline{\lim} \, \|x_n - x_0\| \quad \text{and} \quad \|x_{n_k} - x\| \to \underline{\lim} \, \|x_n - x\| \, ,$$

we would have

$$\|x_{n_k} - y\| = \|\tfrac{1}{2}(x_{n_k} - x) + \tfrac{1}{2}(x_{n_k} - x_0)\|$$

$$\leq \max (\|x_{n_k} - x\|, \|x_{n_k} - x_0\|)(1 - \delta(\|x - x_0\|)) \, .$$

Therefore, $\varliminf \|x_n-y\| < \varliminf \|x_n-x_0\|$, which is impossible.

2.3.3. <u>Lemma</u>. Let T be a nonexpansive mapping on a uniformly convex space X with weakly continuous duality mapping. If $\{x_n\}$ converges weakly to x_0 and $\{(I-T)x_n\}$ converges strongly to y_0 , then $(I-T)x_0 = y_0$.

$\quad$ <u>Proof</u>. We have $\varliminf \|x_n-x_0\| \geq \varliminf \|Tx_n-Tx_0\|$ $= \varliminf \|x_n-y_0-Tx_0\|$. Hence by 2.3.2 we have $x_0 = y_0 + Tx_0$.

$\quad$ The statement of the next theorem is more general than the theorem of Opial [25], but the proof, except for very slight modifications, is the same as that given by Opial.

2.3.4. <u>Theorem</u>. Let T be a mapping of a closed convex subset E of a uniformly convex Banach space into itself with a nonempty set of fixed points. Suppose $\{x_n\}$ is a sequence in E satisfying:

$\quad$ (a) $\|x_{n+1}-y\| \leq \|x_n-y\|$ for each fixed point y of T and for all n .

$\quad$ (b) $\{(I-T)x_n\}$ converges strongly to 0 .

Then $\{x_n\}$ converges weakly to a fixed point of T .

Proof. Let F be the fixed point set of T .
For $y \in F$ let $d(y) = \lim \|x_n - y\|$. Since d is
lower semicontinuous and F is closed and convex,
the set $F_d = \{y \in F : d(y) \leq d\}$ is closed, convex and
bounded. Since X is reflexive there is a smallest
δ for which F_δ is nonempty (1.1.5). Then
$F_\delta = \{y_0\}$, since if F_δ contained two points then the
average of these two points would belong to an F_d
with $d < \delta$ by uniform convexity. If $\{x_n\}$ does not
converge weakly to y_0 then by 1.1.4 there would
exist a subsequence $\{x_{n_k}\}$ converging weakly to
$y \neq y_0$. By (b) $(I-T)x_{n_k} \to 0$, hence by 2.3.3 y
is a fixed point of T . By 2.3.2 we then have

$$\delta = d(y_0) = \lim \|x_{n_k} - y_0\| > \lim \|x_{n_k} - y\| = d(y) ,$$

a contradiction.

If T is nonexpansive and $\{v_n\}$ is defined by
2.0.1, then setting $x_n = v_n$ we see by 2.2.3 and 2.2.2
that (a) and (b) of the previous theorem are satisfied.
Hence we have

2.3.5. Theorem. Let T be a nonexpansive mapping

of a closed convex subset E of a uniformly convex
Banach space with weakly continuous duality mapping
into itself. If T has a fixed point, then the process
2.0.1, where $\{\lambda_n\}$ is a D-sequence, converges weakly
to a fixed point of T for arbitrary v_1 in E .

Note that a fixed point of T is guaranteed if
either E is bounded (1.1.8) or if T(E) is relatively
compact (1.1.7). Also, since weak and strong con-
vergence are equivalent in finite dimensional spaces,
the proof of 2.2.9 shows that for the weak convergence
of 2.0.1 for each T and E satisfying the hypotheses
of 2.3.5 it is necessary that $\{\lambda_n\}$ be a D-sequence.
Finally, we note that Browder [3] has shown that the
spaces ℓ^p with $1 < p < \infty$ have weakly continuous
duality mappings, however, Opial [25] has pointed out
that the spaces L^p with $1 < p \neq 2$ do not have weakly
continuous duality mappings.

Section 3. A Theorem on Variational Inequalities.

Lions and Stampacchia [21] have made extensive in-
vestigations of problems of the following type.

Problem: Let K be a closed bounded convex subset
of a real Hilbert space H . Suppose A is a mapping

from K into H and f ∈ H . Find u_0 ∈ K such that
$(Au_0 - f, v - u_0) \geq 0$ for all v in K .

Such an inequality is called a variational inequality
and u_0 , if it exists, is called a solution of the
variational inequality. In particular if A is the
identity mapping we have

2.3.1. $(u_0 - f, v - u_0) \geq 0$ for all v in K .

It is well known that this is equivalent to saying
that u_0 is the projection of f on K , in fact,
the inequality says that the ''angle'' between $v - u_0$
and $f - u_0$ can never be acute for any v in K .
Other motivating examples and references may be found
in the paper of Stampacchia [29].

The next lemma is well known and follows easily
from 2.3.1.

2.3.2. Lemma. The operation of projection onto a
closed convex set in a Hilbert space is nonexpansive.

Proof. Let P be such a projection and x,y be
in H . Applying 2.3.1 with f = x and v = Py and
then with f = y and v = Px we obtain

$$(Px, Py - Px) \geq (x, Py - Px)$$

and

$$-(Py,Py-Px) \geq (-y,Py-Px) .$$

Adding these inequalities we have

$$\|Py-Px\|^2 \leq (y-x,Py-Px) ,$$

hence the result.

We now show that solutions of variational inequalities involving β-monotone operators (see 1.1.10) may be approximated by using convex iteration procedures.

2.3.2. <u>Theorem</u>. Let K be a closed bounded convex set in a real Hilbert space H and $A:K \to H$ be a β-monotone operator with constant c . Then a solution of the variational inequality $(Au_f-f,v-u_f) \geq 0$, for all v in K , exists for each f in H and this solution may be approximated by the convex iteration procedure.

<u>Proof</u>. The variational inequality is equivalent to $(2cAu_f-2cf,v-u_f) \geq 0$ for all v in K . i.e.

$$(u_f-(u_f-2cAu_f+2cf),v-u_f) \geq 0 \text{ for all } v \text{ in } K .$$

Hence, $u_f = P(u_f-2cAu_f+2cf)$ where P is the pro-

jection on K . Hence we seek a fixed point of the mapping

$$Tx = P(x-2cAx+2cf) .$$

Now,

$$(Ax_1-Ax_2, x_1-x_2) \geq c\|Ax_1-Ax_2\|^2$$

implies that

$$(2cAx_1-2cAx_2, x_1-x_2) \geq \tfrac{1}{2}\|2cAx_1-2cAx_2\|^2$$

Therefore,

$$\|(x_1-2cAx_1+2cf)-(x_2-2cAx_2+2cf)\|^2$$

$$= \|x_1-x_2\|^2-2(x_1-x_2, 2cAx_1-2cAx_2)+\|2cAx_1-2cAx_2\|^2$$

$$\leq \|x_1-x_2\|^2-2\cdot\tfrac{1}{2}\|2cAx_1-2cAx_2\|^2+\|2cAx_1-2cAx_2\|^2$$

$$= \|x_1-x_2\|^2 .$$

Hence the map $x \to x-2cAx+2cf$ is nonexpansive. But P is nonexpansive by 2.3.2, therefore T is non-expansive. Since T maps the closed bounded convex set K into itself a fixed point exists by 1.1.7 and may be weakly approximated by using 2.3.5. If K spans a finite dimensional space then 2.2.4 may be applied to obtain an approximating sequence in the strong topology.

<u>Remark</u>. The method described in the above theorem is not effective in general since the element Px is not always computable. However, if K is a ball or a finite dimensional linear manifold, the procedure is effective.

CHAPTER III

ITERATIVE SOLUTION OF

LINEAR OPERATOR EQUATIONS

In the present chapter we consider the problem of approximating solutions to linear operator equations of the type

$$u - Tu = f$$

where T is a linear nonexpansive or asymptotically bounded (see 1.1.11) mapping. The main tool is the Eberlein Ergodic theorem. We apply these results to obtain theorems on the approximation of solutions of certain linear operator equations of the type $Au = g$, where A is a positive linear operator on a Hilbert space H and equations of the type $Bu = Pg$ where B is an arbitrary bounded linear on H and P is the projection onto the closure of the range of B. We also obtain a theorem on the approximation in L^p norm of solutions of a certain functional equation in L^∞.

Section 1. <u>The Operator Equation $u - Tu = f$</u>.

A point u is a solution of the equation

u - Tu = f if and only if u is a fixed point of the mapping T_f where $T_f x = Tx + f$. Hence we consider the iterative sequence

3.1.0. $v_{n+1} = (1-\lambda_n)v_n + \lambda_n T_f v_n$, $n = 1,2,\ldots$

where v_1 is arbitrary and $\{\lambda_n\}$ is a D-sequence (see 2.1.2). It is easy to see that this can also be written as

3.1.1. $v_n = A_n(T)v_1 + B_n(T)f$

where

3.1.2. $A_1(T) = I$; $A_{n+1}(T) = ((1-\lambda_n)I + \lambda_n T)A_n(T)$

3.1.3. $B_1(T) = 0$; $B_{n+1}(T) = ((1-\lambda_n)I + \lambda_n T)B_n(T) + \lambda_n I$.

3.1.4. <u>Lemma</u>. $I - A_n(T) = (I-T)B_n(T)$ for all n .

$\underline{Proof}$. It is clear that $I - A_1(T) = (I-T)B_1(T)$.
Suppose that $I - A_k(T) = (I-T)B_k(T)$, then

$$(I-T)B_{k+1}(T) = ((1-\lambda_k)I + \lambda_k T)(I-T)B_k(T) + (I-T)\lambda_k I$$

$$= ((1-\lambda_k)I + \lambda_k T)(I - A_k(T)) + \lambda_k(I-T)$$

$$= I - ((1-\lambda_k)I + \lambda_k T)A_k(T)$$

$$= I - A_{k+1}(T) .$$

3.1.5. <u>Lemma</u>. Let T be an asymptotically bounded linear operator on a Banach space X; $G = \{I,T,T^2,...\}$. A sequence $\{J_n\} \subseteq G^*$ is a system of almost invariant integrals for G if and only if $\{(I-T)J_n\}$ converges to zero in the strong operator topology, i.e., $(I-T)J_n x \to 0$ for each x in X.

Proof. Suppose $(I-T)J_n$ converges to the zero operator in the strong operator topology. We must verify that conditions (a), (b) and (c) of 1.2.1 are satisfied. Condition (a) is immediate since $J_n \in G^*$ for all n. Since T is asymptotically bounded and $J_n \in G^*$, $\|J_n\| \leq M$ for all n. Hence, condition (b) is satisfied. Since $(I-T)J_n x \to 0$ a trivial induction argument shows that $\lim (S-I)J_n x = \lim J_n(S-I)x = 0$

for all x in X and S in G, therefore, condition (c) holds. Conversely, if condition (c) holds we immediately have that $(I-T)J_n x \to 0$ for each x in X.

3.1.6. <u>Theorem</u>. Let X be a uniformly convex Banach space. For T a nonexpansive linear operator on X, let $G(T) = \{I,T,T^2,...\}$ and let $\{A_n(T)\}$ be defined by 3.1.2. If $\{\lambda_n\}$ is a D-sequence, then $\{A_n(T)\}$ is a system of almost invariant integrals for $G(T)$.

Proof. If $\{\lambda_n\}$ is a D-sequence, then $(I-T)A_n(T)$ converges to the zero operator in the strong operator topology by 2.2.2 (in fact, T clearly has a fixed point and $v_n = A_n(T)v_1$) . The result now follows from 3.1.5.

3.1.7. Corollary. Let H be a Hilbert space of real dimension at least two. Then $\{A_n(T)\}$ is a system of almost invariant integrals for $G(T)$ irrespective of the linear nonexpansive operator T if and only if $\{\lambda_n\}$ is a D-sequence.

Proof. The sufficiency follows from 3.1.6. For the converse, we may construct as in 2.2.9 a linear isometry T on a subspace M of two real dimensions with the property that $(I-T)A_n(T)x \to 0$ for each x in M implies that $\{\lambda_n\}$ is a D-sequence. This mapping may be linearly extended to H without changing its norm by the Hahn-Banach Theorem. Hence the result follows.

3.1.8. Theorem. If T is an asymptotically bounded linear operator on a reflexive Banach space and $\lambda_n = \lambda$ for all n where $0 < \lambda < 1$, then $\{A_n(T)\}$ in 3.1.1 is a system of almost invariant integrals for $G(T)$.

<u>Proof</u>. In this case we note that $A_n(T) =$

$\sum\limits_{k=0}^{n} \binom{n}{k}(1-\lambda)^{n-k}\lambda^k T^k$. Hence, if $\|T^n\| \leq M$ for all n ,

then

$$\|(I-T)A_n(T)\| \leq (1-\lambda)^n M + \lambda^n M$$

$$+ M \sum\limits_{k=1}^{n} |\binom{n}{k}(1-\lambda)^{n-k}\lambda^k - \binom{n}{k-1}(1-\lambda)^{n-k+1}\lambda^{k-1}| .$$

It is easy to see that the terms in absolute values are

nonnegative for $k \leq [\lambda(n+1)]$ and nonpositive otherwise.

Hence the sum may be writen as

$$\sum\limits_{k=1}^{[\lambda(n+1)]} \{\binom{n}{k}(1-\lambda)^{n-k}\lambda^k - \binom{n}{k-1}(1-\lambda)^{n-k+1}\lambda^{k-1}\}$$

$$+ \sum\limits_{k=[\lambda(n+1)]+1}^{n} \{\binom{n}{k-1}(1-\lambda)^{n-k+1}\lambda^{k-1} - \binom{n}{k}(1-\lambda)^{n-k}\lambda^k\}$$

$$= 2\binom{n}{[\lambda(n+1)]}(1-\lambda)^{n-[\lambda(n+1)]}\lambda^{[\lambda(n+1)]} - (1-\lambda)^n - \lambda^n .$$

Therefore,

$$\|(I-T)A_n(T)\| \leq 2M\binom{n}{[\lambda(n+1)]}(1-\lambda)^{n-[\lambda(n+1)]}\lambda^{[\lambda(n+1)]}$$

$$\leq \frac{2M}{1-\lambda} \binom{n+1}{[\lambda(n+1)]}(1-\lambda)^{n+1-[\lambda(n+1)]}\lambda^{[\lambda(n+1)]} .$$

By Stirling's formula, $j! \sim \sqrt{2\pi j} \; j^j e^{-j}$, we have

$$\binom{n}{[\lambda n]} \sim \frac{\sqrt{n}}{\sqrt{(n-[\lambda n])2\pi[\lambda n]}} \frac{n^n}{(n-[\lambda n])^{n-[\lambda n]}[\lambda n]^{[\lambda n]}}$$

$$\leq (2\pi(1-\lambda)[\lambda n])^{-\frac{1}{2}}(1-\lambda)^{[\lambda n]-n}\lambda^{-[\lambda n]}(\frac{\lambda n}{[\lambda n]})^{[\lambda n]} .$$

Since $(\frac{\lambda n}{[\lambda n]})^{[\lambda n]} \leq (\frac{\lambda n}{\lambda n-1})^{\lambda n} \to e$, we see that

$\lim \|(I-T)A_n(T)\| = 0$. The result now follows from 3.1.5.

We shall obtain a common generalization of 3.1.6 and 3.1.8 in the final chapter.

The proofs of the next two theorems are specializations of arguments of Dotson [12, Th.3] which in turn were appearently modeled on a proof of Browder and Petryshyn [6].

3.1.9. **Theorem.** Let T be an asymptotically bounded mapping on a reflexive Banach space X and assume that {$A_n(T)$} is a system of almost invariant integrals for G(T) . If f is in the range of the operator I-T , then the sequence 3.1.1 converges strongly to a solution of the equation u - Tu = f .

Proof. We first note that G(T) is asymptotically

convergent (see 1.2.3), in fact, for each x in X the sequence $\{A_n(T)x\}$ is bounded and hence contains a weakly convergent subsequence by 1.1.4. Theorem 1.2.2 then shows that $\lim A_n(T)x = T_\infty x$ exists. By assumption there is a w such that $(I-T)w = f$. We then have $B_n(T)f = (I-T)B_n(T)w = (I-A_n(T))w$. Therefore $B_n(T)f \to (I-T_\infty)w$ and $v_n = A_n(T)v_1 + B_n(T)f \to w+T_\infty(v_1-w)$. But $(I-T)(w+T_\infty(v_1-w))$ $= f+(I-T)T_\infty(v_1-w) = f$ by 1.2.4.

3.1.10. <u>Theorem</u>. Under the assumptions of 3.1.9, if for some v_1 the sequence $\{v_n\}$ clusters weakly at y and $\Sigma\lambda_n = \infty$, then $y - Ty = f$ and $\{v_n\}$ converges strongly to y .

<u>Proof</u>. Suppose $v_{n_k} \overset{w}{\to} y$. Since

$A_{n_k}(T)v_1 \to T_\infty v_1$ we have $B_{n_k}(T)f \overset{w}{\to} y - T_\infty v_1$.

Now, $T_\infty B_1(T) = 0$ and

$$T_\infty B_{n+1}(T) = T_\infty((1-\lambda_n)I + \lambda_n T)B_n(T) + \lambda_n T_\infty$$

$$= T_\infty B_n(T) + \lambda_n T_\infty$$

hence $T_\infty B_{n_k}(T)f = (\overset{n_k-1}{\underset{i=1}{\Sigma}} \lambda_i)T_\infty f$. Since T_∞ is weakly

continuous and $B_{n_k}(T)f$ is weakly convergent,

$\{T_\infty B_{n_k}(T)\}$ is bounded. Hence $T_\infty f = 0$ since

$\Sigma \lambda_n = \infty$. Therefore, $(I-T)B_{n_k}f = f-A_{n_k}f \to f-T_\infty f = f$.

Also $(I-T)B_{n_k}f \overset{W}{\to} (I-T)(y-T_\infty v_1) = y-Ty$, and hence

$y - Ty = f$. The rest follows from 3.1.9.

Consider now a finite measure space (S,Σ,μ) and a measure preserving map φ [14,p.667] of S . As an application of the above results we show that convex iteration procedures can be used to approximate ~~in~~ L^p norm a solution $h \in L^\infty$ of the equation

3.1.11. $\quad h(s) - h(\varphi(s)) = f(s)$ a.e.

where $f \in L^\infty$ is fixed. F. E. Browder [4] has shown that a solution to this equation exists if and only if

$$\mu - \text{ess. sup. } | \sum_{j=0}^{k} f(\varphi^j(s))| \text{ is uniformly bounded for}$$

all k . Accordingly we consider the process

3.1.12. $\quad v_{n+1} = (1-\lambda_n)v_n + \lambda_n(v_n \circ \varphi + f)$

where $v_1 \in L^\infty$ is arbitrary and $\{\lambda_n\}$ is a D-sequence.

3.1.13. <u>Theorem</u>. Let (S,Σ,μ) be a finite measure space [14,p.126] φ a measure preserving map of S .

Suppose $f \in L^\infty(S,\Sigma,\mu)$ is such that

μ- ess. sup. $|\sum_{j=0}^{k} f(\varphi^j(s))|$ is uniformly bounded.

Then there exists $h \in L^\infty(S,\Sigma,\mu)$ satisfying 3.1.11 and the sequence $\{v_n\}$ of 3.1.12 converges to a solution of 3.1.11 in the L^p norm for $1 < p < \infty$.

Proof. Fix p with $1 < p < \infty$ and define $T:L^p \to L^p$ by $Tg = g \circ \varphi$. Since φ is measure preserving, T is a linear isometry. By Browder's theorem $f \in (I-T)L^p$ and since L^p is uniformly convex we have $v_n \to h \in L^p$ in the L^p norm by 3.1.9, where h is a solution of 3.1.11. Extracting an almost everywhere convergent subsequence we see that $h \in L^\infty(S,\Sigma,\mu)$. Hence the theorem is proved.

3.1.14. Theorem. In the context of 3.1.13,

μ - ess. sup. $|\sum_{j=0}^{k} f(\varphi^j(s))|$ is uniformly bounded if and only if there is a $v_1 \in L^\infty$ such that the sequence $\{v_n\}$ is bounded in L^p norm for some p with $1 < p < \infty$.

Proof. The necessity follows immediately from the previous theorem. The sufficiency follows from Browder's theorem, theorem 3.1.10 and the Eberlein-Šmulian theorem.

de Figueiredo and Karlovitz [9] have given another
iterative procedure for approximating solutions of 3.1.11
in the L^p norm, however, the intial point of their
sequence must be the function f and the convergence
of the process is not monotone, as it is for the convex
iterates (2.2.3).

Section 2. Operator Equations in Hilbert Space.

We now specailize the considerations of the
previous section to Hilbert space. Recall that a
bounded linear operator A on a Hilbert space is
called positive (denoted $A \geq 0$) if $(Ax,x) \geq 0$
for every x .

3.2.1. Theorem. If T is nonexpansive and self-adjoint
and f is in the range of I-T then the sequence
$\{v_n\}$ of 3.1.1 converges to the solution of u - Tu = f
which is closest to v_1 .

Proof. Let $M = \{u:u-Tu = f\}$. By 3.1.7
$A_n(T)v_1 \to v \in N = \{x:x-Tx = 0\}$. If $v_1 \in N$ then
$A_n(T)v_1 = v_1$ for all n and thus $A_n(T)v_1 \to v_1$.
If v_1 is in the range of I-T , then $A_n(T)v_1$
converges to 0 by 3.1.5. Suppose $v_1 = x+y$ where
x is in the range of I-T and $\|y\| < \epsilon$, then

$\|\lim_n A_n(T)v_1\| = \|\lim_n A_n(T)y\| < \epsilon$. Hence we see that

if v_1 is in the closure of the range of $I-T$, then

$A_n(T)v_1$ converges to 0 . Since T is self-adjoint,

$N^{\perp}$ is the closure of the range of $I-T$, therefore,

we see that $A_n(T)$ converges to P_N , the projection

on N , in the strong operator topology.

As was seen in the proof of 3.1.9 we then have

$B_n(T)f \to (I-P_N)u$ for some u in M . Therefore,

we have $v_n \to P_N v_1 + P_{N^{\perp}} u$ but this last element is

easily seen to be $P_M v_1$, the point in M nearest to

the initial point v_1 .

3.2.2. **Theorem.** Let $A \geq 0$ be a bounded linear

operator on a Hilbert space H . Let $f \in H$ and let

$\{\mu_n\}$ be a D-sequence. Define $\{v_n\}$ by

$$v_{n+1} = v_n - \lambda_n(Av_n + f)$$

where $\lambda_n = 2\mu_n/\|A\|$.

(a) If $f \in R(A)$, then $\{v_n\}$ converges to the solution

 of $Au = f$ which is nearest to v_1 .

(b) If $f \notin R(A)$, then for each v_1 in H , the

 sequence $\{v_n\}$ contains no weakly convergent

 subsequence.

Proof. Note that

$$v_n - \lambda_n(Av_n + f) = (1-\mu_n)v_n + \mu_n(Tv_n + \tilde{f})$$

where $T = I - (2/\|A\|)A$ and $\tilde{f} = (2/\|A\|)f$. Hence $T = T^*$ and since $0 \leq A \leq \|A\|I$, we have $-I \leq T \leq I$, i.e. $\|T\| \leq 1$. Therefore by 3.1.9, if $\tilde{f} \in R(I-T)$ then $v_n \to u$ where u is the solution of $u - Tu = \tilde{f}$ nearest to v_1 . But this is none other that the solution of $Au = f$ which is nearest to v_1 . Since $\tilde{f} \in R(I-T)$ if and only if $f \in R(A)$, part (a) follows. Also $f \notin R(A)$ implies $\tilde{f} \notin R(I-T)$ and hence $\{v_n\}$ has no weakly convergent subsequence by 3.1.10

It may happen that the equation $Bu = h$ has no solution, i.e. $h \notin R(B)$. In this case it might be of interest to solve the equation $Bu = Ph$, were P is the projection onto the closure of the range of B .

3.2.3. Corollary. Let B be a bounded linear operator on a Hilbert space H and let P be the projection of H onto the closure of the range of B . For h in H and $\{\mu_n\}$ a D-sequence, define $\{v_n\}$ by

$$v_{n+1} = v_n - \lambda_n(B^*Bv_n + h)$$

where $\lambda_n = (2/\|B^*B\|)\mu_n$.

(a) If the equation $Bu = Ph$ has a solution, then $\{v_n\}$ converges to a solution for each v_1 in H .

(b) If the equation $Bu = Ph$ has no solution, then for each v_1 in H , the sequence $\{v_n\}$ contains no weakly convergent subsequence.

Proof. We denote the null space of an operator A by $N(A)$. If $Bu = Ph$, then $B*Bu = B*Ph = B*(Ph+(I-P)h) = B*h$ since $I-P$ projects onto $\overline{R(B)}^\perp = N(B*)$. Conversely, if $B*Bu = B*h$ then $Bu - h \in N(B*)$, therefore $P(Bu-h) = 0$, i.e., $Bu = Ph$. Consequently, we may apply theorem 3.2.2 with $A = B*B$ and $f = B*h$.

Note that if condition (b) of 3.2.2 or 3.2.3 holds then the sequence $\{v_n\}$ is unbounded for each v_1 by the Eberlein-Šmulian theorem.

CHAPTER IV

A CLASS OF SUMMABILITY METHODS
RELATED TO CONVEX ITERATION PROCEDURES.

We now consider again the process 2.0.1, but we label the initial point v_0,

$$v_{n+1} = (1-\lambda_{n+1})v_n + \lambda_{n+1}Tv_n .$$

If T is affine (i.e., $T((1-t)x+ty) = (1-t)Tx + tTy$ for $0 \le t \le 1$) we have

$$v_n = \prod_{k=1}^{n} ((1-\lambda_k)I + \lambda_k T)v_0$$

$$= \sum_{k=0}^{n} a_{nk}T^k v_0 .$$

Where,

$$[a_{nk}] = \begin{bmatrix} 1 & 0 & 0 \, .. \\ 1-\lambda_1 & \lambda_1 & 0 \, .. \\ (1-\lambda_1)(1-\lambda_2) & \lambda_1(1-\lambda_2)+\lambda_2(1-\lambda_1) & \lambda_1\lambda_2 \, .. \\ . & . & . \, .. \\ . & . & . \, .. \end{bmatrix} .$$

It is easy to see that the matrix elements a_{nk} are given by the following iterative formulae:

$$a_{00} = 1 \, , \quad a_{0k} = 0 \quad \text{if} \quad k > 0$$

4.0.1. $\quad a_{n+1,0} = (1-\lambda_{n+1})a_{n0}$

4.0.2. $\quad a_{n+1,k+1} = (1-\lambda_{n+1})a_{n,k+1} + \lambda_{n+1}a_{nk}$.

Given a sequence $\Lambda = (\lambda_1, \lambda_2, \dots)$ with $0 < \lambda_n < 1$, the matrix above is denoted by $\Gamma(\Lambda)$. In this chapter we will show that the summability method associated with $\Gamma(\Lambda)$ is permanent if and only if $\Sigma\lambda_n = \infty$. This class of summability methods will be studied in detail. If $\lambda_n = \lambda$ for all n , then this is the Euler-Knopp method [16] with parameter λ and will be denoted by Γ_λ (actually, Knopp considered the special case $\lambda_n = 2^{-p}$ for $n = 1,2,\dots$ where p is a fixed positive integer).

Section 1. Basic Properties.

4.1.1. Definition. Let $\Lambda = (\lambda_1, \lambda_2, \dots)$ be a sequence of numbers with $0 < \lambda_n < 1$ for all n . The Γ-method associated with Λ is the summability method associated with the matrix $\Gamma(\Lambda)$ defined by 4.0.1 and 4.0.2.

Hereafter, unless otherwise specified, the symbol Λ will denote a sequence $(\lambda_1, \lambda_2, \dots)$ where $0 < \lambda_n < 1$

for all n. We now prove a numerical lemma which will be useful in the sequel.

4.1.2. <u>Lemma</u>. Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers such that $\lim_n b_n = 0$ and

$$a_{n+1} = (1-t_{n+1})a_n + t_{n+1}b_n$$

where $0 \leq t_n \leq 1$ for n sufficiently large and $\Sigma t_n = \infty$. Then $\lim_n a_n = 0$.

<u>Proof</u>. First we claim that 0 is a cluster point of $\{a_n\}$. For this purpose we may assume that $|a_n| > |b_n|$ for all n sufficiently large. Choose N large enough so that $0 \leq t_n \leq 1$ and $|a_n| > |b_n|$ for $n \geq N$. If 0 is not a cluster point of $\{a_n\}$ then there is an $\epsilon > 0$ such that $|a_n| - |b_n| > \epsilon$ for $n \geq N$. But then we have

$$|a_{n+1}| \leq (1-t_{n+1})|a_n| + t_{n+1}|b_n|$$

and hence $|a_n| - |a_{n+1}| \geq t_{n+1}(|a_n|-|b_n|) > t_{n+1}\epsilon$ for $n \geq N$. Therefore for any $M > N$,

$$|a_N| \geq |a_N|-|a_{M+1}| = \sum_{n=N}^{M}(|a_n|-|a_{n+1}|)$$

$$\geq \epsilon \sum_{n=N}^{M} t_{n+1} .$$

This is a contradiction since $\Sigma t_n = \infty$. Thus $\{a_n\}$

This gives a contradiction since $\Sigma t_n = \infty$. Thus $\{a_n\}$ clusters at 0 . For a given $\epsilon > 0$, choose M large enough so that $|a_M| < \epsilon$, $0 \leq t_n \leq 1$, and $|b_n| < \epsilon$ for $n \geq M$. Then $|a_{M+1}| \leq (1-t_{n+1})|a_M| + t_{n+1}|b_M| < \epsilon$. Similarly, $|a_{M+p}| < \epsilon$ for all positive integers p , i.e., $\lim_n a_n = 0$.

4.1.3. <u>Theorem</u>. The method $\Gamma(\Lambda)$ is permanent if and only if $\Sigma\lambda_n = \infty$.

<u>Proof</u>. If $\Gamma(\Lambda)$ is permanent, then

$$0 = \lim_n a_{n0} = \prod_{i=1}^{\infty} (1-\lambda_i) .$$ Hence $\Sigma\lambda_n = \infty$ by 1.4.1.

Conversely, if $\Sigma\lambda_n = \infty$ then $\lim_n a_{n0} = 0$. It now follows from 4.0.2, 4.1.2 and induction that $\lim_n a_{nk} = 0$ for each k . It is easy to see that $\sum_{k=0}^{n} a_{nk} = 1$ for each n . Since $a_{nk} \geq 0$ for all n and k , it follows that $[a_{nk}]$ satisfies the Toeplitz conditions (1.3.2), i.e., $\Gamma(\Lambda)$ is permanent.

An alternate discription of a Γ-method which is sometimes useful is the following:

Given a sequence $(s_0, s_1, \ldots)$ define the ''operator''
σ on the terms of the sequence by $\sigma s_n = s_{n+1}$. The
method $\Gamma(\Lambda)$ then transforms the sequence (s_n) into
the sequence (s_n') where $s_0' = s_0$ and

$$s_n' = \prod_{k=1}^{n} ((1-\lambda_k)+\lambda_k \sigma) s_0 \quad \text{for} \quad n > 0 \ . \quad \text{This can also be}$$

written as

$$s_{n+1}' = (1-\lambda_{n+1}) s_n' + \lambda_{n+1} \sigma s_n'$$

where $\quad \sigma s_n' = \sum_{k=0}^{n} a_{nk} \sigma s_k = \sum_{k=0}^{n} a_{nk} s_{k+1}$.

4.1.4. <u>Definition</u>. A summability method A is
called <u>left</u> <u>translative</u> if given any sequence
$(s_0, s_1, \ldots)$ which is A-summable then the sequence
$(s_1, s_2, \ldots)$ is A-summable to the same value. A is
called <u>right</u> <u>translative</u> if given a sequence
$(s_0, s_1, \ldots)$ which is A-summable then the sequence
$(a, s_0, \ldots)$ is A-summable to the same value for
arbitrary a .

4.1.5. <u>Theorem</u>. The method $\Gamma(\Lambda)$ is right translative
if and only if $\sum_{n=1}^{\infty} \lambda_n = \infty$.

<u>Proof</u>. The sequence $(1, 0, 0, \ldots)$ is transformed

by $\Gamma(\Lambda)$ into the sequence whose nth term is

$\prod\limits_{k=1}^{n} (1-\lambda_k)$. If $\Gamma(\Lambda)$ is right translative, then this

sequence must have the same limit as the $\Gamma(\Lambda)$-transform

of the sequence $(0,0,0,\ldots)$, i.e. $\prod\limits_{k=1}^{\infty} (1-\lambda_k) = 0$.

Hence $\sum\limits_{n=1}^{\infty} \lambda_n = \infty$.

Conversely, suppose that $\sum\limits_{n=1}^{\infty} \lambda_n = \infty$ and that

the $\Gamma(\Lambda)$-transform of the sequence $(s_1, s_2, \ldots)$

converges to s . Let r_{k+1} denote the kth element

of the $\Gamma(\Lambda)$-transform of the sequence $(s_1, s_2, \ldots)$

for $k = 0, 1, \ldots$ and let r_0 be arbitrary. Let

$(s_k') = \Gamma(\Lambda)(r_0, s_1, \ldots)$, then $s_0' = r_0$ and

$s_{n+1}' = (1-\lambda_{n+1})s_n' + \lambda_{n+1}\sigma s_n' = (1-\lambda_{n+1})s_n' + \lambda_{n+1}r_{n+1}$.

It follows that $(s_k') = B(r_k)$, where

$$
B = \begin{bmatrix}
1 & 0 & 0 & \cdots \\
1-\lambda_1 & \lambda_1 & 0 & \cdots \\
(1-\lambda_1)(1-\lambda_2) & (1-\lambda_2)\lambda_1 & \lambda_2 & \cdots \\
\cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot &
\end{bmatrix},
$$

i.e., $b_{00} = 1$, $b_{0k} = 0$ if $k > 0$, $b_{ii} = \lambda_i$ if $i > 0$ and $b_{n+1,k} = (1-\lambda_{n+1})b_{nk}$ for $n > k$. For this matrix, $b_{nk} \geq 0$ for all n and k ,

$\sum_{k=0}^{n} b_{nk} = 1$ and $\lim_n b_{nk} = 0$ for all k since

$\prod_{k=1}^{\infty} (1-\lambda_k) = 0$. Therefore, the matrix B is permanent

and hence $\Gamma(\Lambda)(r_0, s_1, \ldots)$ converges to s , i.e., $\Gamma(\Lambda)$ is right translative.

4.1.6. <u>Theorem</u>. The method $\Gamma(\Lambda)$ is left translative if and only if $\underline{\lim} \lambda_n > 0$.

<u>Proof</u>. Suppose that $\underline{\lim} \lambda_n > 0$ and that $(s_n') = \Gamma(\Lambda)(s_0, s_1, \ldots)$ is convergent. Let $(r_n) = \Gamma(\Lambda)(s_1, s_2, \ldots)$, then

$$s'_{n+1} - s_n' = (1-\lambda_{n+1})s_n' + \lambda_{n+1}\sigma s_n' - s_n'$$
$$= \lambda_{n+1}(r_n - s_n') .$$

Since (s_n') converges and $\underline{\lim} \lambda_n > 0$, we have $\lim (r_n - s_n') = 0$, i.e., $\Gamma(\Lambda)$ is left translative.

Conversely, suppose there is a subsequence (λ_{n_k}) of Λ with $\lim_k \lambda_{n_k} = 0$. By 1.4.2, there is a

divergent sequence (ϵ_k) such that $\sum\limits_{k=0}^{\infty} \epsilon_k \lambda_{n_k}$

converges. Define a sequence (s_n') as follows:

$$
s_n' = \begin{cases} 0 & , \text{ for } n < n_0 \\ s_{n_k-1}' + \epsilon_k \lambda_{n_k} & , \text{ for } n = n_k \\ s_{n_k}' & , \text{ for } n_k < n < n_{k+1} \end{cases}
$$

Then $\lim\limits_{n} s_n' = \sum\limits_{k=0}^{\infty} \epsilon_k \lambda_{n_k}$, i.e., the sequence

$(s_n) = \Gamma(\wedge)^{-1}(s_n')$ is $\Gamma(\wedge)$-summable. Since

$s_{n+1}' = (1-\lambda_{n+1})s_n' + \lambda_{n+1}\sigma s_n'$, we have

$\sigma s_{n_k-1}' = s_{n_k-1}' + \epsilon_k$. Therefore the subsequence

$(\sigma s_{n_k-1}')$ of $\Gamma(\wedge)(s_1, s_2, \ldots)$ diverges. It follows

that $\Gamma(\wedge)$ is not left translative.

4.1.7. <u>Definition</u>. A <u>Banach limit</u> (see e.g. [14, p.73])
is a linear functional L defined on the space ℓ^{∞} of
all bounded real sequences such that (a) $L(x_n) \geq 0$
if $x_n \geq 0$ for all n , (b) $L(x_n) = L(x_{n+1})$ and
(c) $L(x_n) = 1$ if $x_n = 1$ for all n .

The next corollary follows immediately from

previous considerations and the Hahn-Banach theorem.

4.1.8. <u>Corollary</u>. There is a Banach limit whose restriction to $\Gamma(\Lambda)^{\#} \cap \ell_{\infty}$ is $\Gamma(\Lambda)$ if and only if $\underline{\lim} \lambda_n > 0$.

4.1.9. <u>Theorem</u>. Let $\Lambda = (\lambda_1, \lambda_2, \ldots)$ and $\Delta = (\delta_1, \delta_2, \ldots)$ be two sequences with $\lambda_n > 0$ and $\delta_n > 0$. Let $\Gamma(\Lambda)$ and $\Gamma(\Delta)$ be the associated Γ-methods. Then $\Gamma(\Lambda)\Gamma(\Delta)$ is a Γ-method if and only if $\delta_n = \delta$ for all n and in this case $\Gamma(\Lambda)\Gamma(\Delta) = \Gamma(\delta\Lambda)$, where $\delta\Lambda = (\delta\lambda_1, \delta\lambda_2, \ldots)$.

<u>Proof</u>. First we note that the product matrix represents the composed transformation since both matrices are lower triangular. Let $[a_{nk}] = \Gamma(\Lambda)$ and $[b_{nk}] = \Gamma(\Delta)$ and suppose that $[a_{nk}][b_{nk}] = [c_{nk}]$ is a Γ-method. Then it is easy to see that $c_{00} = 1$ and $c_{nn} = \prod_{i=1}^{n} \lambda_i \delta_i$ for $n > 0$. Also,

$$c_{n+1,0} = \sum_{j=0}^{n+1} a_{n+1,j} b_{j0}$$

$$= a_{n+1,0} + \sum_{j=0}^{n} a_{n+1,j+1} b_{j+1,0} .$$

Hence

$$c_{n+1,0} = (1-\lambda_{n+1})a_{n0}$$

$$+ \sum_{j=0}^{n} [(1-\lambda_{n+1})a_{n,j+1}b_{j+1,0} + \lambda_{n+1}(1-\delta_{j+1})a_{nj}b_{j0}]$$

$$= (1-\lambda_{n+1})c_{n0} + \sum_{j=0}^{n} \lambda_{n+1}(1-\delta_{j+1})a_{nj}b_{j0}$$

$$= c_{n0} - \lambda_{n+1} \sum_{j=0}^{n} \delta_{j+1}a_{nj}b_{j0} \ .$$

But since $[c_{nk}]$ is by assumption a Γ-matrix, we have $c_{n+1,0} = (1-\delta_{n+1}\lambda_{n+1})c_{n0}$, by 4.0.1. Substituting this above we have

$$\delta_{n+1}c_{n0} = \sum_{j=0}^{n} \delta_{j+1}a_{nj}b_{j0}$$

for all n . From this it follows that $\delta_n = \delta_1 = \delta$ for all n and $c_{n+1,0} = (1-\delta\lambda_{n+1})c_{n0}$.

Conversely, if $\delta_n = \delta$ for all n , then as above we have $c_{n+1,0} = (1-\delta\lambda_{n+1})c_{n0}$. Also,

$$c_{n+1,k+1} = \sum_{j=0}^{n+1} a_{n+1,j}b_{j,k+1}$$

$$= \sum_{j=0}^{n} a_{n+1,j+1}b_{j+1,k+1} \quad (\text{since } b_{0,k+1} = 0)$$

$$= \sum_{j=0}^{n} [(1-\lambda_{n+1})a_{n,j+1} + \lambda_{n+1}a_{nj}]b_{j+1,k+1} \ .$$

Hence

$$c_{n+1,k+1} = (1-\lambda_{n+1})\sum_{j=0}^{n} a_{n,j+1}b_{j+1,k+1} + \lambda_{n+1}(1-\delta)\sum_{j=0}^{n} a_{nj}b_{j,k+1}$$

$$+ \lambda_{n+1}\delta\sum_{j=0}^{n} a_{nj}b_{jk}$$

$$= (1-\lambda_{n+1})c_{n,k+1} + \lambda_{n+1}(1-\delta)c_{n,k+1} + \lambda_{n+1}\delta c_{nk}$$

$$= (1-\delta\lambda_{n+1})c_{n,k+1} + \delta\lambda_{n+1}c_{nk} .$$

Hence $[c_{nk}] = \Gamma(\delta\Lambda)$, by 4.0.1 and 4.0.2.

4.1.10. <u>Theorem</u>. The inverse of the method $\Gamma(\Lambda)$ is the method B given by the matrix $[b_{nk}]$ where

$$b_{n0} = (1-\lambda_1^{-1})^n , \quad b_{nk} = 0 \quad \text{for} \quad k > n$$

and

$$b_{n+1,k+1} = (1-\lambda_{k+2}^{-1})b_{n,k+1} + \lambda_{k+1}^{-1}b_{nk} , \quad 0 \le k \le n .$$

<u>Proof</u>. Let $(s_k') = \Gamma(\Lambda)(s_k)$; we show that $B(s_k') = (s_k)$. First, it is clear that $b_{00}s_0' = s_0$. Suppose $\sum_{k=0}^{n} b_{nk}s_k' = s_n$. Then,

$$\sum_{k=0}^{n+1} b_{n+1,k}s_k' = (1-\lambda_1^{-1})b_{n0}s_0' + \sum_{k=1}^{n+1} b_{n+1,k}s_k'$$

$$= (1-\lambda_1^{-1})b_{n0}s_0' + \sum_{k=1}^{n+1}[(1-\lambda_{k+1}^{-1})b_{nk}s_k' + \lambda_k^{-1}b_{n,k-1}s_k']$$

$$= b_{n0}s_0' + \sum_{k=1}^{n} b_{nk}s_k' - \lambda_1^{-1}b_{n0}s_0' + \sum_{k=1}^{n+1}[\lambda_k^{-1}b_{n,k-1} - \lambda_{k+1}^{-1}b_{nk}]s_k'$$

$$= s_n + \sum_{k=0}^{n} \frac{b_{nk}}{\lambda_{k+1}} (s'_{k+1} - s'_k) \ .$$

We also have,

$$s'_{k+1} - s'_k = \lambda_{k+1}(\sigma s'_k - s'_k) \ .$$

Therefore,

$$\sum_{k=0}^{n+1} b_{n+1,k} s'_k = s_n + \sum_{k=0}^{n} b_{nk}(\sigma s'_k - s'_k)$$

$$= s_n + \sigma(\sum_{k=0}^{n} b_{nk} s'_k) - \sum_{k=0}^{n} b_{nk} s'_k$$

$$= s_n + \sigma s_n - s_n = s_{n+1} \ .$$

Hence the theorem is proved by induction.

The next theorem is technical in character and will be used in the subsequent section.

4.1.11 Theorem. Let $\Gamma(\Delta)$ and $\Gamma(\Lambda)$ be two Γ-methods. Then $D = \Gamma(\Delta)\Gamma(\Lambda)^{-1}$ is given by: $d_{00} = 1$, $d_{0,k} = 0$ for $k > 0$,

$$d_{n+1,0} = (1 - \delta_{n+1}\lambda_1^{-1})d_{n0} \ ,$$

$$d_{n+1,k+1} = (1 - \delta_{n+1}\lambda_{k+2}^{-1})d_{n,k+1} + \delta_{n+1}\lambda_{k+1}^{-1}d_{nk} \ .$$

Proof. Let $[a_{nk}] = \Gamma(\Delta)$; $[b_{nk}] = \Gamma(\Lambda)^{-1}$.

Then

$$d_{n+1,k+1} = a_{n+1,0}b_{0,k+1} + \sum_{i=1}^{n+1} a_{n+1,i}b_{i,k+1}$$

$$= (1-\delta_{n+1})\sum_{i=1}^{n+1} a_{ni}b_{i,k+1} + \delta_{n+1}\sum_{i=1}^{n+1} a_{n,i-1}b_{i,k+1}$$

$$= (1-\delta_{n+1})d_{n,k+1} + \delta_{n+1}\sum_{i=1}^{n+1} a_{n,i-1}b_{i,k+1}$$

$$= (1-\delta_{n+1})d_{n,k+1} + \delta_{n+1}\sum_{i=1}^{n+1}[(1-\lambda_{k+2}^{-1})b_{i-1,k+1}a_{n,i-1}$$

$$+\lambda_{k+1}^{-1}a_{n,i-1}b_{i-1,k}]$$

$$= d_{n,k+1} - \delta_{n+1}\sum_{i=1}^{n+1}\lambda_{k+2}^{-1}a_{n,i-1}b_{i-1,k+1}$$

$$+\delta_{n+1}\sum_{i=1}^{n+1}\lambda_{k+1}^{-1}a_{n,i-1}b_{i-1,k}$$

$$= (1-\delta_{n+1}\lambda_{k+2}^{-1})d_{n,k+1} + \delta_{n+1}\lambda_{k+1}^{-1}d_{nk} .$$

A similar calculation establishes the other assertion.

Section 2. <u>Relative Strength of Various Methods.</u>

In this section we compare the relative strength of various subclasses of permanent Γ-methods.

4.2.1. <u>Theorem</u>. Let $\Delta = (\delta_1,\delta_2,\ldots)$ and $\Lambda = (\lambda_1,\lambda_2,\ldots)$ be sequences with $0 \leq \delta_n$, $0 < \lambda_n$ and $\Sigma\delta_n = \infty$. Suppose that for all n sufficiently

large, $\delta_n \leq \lambda_k$ for $0 \leq k \leq n$, then $\Gamma(\Delta) \geq \Gamma(\Lambda)$.

$\underline{Proof}$. Let $D = \Gamma(\Delta)\Gamma(\Lambda)^{-1}$, then it suffices to show that D is permanent. By 4.1.9 we have

$$d_{n+1,0} = (1-\delta_{n+1}\lambda_1^{-1})d_{n0}$$

$$d_{n+1,k+1} = (1-\delta_{n+1}\lambda_{k+2}^{-1})d_{n,k+1}+\delta_{n+1}\lambda_{k+1}^{-1} d_{nk} \ .$$

A routine calculation shows that $\sum\limits_{k=0}^{n} d_{nk} = 1$ for all n . Choose N so that $\delta_n \leq \lambda_k$ for $0 \leq k \leq n$ and $n \geq N$, and let $M = \max\limits_{0 \leq n \leq N} \sum\limits_{k=0}^{n} |d_{nk}|$. Then,

$$\sum_{k=0}^{N+1} |d_{N+1,k}| \leq (1-\delta_{N+1}\lambda_1^{-1})|d_{N0}|$$

$$+ \sum_{k=0}^{n} [\,(1-\delta_{N+1}\lambda_{k+2}^{-1})|d_{N,k+1}|+\delta_{N+1}\lambda_{k+1}^{-1}|d_{Nk}|\,]$$

$$= |d_{N0}|+\sum_{k=0}^{N-1} \delta_{N+1}\lambda_{k+2}^{-1}|d_{N,k+1}|$$

$$+ \sum_{k=0}^{N-1} (1-\delta_{N+1}\lambda_{k+2}^{-1})|d_{N,k+1}| = \sum_{k=0}^{N} |d_{Nk}| \leq M \ .$$

It follows by induction that $\sum\limits_{k=0}^{n} |d_{nk}| \leq M$ for all n .

We note that $\lim d_{n0} = 0$ since $\Sigma\delta_n = \infty$. If we

suppose that $\lim_n d_{nk} = 0$ and apply 4.1.2 with

$$a_n = d_{n,k+1} \; , \quad b_n = \lambda_{k+2}\lambda_{k+1}^{-1}d_{nk} \quad \text{and} \quad t_n = \delta_n\lambda_{k+2}^{-1} \; ,$$

we see that $\lim_n d_{n,k+1} = 0$. Therefore D satisfies

the Toeplitz conditions (1.3.2) and the theorem is

proved.

4.2.2. <u>Corollary</u>. If $\Lambda = (\lambda_1,\lambda_2,\ldots)$ where $\lambda_n > 0$

and $\lambda_n \leq \lambda$ for almost all n , then $\Gamma(\Lambda) \geq \Gamma_\lambda$.

4.2.3. <u>Example</u>. Let $\Lambda = (\lambda_1,\lambda_2,\ldots)$ where $0 \leq \lambda_n \leq 1$

$\lim \lambda_n = 0$. Suppose $0 < \lambda < 1$ and choose a such

that $a < (\lambda-2)/\lambda$. It is easy to check that

$\Gamma_\lambda(1,a,a^2,\ldots) = \{((1-\lambda)+\lambda a)^n\}_n$ and

$\Gamma(\Lambda)(1,a,a^2,\ldots) = \{\prod_{i=1}^{n}((1-\lambda_i)+\lambda_i a)\}_n$. Since

$|(1-\lambda)+\lambda a| > 1$, the sequence (a_n) is not Γ_λ-summable.

However, if $\Sigma\lambda_n = \infty$, then (a_n) is $\Gamma(\Lambda)$-summable

to 0 by 1.4.1.

4.2.4. <u>Corollary</u>. Let $\Lambda = (\lambda_1,\lambda_2,\ldots)$ be a sequence

with $\lambda_n \to 0$ and $\Sigma\lambda_n = \infty$, then $\Gamma(\Lambda) > \Gamma_\lambda$ for each

λ with $0 < \lambda < 1$.

Proof. This follows from 4.2.2 and the previous example.

4.2.5. Corollary. If $\Lambda = (\lambda_1, \lambda_2, \ldots)$ where $0 < \lambda < \lambda_n$ for all n, then $\Gamma_\lambda \geq \Gamma(\Lambda)$.

It is known that each of the Euler-Knopp methods Γ_λ is weaker than Borel's exponential method [32, p.131], hence we have:

4.2.6. Corollary. If $\Lambda = (\lambda_1, \lambda_2, \ldots)$ where $\underline{\lim} \lambda_n > 0$, then $\Gamma(\Lambda)$ is weaker than Borel's exponential method.

4.2.7. Corollary. If $\Lambda = (\lambda_1, \lambda_2, \ldots)$ and $\Delta = (\delta_1, \delta_2, \ldots)$ are sequences with $0 < \lambda_n < 1$, $0 < \delta_n < 1$, $\underline{\lim} \lambda_n > 0$ and $\underline{\lim} \delta_n > 0$, then the methods $\Gamma(\Lambda)$ and $\Gamma(\Delta)$ are consistent.

Proof. Under these assumptions there is a $\lambda > 0$ such that $\lambda < \lambda_n$ and $\lambda < \delta_n$ for all n . Therefore, by 4.2.6, $\Gamma(\Lambda) \leq \Gamma_\lambda$ and $\Gamma(\Delta) \leq \Gamma_\lambda$. Hence $\Gamma(\Lambda)$ and $\Gamma(\Delta)$ cannot sum the same sequence to different limits.

We now give a theorem which provides a necessary

condition for Γ-summability.

4.2.8. **Theorem.** If the sequence (s_n) is $\Gamma(\wedge)$-summable,

then $s_n = O(\prod_{i=1}^{n}(2\lambda_i^{-1}-1))$.

Proof. If the sequence (s_n) is $\Gamma(\wedge)$-summable,

then the transformed sequence (s_n') is bounded, say

$|s_n'| \leq M$ for all n . Also,

$$s_n' = \prod_{i=1}^{n}((1-\lambda_i) + \lambda_i\sigma)a_0$$

$$= (\prod_{i=1}^{n}\lambda_i)(\prod_{i=1}^{n}(t_i+\sigma))a_0 ,$$

where $t_i = \lambda_i^{-1}-1$. Therefore,

$$\prod_{i=1}^{n}(t_i+\sigma)a_0 = O(\prod_{i=1}^{n}(t_i+1)) .$$

Since $s_n = \sigma^n s_0 = \prod_{i=1}^{n}(t_i+\sigma-t_i)a_0$, it follows that

$$|s_n| \leq M\prod_{i=1}^{n}(2t_i+1)$$

for n large. But this is equivalent to

$s_n = O(\prod_{1}^{n}(2\lambda_i^{-1}-1))$.

By making an obvious modification in the preceeding

argument one obtains:

4.2.9. <u>Corollary</u>. If (s_n) is $\Gamma(\Lambda)$-summable to 0,

then $s_n = o(\prod_{i=1}^{n} (2\lambda_i^{-1}-1))$.

4.2.10. <u>Theorem</u>. Let $\Delta = (\delta_1, \delta_2, \ldots)$ and
$\Lambda = (\lambda_1, \lambda_2, \ldots)$ be sequences with $0 < \delta_n < 1$;
$0 < \lambda_n < 1$, $\Sigma \delta_n = \infty$. Suppose that for all n
sufficiently large, $\delta_n \leq \lambda_k$ for $0 \leq k \leq n$.
If $\lim \lambda_n \delta_n^{-1} = \infty$, then $\Gamma(\Delta) > \Gamma(\Lambda)$.

Proof. By 4.2.1 we have that $\Gamma(\Delta) \geq \Gamma(\Lambda)$.
To prove the theorem we must exhibit a sequence which
is $\Gamma(\Delta)$-summable but not $\Gamma(\Lambda)$-summable. Let

$b_n = \prod_{k=1}^{n} \lambda_k^{-\frac{1}{2}} \delta_k^{\frac{1}{2}}$, then we have $b_n \to 0$. If we let

$s_0' = 1$ and $s_n' = (-1)^{n+1} b_n$ for $n > 0$, then the
sequence $(s_n) = \Gamma(\Delta)^{-1}(s_n')$ is $\Gamma(\Delta)$-summable to 0 .
It is easy to see by using 4.1.10 that for n odd,
the kth element of the nth row of the matrix
$\Gamma(\Delta)^{-1}$ has sign $(-1)^{k+1}$, $k = 0,1,\ldots$. Since the
diagonal element of this row is $\prod_{k=1}^{n} \delta_k^{-1}$, we obtain

$s_n > b_n \prod_{k=1}^{n} \delta_k^{-1}$, for n odd. Therefore,

$$s_n / \prod_{i=1}^{n} (2\lambda_i^{-1} - 1) > b_n \prod_{i=1}^{n} \lambda_i \delta_i^{-1} (2-\lambda_i)^{-1}$$

$$\geq \prod_{i=1}^{n} \lambda_i^{\frac{1}{2}} \delta_i^{-\frac{1}{2}} 2^{-1} \to \infty$$

since $\lambda_n^{\frac{1}{2}} \delta_n^{-\frac{1}{2}} \to \infty$. Hence $s_n \neq O(\prod_{i=1}^{n} (2\lambda_i^{-1} - 1))$,

and $(s_n) \notin \Gamma(\Lambda)^{\#}$, by 4.2.8.

4.2.11. <u>Theorem</u>. Let $\Lambda = (\lambda_1, \lambda_2, \ldots)$ where $\lambda_n \to 0$

and $\Sigma \lambda_n = \infty$. Then $\Gamma(\Lambda)^{\#} \underset{\neq}{\supset} \underset{0 < \lambda < 1}{\cup} \Gamma_\lambda^{\#}$.

<u>Proof</u>. By 4.2.2 we have $\Gamma(\Lambda)^{\#} \supseteq \underset{0 < \lambda < 1}{\cup} \Gamma_\lambda^{\#}$.

To prove the theorem we must find a single sequence
which is $\Gamma(\Lambda)$-summable but which is not summable
by any of the methods Γ_λ . Let $s_0' = 1$,

$$s_n' = (-1)^{n+1} \prod_{i=1}^{n} \lambda_i^{\frac{1}{2}} , \text{ then } (s_n) = \Gamma(\Lambda)^{-1} (s_n') \text{ is}$$

$\Gamma(\Lambda)$-summable and for n odd we have as in 4.2.10,

$$s_n > \prod_{i=1}^{n} \lambda_i^{-\frac{1}{2}} . \text{ Now for } \underline{\text{any}} \ \lambda \text{ with } 0 < \lambda < 1 \text{ and } n \text{ odd}$$

we have

$$s_n / (2\lambda^{-1} - 1)^n > \prod_{i=1}^{n} \lambda_i^{-\frac{1}{2}} (2\lambda^{-1} - 1) \to \infty$$

since $\lambda_n \to 0$. Therefore (s_n) is not Γ_λ-summable for any λ by 4.2.8.

4.2.12. <u>Example.</u> Let $\Delta_p = (2^{-p}, 3^{-p}, \ldots)$ and $\Lambda_p = ((2^p \log 2^p)^{-1} , (3^p \log 3^p)^{-1} , \ldots)$. Then for $0 < q < p \leq 1$ we have $\Gamma(\Delta_p) > \Gamma(\Delta_q)$ and $\Gamma(\Lambda_p) > \Gamma(\Lambda_q)$. Also, for $0 < p \leq 1$ and $0 < \lambda < 1$,

$$\Gamma(\Lambda_p) > \Gamma(\Delta_p) > \Gamma_\lambda ,$$
$$\Gamma(\Lambda_p)^{\#} \underset{\neq}{\supset} \underset{0<\lambda<1}{\cup} \Gamma_\lambda^{\#} ,$$

and

$$\Gamma(\Delta_p)^{\#} \underset{\neq}{\supset} \underset{0<\lambda<1}{\cup} \Gamma_\lambda^{\#} .$$

These assertions follow from 4.2.10 and 4.2.11.

Finally, we give a theorem which relates a Γ-method with the method obtained from it by shifting the sequence of parameters (λ_n) .

4.2.13. <u>Theorem.</u> Let $\Lambda = (\lambda_1, \lambda_2, \ldots)$ and $\Lambda' = (\lambda_2, \lambda_3, \ldots)$, then $\Gamma(\Lambda) \geq \Gamma(\Lambda')$ if and only if $\underline{\lim} \lambda_n > 0$.

<u>Proof.</u> Let

$(t_k) = \Gamma(\wedge')((1-\lambda_1)s_0 + \lambda_1 s_1, (1-\lambda_1)s_1 + \lambda_1 s_2, \ldots)$

and $(r_k) = \Gamma(\wedge)(s_0, s_1, \ldots)$. It is easy to see

that $(r_{n+1}) = (t_n) = (1-\lambda_1)\Gamma(\wedge')(s_0, s_1, \ldots)$

$+ \lambda_1 \Gamma(\wedge')(s_1, s_2, \ldots)$. From this it follows that

$\Gamma(\wedge) \geq \Gamma(\wedge')$ if and only if $\Gamma(\wedge')$ is left translative,

i.e., $\varliminf \lambda_n > 0$ (4.1.6).

Section 3. Summability of Series.

We now consider the problem of summing series

with the methods introduced in this chapter. Of course,

a series is said to be summable by a method A if its

sequence of associated partial sums is summable by A .

If $\overset{\infty}{\underset{k=0}{\Sigma}} a_k$ is a series which is summable by a method A ,

we denote the limit of the A-transform of its sequence

of associated partial sums by $A(\overset{\infty}{\underset{k=0}{\Sigma}} a_k)$.

4.3.1. Theorem. Let $\wedge = (\lambda_1, \lambda_2, \ldots)$, where

$\Sigma \lambda_n = \infty$. If the series $\overset{\infty}{\underset{k=0}{\Sigma}} a_k$ is $\Gamma(\wedge)$-summable,

then $\Gamma(\wedge)(\overset{\infty}{\underset{k=0}{\Sigma}} a_k) = \overset{\infty}{\underset{k=0}{\Sigma}} \lambda_{k+1} a_k'$, where $(a_k') = \Gamma(\wedge)(a_k)$.

Proof. By definition $\Gamma(\wedge)(\overset{\infty}{\underset{k=0}{\Sigma}} a_k)$ is the limit

of the sequence $\Gamma(\wedge)(s_0,s_1,\ldots)$ where $(s_0,s_1,\ldots)$ is the sequence of partial sums associated with $\sum_{k=0}^{\infty} a_k$. By 4.1.5 this is the same as the limit of $\Gamma(\wedge)(0,s_0,s_1,\ldots) = (s_k')$. If $\sigma 0 = s_0$ and $\sigma s_i = s_{i+1}$ then $\sigma s_k' - s_k'$ is the kth element of the sequence $\Gamma(\wedge)(s_0,s_1,\ldots)-\Gamma(\wedge)(0,s_0,\ldots) = \Gamma(\wedge)(a_0,a_1,\ldots) = (a_k')$. It is clear that $s_1' = \lambda_1 s_0 = \lambda_1 a_0'$. Suppose by induction that $s_k' = \sum_{i=0}^{k-1} \lambda_{i+1} a_i'$, then

$$s'_{k+1} = (1-\lambda_{k+1})s_k' + \lambda_{k+1}\sigma s_k'$$

$$= \sum_{i=0}^{k-1} \lambda_{i+1} a_i' + \lambda_{k+1}(\sigma s_k' - s_k')$$

$$= \sum_{i=0}^{k-1} \lambda_{i+1} a_i' + \lambda_{k+1} a_k' \; .$$

Hence the theorem is proved.

4.3.2. <u>Theorem</u>. Let $\wedge = (\lambda_1,\lambda_2,\ldots)$, where $\sum \lambda_n = \infty$. If the series $\sum_{k=0}^{\infty} a_k$ is $\Gamma(\wedge)$-summable, then the sequence (a_k) is $\Gamma(\wedge)$-summable to zero.

<u>Proof</u>. As was seen in the previous proof, $(a_k') = \Gamma(\wedge)(s_0,s_1,\ldots)-\Gamma(\wedge)(0,s_0,\ldots)$. Since $\Gamma(\wedge)$

is right translative it follows that $a_n' \to 0$, i.e.,
(a_n) is $\Gamma(\wedge)$-summable to zero.

We have seen that if a series $\sum_{k=0}^{\infty} a_k$ is summable
to a by a permanent method $\Gamma(\wedge)$, then the series
$\sum_{k=0}^{\infty} \lambda_{k+1} a_k'$ converges to a . However, the later series
may converge even if $\sum_{k=0}^{\infty} a_k$ is not $\Gamma(\wedge)$-summable.
Accordingly, we make the following definition.

4.3.3. <u>Definition</u>. Let $\wedge = (\lambda_1, \lambda_2, \ldots)$ be a sequence
of numbers with $0 < \lambda_n < 1$. A series $\sum_{k=0}^{\infty} a_k$ is said
to be $\widetilde{\Gamma}(\wedge)$-summable to a if the series
$\sum_{k=0}^{\infty} \lambda_{k+1} a_k'$ converges to a , where $(a_k') = \Gamma(\wedge)(a_k)$.

4.3.4. <u>Theorem</u>. Let $\wedge = (\lambda_1, \lambda_2, \ldots)$, where $\Sigma \lambda_n = \infty$.
A series $\sum_{k=0}^{\infty} a_k$ is $\Gamma(\wedge)$-summable if and only if
$\sum_{k=0}^{\infty} a_k$ is $\widetilde{\Gamma}(\wedge)$-summable and (a_k) is $\Gamma(\wedge)$-summable.

<u>Proof</u>. The necessity follows from 4.3.1 and 4.3.2.
Conversely, if $\sum_{k=0}^{\infty} a_k$ is $\widetilde{\Gamma}(\wedge)$-summable, then as was

seen in 4.3.1 its $\widetilde{\Gamma}(\wedge)$-sum is the limit of the sequence $\Gamma(\wedge)(0, s_0, \ldots)$. Since $(a_k') = \Gamma(\wedge)(s_0, s_1, \ldots) - \Gamma(\wedge)(0, s_0, \ldots)$, if $\lim_n a_n'$ exists, then $\Gamma(\wedge)(s_0, s_1, \ldots)$ is convergent, i.e., $\sum\limits_{k=0}^{\infty} a_k$ is $\Gamma(\wedge)$-summable.

We note that under the conditions of the previous theorem that

$$\Gamma(\wedge)(\sum_{k=0}^{\infty} a_k) = \widetilde{\Gamma}(\wedge)(\sum_{k=0}^{\infty} a_k) + \lim \Gamma(\wedge)(a_n)$$

The proof of the next lemma is a trivial exercise in mathematical induction.

4.3.5. **Lemma.** If (λ_n) is any sequence of numbers, then

$$\lambda_1 + \sum_{k=1}^{n} \lambda_{k+1} \prod_{i=1}^{k} (1-\lambda_i) = 1 - \prod_{i=1}^{n+1} (1-\lambda_i) .$$

4.3.6. **Theorem.** The method $\widetilde{\Gamma}(\wedge)$ is a permanent series to series transformation if and only if $\Sigma \lambda_n = \infty$.

Proof. If $\sum\limits_{k=0}^{\infty} a_k = a$ and $\Sigma \lambda_n = \infty$, then

$\sum\limits_{k=0}^{\infty} a_k$ is $\Gamma(\wedge)$-summable to a and hence $\widetilde{\Gamma}(\wedge)$-summable

to a by 4.3.1.

Conversely, suppose $\widetilde{\Gamma}(\wedge)$ is permanent. Let

$[d_{nk}] = \Gamma(\wedge)$ and let $a_0 = 1$, $a_k = 0$ for $k > 0$.

Then $\sum\limits_{k=0}^{\infty} a_k = 1$ and hence

$$1 = \sum_{k=0}^{\infty} \lambda_{k+1} a'_k = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \lambda_{k+1} d_{kj} a_j \right)$$

$$= \sum_{k=0}^{\infty} \lambda_{k+1} d_{k0} = \lambda_1 + \sum_{k=1}^{\infty} \lambda_{k+1} \prod_{i=1}^{k} (1-\lambda_i)$$

$$= 1 - \prod_{i=1}^{\infty} (1-\lambda_i)$$

by 4.3.5. It follows from 1.4.1 that $\Sigma \lambda_n = \infty$

4.3.7. **Theorem.** The permanent series to series

methods $\Gamma(\wedge)$ and $\widetilde{\Gamma}(\wedge)$ are equivalent if and only

if $\underline{\lim} \, \lambda_n > 0$.

Proof. By 4.3.1 $\Gamma(\wedge)$ is weaker than $\widetilde{\Gamma}(\wedge)$.

By definition, $\widetilde{\Gamma}(\wedge)(\sum\limits_{k=0}^{\infty} a_k)$ is the limit of the

sequence $\Gamma(\wedge)(0, s_0, \ldots)$. This limit is the same as

the limit of the sequence $\Gamma(\wedge)(s_0, s_1, \ldots)$ only if

$\underline{\lim} \, \lambda_n > 0$, by 4.1.6.

4.3 8. **Theorem.** Each of the permanent $\widetilde{\Gamma}(\wedge)$ methods is absolutely permanent

Proof. The method $\widetilde{\Gamma}(\wedge)$ transforms the series $\sum\limits_{k=0}^{\infty} a_k$ into the series $\sum\limits_{n=0}^{\infty} (\sum\limits_{k=0}^{\infty} \lambda_{n+1} d_{nk} a_k)$ where $[d_{nk}] = \Gamma(\wedge)$. Since $\lambda_{n+1} d_{nk} \geq 0$ for all n and k , it suffices (by 1.3.6) to show that

$$\sum\limits_{n=0}^{\infty} \lambda_{n+1} d_{nk} = 1 \quad \text{for} \quad k = 0,1,2,\dots .$$ It was shown in the proof of 4.3.6 that $\sum\limits_{n=0}^{\infty} \lambda_{n+1} d_{n0} = 1$ if and only if $\Sigma \lambda_n = \infty$, i.e., if and only if $\widetilde{\Gamma}(\wedge)$ is permanent. Suppose that $\sum\limits_{n=0}^{\infty} \lambda_{n+1} d_{nk} = 1$. Since

$$d_{n+1,k+1} = (1-\lambda_{n+1}) d_{n,k+1} + \lambda_{n+1} d_{nk}$$

and since $[d_{nk}]$ is permanent, we have

$$0 = \lim_{n} d_{n,k+1} = \sum\limits_{n=0}^{\infty} (d_{n+1,k+1} - d_{n,k+1})$$

$$= -\sum\limits_{n=0}^{\infty} \lambda_{n+1} d_{n,k+1} + \sum\limits_{n=0}^{\infty} \lambda_{n+1} d_{nk} .$$

Therefore $\sum\limits_{n=0}^{\infty} \lambda_{n+1} d_{n,k+1} = 1$ and the theorem is proved by induction.

It follows from 4.2.8 that if $\underline{\lim} \, \lambda_n > 0$ and

the series $\sum_{k=0}^{\infty} a_k$ is $\Gamma(\Lambda)$-summable, then the power

series $\sum_{k=0}^{\infty} a_k z^k$ has a positive radius of convergence.

The class of Nörlund means also has this property,

thus none of these methods can sum a series which is

''too divergent''. The next example shows that this

is not necessarily the case with the methods $\widetilde{\Gamma}(\Lambda)$

when $\lambda_n \to 0$.

4.3.9. <u>Example</u>. Let $\Delta = (2^{-1}, 3^{-1}, \ldots)$, then there

exists a series $\sum_{k=0}^{\infty} a_k$ which is $\widetilde{\Gamma}(\Delta)$-summable and

for which the power series $\sum_{k=0}^{\infty} a_k z^k$ has zero radius

of convergence. In fact, let $a_n' = (-1)^{n+1}$ and let

$(a_n) = \Gamma(\Delta)^{-1}(a_n')$. Then $\widetilde{\Gamma}(\Delta)(\Sigma a_n) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1}$

converges, that is, $\sum_{k=0}^{\infty} a_k$ is $\widetilde{\Gamma}(\Delta)$-summable. Let

$[b_{nk}] = \Gamma(\Delta)^{-1}$, then for all n

$|a_{n+1}| = \sum_{k=0}^{n+1} |b_{n+1,k}|$ (see the argument of 4.2.10).

By the iterative formula for $b_{n+1,k+1}$ (4.1.10) and

the fact that b_{nk} and $b_{n,k+1}$ differ in sign we have

$$\sum_{k=0}^{n+1} |b_{n+1,k}| = 1 + \sum_{k=0}^{n} |b_{n+1,k+1}|$$

$$= 1 + \sum_{k=0}^{n} (k+2) \{ |b_{nk}| + |b_{n,k+1}| \}$$

$$= \sum_{n=0}^{n} (k+1) |b_{nk}| + \sum_{k=0}^{n} (k+2) |b_{n,k+1}|$$

$$> 2 \sum_{k=0}^{n} (k+1) |b_{nk}| .$$

Using 4.1.10, one can show that $|b_{2n,k-1}| < |b_{2n,2n-k}|$ for $1 \leq k < n$. It follows that

$$\sum_{k=n}^{2n} (k+1-n) |b_{2n,k}| > \sum_{k=0}^{n-1} (n-k-1) |b_{2n,k}| ,$$

and hence

$$\sum_{k=n}^{2n} (k+1) |b_{2n,k}| > \sum_{k=0}^{2n} n |b_{2n,k}| - \sum_{k=0}^{n-1} (k+1) |b_{2n,k}| .$$

Therefore, for n even we have

$$|a_{n+1}| > 2 \cdot \frac{n}{2} \sum_{k=0}^{n} |b_{n,k}| = n|a_n| ,$$

i.e., $\sum_{k=0}^{\infty} a_k z^k$ has zero radius of convergence.

We recall that the classical Abel limit theorem

says that if a series $\sum\limits_{n=0}^{\infty} a_n z_1^n$ converges then for each

z with $|z| < |z_1|$, the series $\sum\limits_{n=0}^{\infty} a_n z^n$ is

absolutely convergent and

$$\lim_{t \to 1^-} \sum_{n=0}^{\infty} a_n (tz_1)^n = \sum_{n=0}^{\infty} a_n z_1^n \quad .$$

We now prove an analogous theorem for the summability methods studied in this chapter. In the special case of the Euler-Knopp method such an Abelian theorem was proved by Knopp [16]. The proof given here is modeled on the proof of the classical Abel limit theorem and does not appear to be more difficult than the proof given by Knopp for the smaller class of Euler-Knopp methods.

4.3.10. <u>Lemma</u>. Suppose $|a_n'| \le M$ for all n where $(a_n') = \Gamma(\wedge)(a_n)$. Then

$$|(a_n t^n)'| \le M \prod_{i=1}^{n} (1-\lambda_i(1-t)) \quad \text{for} \quad n \ge 1 \quad .$$

<u>Proof</u>. By 4.1.10 we have

$$(a_1 t)' = (1-\lambda_1)a_0 + \lambda_1 a_1 t$$

$$= (1-\lambda_1)a_0' + \lambda_1 t[(1-\tfrac{1}{\lambda_1})a_0' + \tfrac{1}{\lambda_1} a_1']$$

$$= (1-\lambda_1)(1-t)a_0' + a_1' t \quad .$$

Hence $|(a_1 t)'| \le M[1-\lambda_1(1-t)]$. Let $\sigma(a_n t^n)$

$= a_{n+1} t^{n+1}$, then given that

$$|(a_n t^n)'| \le M \prod_{i=1}^{n} (1-\lambda_i(1-t))$$

we have

$$|(a_{n+1} t^{n+1})'| = |(1-\lambda_{n+1})(a_n t^n)' + \lambda_{n+1}\sigma(a_n t^n)'|$$

$$\le (1-\lambda_{n+1}) M \prod_{i=1}^{n} ((1-\lambda_i)+\lambda_i t)$$

$$+\lambda_{n+1} tM \prod_{i=1}^{n} ((1-\lambda_i)+\lambda_i t)$$

$$= M \prod_{i=1}^{n+1} (1-\lambda_i(1-t)) .$$

4.3.11. <u>Theorem</u>. Let $\Gamma(\wedge)$ be permanent. If for

some complex number z_1 the series $\sum_{n=0}^{\infty} a_n z_1^n$ is

$\Gamma(\wedge)$-summable, then the series $\sum_{n=0}^{\infty} a_n z^n$ is absolutely

and uniformly $\Gamma(\wedge)$-summable on compact subsets of the

disc $\{z: |z| < |z_1|\}$ and

$$\lim_{t \to 1^-} \Gamma(\wedge)(\sum_{n=0}^{\infty} a_n (tz_1)^n) = \Gamma(\wedge)(\sum_{n=0}^{\infty} a_n z_1^n) .$$

<u>Proof</u>. It is easy to see that we may consider

real values of z only and take $z_1 = 1$. By 4.3.10

and 4.3.5 we have

$$\sum_{n=0}^{\infty} \lambda_{n+1} |(a_n t^n)'| \leq M \sum_{n=0}^{\infty} \lambda_{n+1} \prod_{i=1}^{n} [1-\lambda_i(1-t)]$$

$$= \frac{M}{1-t} \quad .$$

Hence the first conclusion of the theorem holds.

Let $a = \Gamma(\wedge)(\sum_{n=0}^{\infty} a_n)$ and $f(t) = \Gamma(\wedge)(\sum_{n=0}^{\infty} a_n t^n)$,

where $0 \leq t < 1$. Then

$$\frac{1}{1-t} \cdot f(t) = \Gamma(\wedge)(\frac{1}{1-t} \sum_{n=0}^{\infty} a_n t^n)$$

$$= \Gamma(\wedge)(\sum_{n=0}^{\infty} t^n \cdot \sum_{n=0}^{\infty} a_n t^n)$$

$$= \Gamma(\wedge)(\sum_{n=0}^{\infty} s_n t^n)$$

where $s_n = \sum_{k=0}^{n} a_k$. Also,

$$\frac{1}{1-t} \cdot a = \Gamma(\wedge)(\sum_{n=0}^{\infty} t^n) \cdot a$$

$$= \Gamma(\wedge)(\sum_{n=0}^{\infty} a t^n) \quad .$$

Therefore

$$a - f(t) = (1-t)\Gamma(\wedge)(\sum_{n=0}^{\infty} (a-s_n) t^n) \quad .$$

Let $\epsilon > 0$ and choose N such that $n > N$ implies $|(a-s_n)'| < \epsilon/2$ (4.3.2). Then

$$|\Gamma(\wedge)(\sum_{n=0}^{\infty}(a-s_n)t^n)|$$

$$= |\sum_{n=0}^{N}(a-s_n)t^n + t^{N+1}\Gamma(\wedge)(\sum_{n=0}^{\infty}(a-s_{N+n+1})t^n)|$$

$$\leq |\sum_{n=0}^{N}(a-s_n)t^n| + t^{N+1}\sum_{n=0}^{\infty}\epsilon/2\lambda_{n+1}\prod_{i=1}^{n}(1-\lambda_i(1-t))$$

$$= |\sum_{n=0}^{N}(a-s_n)t^n| + \frac{\epsilon\,t^{N+1}}{2(1-t)}\ .$$

Hence, $\quad|a-f(t)| = (1-t)|\sum_{n=0}^{N}(a-s_n)t^n| + \epsilon/2t^{N+1}$

$$\leq (1-t)|\sum_{n=0}^{N}(a-s_n)t^n| + \epsilon/2\ .$$

Therefore $|a-f(t)| < \epsilon$ if t is sufficiently close to 1 and the theorem is proved.

The next corollary is a simple partial converse of the theorem.

4.3.12. <u>Corollary</u>. Suppose $a_n \geq 0$ for all n and that $\sum_{n=0}^{\infty} a_n t^n$ is $\Gamma(\wedge)$-summable for $t < 1$ and t sufficiently close to 1 . If $\lim_{t\to 1^{-}} \Gamma(\wedge)(\sum_{n=0}^{\infty} a_n t^n) < \infty$,

then $\sum_{n=0}^{\infty} a_n$ is $\Gamma(\Lambda)$-summable.

Proof. First note that $a_n' \geq 0$ for all n .

Suppose $\sum_{n=0}^{\infty} \lambda_{n+1} a_n' = \Gamma(\Lambda)(\sum_{n=0}^{\infty} a_n) = \infty$, then given

M , there is an N such that $\sum_{n=0}^{N} \lambda_{n+1} a_n' > M + 1$.

Since $\lim\limits_{t \to 1^-} (a_n t^n)' = a_n'$ for each n , there is a

$t_0 < 1$ such that $t > t_0$ implies that

$\sum_{n=0}^{N} \lambda_{n+1}(a_n t^n)' > M$, i.e., $\lim\limits_{t \to 1^-} \Gamma(\Lambda)(\sum_{n=0}^{\infty} a_n t^n) = \infty$, a

contradiction.

Section 4. Applications to Iteration.

We now give some applications to the iterative solution of linear operator equations of the type $u - Tu = f$. Browder and Petryshyn [6] have approached this problem by use of the classical Picard iteration procedure

4.4.1. $x_{n+1} = Tx_n + f$.

4.4.2. Theorem. If T is a linear operator on a normed linear space X and if the sequence 4.4.1

with initial point x_1 converges, then the convex iteration procedure 2.0.1 with $v_1 = x_1$ converges to the same limit if $\Sigma \lambda_n = \infty$.

Proof. Since $\{v_n\}$ is the $\Gamma(\Lambda)$ - transform of $\{x_n\}$, the theorem follows by 4.1.3.

We now give the promised improvements of 3.1.6 and 3.1.8. For this purpose we reconsider the process $\Gamma(\Lambda)$ in terms of probability theory (see section 1.4). If we consider a sequence of independent random trials in which the probability of a success on the ith trial is λ_i and if $[a_{nk}] = \Gamma(\Lambda)$, then it is easy to see that a_{nk} is the probability of k successes in the first n trials. Equivalently, if X_i is the random variable indicating the outcome of the ith trial (i.e., $P(X_i = 1) = \lambda_i$ and $P(X_i = 0) = 1 - \lambda_i$) and if $S_n = X_1 + \ldots + X_n$, then $P(S_n = k) = a_{nk}$.

4.4.3. Lemma. Let $\Lambda = (\lambda_1, \lambda_2, \ldots)$ be a D-sequence and let $[a_{nk}] = \Gamma(\Lambda)$. For each positive integer n there is a nonnegative integer m(n) such that $a_{n,k-1} \leq a_{nk}$ for $k \leq m(n)$ and $a_{nk} > a_{n,k+1}$ for

for $m(n) \leq k \leq n-1$. (If $a_{n,k-1} \leq a_{nk}$ for $1 \leq k \leq n$

we take $m(n) = n$ and if $a_{n,k-1} > a_{nk}$ for $1 \leq k \leq n$

we take $m(n) = 0$) .

Proof. We first note that the monotonic cases
given in the parenthetical remark cannot occur for
n sufficiently large. In fact, since Λ is a
D-sequence, $\Sigma \lambda_n = \infty$ and $\Sigma(1-\lambda_n) = \infty$. Now $a_{n0} \geq a_{n1}$
for all n only if

$$\prod_{i=1}^{n} (1-\lambda_i) \geq \sum_{j=1}^{n} \lambda_j \prod_{\substack{i=1 \\ i \neq j}}^{n} (1-\lambda_i)$$

for all n.. But this implies that $1 \geq \sum_{i=1}^{n} \frac{\lambda_j}{1-\lambda_j}$

$\geq \sum_{j=1}^{n} \lambda_j$, which is impossible since $\Sigma \lambda_n = \infty$. Also,

if $a_{n,n-1} \leq a_{nn}$ for all n , then

$$\sum_{j=1}^{n} (1-\lambda_j) \prod_{\substack{i=1 \\ i \neq j}}^{n} \lambda_i \leq \prod_{i=1}^{n} \lambda_i$$

for all n . This implies that $\sum_{j=1}^{\infty} (1-\lambda_j) \leq \sum_{j=1}^{n} \frac{1-\lambda_j}{\lambda_j} \leq 1$,

a contradiction.

Suppose first that $1-\lambda_1 \leq \lambda_1$, then $m(1) = 1$.

We prove the theorem by induction on n. If $a_{n,k-1} \leq a_{n,k}$ for $k \leq m(n)$ and $a_{nk} > a_{n,k+1}$ for $m(n) \leq k \leq n-1$, then using 4.0.1 and 4.0.2 it is easy to show that $a_{n+1,k-1} \leq a_{n+1,k}$ for $k \leq m(n)$ and $a_{n+1,k} > a_{n+1,k+1}$ for $m(n) + 1 \leq k \leq n+1$. Therefore $m(n+1) = m(n)$ or $m(n) + 1$ depending on whether $a_{n+1,m(n)} > a_{n+1,m(n)+1}$ or $a_{n+1,m(n)} \leq a_{n+1,m(n)+1}$. The case $1-\lambda_1 > \lambda_1$ is handled similarly.

4.4.4. <u>Theorem</u>. If $\wedge$ is a D-sequence and $[a_{nk}] = \Gamma(\wedge)$, then $\lim_n a_{nk} = 0$ uniformly in k.

<u>Proof</u>. We show that $\lim_n a_{n,m(n)} = 0$. Let $\epsilon > 0$ and choose N large enough so that $n \geq N$ implies

$$P(t_1 < \frac{S_n - \mu_n}{\sigma_n} < t_2) < \frac{1}{\sqrt{2\pi}} \int_{t_1}^{t_2} \exp(-u^2/2)du + \epsilon/2$$

for all $t_1 < t_2$ and $\sigma_n^{-1} < \epsilon\sqrt{\frac{\pi}{2}}$ (see 1.4.3). Then for any $n \geq N$ if we let $t_1 = (m(n) - \frac{1}{2} - \mu_n)/\sigma_n$ and $t_2 = (m(n) + \frac{1}{2} - \mu_n)/\sigma_n$, we have

$$a_{n,m(n)} = P(S_n = m(n)) = P(m(n) - \tfrac{1}{2} < S_n < m(n) + \tfrac{1}{2})$$

$$= P(t_1 < \frac{S_n - \mu_n}{\sigma_n} < t_2)$$

$$< \frac{1}{\sqrt{2\pi}} \int_{t_1}^{t_2} \exp(-u^2/2)\, du + \epsilon/2$$

$$< \frac{1}{\sqrt{2\pi}} (t_2 - t_1) + \epsilon/2 = \frac{1}{\sqrt{2\pi}} \sigma_n^{-1} + \epsilon/2 < \epsilon .$$

We note that the condition $\Sigma \lambda_n = \infty$ is not sufficient to guarantee the uniform convergence of the columns of $\Gamma(\Lambda)$ to zero. In fact, if $\lambda_n = 1 - (n+1)^{-2}$, then $\Sigma \lambda_n = \infty$ but (λ_n) is not a D-sequence and

$$\lim_n a_{nn} = \prod_{i=1}^{\infty} (1 - (i+1)^{-2}) > 0 .$$

4.4.5. __Theorem__. If T is an asymptotically bounded linear operator on a reflexive Banach space and $\Lambda = \{\lambda_n\}$ is a D-sequence, then $\{A_n(T)\}$ defined in 3.1.2 is a system of almost invariant integrals for $G(T)$.

Proof. If $[a_{nk}] = \Gamma(\Lambda)$ and $\|T^n\| \leq M$, then

$$\|(I-T)A_n(T)\| = \|\sum_{k=0}^{n} a_{nk} T^k - \sum_{k=0}^{n} a_{nk} T^{k+1}\|$$

$$\leq M(a_{n0} + a_{nn} + \sum_{k=1}^{n} |a_{nk} - a_{n,k-1}|)$$

$$= 2Ma_{nm}(n) .$$

Therefore $\lim_{n} \|(I-T)A_n(T)\| = 0$ by 4.4.4 and the

theorem follows by 3.1.5.

4.4.6. Corollary. Let T be an asymptotically bounded linear operator on a reflexive Banach space X and let $\{\lambda_n\}$ be a D-sequence. (a) If f is in the range of $I-T$, then the sequence 3.1.0 converges to a solution u of $u - Tu = f$ for any $v_1 \in X$. (b) If f is not in the range of $I-T$, then for any $v_1 \in X$ the sequence 3.1.0 contains no weakly convergent subsequence.

Proof. The conclusions of the theorem follow directly from 3.1.9 and 3.1.10.

Browder and Petryshyn [6] have given an approximation theorem similar to 4.4.6 for the iterative sequence 4.4.1 but under the stronger assumption that T is asymptotically convergent, i.e., $\{T^n x\}$ converges for each $x \in X$. If $\lambda_n = \lambda$ for all n, $\|T\| \leq 1$ and X is uniformly convex this corollary specializes to give a result of Dotson [13].

Finally, we give a simple consistency theorem.

4.4.7. <u>Theorem</u>. Under the assumptions of 4.4.6 (a)
the sequence 3.1.0 converges to a solution of
u - Tu = f which is independent of the D-sequence
so long as the D-sequence is bounded away from O .

<u>Proof</u>. This follows directly from 4.2.7.

Section 5. <u>Some Open Problems</u>.

In this final section we mention a few open
questions concerning the summability methods studied
in this chapter.

We have shown in 4.2.7 that a certain subclass
of Γ-methods is consistent.

4.5.1. Is the class of all permanent Γ-methods con-
sistent?

A permanent invertible summability method $[a_{nk}]$
is called <u>perfect</u> if the conditions

$$\sum_{n=0}^{\infty} |t_n| < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} t_n a_{nk} = 0 , \quad k = 0,1,\ldots$$

imply that $t_n = 0$ for $n = 0,1,\ldots$. A perfect method
A has the property that if B is a permanent method

with $A^{\#} \subset B^{\#}$, then A and B are consistent [1,p.95].
Mazur [23] has given a simple argument which shows that
the Euler-Knopp methods are perfect.

4.5.2. Is every permanent Γ-method perfect?

The author has made several attempts at proving
4.5.1 and 4.5.2 and at this time he feels that perhaps
counterexamples are in order.

Knopp has given a Cauchy product theorem for
Euler-Knopp methods, however, the author has been
unsuccessful in proving a Cauchy product theorem for
the general method. There are several possibilities.

4.5.3. (a) If $\sum\limits_{k=0}^{\infty} a_k$ is Γ(Λ)-summable to a ,

$\sum\limits_{k=0}^{\infty} b_k$ is Γ(Λ)-summable to b and

the Cauchy product series $\sum\limits_{k=0}^{\infty} c_k$ is

Γ(Λ)-summable to c , is ab = c ?

(b) If $\sum\limits_{k=0}^{\infty} a_k$ is Γ(Λ)-summable to a ,

$\sum\limits_{k=0}^{\infty} b_k$ is Γ(Λ)-summable to b and if

one of the series is absolutely

Γ(Λ)-summable, is the Cauchy product series

$\Gamma(\wedge)$-summable, say to c , with ab = c ?

(c) If $\sum\limits_{k=0}^{\infty} a_k$ is absolutely $\Gamma(\wedge)$-summable

to a and $\sum\limits_{k=0}^{\infty} b_k$ is absolutely

$\Gamma(\wedge)$-summable to b , is the Cauchy

product series absolutely $\Gamma(\wedge)$-summable,

say to c , with ab = c ?

An interesting and difficult problem which has not been considered in this paper is the problem of Tauberian theorems. By a Tauberian condition for a summability method A we mean a growth condition on the sequence of terms (a_k) which along with the assumption of A-summability of the series $\sum\limits_{k=0}^{\infty} a_k$ insures that the series is convergent in the ordinary sense. Knopp [17] has given a Tauberian condition for Euler-Knopp summability.

4.5.4. Is there a Tauberian condition for $\Gamma(\wedge)$-summability?

A class of summability methods is said to be adequate for bounded sequences if given any bounded sequence there is a member of the class which sums it.

4.5.5. Is the class of Γ-methods adequate for bounded sequences?

Mazur and Orlicz [24] have shown that the convergence field of a triangular invertible summability method is a Banach space under a suitable norm.

4.5.6. What is the structure of the Banach spaces $\Gamma(\wedge)^{\#}$?

Finally, we mention that except for special cases (i.e., 4.2.4 and 4.2.6) we have not compared the methods $\Gamma(\wedge)$ with known methods of summability.

4.5.7. How do the methods $\Gamma(\wedge)$ compare with the known methods of summability (e.g. Riesz, Nörlund, etc.)?

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EXAMINATION AND THESIS REPORT

Candidate: Charles William Groetsch

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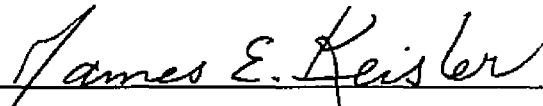Title of Thesis: Convex Iteration Procedures and a Related Class of Summability Methods

Approved:

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Dean of the Graduate School

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