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CONVEX POLYTOPES, COXETER ORBIFOLDS AND TORUS ACTIONS

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0. Introduction. An n-dimensional convex polytope is *simple* if the number of codimension-one faces meeting at each vertex is n. In this paper we investigate certain group actions on manifolds, which have a simple convex polytope as orbit space. Let P^n denote such a simple polytope. We have two situations in mind.

(1) The group is Z_2^n, M^n is n-dimensional and $M^n/Z_2^n \simeq P^n$.

(2) The group is T^n, M^{2n} is $2n$-dimensional and $M^{2n}/T^n \simeq P^n$.

Up to an automorphism of the group, the action is required to be locally isomorphic to the standard representation of Z_2^n on $\mathbb{R}^n$ in the first case, or T^n on $\mathbb{C}^n$ in the second case. In the first case, we call M^{2n} a "small cover" of P^n; in the second, it is a "toric manifold" over P^n. First examples are provided by the natural actions of Z_2^n and T^n on $\mathbb{R}P^n$ and $\mathbb{C}P^n$, respectively. In both cases the orbit space is an n-simplex.

Associated to a small cover of P^n, there is a homomorphism $\lambda\colon Z_2^m \to Z_2^n$, where m is the number of codimension-one faces of P^n. The homomorphism λ specifies an isotropy subgroup for each codimension-one face. We call it a "characteristic function" of the small cover. Similarly, the characteristic function of a toric manifold over P^n is a map $\mathbb{Z}^m \to \mathbb{Z}^n$. A basic result is that small covers and toric manifolds over P^n are classified by their characteristic functions (see Propositions 1.7 and 1.8).

The algebraic topology of these manifolds is very beautiful. The calculation of their homology and cohomology groups is closely related to some well-known constructions in commutative algebra and the combinatorial theory of convex polytopes. We discuss some of these constructions below.

Let f_i denote the number of i-faces of P^n and let h_j denote the coefficient of t^{n-j} in $\sum f_i(t-1)^i$. Then $(f_0, \dots, f_n)$ is called the *f-vector* and $(h_0, \dots, h_n)$ the *h-vector* of P^n. The f-vector and the h-vector obviously determine one another. The Upper Bound Theorem, due to McMullen, asserts that the inequality $h_i \leqslant \binom{m-n+i-1}{i}$, holds for all n-dimensional convex polytopes with m faces of codimension one. In 1971 McMullen conjectured simple combinatorial conditions on a sequence $(h_0, \dots, h_n)$ of integers necessary and sufficient for it to be the h-vector of a simple convex polytope. The sufficiency of these conditions was proved by Billera and Lee and necessity by Stanley (see [Brønsted] for more details and references). Research on

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such problems, particularly that of R. Stanley, has led to the injection of some heavy-duty commutative algebra and algebraic geometry into the theory of convex polytopes. We focus on the following three aspects of this.

(A) The proof of the Upper Bound Theorem in [Brønsted] is free of machinery. The first step in the argument is that one can choose a vector in $\mathbb{R}^n$ which is generic in the sense that it is never tangent to a proper face of P^n. The choice of such a vector allows one to attach an integer-valued index to each vertex (called the "in-valence") so that the number of vertices of index i is h_i.

(B) The Upper Bound Theorem was reproved and generalized in [Stanley1], by studying the "face ring" of the simplicial complex K, dual to the boundary complex of P^n. The fundamental result here is Reisner's Theorem [Reisner]. This states that the face ring is Cohen-Macaulay if and only if K satisfies certain homological conditions (see Theorem 5.1). Such a K is called a "Cohen-Macaulay complex".

(C) In [Stanley1], the necessity of McMullen's condition is established by studying a certain quasi-smooth projective variety associated to P^n, called a "toric variety". McMullen's conditions then follow from the existence of a Kähler class.

Aspects of (A), (B) and (C) have the following topological interpretations.

(A') A generic vector for P^n can be used to define a cell structure on any small cover M^n with one cell of dimension i for each vertex of P^n of index i. Thus the number of i-cells is h_i. This cell structure is perfect in the sense of Morse theory: the h_i are the mod 2 Betti numbers of M^n. (In fact a generic vector can be used to produce a perfect Morse function on M^n with one critical point of index i for each vertex of in-valence i.) Similar statements hold for toric manifolds over P^n, except that in this case all cells are even-dimensional and the h_i are the even Betti numbers of M^{2n}. These facts are proved in Section 3. (A version of this Morse-theoretic argument appears already in [Khovanskii].)

(B') A standard construction in transformation groups is the Borel construction, (or the homotopy quotient). We denote by BP^n the result of applying this construction to a small cover M^n of P^n; that is,

$$BP^n = EZ_2^n \times_{Z_2^n} M^n$$

where EZ_2^n is a contractible CW-complex on which Z_2^n acts freely. A definition of BP^n can be given which is independent of the existence of a small cover; in fact, BP^n depends only on P^n. The fundamental group of BP^n is a Coxeter group (cf. Lemma 4.4). We prove in Theorem 4.8 that the cohomology ring of BP^n (with coefficients in $\mathbb{Z}_2$) can be identified with the Stanley-Reisner face ring of K. The orbit space of EZ_2^n, denoted by BZ_2^n, is homotopy equivalent to the n-fold Cartesian product of $\mathbb{R}P^\infty$; its $\mathbb{Z}_2$ cohomology ring is $\mathbb{Z}_2[x_1, \ldots, x_n]$. The small cover M^n induces a fibration $M^n \to BP^n \to BZ_2^n$. The Morse theoretic argument in (A') shows that the spectral sequence of this fibration degenerates: $H^*(BP^n) \simeq H^*(M^n) \otimes H^*(BZ_2^n)$, (the coefficients are in $\mathbb{Z}_2$). This proves that the face ring of K is Cohen-Macaulay, without resort to Reisner's Theorem. There are deeper connections to (B). By definition, a graded algebra over a field is Cohen-Macaulay if and only if it admits a "regular sequence" (see Section 5). Since BZ_2^n is the Eilenberg-Maclane space $K(\mathbb{Z}_2^n, 1)$, the homotopy class of the map $BP^n \to BZ_2^n$ is given by a sequence $\lambda_1, \ldots, \lambda_n$ of elements of $H^1(BP^n; \mathbb{Z}_2^n)$. When the map is the projection map of the fibration $M^n \to BP^n \to BZ_2^n$, the sequence $(\lambda_1, \ldots, \lambda_n)$ can be identified with the dual map to the characteristic function of M^n; furthermore, $\lambda_1, \ldots, \lambda_n$ is a regular sequence of degree-one elements for the face ring. The converse is also true. Thus, the set of characteristic functions for small covers is naturally identified with the set of regular sequences of degree-one elements for the face ring. The $\mathbb{Z}_2$ cohomology ring of M^n is the quotient of the face ring by the ideal generated by the λ_i (cf. Theorem 4.14). Therefore, although it follows from (A') that the additive structure of $H^*(M^n; \mathbb{Z}_2^n)$ depends only on P^n, the ring structure depends on the characteristic function. Again, similar statements hold for toric manifolds. In this case the cohomology of the Borel construction can be identified with the face ring over $\mathbb{Z}$. Explanations for the statements in this paragraph can be found in Sections 4 and 5.

(C') Nonsingular toric varieties are toric manifolds in the sense of this paper; however, the converse does not hold. For example, $\mathbb{C}P^2 \# \mathbb{C}P^2$ is a toric manifold, but it does not admit an almost complex structure. In fact, the characteristic function of a toric variety is encoded in the "fan" (see [Oda] for definition) as follows. The codimension-one faces of the quotient M^{2n}/T^n are in one-to-one correspondence with the vertices of the triangulation of the sphere given by the fan. The value of the characteristic function at a face F is the primitive integral vector on the ray passing through the vertex corresponding to F. This family of characteristic functions is rather restricted. (In Section 7, we describe the analogous statement in the symplectic category, based on a paper by Delzant.) Thus, a Kähler class is one important tool which is missing in the case of general toric manifolds.

With regard to (C'), it should be mentioned that most of the results of this paper are well known in the case of toric varieties. However, it follows from this paper that, to a large extent, these results on cohomology of toric varieties are topological, and algebraic geometry need not be brought into the picture (unless one wants information about algebraic objects like the Chow ring, etc.).

The notions of f-vector, h-vector and face ring make sense for arbitrary finite simplicial complexes. This suggests that one should generalize the concepts of small cover and toric manifold as well as the results described in (B'). Such a generalization is accomplished in Sections 2, 4, and 5. Suppose that K is an $(n-1)$-dimensional simplicial complex. One can define an n-dimensional polyhedral complex P which is "dual" to K. The concepts of a small cover or toric space over P make sense. As in (B'), one defines a space BP, the cohomology ring of which is the face ring of K. If Y is a small cover of P, then there is a fibration $Y \to BP \to BZ_2^n$, and the projection map is determined by a sequence $\lambda_1, \ldots, \lambda_n$ of elements in $H^1(BP; \mathbb{Z}_2)$. It follows from Reisner's Theorem that the spectral sequence of the fibration degenerates if and only if K is Cohen-Macaulay over $\mathbb{Z}_2$ (cf. Theorem 5.9). One can deduce from this that if Y is a small cover of the dual of a Cohen-Macaulay complex, then the h-vector gives the mod 2 Betti numbers of Y. (A Morse theoretic proof of this fact, along the lines of (A'), would yield a new proof of Reisner's Theorem.)

The identification of the face ring of an arbitrary finite simplicial complex with the cohomology ring of a space has a concrete application: it allows us, in Theorem 4.11, to give a simple formula for the $\mathbb{Z}_2$ cohomology ring of any right-angled Coxeter group.

In Section 6 we consider the tangent bundle of small covers and toric manifolds. We prove that that tangent bundle of a small cover M^n is stably isomorphic to a sum of real line bundles. It follows that the Pontriagin classes of M^n vanish and that one can give an explicit formula (see Corollary 6.9) for the Steifel-Whitney classes. Similarly, the tangent bundle of a toric manifold is stably isomorphic, as a real vector bundle, to a sum of complex line bundles, and this leads to formulae for its characteristic classes.

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1. Definitions, examples and constructions. An n-dimensional convex polytope P^n is *simple* if precisely n codimension-one faces meet at each vertex. Equivalently, P^n is simple if the dual of its boundary complex is an $(n-1)$-dimensional simplicial complex. For example, a dodecahedron is simple; an icosahedron is not.

As $d = 1$ or 2, let $\mathbb{F}_d$ stand for $\mathbb{R}$ or $\mathbb{C}$, respectively, and let G_d stand for Z_2 or S^1 respectively. (G_d is regarded as the group of elements of $\mathbb{F}_d$ of norm 1.) The natural action of G_d on $\mathbb{F}_d$ is called *the standard one-dimensional representation.* The orbit space of that action is naturally identified with $\mathbb{R}_+$ and the orbit map is the norm. Let $\mathbb{F}_d^n$ and G_d^n denote the n-fold cartesian products. The natural action of G_d^n on $\mathbb{F}_d^n$ is again called *the standard representation.* The orbit space is $\mathbb{R}_+^n$.

Suppose that G_d^n acts on a manifold M^{dn}. A *local isomorphism* of M^{dn} with the standard representation consists of

(i) an automorphism $\theta\colon G_d^n \to G_d^n$,

(ii) G_d^n-stable open sets V in M^{dn} and W in $\mathbb{F}_d^n$ and

(iii) a θ-equivariant homeomorphism $f\colon V \to W$ (i.e., $f(gv) = \theta(g)f(v)$).

One says that M^{dn} is *locally isomorphic to the standard representation* if each point of M is in the domain of some local isomorphism.

Now let P^n be a simple convex polytope. By a *G_d^n-manifold over P^n* we will mean

(a) a G_d^n-action on a manifold M^{dn} locally isomorphic to the standard representation

(b) a projection map $\pi\colon M^{dn} \to P^n$ such that the fibers of π are the G_d^n orbits.

Note that (b) says that the orbit space of G_d^n on M^{dn} is homeomorphic to P^n.

Suppose that $\pi_1\colon M_1^{dn} \to P^n$, $\pi_2\colon M_2^{dn} \to P^n$ are two G_d^n-manifolds over P^n. *An equivalence over P^n* is an automorphism θ of G_d^n, together with a θ-equivariant homeomorphism $f\colon M_1 \to M_2$, which covers the identity on P^n.

When $d = 1$, G_d^n is Z_2^n. In this case P^n inherits the structure of an orbifold. In fact it is a *right-angled Coxeter orbifold*, which means nothing more than that it is locally isomorphic to the orbifold $\mathbb{R}^n/Z_2^n$. The map $\pi\colon M^n \to P^n$ is then a regular orbifold covering of P^n with Z_2^n as the group of covering transformations. We will call $\pi\colon M^{dn} \to P^n$ a *small cover* of the orbifold P^n. "Small" here refers to the fact that any cover of P^n by a manifold (=nonsingular orbifold) must have at least 2^n sheets. It can be proved that any 2^n-sheeted orbifold covering of P^n by a manifold is regular with the group of covering transformations Z_2^n.

When $d = 2$, $G_d^n = T^n$, and an interesting class of examples arises in algebraic geometry under the name of "nonsingular toric varieties". Good references are [Danilov] and [Oda]. We will say that $\pi: M^{2n} \to P^n$ is a *toric manifold.*

Example 1.1 (*Products of d-spheres*). The group Z_2 acts on S^1 by a complex conjugation. The orbit space is an interval I. The group S^1 acts on S^2 by rotation about an axis. The orbit space is again the interval. Thus, S^d is a G_d-manifold over I. Taking the n-fold product of this example, we have a G_d^n-manifold over the n-cube. When $d = 1$, Z_2^n acts on $S^1 \times \cdots \times S^1$ as a group generated by reflections. (A reflection is a locally smooth involution whose fixpoint set separates the manifold.) We shall see later that this reflection group property means that $\pi: T^n \to I^n$ is trivial in a certain sense. The next examples are not trivial in this sense.

Example 1.2 (*Projective spaces*). Identify G_d^n with G_d^{n+1}/D where D is the diagonal subgroup $D = \{(g, g, \ldots, g) \in G_d^{n+1}\}$. Then G_d^n acts on the projective space $\mathbb{F}_d P^n$ in the usual manner. It is easy to check that this action is locally isomorphic to the standard representation. Furthermore, $\mathbb{C}P^n/T^n = \mathbb{R}P^n/Z_2^n$ is an n-simplex. Thus, the projective space is a G_d^n-manifold over the simplex. Note that Z_2^n, $n \geqslant 2$, does not act on $\mathbb{R}P^n$ as a group generated by reflections, since the fixed set of a basic generator of Z_2^n is a disjoint union of a point and a codimension-one projective space, and hence does not separate.

Notation. If F is a k-face of P^n, then denote $\pi^{-1}(F)$ by M_F^{dk}.

LEMMA 1.3. *Suppose that $\pi: M^{dn} \to P^n$ is a G_d^n-manifold over P^n. Let F^k be a k-face of P^n. Then*

(i) *M_F^{dk} is a connected submanifold of M^{dn} of dimension dk and*

(ii) *$M_F^{dk} \to F^k$ is naturally a G_d^k-manifold over F^k.*

Proof. That M_F^{dk} is a submanifold of dimension dk follows from the fact that M^{dn} is locally isomorphic to $\mathbb{F}_d^n$. Let us show that M_F^{dk} is connected. Let $\operatorname{int}(F^k)$ denote the relative interior of F^k. First, suppose that $d = 2$. Then $\pi^{-1}(\operatorname{int}(F^k)) \to \operatorname{int}(F^k)$ is a trivial T^n/T^{n-k} bundle. Since $\pi^{-1}(\operatorname{int}(F^k)) = T^k \times \operatorname{int}(F^k)$ is connected, so is its closure M_F^{dk}. Suppose $d = 1$. Then $\pi^{-1}(\operatorname{int}(F^k))$ is homeomorphic to $Z_2^k \times \operatorname{int}(F^k)$. Hence, M_F^{dk} consists of 2^k copies of F^k glued together on their boundaries. At any vertex of F^k all 2^k copies meet; hence, M_F^{dk} is connected. This proves (i).

Let $x \in \pi^{-1}(\operatorname{int}(F^k)) \subset M_F^{dk}$ and let G_x denote the isotropy subgroup at x. Then G_x is isomorphic to G_d^{n-k}. The intersection of the fixed set of G_x with M_F^{dk} is an open and closed submanifold of M_F^{dk}; since it is connected, G_x fixes M_F^{dk}. Hence, $G_x \subset G_y$ for any $y \in M_F^{dk}$. Identifying G_x with G_d^{n-k} and G_d^k with G_d^n/G_d^{n-k}, we obtain an G_d^k-action on M_F^{dk} with orbits the fibers of $\pi: M_F^{dk} \to F^k$.

Remark. It is easy to see that any G_d^n-manifold M^{dn} can be given a smooth structure in which the G_d^n-action is smooth.

LEMMA 1.4. *Suppose that $M^{dn} \to P^n$ is a G_d^n-manifold over the simple convex polytope P^n. There is a continuous map $f\colon G_d^n \times P^n \to M^{dn}$ so that for each $p \in P^n$, f maps $G_d^n \times p$ onto $\pi^{-1}(p)$.*

Proof. This is fairly obvious for $d = 1$. For in this case, consider the restriction of π to the interior of P^n, $\pi^{-1}(\mathrm{int}(P^n)) \to \mathrm{int}(P^n)$. Since $\mathrm{int}(P^n)$ is simply connected, this is a trivial covering. Let $\mathrm{int}(C)$ be a component of $\pi^{-1}(\mathrm{int}(P^n))$ and let C denote its closure. Then it is easy to check that $\pi|_C\colon C \to P^n$ is a homeomorphism. Let $s = (\pi|_C)^{-1}$. The map $f\colon Z_2^n \times P^n \to M^n$ can then be defined by $f(g, p) = gs(p)$, where $g \in Z_2^n$ and $p \in P^n$.

In the case of toric manifolds we must proceed somewhat differently. We may assume that T^n acts smoothly on M^{2n}. In general, when a compact Lie group G acts smoothly on a manifold M with orbit space B, there is a procedure for "blowing up the singular strata" of M to obtain a smooth G-manifold with boundary $\hat{M}$ with only principal orbits and with the orbit space $\hat{B}$. The manifold $\hat{M}$ is equivariantly diffeomorphic to the complement of the union of tubular neighbourhoods of the singular strata. However, the definition of $\hat{M}$ is more canonical. Roughly speaking, one begins by removing the minimal stratum and replacing it by the sphere bundle of its normal bundle. One continues in this fashion, blowing up minimal strata, until only the top stratum is left. It is clear from this construction that there is a natural map $\hat{M} \to M$ which is the identity on top stratum, and which collapses each sphere bundle to its base space. The details of the construction can be found in [Davis1], p. 344. Now suppose that M^{2n} is a smooth T^n-manifold locally modelled on the standard representation with the orbit space B. Then B is an n-manifold with corners. In this case, $\hat{B}$ (the orbit space of $\hat{M}$) is canonically identified with B. In our particular case, $B = P^n$, a convex polytope. Since P^n is contractible, $\hat{\pi}\colon \hat{M}^{2n} \to P^n$ is trivial T^n bundle; i.e., there is an equivariant diffeomorphism $\phi\colon T^n \times P^n \to \hat{M}^{2n}$ inducing the identity on P^n. Composing ϕ with the natural collapse $\hat{M}^{2n} \to M^{2n}$, we obtain the map f.

For $d = 1, 2$, let R_d denote the ring $\mathbb{Z}_2$ or $\mathbb{Z}$, respectively, and let R_d^n denote the free R_d-module of rank n. By a *k-dimensional unimodular subspace* of R_d^n, we will mean a submodule of rank k which is a direct summand. (If $d = 1$, any submodule is a direct summand). The set of subgroups of G_d^n, which are isomorphic by an element of $\mathrm{Aut}(G_d^n)$ to the standard copy of G_d^k, is naturally parametrized by the set of k-dimensional unimodular subspaces of R_d^n. In particular, the subgroups of G_d^n which are isomorphic to G_d^1 are parametrized by the lines in R_d^n, i.e., by elements of PR_d^n, the set of primitive vectors in R_d^n modulo $\{\pm 1\}$. We note that $PR_1^n = R_1^n - \{0\} = \mathbb{Z}_2^n - \{0\}$. Thus, for $d = 1$, rank-one subgroups of Z_2^n correspond to nonzero vectors in R_1^n. For $d = 2$ we can for now safely ignore ± 1; thus, a rank-one subgroup of T^n is determined by a primitive vector in $R_2^n = \mathbb{Z}^n$.

Suppose that P^n is a simple convex polytope and that $\pi\colon M^{dn} \to P^n$ is a G_d^n-manifold over P^n. Let F^k be a k-face of P^n. By Lemma 1.3, for any $x \in \pi^{-1}(\mathrm{int}\, F^k)$, the isotropy group at x is independent of the choice of x; denote it by G_F. If F^{n-1} is a codimension-one face, G_F is a rank-one subgroup; hence, it is determined by a primitive vector $v \in R_d^n$, (well defined up to a sign). In this way we obtain a function λ from the set of codimension-one faces of P^n, denoted by $\mathscr{F}$, to primitive vectors in R_d^n. This function $\lambda: \mathscr{F} \to R_d^n$ is called *the characteristic function of* M^{dn}. The characteristic function obviously determines the group associated to any face. In fact, suppose F is a face of codimension l. Since P^n is simple, $F = F_1 \cap \cdots \cap F_l$, where F_i are the codimension-one faces which contain F. Then G_F is the subgroup corresponding to the subspace spanned by $\lambda(F_1), \ldots, \lambda(F_l)$. We therefore see that the characteristic function satisfies the following condition, which we will call $(*)$.

$(*)$ Let $F = F_1 \cap \cdots \cap F_l$ be any codimension-l face of P. Then $\lambda(F_1), \ldots, \lambda(F_l)$ span an l-dimensional unimodular subpspace of R_d^n.

In particular, if $F_1, \ldots, F_n$ are the codimension-one faces meeting at some vertex of P^n then $\lambda(F_1), \ldots, \lambda(F_n)$ is a basis. We will now show that the characteristic function determines the G_d^n-manifold over P^n and that any function satisfying $(*)$ can be realized as the characteristic function of some such G_d^n-manifold over P^n.

1.5. *The basic construction.* Let P^n be a simple convex polytope, $\mathscr{F}$ the set of codimension-one faces of P^n and $\lambda: \mathscr{F} \to R_d^n$ a function satisfying condition $(*)$. The function λ provides the necessary information to reverse the construction in the proof of the Lemma 1.4: one starts with $G_d^n \times P^n$ and then uses λ to "blow down" each $G_d^n \times F$ to a singular stratum. The construction goes as follows. For each face F of P^n, let G_F be the subgroup of G_d^n determined by λ and F. (Thus, if $F = F_1 \cap \cdots \cap F_l$, G_F is the rank-l subgroup determined by the span of $\lambda(F_1), \ldots, \lambda(F_l)$). For each point $p \in P^n$, let $F(p)$ be the unique face of P^n which contains p in its relative interior. We define an equivalence relation on $G_d^n \times P^n$: $(g, p) \sim (h, q)$ if and only if $p = q$ and $g^{-1}h \in G_{F(p)}$. We denote the quotient space $G_d^n \times P^n/\sim$ by $M^{dn}(\lambda)$ and the quotient map by $f: G_d^n \times P^n \to M^{dn}(\lambda)$. The action of G_d^n by the left translations descends to a G_d^n-action on $M^{dn}(\lambda)$, and the projection onto the second factor of $G_d^n \times P^n$ descends to a projection $\pi: M^{dn}(\lambda) \to P^n$. We claim that this is a G_d^n-manifold over P^n. To show this, we need to see that $M^{dn}(\lambda)$ is a manifold and that the action is locally isomorphic to the standard representation. Both these facts are local and follow easily from the next lemma, the proof of which we omit.

LEMMA 1.6. *Let $i: \mathbb{R}_+^n \to \mathbb{F}_d^n$ be the inclusion. Define an equivalence relation on $G_d^n \times \mathbb{R}_+^n$ by: $(g, x) \sim (h, y) \Leftrightarrow x = y$ and $g^{-1}h \in G_{i(x)}$ where $G_{i(x)}$ denotes the isotropy subgroup of G_d^n at $i(x)$. Then the natural map $G_d^n \times \mathbb{R}_+^n \to \mathbb{F}_d^n$ given by $(g, x) \to gi(x)$ descends to a homeomorphism $(G_d^n \times \mathbb{R}_+^n)/\sim \to \mathbb{F}_d^n$.*

We summarize the above construction in the following result.

PROPOSITION 1.7. *Let P^n be a simple convex polytope, $\mathscr{F}$ the set of codimension-one faces, and $\lambda: \mathscr{F} \to R_d^n$ a function satisfying $(*)$. Then the result of Construction 1.5, $\pi: M^{dn}(\lambda) \to P^n$, is a G_d^n-manifold over P^n.*

The next result also follows from Lemma 1.6.

PROPOSITION 1.8. *Let $\pi: M^{dn} \to P^n$ be a G_d^n-manifold over P^n, let $\lambda: \mathcal{F} \to R_d^n$ be its characteristic function, and let $\pi: M^{dn}(\lambda) \to P^n$ be the result of applying the Construction 1.5 to λ. Let $f: G_d^n \times P^n \to M^{dn}$ be the map constructed in Lemma 1.4. Then f descends to an equivariant homeomorphism $M^{dn}(\lambda) \to M^{dn}$ covering the identity on P^n. Thus M^{dn} is determined up to equivalence over P^n by its characteristic function.*

COROLLARY 1.9. *Let $\pi: M^{2n} \to P^n$ be a toric manifold. Then there is an involution τ on M^{2n} called "conjugation", with fixed point set M^n, such that $\pi|_{M^n}: M^n \to P^n$ is a small cover. Moreover, the characteristic function of M^n is the* mod 2 *reduction of the characteristic function of M^{2n}.*

Proof. We may assume that $M^{2n} = M^{2n}(\lambda)$ where λ is the characteristic function of M^{2n}. Consider the involution $\bar{\tau}$ on $T^n \times P^n$ defined by $(g, p) \to (g^{-1}, p)$. The fixpoint set of $\bar{\tau}$ is $Z_2^n \times P^n$. One checks easily that $\bar{\tau}$ descends to the involution τ on $M^{2n}(\lambda)$ with the fixpoint set homeomorphic to $M^n(\bar{\lambda})$, where $\bar{\lambda}: \mathcal{F} \to \mathbb{Z}_2^n$ is the mod 2 reduction of λ.

Remark. When $M^{2n} = \mathbb{C}P^n$, τ is the complex conjugation, and M^n is $\mathbb{R}P^n$.

1.10 *Cartesian products.* The cartesian product of a G_d^n-manifold over P^n and a G_d^m-manifold over P^m is a G_d^{n+m}-manifold over $P^n \times P^m$.

1.11. *Equivariant connected sums.* Suppose that $\pi_1: M_1^{2n} \to P_1^n$ and $\pi_2: M_2^{2n} \to P_2^n$ are G_d^n-manifolds. Let v_i be a vertex of P_i and V_i be its preimage in M_i^{dn}. Changing the action by an element of $Aut(G_d^n)$ if necessary, we may assume that G_d^n-actions are equivalent in a neighbourhood of V_i. One can then perform the connected sum equivariantly near the fixpoint; the result is a G_d^n-manifold $M_1 \# M_2$. Its quotient space, $P_1 \# P_2$, is formed by removing a small ball around v_i from P_i and gluing the results together. This space is not canonically identified with a simple convex polytope but is almost as good in that its boundary complex is dual to some PL triangulation of S^{n-1}.

1.12. *Pullbacks.* Suppose that Q^n and P^n are simple convex polytopes and that $f: Q^n \to P^n$ is a continuous map which takes each face of Q^n to a face of the same dimension in P^n. Let $\pi: M^{dn} \to P^n$ be a G_d^n-manifold over P^n. Then $f^*(M^{dn})$, the *pullback* of M^{dn}, is the fiber product of Q^n and M^{dn}. In other words, $f^*(M^{dn})$ is the subset of $M^{dn} \times Q^n$ consisting of all (x, q) such that $f(q) = \pi(x)$. The space $f^*(M^{dn})$ is a G_d^n-stable subspace of $M^{dn} \times Q^n$, where G_d^n acts on the first factor. Projection of $M^{dn} \times Q^n$ onto Q^n restricts to a projection $\pi': f^*(M^{dn}) \to Q^n$. It can be checked that $f^*(M^{dn})$ is a manifold and that the G_d^n-action is locally isomorphic to the standard representation. Thus, $\pi': f^*(M^{dn}) \to Q^n$ is a G_d^n-manifold over Q^n. Let $\mathcal{F}_Q$ and $\mathcal{F}_P$ denote the set of codimension-one faces of Q^n and P^n, respectively, and let $\hat{f}: \mathcal{F}_Q \to \mathcal{F}_P$ be the map induced by f. If $\lambda: \mathcal{F}_P \to \mathcal{R}_d^n$ is the characteristic function of M^{dn}, then $\lambda \circ \hat{f}$ is the characteristic function of $f^*(M^{dn})$.

Remark 1.13. The notions of the pullback and characteristic function can be reformulated in terms of the dual complex of the boundary complex of a simple convex polytope P^n. This dual complex is an $(n-1)$-dimensional simplicial complex K_P, defined as follows. The vertex set of K_P is $\mathscr{F}_P$, the set of codimension-one faces of P. A set of $(k+1)$ elements in $\mathscr{F}_P$, $\{F_0, \dots, F_k\}$ span a k-simplex in K_P if and only if $F_0 \cap \cdots \cap F_k \neq \emptyset$. (This intersection is then a face of P^n of codimension $(k+1)$). Thus, the k-simplices of K_P are naturally identified with the codimension-$(k+1)$ faces of P^n. One can find a face preserving map $Q^n \to P^n$, as in 1.12 if and only if there is a simplicial map $\hat{f}\colon K_Q \to K_P$ which restricts to an isomorphism on each simplex. Such a simplicial map is called *nondegenerate*. The characteristic function λ is a function from the vertex set of K_P to R_d^n. If λ satisfies $(*)$, then the condition that $\hat{f}$ be a nondegenerate simplicial map implies that $\lambda \circ \hat{f}$ also satisfies $(*)$. Thus, if $\pi\colon M^{dn} \to P^n$ is a G_d^n-manifold, Q^n is another simple polytope, and $\hat{f}\colon K_Q \to K_P$ is a nondegenerate simplicial map, then $f^*(M^{dn})$ is the G_d^n-manifold over Q^n determined by the characteristic function $\lambda \circ \hat{f}\colon \mathscr{F}_Q \to R_d^n$.

The proof of the next lemma is straightforward and is omitted.

LEMMA 1.14. *Let $\pi\colon M^{dn} \to P^n$ be a G_d^n-manifold over P^n. The following statements are equivalent.*

(i) *M^{dn} is a pullback of the linear model $\mathbb{F}_d^n \to \mathbb{R}_+^n$.*

(ii) *The image of the characteristic function λ is a basis for R_d^n.*

(iii) *There is a nondegenerate simplicial map $\hat{f}$ from K_P to an $(n-1)$-simplex Δ^{n-1}, and a bijection θ of the vertex set of Δ^{n-1} with a basis for R_d^n so that $\lambda = \theta \circ \hat{f}$.*

Moreover, in the case $d = 1$, any one of these conditions is equivalent to the condition that Z_2^n act on M^n as a group generated by reflections.

Example 1.15. (*Pullbacks from the linear model.*) There are many simple convex polytopes P^n, such that K_P admits a nondegenerate simplicial map onto Δ^{n-1}. Here are two types of examples, discussed in [Davis3].

(1) Suppose that P^n is such that K_P is the barycentric subdivision of a convex polytope Q^n; i.e., P^n is the dual of the barycentric subdivision of Q^n. The vertices of K_P are then identified with faces of Q^n. Regard Δ^{n-1} as the simplex on $\{0, 1, \dots, n-1\}$ and define D to be the function which associates to each vertex of K_P the dimension of the corresponding face of Q^n. Clearly, D extends to a nondegenerate simplicial map $D\colon K_P \to \Delta^{n-1}$.

(2) Suppose that K_P is a Coxeter complex of a finite Coxeter group W. Thus, K_P is a triangulation of S^{n-1} and P^n is the dual. Moreover, the orbit map $S^{n-1} \to S^{n-1}/W$ is a nondegenerate simplicial map $K_P \to \Delta^{n-1}$. An explicit geometric realization is as follows. Let W act on $\mathbb{R}^n$ as a linear orthogonal group generated by reflections and let $x \in \mathbb{R}^n$ be a point in the interior of some chamber. Then P^n can be identified with the convex hull of the orbit $W(x)$.

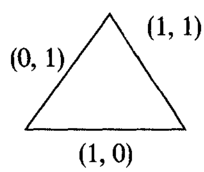
Both types of examples yield G_d^n-manifolds $M^{dn} \to P^n$ which pullback from the linear model. An important special case is when P^n is a "permutohedron". This is the specialization of (2), where the Coxeter group is the symmetric group on $n+1$ letters. It can also be regarded as a specialization of (1), where $Q^n = \Delta^n$, so that K_P is the barycentric subdivision of the boundary of an n-simplex. As was pointed out by [Tomei], the resulting G_d^n-manifolds arise in nature: when $d = 1$, the Z_2^n-manifold M^n over the permutohedron P^n can be identified with an isospectral manifold of tridiagonal real symmetric $(n + 1) \times (n + 1)$ matrices; when $d = 2$, the toric manifold M^{2n} is an isospectral manifold of tridiagonal hermitian matrices. (See also [Bloch, et. al.], [Davis 3] and [Fried].)

The following two remarks are useful when considering the examples that follow.

Remark 1.16. Suppose that $\lambda: \mathscr{F} \to R_d^n$ is a characteristic function and that $F_1, \ldots, F_n$ are the codimension-one faces meeting at some vertex. After changing λ by an element of $Aut(G_d^n)$, we may assume that $\{\lambda(F_1), \ldots, \lambda(F_n)\}$ is the standard basis for R_d^n.

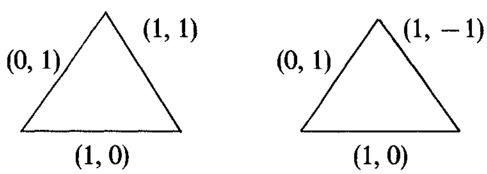
Remark 1.17. Orientations are important when $d = 2$. An orientation for M^{2n} is determined by an orientation of P^n and an orientation of the torus T^n.

Example 1.18. (P^2 *is a triangle*). Suppose $P^2 = \Delta^2$, a two simplex. When $d = 1$, there is essentially only one possible characteristic function λ, namely the one indicated below.



The resulting small cover is $\mathbb{R}P^2$.

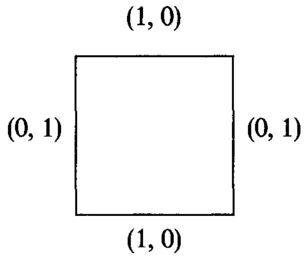
When $d = 2$, there are two possibilities.

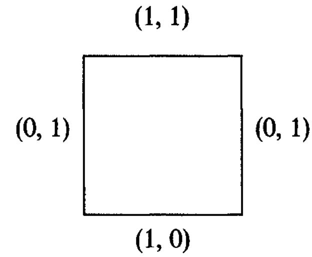


The first corresponds to the usual T^2-action on $\mathbb{C}P^2$ with its standard orientation; the second to the same action with the reverse orientation of $\mathbb{C}P^2$, which we denote by $\mathbb{C}\bar{P}^2$.

The same considerations apply whenever P^n is an n-simplex: any small cover of Δ^n is $\mathbb{R}P^n$, while any toric manifold over σ^n is either $\mathbb{C}P^n$ or $\mathbb{C}\bar{P}^n$.

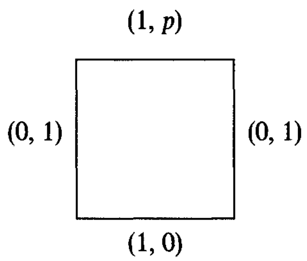
Example 1.19. (P^2 *is a square*). Suppose that P^2 is combinatorially a square. When $d = 1$, up to symmetries of P^2 and automorphisms of Z_2^2, there are essentially only two possibilities for λ:

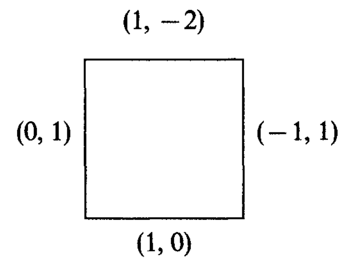




The first is T^2 with Z_2^2 acting as a reflection group; the second one is the Klein bottle and is equivariantly homeomorphic to $\mathbb{R}P^2 \# \mathbb{R}P^2$ (cf. 1.11).

When $d = 2$, there are more possibilities, for example





The first picture describes an infinite family of T^2-manifolds, M_p^4. The second picture is the equivariant connected sum $\mathbb{C}P^2 \# \mathbb{C}P^2$. [Orlik and Raymond, p. 552] have considered these T^2-manifolds. They show that M_p^4 is homeomorphic either to $S^2 \times S^2$ or to the nontrivial S^2-bundle over S^2 (as p is even or odd). The manifolds M_p^4 are equivariantly distinct; they are equivariantly homeomorphic to the natural actions on the Hirzebruch surface $\mathbb{P}(L(p) \oplus L(p))$, where $L(p)$ denotes the complex line bundle over S^2 with the first Chern class p and $\mathbb{P}(L(p) \oplus L(p))$ is the associated projective bundle (see [Oda]). Note that M_1^4 is $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$.

Example 1.20. (*P^2 is an m-gon*). If $d = 1$, then $M^2 \to P^2$ is a surface tiled by 4 copies of an m-gon. Thus, $\chi(M^2) = 4 - m$. If m is odd, M^2 is the connected sum of $m - 2$ copies of $\mathbb{R}P^2$. If m is even, M^2 can be either the connected sum of $m - 2$ copies of $\mathbb{R}P^2$ or the connected sum of $(m - 2)/2$ copies of T^2. The equivariant connected sum construction applied to Examples 1.18 and 1.19 shows that both possibilities can occur.

The case $d = 2$ has been completely decided by [Orlik and Raymond, p. 553]. They show that any such toric manifold $M^4 \to P^2$ is an equivariant connected sum of copies of Examples 1.18 and 1.19. Thus, any toric four-manifold is (nonequivariantly) homeomorphic to a connected sum of copies of $\mathbb{C}P^2$, $\overline{\mathbb{C}P}^2$, and $S^2 \times S^2$.

Example 1.21. (*3-dimensional polytopes*). Suppose that P^3 is a simple convex polytope of dimension 3 and let K_P be the triangulation of S^2 dual to ∂P^3. The Four Color Theorem states that there is a nondegenerate simplicial map from K_P to the $\partial\Delta^3$, the boundary of the tetrahedron. Thus, we can find a map $f\colon P^3 \to \Delta^3$ as in 1.12 and use it to pullback the G_d^3-manifold $\mathbb{F}_d P^3 \to \Delta^3$. It follows that every simple 3-dimensional polytope arises as the base space of some G_d^3-manifold. Let us consider the 3-manifolds which arise as small covers of such a P^3. If P^3 has no triangles or squares as faces, then it follows from Andreev's Theorem (cf. [Andreev] or [Thurston]) that it can be realized as a right-angled polytope in the hyperbolic 3-space. Hence, any small cover carries a hyperbolic structure. If P^3 has no triangular faces, M^3 can be decomposed into Seifert-fibered or hyperbolic pieces glued along tori or Klein bottles arising from square faces. In particular, such an M^3 is aspherical. If P^3 has triangular faces, then M^3 has a decomposition into such pieces glued along projective planes. Of course, all this fits with Thurston's Conjecture.

Nonexamples 1.22. (*Duals of cyclic polytopes*). For each integer $n \geqslant 4$ and $k \geqslant n+1$, there is an n-dimensional convex polytope with k vertices denoted by C_k^n and called a cyclic polytope, defined as the convex hull of k points on the curve $\gamma(t) = (t, t^2, \dots, t^{n-1})$. This is a simplicial polytope (i.e., its boundary complex is a simplicial complex). Moreover, it is m-neighbourly for $m = [n/2]$. This means that the $(m-1)$-skeleton of C_k^n coincides with the $(m-1)$-skeleton of a $(k-1)$-simplex (see [Brønsted, §13]). In particular, for $n \geqslant 4$, the 1-skeleton of C_k^n is a complete graph. Let Q_k^n be the simple polytope dual to C_k^n. We claim that for $n \geqslant 4$ and large values of k, the polytope Q_k^n admits no small cover. Indeed, let $\mathscr{F}_k$ denote the set of codimension-one faces of Q_k^n, and consider a characteristic function $\lambda\colon \mathscr{F}_k \to \mathbb{Z}_2^n$. For any two codimension-one faces F_1 and F_2, $\lambda(F_1)$ and $\lambda(F_2)$ are distinct nonzero vectors in $\mathbb{Z}_2^n$, since $F_1 \cap F_2 \neq \varnothing$. Hence, there can be no such function λ when $k \geqslant 2^n$. Therefore, the polytope Q_k^n admits no small cover whenever $n \geqslant 4$ and $k \geqslant 2^n$. This also implies that such a polytope cannot be the base space of a toric manifold, since if $M^{2n} \to Q_k^n$ is a toric manifold then the fixed set of the conjugation τ on M^{2n} is a small cover.

On the other hand, every simple polytope is the base space of some toric variety in the sense of algebraic geometers (see [Oda]).

COROLLARY 1.23. *Any toric variety over the dual of a 2-neighbourly polytope with more than 2^n vertices, (such as Q_k^n, $n \geqslant 4$ and $k \geqslant 2^n$), is singular.*

2. The universal G_d^n-space. Construction 1.5 can be generalized to cases where the base space is more general than a simple polytope. All that is necessary is that the base space have a face structure which is in some sense dual to a simplicial complex. We discuss such a generalization below.

Let K be a simplicial complex of dimension $n-1$ and let K' denote its barycentric subdivision. For each simplex $\sigma \in K$, let F_σ denote the geometric realization of the poset $K_{\geqslant\sigma} = \{\tau \in K \mid \sigma \leqslant \tau\}$. Thus, F_σ is the subcomplex of K' consisting of all simplices of the form $\sigma = \sigma_0 < \sigma_1 < \cdots < \sigma_k$. We note that F_σ is a cone (on the geometric realization of $K_{>\sigma}$). If σ is a $(k-1)$-simplex, then we say that F_σ is a *face of codimension k*. Let P_K denote the cone on K. The polyhedron P_K together with its decomposition into "faces" $\{F_\sigma\}_{\sigma\in K}$ will be called a *simple polyhedral complex.*

Let $\mathrm{Ver}(K)$ denote the set of vertices of K and let λ: $\mathrm{Ver}(K) \to R_d^n$ be a function. For each simplex σ in K, let E_σ denote the span of $\lambda(v)$, $v \in \sigma$. Condition (*) of Section 1 translates as follows: for each $(k-1)$-simplex $\sigma \in K$, E_σ is a k-dimensional unimodular subspace of R_d^n.

Suppose that λ: $\mathrm{Ver}(K) \to R_d^n$ is any function satisfying (*). Construction 1.5 goes through essentially without change. We put $Y = (G_d^n \times P_K)/_\sim$, where the equivalence relation $\sim$ is defined exactly as in 1.5; as before, we have a G_d^n-action on Y with the orbit map π: $Y \to P_K$. We will say that π: $Y \to P_K$ is a *G_d^n-space over P_K*.

It is interesting to note that there is an universal G_d^n-space, such that any G_d^n-space can be obtained as a pullback via the suitable map. The base space of the universal G_d^n-space is dual to a certain $(n-1)$-dimensional simplicial complex K_d^n which we will now describe. The vertex set of K_d^n is PR_d^n, the set of lines in R_d^n. A k-simplex σ in K_d^n is a collection of lines $\{l_0, \dots, l_k\}$, $l_i \in PR_d^n$, which span a $(k+1)$-dimensional unimodular subspace E_σ of R_d^n. The dual of K_d^n will be denoted by U_d^n (instead of $P_{K_d^n}$). There is a tautological characteristic function λ on $\mathrm{Ver}(K_d^n)$ defined by arbitrarily choosing a primitive vector for each line in $\mathrm{Ver}(K_d^n)$ (for $d = 1$ there is no choice involved). The resulting G_d^n-space is denoted by Y_d^n.

Suppose that π: $Y \to P_K$ is a G_d^n-space over a simple polyhedral complex P_K with characteristic function λ: $\mathrm{Ver}(K) \to R_d^n$. The function λ yields a nondegenerate simplicial map $\hat{\lambda}$: $K \to K_d^n$ defined by taking each vertex $v \in K$ to the line spanned by $\lambda(v)$. (It is a tautology that λ satisfies (*) if and only if $\tilde{\lambda}$ is a nondegenerate simplicial map.) The following result is now obvious.

PROPOSITION 2.1. *Let P_K be the n-dimensional simple polyhedral complex dual to K. Any G_d^n-space π: $Y \to P_K$ is equivalent to the pullback of the universal G_d^n-space $Y_d^n \to U_d^n$ via some nondegenerate simplicial map f: $K \to K_d^n$. In fact the set of equivalence classes of G_d^n-spaces over P_K is in this way bijective with the set of nondegenerate simplicial maps f: $K \to K_d^n$ modulo the natural action of* $\mathrm{Aut}(G_d^n)$.

The complex K_d^n has remarkable homotopical properties. In the terminology of Section 5, the theorem that follows means that K_d^n is a Cohen-Macaulay complex.

THEOREM 2.2. *The complex K_d^n is $(n-1)$-dimensional and $(n-2)$-connected. Moreover, for each i-simplex $\sigma \in K_d^n$, the link of σ in K_d^n is $(n-i-2)$-dimensional and $(n-i-3)$-connected.*

Proof. The complex K_d^n is closely related to some complexes which have been considered in algebraic K-theory. Let R be an associative ring. A sequence of vectors $(v_1, \dots, v_k)$ in R^n is *unimodular* if it spans a k-dimensional direct summand of R^n. The set of such sequences is a poset, where the partial ordering is the relation of being a subsequence. The simplicial complex $X(R^n)$ associated to this poset was first studied by Quillen. The main result of [vanderKallen, p. 274] implies that $X(R_d^n)$ is $(n-2)$-connected for $R_d = \mathbb{Z}$ *or* $\mathbb{Z}_2$ and that the similar statement holds for links. Let K_d' denote the barycentric subdivision of K_d^n. We claim that K_d' is a retract of $X(R_d^n)$ (and that the similar statement holds for links). The theorem follows immediately from this claim. To prove the claim, we first note that there is a simplicial map $r\colon X(R_d^n) \to K_d'$ defined on the vertices by $(v_1, \ldots, v_k) \to \{l(v_1), \ldots, l(v_k)\}$, where $l(v)$ denotes the line determined by v. Suppose $d = 1$. Order the vectors of $\mathbb{Z}_2^n$ arbitrarily. Define $i\colon K_1' \to X(\mathbb{Z}_2^n)$ by sending $\{l_1, \ldots, l_k\}$ to $(v(l_1), \ldots, v(l_k))$ where $v(l)$ is the unique nonzero vector on the line l and where $(v(l_1), \ldots, v(l_k))$ is written in increasing order. Obviously, $r \circ i = \mathrm{id}$, so that K_1' is a retract of $X(\mathbb{Z}_2^n)$. For $d = 2$, there are exactly two primitive vectors on a given line l in $\mathbb{Z}^n$, and we choose any convention for picking one $v(l)$. After ordering the set of primitive vectors in $\mathbb{Z}^n$, we can define $i\colon K_2' \to X(\mathbb{Z}^n)$, as before. Thus, K_d' is a retract of $X(R_d^n)$ and a similar argument works for links.

3. A perfect cell structure for G_d^n-manifolds. Let K be a finite simplicial complex of dimension $n - 1$. For $0 \leqslant i \leqslant n - 1$, let f_i be the number of i-simplices in K. Then $(f_0, \ldots, f_{n-1})$ is the *f-vector* of K. Define a polynomial $\Psi_K(t)$ of degree n by

$$\Psi_K(t) = (t-1)^n + \sum_{i=0}^{n-1} f_i(t-1)^{n-1-i}$$

and let h_i be the coefficients of t^{n-i} in $\Psi_K(t)$; i.e.,

$$\Psi_K(t) = \sum_{i=0}^{n} h_i t^{n-i}.$$

Then $(h_0, \ldots, h_n)$ is called the *h-vector* of P. Obviously, $h_0 = 1$, $h_n = \Psi_K(0) = (-1)^n(1 - \chi(K))$, and $\sum_{i=0}^{n} h_i = \Psi_K(1) = f_{n-1}$.

If we specialize this to the case where K is the boundary complex of a simplicial polytope and P^n is the dual simple polytope, then f_i is the number of faces of P^n of codimension $i + 1$, $h_n = 1$, and $\sum h_i$ is the number of vertices of P^n.

The main result of this section is the following.

THEOREM 3.1. *Let $\pi\colon M^{dn} \to P^n$ be a G_d^n-manifold over a simple convex polytope P^n.*

(i) *Suppose $d = 1$. Let $b_i(M^n)$ be the i^{th} mod 2 Betti number of M^n; i.e., $b_i(M^n) = \dim_{\mathbb{Z}_2} H_i(M^n; \mathbb{Z}_2)$. Then $b_i(M^n) = h_i(P^n)$*

(ii) *Suppose $d = 2$. The homology of M^{2n} vanishes in odd dimensions and is free abelian in even dimension. Let $b_{2i}(M^{2n})$ denote the rank of $H_{2i}(M^{2n}; \mathbb{Z})$. Then $b_{2i} = h_i$.*

Remark 3.2. If P^n admits a small cover, then the h_i are the mod 2 Betti numbers of an n-manifold; hence they satisfy Poincaré duality

$$h_i = h_{n-i}.$$

These equations are the Dehn-Sommerville Relations for the simple convex polytopes (cf. [Brønsted, p. 121]). Hence, they hold for P^n independently of the existence of a small cover.

Remark 3.3. Let $h(t) = h_0 + \cdots h_n t^n$ so that $h(t) = t^n \Psi_K(t^{-1}) = \Psi_K(t)$, where the last equation holds by Poincaré duality. Let $\mathscr{P}_{M^{dn}}(t)$ be the Poincaré polynomial of

M^{dn} with respect to coefficients in $\mathbb{Z}_2$ or $\mathbb{Q}$ as $d = 1$ *or* 2. Theorem 3.1 can then be summarized by the equation

$$\mathscr{P}_{M^{dn}}(t) = h(t^d).$$

Remark 3.4. It is somewhat surprising that the Betti numbers of M^{dn} depend on P^n only. It turns out that the structure of the cohomology ring does depend on the characteristic function λ. (When $n = 2$, one can see this by considering G_d^2-manifolds over the square.) We shall calculate the cohomology ring of M^{dn} in the Theorem 4.14 of the next section.

Remark 3.5. For $d = 1$, one can only hope to prove Theorem 3.1 for *mod* 2 Betti numbers. One can see this from 2-dimensional examples, say, the small covers of a square by either a torus or a Klein bottle (cf. Example 1.19). Thus, the rational Betti numbers of M^n are not determined by P^n.

Remark 3.6. Theorem 3.1 does not hold when P is a general simple polyhedral complex, for example, when K is the disjoint union of two triangulated circles. In Section 5 we will show that the theorem holds whenever K is a "Cohen-Macaulay" complex.

Proof of Theorem 3.1. A stronger result than Theorem 3.1 is true: the manifold M^{dn} has a cell structure which is perfect in the sense of Morse theory, with one cell for each vertex of P^n and with exactly h_i cells of dimension di. The first step in our description of this cell structure makes essential use of the fact that P^n has the combinatorial type of a convex polytope. Realize the P^n as a convex polytope in $\mathbb{R}^n$ and then choose a vector w in $\mathbb{R}^n$ which is generic in the sense that it is tangent to no proper face of P^n. (This is also the first step in the proof of the Upper Bound Theorem for convex polytopes given in [Brønsted, §18]). Choose an inner product in $\mathbb{R}^n$ and let $\phi\colon \mathbb{R}^n \to \mathbb{R}$ be the linear functional dual to w; i.e., $\phi(x) = \langle x, w\rangle$.

One could produce the cell structure on M^{dn} by modifying w to get a vector field on P^n, which is tangent to each face and which vanishes at the vertices. Pulling back such a vector field to M^{dn}, one obtains a gradient-like vector field on M^{dn}, the ascending submanifolds of which give the desired cell structure. (Equivalently, one could modify ϕ and then pull it back to obtain a perfect Morse function on M^{dn}.) However, it is unnecessary to make these modification in order to describe the cell structure: one can describe it directly in terms of w (or ϕ).

We think of ϕ as a height function on P^n. Using ϕ, one makes the 1-skeleton of P^n into a directed graph (as in [Brønsted]): orient each edge so that ϕ increases along it. (Since ϕ is generic, its restriction to an edge is never constant.) Since P^n is simple, each vertex is incident to precisely n edges. For each vertex v of P^n, let $m(v)$ denote the number of incident edges which point towards v (so that $n - m(v)$ edges point away). Let F be any face of P^n of dimension >0. Since ϕ is linear, $\phi|_F$ assumes its maximum (or minimum) at a vertex. Since ϕ is generic, this vertex is unique. Hence, each face F of P^n has a unique "top" vertex and a unique "bottom" vertex. For each vertex v, let F_v be the smallest face of P^n which contains the inward pointing edges incident to v. Clearly, $\dim F_v = m(v)$ and if F' is a face of P^n with top vertex v then F' is a face of F_v. (Compare [Brønsted p. 114]).

Let $\hat{F}_v$ denote the union of the relative interiors of those faces F', whose top vertex is v. In other words, $\hat{F}_v$ is just F_v with some faces in the boundary deleted (namely we delete all faces not incident to v).

The space $\hat{F}_v$ is diffeomorphic to the "quadrant" $\mathbb{R}_+^{m(v)}$. A simple combinatorial argument ([Brønsted, p. 115]) shows that the number of vertices v with $m(v) = i$ is h_i.

Suppose π: $M^{dn} \to P^n$ is a G_d^n-manifold. We are now in the position to describe the cell structure on M^{dn}. For each vertex of P^n, put

$$e_v = \pi^{-1}(\hat{F}_v) \qquad M_v = \pi^{-1}(F_v) = M_{F_v}.$$

Since $\hat{F}_v$ is diffeomorphic to $\mathbb{R}_+^{m(v)}$, e_v is equivalent to $\mathbb{F}_d^{m(v)}$; i.e., it is a cell of dimension $dm(v)$. The closure of e_v is obviously M_v, which, by Lemma 1.3, is a $dm(v)$-manifold. When $d = 2$, all the cells are of even dimension; hence the cell structure is perfect, and the homology of M^{2n} is as claimed in Theorem 3.1. (Lemma 1 of [Khovanskii] is a similar result which is proved by essentially the same argument as above.)

In general to prove that a cell structure is perfect with respect to $\mathbb{Z}_2$ coefficients, it is sufficient to show that the closure of each cell is a (pseudo) manifold. To see this, note that a manifold is a mod 2 cycle. Hence, if the closure of a cell is a manifold, then its attaching map is trivial on $\mathbb{Z}_2$ homology; thus with regard to $\mathbb{Z}_2$ homology, it looks like a wedge of spheres; i.e., the cell-structure is perfect. This completes the proof of Theorem 3.1.

COROLLARY 3.7. *Let π: $M^n \to P^n$ be a small cover of a convex polytope. Then Z_2^n acts trivially on $H^*(M^n; \mathbb{Z}_2)$.*

Proof. Each cell in the perfect cell structure described above is Z_2^n-stable.

We note that if the closure of each cell is an orientable manifold, then the same reasoning shows that the cell-structure is perfect with respect to integral coefficients. We record this observation in the following result.

COROLLARY 3.8. *Let π: $M^n \to P^n$ be a small cover of a simple convex polytope and suppose that for each face F, the submanifold M_F $(=\pi^{-1}(F))$ is orientable. Then $H_i(M^n; \mathbb{Z})$ is a free abelian of rank h_i.*

If $d = 2$, then M^{2n} has no odd-dimensional cells; hence, we have the following.

COROLLARY 3.9. *Any toric manifold over a simple convex polytope is simply connected.*

We now make a start on the study of the cohomology ring of a G_d^n-manifold. A more complete description is given in Theorem 4.14. Two faces F and F' of P^n intersect *transversely* if $\operatorname{codim}(F \cap F') = \operatorname{codim} F + \operatorname{codim} F'$. Since P^n is simple, it satisfies the following two properties.

(1) If F is a k-face of P^n and v is a vertex of F, then there is a face F' of complementary dimension $n-k$ such that $F \cap F' = v$.

(2) If F is a face of codimension l, then F is the transverse intersection of l faces of codimension one.

For each k-face F of P^n, we have a connected dk-dimensional submanifold M_F of M^{dn}. Theorem 3.1 shows that the homology in degree dk of M^{dn} is generated by classes of the form $[M_F]$, where F is a k-face. We consider some simple consequences of Poincaré duality. As is well known, the cup product is Poincaré dual to intersection. Hence, the cup product of the dual of $[M_F]$ with the dual of $[M_{F'}]$ is the dual of $[M_{F\cap F'}]$, if F and F' intersect transversely, and zero otherwise. From properties (1) and (2) above we get the following result.

PROPOSITION 3.10. *Let $\pi\colon M^{dn} \to P^n$ be a G_d^n-manifold over the simple convex polytope P^n.*

(i) *For each face F of P^n, the class $[M_F]$ is not zero in $H_*(M^{dn})$.*

(ii) *The cohomology ring $H^*(M^{dn})$ is generated by d-dimensional classes.* (*Coefficients are in $\mathbb{Z}_2$ or $\mathbb{Z}$ as $d = 1, 2$.*)

Proof. (i) By property (1) above, there is a face F' which intersects F transversely in a vertex. Since the classes $[M_F]$ and $[M'_F]$ are dual under intersection, they are both nonzero.

(ii) The cohomology in degree dl is generated by Poincaré duals of classes of the form $[M_F]$, codim $F = l$. By property (2), F is the transverse intersection of l faces of codimension one. Hence, the Poincaré dual of $[M_F]$ is the product of l cohomology classes of lowest dimension d.

4. The Borel construction and the face ring. Suppose that $\pi\colon M^{dn} \to P^n$ is a G_d^n-manifold over a simple convex polytope. The Borel construction (or "the homotopy quotient" of G_d^n on M^{dn}) is the space

$$B_d P^n = EG_d^n \times_{G_d^n} M^{dn}$$

where EG_d^n is a contractible space on which G_d^n acts freely. When $d = 1$, we shall sometimes drop it from the notation and simply write BP, and we sometimes use B_T instead of B_2.

When $d = 1$, P is a right-angled Coxeter orbifold and BP is its classifying space in the sense of [Haefliger]. We shall show that the space $B_d P$ does not depend on the G_d^n-manifold over P, but only on P and its face structure. Indeed, a construction of $B_d P$ can be made independent of the existence of a G_d^n-manifold over P: $B_d P$ can be constructed whenever P is a simple polyhedral complex (cf. Section 2). To establish this we shall present in 4.1 and 4.2 two different constructions of $B_d P$.

The main point of this section is that the cohomology ring $H^*(B_d P)$ can be identified with the face ring, a certain graded ring associated to the combinatorial type of P^n (see [Stanley3]).

We return now to the generality of Section 2. Let K be an $(n-1)$-dimensional simplicial complex and let P be the simple polyhedral complex dual to K. We shall suppose that K is a finite complex.

4.1. *A general construction of B_dP.* Let $\mathscr{F} = \mathrm{Ver}(K)$ denote the set of codimension one faces of P. Suppose $\mathscr{F} = (F_1, \ldots, F_m)$. Let $\{e_1, \ldots, e_m\}$ be the standard basis for R_d^m and define $\theta\colon \mathscr{F} \to R_d^m$ by $\theta(F_i) = e_i$. Using θ, we get a G_d^m-space $\mathscr{Z} = (G_d^m \times P)/\sim$ as in Construction 1.5. The dimension of $\mathscr{Z}$ is $n + m(d-1)$. When $d = 1$ and P^n is a simple polytope, $\mathscr{Z}$ is the universal abelian cover of the right-angled orbifold P^n. (This is also true for more general P provided we replace the word "orbifold" by "orbihedron", cf. [Gromov, §4.5].)

Put

$$B_dP = EG_d^m \times_{G_d^m} \mathscr{Z}.$$

We claim that this agrees with our previous definition. Indeed, suppose $Y \to P$ is the G_d^n-space associated to a characteristic function $\lambda\colon \mathscr{F} \to L_d^n$ Regarding R_d^m as a free module generated by $\mathscr{F}$, the map λ extends linearly to a surjection $\tilde{\lambda}\colon R_d^m \to R_d^n$ so that the following diagram commutes.

$$\begin{array}{ccc} \mathscr{F} & \xrightarrow{\theta} & R_d^m \\ {\scriptstyle id}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{\lambda}} \\ \mathscr{F} & \xrightarrow{\lambda} & R_d^n \end{array}$$

Let H be the subgroup of G_d^m corresponding to the kernel of $\tilde{\lambda}$ (note that $H \simeq G_d^{m-n}$). Clearly, $Y = \mathscr{Z}/H$. Thus,

$$B_dP = EG_d^m \times_{G_d^m} \mathscr{Z} = EG_d^m \times_H \mathscr{Z}/(G_d^m/H) = EG_d^m \times (\mathscr{Z}/H)/(G_d^m/H) = EG \times_G Y$$

where $G = G_d^m/H$. That is, the two Borel constructions are homeomorphic. It follows that B_dP is defined independently of the existence of a small cover or a T^n-space over P and that its homotopy type depends only on P.

A more natural way to see this would be to give a local construction of B_dP. We outline such a construction below.

4.2. *A local construction of B_dP.* Let P be a simple polyhedral complex. Thus, P is the cone on the barycentric subdivision of a simplicial complex K. The cone on the barycentric subdivision of a k-simplex σ is isomorphic to the standard subdivision of a $(k+1)$-cube. It follows that P is naturally a cubical complex; it is decomposed into cubes indexed by the simplices of K.

We regard the k-cube as the oribit space of a G_d^k-action on the dk-disk

$$D^{dk} = \{(x_1, \ldots, x_k) \in \mathbb{F}_d^k \colon |x_i| \leqslant 1\}.$$

For $d = 2$, D^{2k} is the standard polydisk in $\mathbb{C}^k$, for $d = 1$ it is a k-cube. For each $(k-1)$-simplex $\sigma \in K$, let I_σ be the corresponding k-cube in P, and D_σ the dk-disk with G_d^k-action, and let $B_d I_\sigma = EG_d^k \times_{G_d^k} D_\sigma$. If σ is a face of τ, then $B_d I_\sigma$ is canonically identified with a subset of $B_d I_\tau$. In this way the $B_d I_\sigma$ fit together to yield BP.

4.3. *The fundamental group.* Let P be a simple polyhedral complex dual to a simplicial complex K. The right-angled Coxeter group W associated to P is the Coxeter group with one generator for each element of $\mathrm{Ver}(K)$ and relations $s^2 = 1$, $s \in \mathrm{Ver}(K)$, and $(st)^2 = 1$ whenever $\{s, t\} \in \mathrm{Edge}(K)$.

LEMMA 4.4. *Let P be a simple polyhedral complex dual to K and let W be the associated Coxeter group. Then*

(1) *$\pi_1(BP)$ is isomorphic to W and*

(2) *$H_1(BP; \mathbb{Z}_2) \simeq \mathbb{Z}_2^m$, where m is the number of vertices of K (=the number of codimension-one faces of P).*

Proof. Let $\mathscr{X} = (W \times P)/_\sim$, where the equivalence relation is defined by $(w, x) \sim (v, y) \Leftrightarrow x = y$ and $w^{-1}v$ belongs to a subgroup generated by the codimension-one faces which contain x. Since each codimension-one face is connected and since whenever two generators of W commute, the corresponding codimension-two face is nonempty, it follows from [Davis 2, Theorems 10.1 and 13.5] that $\mathscr{X}$ is simply connected. Clearly, $BP = EW \times_W \mathscr{X}$. Hence, we have fibration $BP \to BW$ with simply connected fiber $\mathscr{X}$. Thus, statement (1) follows. Since the abelianization of W is obviously Z_2^m, statement (2) also holds.

COROLLARY 4.5. *Let Y be a small cover of P. Let $\tilde{\lambda}$: $\mathbb{Z}_2^m \to \mathbb{Z}_2^n$ be the map induced by the characteristic function and let ϕ: $W \to \mathbb{Z}_2^n$ the composition of $\tilde{\lambda}$ with the projection $W \to W^{ab}$. Then $\pi_1(Y) = \ker \phi$.*

Proof. Since $BP = EZ_2^n \times_{Z_2^n} Y$, we see that Y is homotopy equivalent to the covering space of BP corresponding to $\ker \phi$.

LEMMA 4.6. *Let P be a simple polyhedral complex. Then $B_T P$ is simply connected.*

Proof. $B_T P$ is the union of contractible spaces, namely the $B_T I_\sigma$, $\sigma \in K$, and any two of these subspaces have nonempty connected intersection.

4.7. *The face ring.* Let K be a simplicial complex with vertex set $\{v_1, \dots, v_m\}$ and let R be a commutative ring. Form the polynomial ring $R[v_1, \dots, v_m]$ where the v_i are regarded as indeterminates. Let I be the homogenous ideal generated by all square free monomials of the form $v_{i_1} \dots v_{i_s}$, where $\{v_{i_1}, \dots, v_{i_s}\}$ does not span a simplex in K. The *face ring* (*or the Stanley-Reisner ring*), denoted by $R(K)$, is $R[v_1, \dots, v_m]/I$. If P is the simple polyhedral complex which is dual to K, then we shall put $R(P) = R(K)$. Of course, we are particularly interested in the coefficient rings $\mathbb{Z}_2$ and $\mathbb{Z}$, denoted by R_1 and R_2, respectively. We shall regard the indeterminates v_i in $R_1[v_1, \dots, v_m]$ as being of degree one, while in $R_2[v_1, \dots, v_m]$ the v_i are of degree two. In this way, $R_d(P)$ becomes a graded ring. Recall that BG_d^m is the m-fold cartesian product $\mathbb{F}_d P^\infty \times \cdots \times \mathbb{F}_d P^\infty$. Thus, we can identify the cohomology ring $H^*(BG_d^m, R_d)$ with the polynomial ring $R_d[v_1, \ldots, v_m]$. By 4.1 we have a fibration $p\colon B_dP \to BG_d^m$ and hence, a homomorphism $p^*\colon H^*(BG_d^m) \to H^*(B_dP; R_d)$.

THEOREM 4.8. *Let P be a simple polyhedral complex and for $d = 1, 2$ let $R_d(P)$ be the associated face ring. The map $p^*\colon H^*(BG_d^m) \to H^*(B_dP; R_d)$ is surjective and induces an isomorphism of graded rings $H^*(B_dP; R_d) \simeq R_d(P)$.*

The proof of this theorem comes down to analyzing the case, where K is the boundary of the simplex. So suppose σ is a $(k-1)$-simplex with the vertices $v_1, \ldots, v_k$. Let P_σ and $P_{\partial\sigma}$ be the polyhedral complexes dual to σ and $\partial\sigma$, respectively. Then B_dP_σ is a D^{dk}-bundle over BG_d^k and $B_dP_{\partial\sigma}$ is the associated sphere bundle. The associated vector bundle is $EG \times_G \mathbb{F}_d^k \to BG$, where $G = G_d^k$. Since G_d^k acts diagonally on $\mathbb{F}_d^k$, this vector bundle is a sum of line bundles $L_1 \oplus \cdots \oplus L_k$ (these are real line bundles if $d = 1$ and complex line bundles if $d = 2$). Let c_i stand for the i^{th} Stiefel-Whitney or Chern class (as $d = 1$ *or* 2). Our notation is such that $c_1(L_i) = v_i \in H^d(BG_d^k; R_d)$. The mod 2 Euler class if $d = 1$ (or the Euler class if $d = 2$) of the vector bundle is given by the formula, $c_k = c_k(L_1 \oplus \cdots \oplus L_k)$. By the Whitney product formula $c_k = v_1 \ldots v_k$. Consider the Gysin sequence

$$\begin{tikzcd} H^*(BG_d^k) \arrow[r, "\cup c_k"] \arrow[d, "id"'] & H^{*+dn}(BG_d^k) \arrow[r] \arrow[d, "id"'] & H^{*+dn}(B_dP_{\partial\sigma}) \\ R_d[v_1, \ldots, v_k] \arrow[r, "v_1 \ldots v_k"] & R_d[v_1, \ldots, v_k]. & \end{tikzcd}$$

From this one deduces the following lemma.

LEMMA 4.9. *With notation as above, $H^*(B_dP_\sigma; R_d) = R_d[v_1, \ldots, v_k]$ and $H^*(B_dP_{\partial\sigma}; R_d) = R_d[v_1, \ldots, v_k]/(v_1 \ldots v_k)$. Thus Theorem* 4.8 *holds when P is dual to a simplex or to a boundary of the simplex.*

Proof of Theorem 4.8. The proof is by induction on the dimension of K. If $\dim K = 0$, then K is a disjoint union of vertices $v_1, \ldots, v_m$ and P is the cone on K. Hence, B_dP is a bouquet of m copies of $\mathbb{F}_dP^\infty$. Thus, in degree zero $H^*(B_dP; R_d)$ is R_d, while in degrees $\geqslant 1$ it is isomorphic to $R_d[v_1] \oplus \cdots \oplus R_d[v_m]$. In other words, $H^*(B_dP; R_d)$ is $R_d[v_1, \ldots, v_m]/I$ where I is the ideal generated by all square free monomials in more than one variable. Thus, the theorem holds if $\dim K = 0$.

Now suppose that $\dim K = n - 1$. By inductive hypothesis, the theorem is true for the $(n-2)$-skeleton of K. We add $(n-1)$-simplices one at a time to the $(n-2)$-skeleton and use Lemma 4.9 and the Mayer-Vietoris sequence to get the conclusion of the Theorem 4.8.

Remark 4.10. Suppose that K is a simplicial complex and that L is a subcomplex. Let P_K and P_L be the corresponding duals. The inclusion $L \subset K$ induces a homomorphism $R_d(K) \to R_d(L)$. If K and L have the same vertex set, the induced map is surjective. We also have an inclusion $i\colon B_dP_L \to B_dP_K$. The proof of Theorem 4.8

shows that i^* induces the canonical map of face rings. More generally, if $f: L \to K$ is a simplicial map, then there is an induced homomorphism $f_\sharp: R(K) \to R(L)$ of face rings. If f is a surjective, nondegenerate simplicial map, then $f_\sharp$ is injective. Thus the face ring provides a contravariant functor from the category of simplicial complexes to the category of graded commutative rings. The map $f: L \to K$ also induces a map $\hat{f}: B_d P_L \to B_d P_K$, well defined up to homotopy. Thus $B_d P_K$ provides a functor from simplicial complexes to the category of homotopy types of topological spaces, such that upon taking cohomology we retrieve the face ring functor.

Lemma 4.4 and Theorem 4.8 can be combined to calculate the cohomology ring of any right-angled Coxeter group. Suppose that (W, S) is a Coxeter system, with W right-angled and with $\{v_1, \ldots, v_m\}$ the set of fundamental generators (= reflections). Define Nerve(W, S) to be the abstract simplicial complex with vertex set S and with simplices the nonempty subsets of S which generate finite subgroups.

THEOREM 4.11. *Let W be a right-angled Coxeter group, as above.*

(1) *With coefficients in $\mathbb{Z}_2$, the cohomology ring of W is isomorphic to the face ring associated to the* Nerve(W, S); *i.e.,*

$$H^*(W; \mathbb{Z}_2) \simeq \mathbb{Z}_2[v_1, \ldots, v_m]/I$$

where I is the ideal generated by all square free monomials of the form $v_{i_1} \ldots v_{i_j}$, where at least two of the v_{i_j} do not commute when regarded as elements of W.

(2) *$H^j(W; \mathbb{Z})$ is 2-torsion for $j \geqslant 1$.*

Proof. Let P be the dual of the Nerve(W, S) and let $\mathscr{X} = (W \times P)/_\sim$ be the space constructed in the proof of Lemma 4.4(i). By [Davis2, §14], $\mathscr{X}$ is contractible. Hence $BP = EW \times_W \mathscr{X}$ is homotopy equivalent to $BW (= K(W, 1))$. Thus (1) follows from Theorem 4.8.

To prove (2), we note that the polyhedron P is acyclic and the fibers of $BP \to P$ are of the form BZ_2^k, for some k. This implies (2).

The face ring $R_1(P)$ is a graded ring and it is generated by elements of degree one. Let $H: \mathbb{N} \to \mathbb{N}$ be the associated Hilbert function; i.e., $H(k)$ is the dimension of the part of $R_1(P)$ in degree k. An easy calculation [Stanley1, Prop. 3.2] shows that

$$(1) \qquad H(k) = \begin{cases} \sum_{i=0}^{k-1} f_i \binom{k-1}{i}, & \text{for } k \geqslant 1 \\ 1 \quad , & \text{for } k = 0. \end{cases}$$

The Poincaré series of $R_1(P)$ is then $\sum_{k=0}^{\infty} H(k) t^k$. Another easy calculation shows that there is an identity of formal power series

$$(2) \qquad (1-t)^n \sum_{k=0}^{\infty} H(k) t^k = h_0 + h_1 t + \cdots + h_n t^n$$

where $(h_0, \ldots, h_n)$ is the h-vector of P defined in the beginning of Section 3.

Next we deduce some consequences of Theorem 4.8 for small covers and toric manifolds and show that (2) has a simple algebraic topological interpretation.

Let $\pi\colon M^{dn} \to P^n$ be a G_d^n-manifold over a simple polytope P^n. We have a projection map $p\colon B_dP \to BG_d^n$ which classifies the principal G_d^n-bundle $EG_d^n \times M^{dn} \to B_dP$. The map p is a Serre fibration with fiber $EG_d^n \times M^{dn}$ which is homotopy equivalent to M^{dn}. The Serre spectral sequence of this fibration has E_2-term

$$E_2^{p,q} = H^p(BG_d^n; H^q(M^{dn})).$$

Here the coefficients of $H^q(M^{dn})$ are in R_d.

THEOREM 4.12. *Let $\pi\colon M^{dn} \to P^n$ be a G_d^n-manifold and let $p\colon B_dP \to BG_d^n$ be the associated fibration. Then the Serre spectral sequence of p degenerates: $E_2^{p,q} = E_\infty^{p,q}$.*

Proof. Let us take coefficients in $\mathbb{Z}_2$ if $d = 1$, or $\mathbb{Q}$ if $d = 2$. First, we must see that the fundamental group of the base $(=BG_d^n)$ acts trivially on the cohomology of the fiber $(=M^{dn})$. When $d = 2$, $BG_d^n = \mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty$ is simply connected. When $d = 1$, $\pi_1(BG_d^n) = Z_2^n$, which acts trivially by Corollary 3.7. Thus, $E_2^{p,q} = H^p(BG_d^n) \otimes H^q(M^{dn})$. The Poincaré series of $H^*(BG_d^n)$ is $1/(1 - t^d)^n$. The Poincaré series of $H^*(M^{dn})$ is $h(t^d)$, where $h(t) = h_0 + h_1 t + \cdots + h_n t^n$ (cf. Remark 3.3). Hence, the Poincaré series of $E_2 (=\sum_k (\sum_{p+q=k} \dim E_2^{p,q}) t^k)$ is $h(t^d)/(1 - t^d)^n$. Comparing this with (2), we have that the Poincaré series of E_2 is $\sum_k H(k) t^{dk}$, i.e., the Poincaré series of the face ring $R_d(P)$. By Theorem 4.8, this is the Poincaré series of $H^*(B_dP)$, i.e., of E_∞. Since E_∞ is in general an iterated subquotient of E_2 and since they have the same Poincaré series, they are equal. (When $d = 2$, we initially only have this conclusion with $\mathbb{Q}$ coefficients, but since both E_2 and E_∞ are free abelian the result also holds in the integral case.)

COROLLARY 4.13. *With the hypotheses as in Theorem 4.12, let $j\colon M^{dn} \to B_dP$ be inclusion of the fiber. Then $j^*\colon H^*(B_dP) \to H^*(M^{dn})$ is onto.*

We shall now use this corollary to determine the ring structure of $H^*(M^{dn})$. We have natural identifications $H_d(B_dP) = H_d(BG_d^m) = R_d^m$ and $H_d(BG_d^n) = R_d^n$. (Recall that m is the number of codimension-one faces of P^n). Moreover, $p_*\colon H_d(B_dP) \to H_d(BG_d^n)$ is naturally identified with the characteristic function $\lambda\colon R_d^m \to R_d^n$. (Here R_d^m is regarded as the free R_d-module on the set of codimension-one faces of P^n).

The map $p^*\colon H^d(BG_d^n) \to H^d(B_dP)$ is then identified with the dual map $\lambda^*\colon R_d^{m*} \to R_d^{n*}$. Regarding the map λ as an $n \times m$ matrix λ_{ij}, the matrix for λ^* is the transpose. Column vectors of λ^* can then be regarded as linear combinations of $v_1, \ldots, v_m$, the indeterminates in the face ring. Put

$$\lambda_i = \lambda_{i1} v_1 + \cdots + \lambda_{im} v_m. \tag{3}$$

We have a short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^d(BG_d^n) & \xrightarrow{p^*} & H^d(B_dP) & \xrightarrow{j^*} & H^d(M^{dn}) & \longrightarrow & 0 \\ & & \| & & \| & & & & \\ & & (R_d^n)^* & \xrightarrow{\lambda^*} & (R_d^m)^*. & & & & \end{array}$$

Let J be the homogeneous ideal in $R_d[v_1, \ldots, v_m]$ generated by the λ_i and let $\bar{J}$ be its image in the face ring. Since j^*: $R_d(P) \to H^*(M^{dn})$ is onto and $\bar{J}$ is in its kernel, j^* induces a surjection $R_d(P)/\bar{J} \to H^*(M^{dn})$.

THEOREM 4.14. *Let π: $M^{dn} \to P^n$ be a G_d^n-manifold. Then $H^*(M^{dn}; R_d)$ is the quotient of the face ring by $\bar{J}$; i.e.,*

$$H^*(M^{dn}; R_d) = R_d[v_1, \ldots, v_m]/I + J.$$

Proof. We know that $H^*(BG_d^n)$ is a polynomial ring on n generators, and $H^*(BP)$ is the face ring. Since the spectral sequence degenerates, $H^*(BP) \simeq H^*(BG_d^n) \otimes H^*(M^{dn})$. Furthermore, p^*: $H^*(BG_d^n) \to H^*(BP)$ is injective and $\bar{J}$ is identified with the image of p^*. Thus, $H^*(M^{dn}) = H^*(BP)/\bar{J} = R_d[v_1, \ldots, v_m]/I + J$.

In the case of toric varieties, the above result is known as the Danilov-Jurkiewicz Theorem.

5. Cohen-Macaulay rings and complexes. We begin this section by reviewing some commutative algebra. Our exposition is taken from [Stanley3].

Suppose that R is an $\mathbb{N}$-graded algebra over a field k. The *Krull dimension* of R is the maximal number of algebraically independent elements of R. Suppose that the Krull dimension of R is n. A sequence $(\lambda_1, \ldots, \lambda_n)$ of homogenous elements of R is called *homogenous system of parameters*, if the Krull dimension of $R/(\lambda_1, \ldots, \lambda_n)$ is zero. A homogenous system of parameters $(\lambda_1, \ldots, \lambda_n)$ is called a *regular sequence* if λ_{i+1} is not a zero divisor in $R/(\lambda_1, \ldots, \lambda_i)$. Equivalently, $\lambda_1, \ldots, \lambda_n$ is a regular sequence if the λ_i are algebraically independent and if R is a finite-dimensional free $k[\lambda_1, \ldots, \lambda_n]$-module. The k-algebra R is *Cohen-Macaulay* if it admits a regular sequence. It can be proved that if R is Cohen-Macaulay then any homogenous system of parameters is a regular sequence.

Suppose that R is generated by elements of degree 1. If k is infinite, then it is a consequence of Noether's Normalization Lemma that there exists a homogeneous system of parameters $\lambda_1, \ldots, \lambda_n$, where each λ_i is of degree one. Such a sequence will be called a *degree-one homogenous system of parameters*. Similarly, if such $\lambda_1, \ldots, \lambda_n$ is a regular sequence, then it is a *degree-one regular sequence*. We shall see in Example 5.3 below, that when k is finite, degree-one regular sequences may fail to exist.

As in Section 4, let K be an $(n-1)$-dimensional simplicial complex on vertices $v_1, \dots, v_m$ and let $k(K)$ be its face ring. The Krull dimension of $k(K)$ is n. In [Reisner] the following fundamental result is proved.

THEOREM 5.1 ([REISNER]). *The following conditions are equivalent.*

(i) *$k(K)$ is Cohen-Macaulay.*

(ii) *$\bar{H}_i(K; k) = 0$ for $i < n-1$ and for each simplex $\sigma \in K$, $\bar{H}_i(\mathrm{Link}(\sigma, K); k) = 0$ for $i < \dim \mathrm{Link}(\sigma, K)$.*

A simplicial complex K satisfying condition (ii) of the theorem is a *Cohen-Macaulay complex over k*. Examples of Cohen-Macaulay complexes are

(a) a triangulation of a sphere,

(b) a spherical building,

(c) the universal complex K_d^n constructed in Section 2.

For the remainder of this section, k will be $\mathbb{Z}_2$ if $d = 1$ or $\mathbb{Q}$ if $d = 2$. Let P be the simple polyhedral complex dual to K and B_dP the space constructed in Section 4. In Theorem 4.8 we identified the cohomology ring of B_dP with the face ring of K. We shall now investigate the connection between G_d^n-spaces over P and degree-one homogenous systems of parameters for $k(K)$.

Suppose $\lambda_1, \dots, \lambda_n$ is a sequence of degree one elements $k(K)$. (If $k = \mathbb{Q}$, after multiplying by a scalar we may assume that each $\lambda_i \in \mathbb{Z}(K)$.) Since $k(K) \simeq H^*(B_dP; k)$, we may view λ_i as an element of $H^d(B_dP; k)$. Such a cohomology class is the same thing as a map to an Eilenberg-Maclane space λ_i: $B_dP \to K(R_d, 1)$, where $K(Z_2, 1) = \mathbb{R}P^\infty = BZ_2$, and $K(\mathbb{Z}, 2) = \mathbb{C}P^\infty = BT$. Thus, $(\lambda_1, \dots, \lambda_n)$ can be regarded as a map λ: $B_dP \to BG_d^n$. The homotopy theoretic fiber of λ need not have the homotopy type of a finite complex; however, it does when λ arises from a G_d^n-space over P.

Let us recall how this works. Suppose Y is a G_d^n-space over P. Its characteristic function can be viewed as a homomorphism λ: $R_d^m \to R_d^n$, (R_d^m is identified with the free R_d-module on the set of codimension-one faces of P.) If $d = 2$, we tensor with $\mathbb{Q}$ to obtain a map $\mathbb{Q}^m \to \mathbb{Q}^n$, which we shall again denote by λ. Taking the Borel construction on Y, we get a fibration

$$Y \to B_dP \to BG_d^n$$

where the projection map $B_dP \to BG_d^n$ is identified with λ as above. These observations lead to the following lemma.

LEMMA 5.2. *Let P be the dual of an $(n-1)$-simplicial complex K and let $Y \to P$ be a G_d^n-space over P with characteristic function $\lambda = (\lambda_1, \dots, \lambda_n)$: $k^m \to k^n$. If $d = 1$, let us further assume that Z_2^n acts trivially on $H^*(Y; \mathbb{Z}_2)$. Then regarding λ_i as a degree-one element of $k(K)$, we have that $(\lambda_1, \dots, \lambda_n)$ is a degree-one homogenous system of parameters for $k(K)$.*

Proof. The cohomology of B_dP is a module over $H^*(BG_d^n)$, as is each E_j in the Serre spectral sequence (with coefficients in k) of the fibration $Y \to B_dP \to BG_d^n$. In particular, $E_2 = H^*(Y) \otimes H^*(BG_d^n)$ is a finite-dimensional free $k[\lambda_1, \ldots, \lambda_n]$-module. Since E_2 is therefore a Noetherian module over $k[\lambda_1, \ldots, \lambda_n]$, the same is true for its iterated subquotient E_∞. Since, by hypothesis, the fundamental group of the base acts trivially on the cohomology of the fiber, the Serre spectral sequence converges and hence, $H^*(B_dP)$ is a finitely generated $k[\lambda_1, \ldots, \lambda_n]$-module. This implies that $(\lambda_1, \ldots, \lambda_n)$ is a homogenous system of parameters.

We do not know if the converse to this lemma is true in general; however, a converse is true for regular sequences. This will be established in Lemma 5.3 below. Before stating it, we need some more terminology.

Let $e_1, \ldots, e_m$ be the standard basis for k^m and denote the dual basis for $(k^m)^*$ by $v_1, \ldots, v_m$. If $\sigma = \{v_{i_1}, \ldots, v_{i_l}\}$ is a $(l-1)$-simplex in K, then let k^σ denote the l-dimensional subspace of k^m spanned by $e_{i_1}, \ldots, e_{i_l}$. A linear map $\lambda = (\lambda_1, \ldots, \lambda_n)$: $k^m \to k^n$ is *nondegenerate over* K if for each $\sigma \in K$, $\lambda|_{k^\sigma}\colon k^\sigma \to k^n$ is injective. We note that if $\lambda\colon k^m \to k^n$ arises from a characteristic function of a G_d^n-space over the dual of K, then it is nondegenerate over K.

If $\lambda_1, \ldots, \lambda_n$ is a degree-one homogenous system of parameters for $k(K)$, then each λ_i is a linear combination of the v_j, say, $\lambda_i = \lambda_{i1}v_1 + \cdots + \lambda_{im}v_m$. Regarding the v_j as linear functions on k^n, we have that $\lambda_i\colon k^m \to k$ is linear; hence, $(\lambda_1, \ldots, \lambda_n)$ is a linear map $\lambda\colon k^m \to k^n$.

LEMMA 5.3. *Suppose that K is an $(n-1)$-dimensional simplicial complex and that each maximal simplex of K is $(n-1)$-dimensional. Let $(\lambda_1, \ldots, \lambda_n)$ be a degree-one regular sequence for $k(K)$ and let $\lambda = (\lambda_1, \ldots, \lambda_n)\colon k^m \to k^n$ be the resulting linear map. Then λ is nondegenerate over K.*

Proof. Suppose λ is degenerate. Then $\lambda|_{k^\sigma}$ is not an isomorphism for some maximal simplex σ. Without loss of generality, we may suppose, that $\sigma = \{v_1, \ldots, v_n\}$. For $1 \leqslant i \leqslant n$, put $\hat{\lambda}_i = \lambda_{i1}v_1 + \cdots + \lambda_{in}v_n$, so that $\lambda|_{k^\sigma} = (\hat{\lambda}_1, \ldots, \hat{\lambda}_n)\colon k^\sigma \to k^n$. Since $\lambda|_{k^\sigma}$ is not onto, some nontrivial linear combination of the $\hat{\lambda}_i$ is zero, say, $\sum c_i\hat{\lambda}_i = 0$. Consider the nonzero element $w = v_1 \ldots v_n$ in $k(K)$. By definition, $v_jw = 0$ whenever $j > n$. Hence, $(\sum c_i\lambda_i)w = (\sum c_i\hat{\lambda}_i)w = 0$. We may assume without loss of generality, that $c_n \neq 0$. The above equation then says that λ_n is a zero divisor in $k(K)/(\lambda_1, \ldots, \lambda_{n-1})$, contradicting the definition of a regular sequence.

Example 5.4. Degree-one regular sequences may fail to exist. Suppose that k is a finite field with q elements. Let K be the boundary complex of the n-dimensional cyclic polytope with m vertices, $n \geqslant 4$. By the argument in 1.22., if $m \geqslant q^n$, then any linear map $k^m \to k^n$ is degenerate over K. Hence, for such a K the face ring $k(K)$ admits no degree-one regular sequence.

Now suppose that $Y \to P$ is a small cover, where P is the dual of K. Let ∂P denote the union of faces of P of codimension $\geqslant 1$ ($\partial P = |K|$). Define an element A in the group ring $\mathbb{Z}_2[Z_2^n]$ by $A = \sum g$, where the summation is taken over all elements $g \in Z_2^n$. If α is a simplex in ∂P, then it is fixed by some nontrivial element $g \in Z_2^n$; since the coefficients are in $\mathbb{Z}_2$, it follows that $A\alpha = 0$, and therefore that A induces a chain map $A\colon C_*(P, \partial P; \mathbb{Z}_2) \to C_*(Y; \mathbb{Z}_2)$. Let A_* be the induced map in homology.

We have a map $c_*: H_*(Y) \to H_*(Y, Y - \text{int } P) \simeq H_*(P, \partial P)$, where the last isomorphism is excision. Obviously, $c_* A_* = \text{id}$. Thus, A_* is a monomorphism. We apply this observation to calculate the top homology of Y.

LEMMA 5.5. *Let K be an $(n-1)$-dimensional simplicial complex, P its dual and $Y \to P$ a small cover. Then $A_*: H_n(P, \partial P; \mathbb{Z}_2) \to H_n(Y; \mathbb{Z}_2)$ is an isomorphism. In particular, Z_2^n acts trivially on the top-dimensional homology of Y with $\mathbb{Z}_2$ coefficients.*

Proof. The proof is by induction on the number of $(n-1)$-simplices in K. If there are no $(n-1)$-simplices, then both $H_n(P, \partial P)$ and $H_n(Y)$ vanish. If K is a single $(n-1)$-simplex σ^{n-1}, then $P = I_\sigma$ is an n-cube, and Y is a linear n-disk. If there is at least one $(n-1)$-simplex, then we can decompose K as $K = K' \cup \sigma$, where σ is an $(n-1)$-simplex and $K' = K - \text{int}(\sigma)$. Let $P = P' \cup I_\sigma$ and $Y = Y' \cup D^n$ be the corresponding decompositions of P and Y. By induction, the result holds for Y'. The theorem follows by comparing the Mayer-Vietoris sequences for P and Y.

Remark 5.6. A similar argument shows that if $Y \to P$ is a T^n-space, then $H_{2n}(Y; \mathbb{Z}) \simeq H_n(P, \partial P; \mathbb{Z})$.

The goal of the remainder of this section is to show that results of Sections 3 and 4 for G_d^n-manifolds over simple polytopes go through for G_d^n-spaces over duals of Cohen-Macaulay complexes. Suppose that K is Cohen-Macaulay, that P is its dual and that $\pi: Y \to P$ is a G_d^n-space. Each $(l-1)$ simplex σ in K corresponds to a codimension-l face F of P, where F is the dual of $\text{Link}(\sigma, K)$. Thus, F is the dual of Cohen-Macaulay complex of dimension $n - l - 1$. Let Y_F denote the inverse image of F in Y. First, we prove the analog of the Corollary 3.7.

LEMMA 5.7. *Suppose that K is an $(n-1)$-dimensional Cohen-Macaulay complex over $\mathbb{Z}_2$, P is its dual, and that $Y \to P$ is a small cover.*

(i) *The homology of Y is "generated by faces" in the following sense: the natural map $\sum H_i(Y_F; \mathbb{Z}_2) \to H_i(Y; \mathbb{Z}_2)$ is onto, where the summation ranges over all i-dimensional faces of P.*

(ii) *The group Z_2^n acts trivially on $H_*(Y; \mathbb{Z}_2)$.*

Proof. (i) In this proof the coefficients are in $\mathbb{Z}_2$. We have a filtration $Y = Y_n \supset Y_{n-1} \supset \cdots \supset Y_0$, where Y_j is the inverse image of the union of all j-dimensional faces of P. By excision, $H_*(Y_j, Y_{j-1}) \simeq \sum H_*((F, \partial F) \times \mathbb{Z}_2^j)$ where the summation is over all j-faces of P. Since each such F is the dual of an $(j-1)$-dimensional Cohen-Macaulay complex, the homology of $(F, \partial F)$ vanishes except in dimension j. Thus, $H_{j-1}(Y_{j-1}) \to H_{j-1}(Y_j)$ is onto and $H_i(Y_{j-1}) \to H_i(Y_j)$ is an isomorphism for $i < j - 1$. This implies, that $H_i(Y_i) \to H_i(Y)$ is onto. Since Y_i is the union of the i-dimensional complexes Y_F, F an i-face, and since these intersect along lower-dimensional complexes, the statement of (i) follows.

(ii) Since each i-face F is dual to a Cohen-Macaulay complex, it follows from Lemma 5.5, that Z_2^i acts trivially on $H_i(Y_F)$, hence so does Z_2^n (the extension of Z_2^i by the stabilizer of F). Thus, the statement of (ii) follows from (i).

Remark 5.8. The proof of (i) shows that the situation is analogous to the standard argument that the cellular homology of a CW complex is isomorphic to its singular homology. Put $C_j = H_j(Y_j, Y_{j-1}; \mathbb{Z}_2)$. Then, $0 \to C_n \to C_{n-1} \to \cdots \to C_0$ is a chain complex and $H_*(C) = H_*(Y)$.

THEOREM 5.9. *Let K be an $(n-1)$-dimensional Cohen-Macaulay complex over $\mathbb{Z}_2$, P its dual and $(\lambda_1, \ldots, \lambda_n)$ a sequence of elements in $H^1(BP)$. Then the following statements are equivalent.*

(1) *$(\lambda_1, \ldots, \lambda_n)$ is a degree-one regular sequence.*

(2) *Regarded as a linear map, $\lambda = (\lambda_1, \ldots, \lambda_n)\colon \mathbb{Z}_2^m \to \mathbb{Z}_2^n$ is the characteristic function of a small cover.*

Morever, if either of these conditions hold, then the Serre spectral sequence of the resulting fibration $\lambda\colon BP \to BZ_2^n$ degenerates: $E_2^{p,q} = E_\infty^{p,q}$. (Again everything is with $\mathbb{Z}_2$ coefficients.)

Proof. The implication (1) $\Rightarrow$ (2) follows from Lemma 5.3. Suppose that (2) holds and that $Y \to P$ is the small cover corresponding to λ. By Lemma 5.7(ii), Z_2^n acts trivially on $H^*(Y)$; hence, Lemma 5.2 implies that $(\lambda_1, \ldots, \lambda_n)$ is a degree-one homogeneous system of parameters. But any homogenous system of parameters is a regular sequence; so (2) $\Rightarrow$ (1).

We consider the Serre spectral sequence of the fibration $Y \to BP \to BZ_2^n$. Since the fundamental group of BZ_2^n acts trivially on $H_*(Y)$, the spectral sequence converges, and $E_2 = H_*(Y) \otimes \mathbb{Z}_2[\lambda_1, \ldots, \lambda_n]$. The fact that $(\lambda_1, \ldots, \lambda_n)$ is a regular sequence means that there are homogenous elements $\eta_1, \ldots, \eta_t$ in $H^*(BP)$ such that every element of $H^*(BP)$ can be written uniquely in the form $\sum \eta_i p_i(\lambda_1, \ldots, \lambda_n)$, where p_i is some polynomial. We claim that this means that every class in $E_2^{*,0} = H^*(Y)$ survives to E_∞ and that we may take $\eta_1, \ldots, \eta_t$ to be a homogenous basis for $H^*(Y)$. By induction, we may suppose that this is true for $E_2^{j,0}$, with $j < p$. Let α be a class in $H^p(Y)$ and d a differential. Then $d\alpha$ has the form $\sum \eta_i p_i(\lambda_1, \ldots, \lambda_n)$. By uniqueness, this expression cannot represent 0 in E_∞; hence, $d\alpha$ must be zero. Thus, every differential vanishes on $H^*(Y)$, and consequently, $H^p(Y)$ must have a basis consisting of the η_i of degree p. It follows that $E_2 = E_\infty$.

A similar argument gives the following result for T^n-spaces.

THEOREM 5.10. *Let K be an $(n-1)$-dimensional Cohen-Macaulay complex, P its dual, and $Y \to P$ a T^n-space over P. Then the Serre spectral sequence with $\mathbb{Q}$ coefficients of $Y \to B_T P \to BT^n$ degenerates: $E_2 = E_\infty$.*

Remark 5.11. Suppose $k(K)$ admits a degree-one homogenous system of parameters $(\lambda_1, \ldots, \lambda_n)$. If the resulting spectral sequence degenerates, then $(\lambda_1, \ldots, \lambda_n)$ is obviously a regular sequence. Hence, $k(K)$ is Cohen-Macaulay if and only if the spectral sequence degenerates.

From the last two theorems one can prove the following analog of Theorem 3.1 and Theorem 4.14 without resorting to the perfect cell structure argument.

THEOREM 5.12. *Let k be $\mathbb{Z}_2$ or $\mathbb{Q}$ as $d = 1, 2$. Let K be an $(n-1)$-dimensional Cohen-Macaulay complex over k, P its dual, and $Y \to P$ a G_d^n-space. Let $(h_0, \ldots, h_n)$ be the h-vector of P defined as in Section 3.*

(i) *Suppose $d = 1$. Let $b_i(Y)$ be the i-th* mod 2 *Betti number of Y. Then $b_i(Y) = h_i$.*

(ii) *Suppose $d = 2$. Then the homology of Y vanishes in odd dimensions and $b_{2i}(Y) = h_i$, where b_{2i} now denotes the rational Betti number.*

The cohomology ring of Y is given by the formula in Theorem 4.14; i.e.,

$$H^*(Y; k) = k[v_1, \ldots, v_m]/I + J$$

where I and J are defined as in the Section 4, and the v_i are of degree d.

Using Theorem 2.2, we have the following application of Theorems 5.9 and 5.12.

COROLLARY 5.13. *As in Section 2, let $Y_1^n \to U_1^n$ be the universal small cover and let K_1^n be the simplicial complex dual to U_1^n.*

(1) *K_1^n is a Cohen-Macaulay complex and the Serre spectral sequence of $Y_1^n \to U_1^n$ degenerates.*

(2) *The h-vector of U_1^n gives the* mod 2 *Betti numbers of Y_1^n.*

6. Vector bundles.

6.1. *The canonical line bundles.* Let K be an $(n-1)$-dimensional simplicial complex and P the simple polyhedral complex dual to K. For each codimension-one face of P, we shall define a line bundle over B_dP, where "line bundle" means a real line bundle if $d = 1$ and a complex line bundle if $d = 2$.

Let $\{v_1, \ldots, v_m\}$ be the vertex set of K and let F_i denote the codimension-one face of P corresponding to v_i. As before, we use the v_i to denote the generators of $H^d(B_dP)$. Recall that $H^*(B_dP) = R_d[v_1, \ldots, v_m]/I$. We follow the notation of Section 4.1: $\mathscr{X} = (G_d^m \times P)/_\sim$ and $B_dP = EG_d^m \times_{G_d^m} \mathscr{X}$. Let $\rho_i\colon G_d^m \to G_d$ be the projection onto the i-th factor and let $\mathbb{F}_d(\rho_i)$ denote the corresponding 1-dimensional representation space of G_d^m. Define a trivial equivariant line bundle $\tilde{L}_i$ over $\mathscr{X}$ by $\tilde{L}_i = \mathbb{F}_d(\rho_i) \times \mathscr{X}$. Taking the Borel construction on $\tilde{L}_i$, we get a line bundle L_i over B_dP

$$L_i = EG_d^m \times_{G_d^m} \tilde{L}_i. \tag{1}$$

The *characteristic class* of L_i, denoted by $c_1(L_i)$, is the first Stiefel-Whitney class if $d = 1$, or first Chern class if $d = 2$, in $H^d(B_dP; R_d)$. It is straightforward to see that

$$c_1(L_i) = v_i. \tag{2}$$

The bundle L_i is called the *canonical line bundle corresponding to F_i*. Since the i-th factor acts freely on the complement of the inverse image of F_i in $\mathscr{X}$, it follows that the restriction of L_i to $B_dP - B_dF_i$ is the trivial line bundle.

6.2. *Line bundles associated to a G_d^n-space.* Suppose that $Y \to P$ is a G_d^n-space with the characteristic function $\lambda: R_d^m \to R_d^n$. We have the fibration

$$Y \xrightarrow{j} B_d P \xrightarrow{p} BG_d^n$$

where the homomorphism $p_*: H_d(B_d P) \to H_d(BG_d^n)$ can be identified with λ. Choose a basis $e_1, \ldots, e_n$ for R_d^n; let $e_1^*, \ldots, e_n^*$ be the dual basis for $H^d(BG_d^n) = (R_d^n)^*$, and let $\hat{e}_i$ be the line bundle over BG_d^n corresponding to e_i^*.

Define a line bundle over $B_d P$ by $E_i = p^*(\hat{e}_i)$. Then

$$c_1(E_i) = \lambda_i \tag{3}$$

where $\lambda_i = \lambda_{i1} v_1 + \cdots + \lambda_{im} v_m$ is the class in $H^d(B_d P)$ defined by the formula (3) of Section 4. On the level of line bundles this means that

$$E_i = L_1^{\otimes \lambda_{i1}} \otimes \cdots \otimes L_m^{\otimes \lambda_{im}}. \tag{4}$$

Now suppose that $Y \to P$ is a pullback of the linear model. This means that the characteristic function λ maps $\{F_1, \ldots, F_m\}$ onto some basis of R_d^n, which we will continue to write as $e_1, \ldots, e_n$. We therefore have a partition of the set of codimension-one faces into n subsets $(\lambda^{-1}(e_i))_{1 \leqslant i \leqslant n}$ and a corresponding partition of the set of indices $\{1, \ldots, m\}$ into subsets $(S_i)_{1 \leqslant i \leqslant n}$, where $S_i = \{j \mid F_j \in \lambda^{-1}(e_i)\}$. Now formula (4) becomes

$$E_i = \prod_{j \in S_i} L_j. \tag{5}$$

For each $1 \leqslant i \leqslant n$, consider the vector bundle D_i over $B_d P$ defined by

$$D_i = \sum_{j \in S_i} L_j. \tag{6}$$

Let $p(i) = \operatorname{Card} S_i - 1$ and let $\varepsilon^{p(i)}$ denote the trivial vector bundle of dimension $p(i)$. Using the fact that the codimension-one faces in $\lambda^{-1}(e_i)$ are pairwise disjoint, one can construct a bundle isomorphism $f: E_i \oplus \varepsilon^{p(i)} \to D_i$. If $F_k \in \lambda^{-1}(e_i)$, then the map f is such that when restricted to $B_d F_k$, it maps E_i to the k-th component of D_i, and it maps $\varepsilon^{p(i)}$ to the other components. (E_i restricts to L_k over $B_d F_k$ and D_i restricts to $L_k \oplus \varepsilon^{p(i)}$). Thus,

$$E_i \oplus \varepsilon^{p(i)} \simeq \sum_{j \in S_i} L_j. \tag{7}$$

It follows that

$$E_1 \oplus \cdots \oplus E_n \oplus \varepsilon^{m-n} \simeq L_1 \oplus \cdots \oplus L_m. \tag{8}$$

This gives the following criterion for deciding if Y is a pullback of the linear model.

PROPOSITION 6.3. *Suppose that the G_d^n-space $Y \to P$ is a pullback of the linear model. Then the bundle $j^*(L_1 \oplus \cdots \oplus L_m)$ is trivial.*

Proof. Since $j^*c_1(E_i) = 0$, E_i pulls back to the trivial line bundle over Y. The proposition then follows from (8).

6.4. *The "tangent bundle" of B_dP.* Now suppose that P^n is a simple convex polytope. Then $\mathscr{Z}$ is a smooth G_d^n-manifold of dimension $n + m(d-1)$. The Borel construction on the tangent bundle of $\mathscr{Z}$ yields a vector bundle τ over B_dP

$$\tau = EG_d^m \times_{G_d^m} T\mathscr{Z}. \tag{9}$$

When $d = 2$, we will only be interested in τ as a stable (real) vector bundle. The reason for this is explained by the next lemma.

LEMMA 6.5. *Let $M^{dn} \to P^n$ be a G_d^n-manifold over P^n and let τ' denote the Borel construction on TM^{dn}.*

(i) *If $d = 1$, then $\tau = \tau'$.*

(ii) *If $d = 2$, then $\tau = \tau' \oplus \varepsilon^{m-n}$, where ε^{m-n} denotes the trivial real vector bundle of dimension $m - n$.*

Proof. We prove (ii); the proof of (i) is similar. Taking notation from Section 4.1, we let H be the subgroup of T^m corresponding to the kernel of $\lambda\colon \mathbb{Z}^m \to \mathbb{Z}^n$. Then H acts freely on $\mathscr{Z}$, $T^m/H \simeq T^n$, and $M^{2n} = \mathscr{Z}/H$. Thus, $\mathscr{Z}$ is a principal H-bundle over M^{2n} with projection map $q\colon \mathscr{Z} \to M^{2n}$. We have that $T\mathscr{Z} = q^*(TM^{2n}) \oplus F$, where F is the tangent bundle along the fibers. In general, the tangent bundle along the fibers of a principal H-bundle is the pullback of a vector bundle over M^{2n}, associated to the principal bundle via the adjoint representation of H. Since in our case H is abelian, this bundle is the trivial bundle of appropriate dimension, and the statement (ii) follows.

THEOREM 6.6. *Let P^n be a simple convex polytope. Then the vector bundles τ and $L_1 \oplus \cdots \oplus L_m$ are stably isomorphic as real vector bundles over B_dP^n.*

Proof. It suffices to show that the bundle $T\mathscr{Z}$ and $\tilde{L}_1 \oplus \cdots \oplus \tilde{L}_m$ are G_d^m-equivariantly isomorphic, after possibly adding trivial bundle of the form ε^k, where $\varepsilon^k = \mathbb{R}^k \times \mathscr{Z}$ with the trivial G_n^m-action on $\mathbb{R}^k$. To this end we define bundle maps $\phi_i\colon \tilde{L}_i \to T\mathscr{Z} \oplus \varepsilon^{dm}$ and $\theta\colon T\mathscr{Z} \to \pi^*TP^n$, where $\pi\colon \mathscr{Z} \to P^n$ is the projection. Let $d\pi\colon T\mathscr{Z} \to TP^n$ be the differential of π. If F is a face of P^n, $x \in \operatorname{int}(F)$, and $z \in \pi^{-1}(x)$, then $d\pi_z$ maps $T_z\mathscr{Z}$ surjectively onto T_xF. Let $\theta\colon T\mathscr{Z} \to \pi^*TP^n$ be the map induced by $d\pi$. (N.B. the rank of θ is not constant). Put $\mathscr{Z}_i = \pi^{-1}(F_i)$. We have that $T\mathscr{Z}|_{\mathscr{Z}_i} \simeq \tilde{L}_i|_{\mathscr{Z}_i} \oplus T\mathscr{Z}_i$. Hence, we can find a bundle map $\alpha_i\colon \tilde{L}_i \to T\mathscr{Z}$ such that α_i takes $\tilde{L}_i|_{\mathscr{Z}_i}$ monomorphically onto the normal bundle of $\mathscr{Z}_i$ and such that α_i is zero on the complement of a small tubular neighbourhood of $\mathscr{Z}_i$. Regard ε^{dm} as $\mathbb{F}_d^m \times \mathscr{Z}$. Since $\tilde{L}_i|_{\mathscr{Z} - \mathscr{Z}_i} \simeq \varepsilon^d|_{\mathscr{Z} - \mathscr{Z}_i}$, we can find a map $\beta_i\colon \tilde{L}_i \to \varepsilon^{dm}$ such that on $\tilde{L}_i|_{\mathscr{Z}_i}$, β_i is the zero map and such that it maps $\tilde{L}_i|_{\mathscr{X}-\mathscr{X}_i}$ monomorphically onto the i-th factor of ε^{dm}. Put $\phi_i = (\alpha_i, \beta_i)\colon \tilde{L}_i \to T\mathscr{X} \oplus \varepsilon^{dm}$ and $\phi = \sum \phi_i\colon \sum \tilde{L}_i \to T\mathscr{X} \oplus \varepsilon^{dm}$. Since the normal bundle of F_i in P^n is a trivial line bundle, there is a map $\mathbb{R} \times P^n \to TP^n$ which takes $\mathbb{R} \times F_i$ onto the normal bundle and is zero off a collared neighbourhood of F_i. Pulling back to $\mathscr{X}$, we get a map $\omega = \sum \omega_i\colon \varepsilon^m \to \pi^* TP^n$.

Now suppose that $d = 1$. Then it is easy to see that we have the short exact sequence of vector bundles

$$0 \longrightarrow \sum_{i=1}^m \tilde{L}_i \xrightarrow{\phi} T\mathscr{X} \oplus \varepsilon^m \xrightarrow{\theta \oplus \omega} \pi^* TP^n \longrightarrow 0.$$

Since P^n is a convex polytope, TP^n is trivial; hence, $\pi^* TP^n \simeq \varepsilon^n$. This proves the result when $d = 1$.

The argument must be modified when $d = 2$ to take into account the fact that the kernel of $\theta\colon T\mathscr{X} \to \pi^* TP^n$ contains the tangent spaces of T^m-orbits. As in Proposition 1.8, there is an equivariant map $T^m \times P^n \to \mathscr{X}$. This induces a bundle map $\gamma\colon \varepsilon^m \to T\mathscr{X}$, which takes each fiber of ε^m surjectively onto the tangent space along the orbit. Thus, when restricted to $\mathscr{X}_i$, γ maps the i-th factor of ε^m to zero. Regard ε^{2m} as $\varepsilon^m \oplus \varepsilon^m$ and let $\gamma_i'\colon \varepsilon \to \varepsilon^{2m}$ be $\omega_i\colon \varepsilon \to \varepsilon^m$ followed by inclusion into the first summand. Put $\gamma' = \sum \gamma_i'\colon \varepsilon^m \to \varepsilon^{2m}$ and $\delta = (\gamma, \gamma')\colon \varepsilon^m \to T\mathscr{X} \oplus \varepsilon^{2m}$. As before it is easy to check that the sequence is exact

$$0 \longrightarrow \sum \tilde{L}_i \oplus \varepsilon^m \xrightarrow{\phi \oplus \delta} T\mathscr{X} \oplus \varepsilon^{2m} \xrightarrow{\theta \oplus 0 \oplus \omega} \pi^* TP^n \longrightarrow 0$$

where the map $0 \oplus \omega\colon \varepsilon^{2m} \to \pi^* TP^n$ is the zero map on the first summand of $\varepsilon^m \oplus \varepsilon^m$ and ω on the second. This completes the proof.

COROLLARY 6.7. *Let P^n be a simple convex polytope and τ the vector bundle over $B_d P$ defined by* (9).

(i) *Suppose $d = 1$. The total Stiefel-Whitney class $\omega(\tau)$ of τ is given by*

$$w(\tau) = \prod_{i=1}^m (1 + v_i).$$

The Pontriagin classes of τ all vanish.

(ii) *Suppose $d = 2$. The total Pontriagin class $p(\tau)$ is given by*

$$p(\tau) = \prod_{i=1}^m (1 - v_i^2).$$

The Stiefel-Whitney classes are given by the formula

$$w(\tau) = \prod_{i=1}^m (1 + v_i) \bmod 2.$$

Proof. (i) By Theorem 6.6, $w(\tau) = w(L_1 \oplus \cdots \oplus L_m) = \prod(1 + v_i)$. The last sentence in (i) follows since a real line bundle has vanishing Pontriagin class.

(ii) The total Pontriagin class of $L_1 \oplus \cdots \oplus L_m$ is the total Chern class of the complexification. Since $L_i \otimes \mathbb{C} = L_i \oplus \bar{L}_i$, we have $c(L_i \otimes \mathbb{C}) = (1 + v_i)(1 - v_i) = 1 - v_i^2$; hence $p(\tau) = \prod(1 - v_i^2)$. The last formula is the standard fact that the total Stiefel-Whitney class of a complex vector bundle is the mod 2 reduction of its total Chern class, cf. [Milnor-Stasheff, Problem 14-B, p. 171].

COROLLARY 6.8. *Let $M^{dn} \to P^n$ be a G_d^n-manifold over a simple convex polytope P^n and let $j\colon M^{dn} \to B_d P^n$ be inclusion of the fiber.*

(i) *If $d = 1$, then*

$$w(M^n) = j^* \prod_{i=1}^{m} (1 + v_i) \qquad \textit{and}$$

$$p(M^n) = 1.$$

(ii) *If $d = 2$, then*

$$w(M^{2n}) = j^* \prod_{i=1}^{m} (1 + v_i) \bmod 2 \qquad \textit{and}$$

$$p(M^{2n}) = j^* \prod_{i=1}^{m} (1 - v_i^2).$$

Remark 6.9. Our "canonical" line bundles do not agree with the algebraic geometers' in the case of toric varieties. For example, if M^{2n} is a nonsingular toric variety, then [Oda, Theorem 3.12] would seem to predict $c(M^{2n}) = j^* \prod(1 + v_i)$, for the total Chern class; however, this is not true. For example, when $M = \mathbb{C}P^1$ we have $j^*(1 + v_1)(1 + v_2) = 1$ (by Proposition 6.3), while $c_1(\mathbb{C}P^1) \neq 0$. The difference is a matter of a sign conventions.

We have one final corollary to Theorem 6.6 and Proposition 6.3.

COROLLARY 6.10. *Suppose that the G_d^n-manifold $M^{dn} \to P^n$ is the pullback of the linear model $\mathbb{F}_d^n \to \mathbb{R}_+^n$. Then M^{dn} is stably parallelizable.*

Remark. In fact this last result could have been proved directly without resorting to Theorem 6.6 and Proposition 6.3. Indeed, it follows from the definition of pullback that M^{dn} is a smooth submanifold of $\mathbb{F}_d^n \times P^n$, with trivial normal bundle (compare [Davis 3, Prop. 1.4]).

COROLLARY 6.11. *Suppose that a small cover $M^n \to P^n$ is a pullback of the linear model. Then $H_i(M^n; \mathbb{Z})$ is free abelian of rank h_i.*

Proof. For each face F of P^n, the submanifold M_F is stably parallelizable and therefore orientable. Hence, the result follows from Corollary 3.8.

7. Further questions and remarks.

7.1. *Toric orbifolds.* Suppose that P^n is a simple convex polytope with m faces of codimension one and that K is the dual simplicial complex. Let $\mathscr{Z}^{m+n}$ be the T^m manifold constructed in Section 4.1: $\mathscr{Z}^{m+n} = T^m \times P^n/\sim$. Further, suppose that a linear map $\lambda\colon \mathbb{Q}^m \to \mathbb{Q}^n$ is nondegenerate over K. Changing λ by homothety, we may assume that it is induced by a homomorphism from $\mathbb{Z}^m$ to $\mathbb{Z}^n$, which we continue to denote by λ. This homomorphism need not satisfy the condition $(*)$ of Section 1: if σ is an $(l-1)$-simplex of K, and $\mathbb{Z}^\sigma$ is the corresponding summand of $\mathbb{Z}^m$, then although $\lambda(\mathbb{Z}^\sigma)$ is of rank l, it need not be a direct summand. Let N be the subgroup of T^m corresponding to the kernel of λ. Then N acts on $\mathscr{Z}^{n+m}$ with finite isotropy groups. We denote the orbit space of N on $\mathscr{Z}^{n+m}$ by $Q^{2n}(\lambda)$ and call it the *toric orbifold* corresponding to λ. (If λ satisfies $(*)$, then N acts freely and $Q^{2n}(\lambda)$ is the toric manifold corresponding to λ.) Singular toric varieties are toric orbifolds in this sense.

A toric orbifold Q^{2n} over P^n leads to a fibration $BQ^{2n} \to B_TP^n \to B\mathbb{Z}^n$, where the projection map $B_TP^n \to B\mathbb{Z}^n$ is induced by λ, and where $BQ^{2n} = EN \times_N \mathscr{Z}^{n+m}$. The space BQ^{2n} is the classifying space for the orbifold Q^{2n} in the sense of [Haefliger]. Since the fibers of $BQ^{2n} \to Q^{2n}$ are rationally acyclic, the rational cohomology rings of BQ^{2n} and Q^{2n} are isomorphic. In particular $H^*(BQ^{2n}; \mathbb{Q})$ is finite dimensional.

It is easy to see that any simple polytope P^n is the base space of some toric orbifold: realize the dual complex K as the boundary complex of a simplicial polytope in $\mathbb{R}^n$. The fact that K is simplicial means that its vertices are in general position. Hence, if we make a small perturbation of the vertices and take the convex hull, the boundary of the resulting polytope will be combinatorially equivalent to K. By making such a small perturbation we can assume that the vertices of K lie in $\mathbb{Q}^n$. This gives a map $\mathrm{vert}(K) \to \mathbb{Q}^n$, which extends linearly to $\lambda\colon \mathbb{Q}^m \to \mathbb{Q}^n$. Since K is embedded in $\mathbb{R}^n$, the map λ is nondegenerate over K. From such a λ, one constructs a toric orbifold as above.

With the exception of the previous paragraph, all of the above obviously extends to the case where K is an arbitrary simplicial complex. The following improved version of Theorem 5.10 is now clear.

THEOREM 7.2. *Suppose that K is an $(n-1)$-dimensional Cohen-Macaulay complex over $\mathbb{Q}$, that P is its dual, and that $(\lambda_1, \dots, \lambda_n)$ is a sequence of elements in $H^2(B_TP; \mathbb{Q})$. The following statements are equivalent.*

(i) *$(\lambda_1, \dots, \lambda_n)$ is a regular sequence for the face ring $\mathbb{Q}(K)$.*

(ii) *$\lambda = (\lambda_1, \dots, \lambda_n)\colon \mathbb{Q}^m \to \mathbb{Q}^n$ is nondegenerate over K.*

(iii) *Up to a homothety, λ is a characteristic function of a toric orbihedron Q^{2n}.*

Moreover, if any of these condition holds, then the resulting spectral sequence, with rational coefficients, degenerates.

7.3. *Hamiltonian actions.* Suppose that M^{2n} is a symplectic manifold with a Hamiltonian action of T^n. (There is rather extensive literature on this subject; the reference most convenient for us is [Delzant], one can find there definitions and further references.)

One defines then a "momentum map" $\mu\colon M^{2n} \to t^*$, where t^* denotes the dual of the Lie algebra of T^n. The momentum map is constant on the T^n-orbits and its image is a simple convex polytope P^n. It can be shown that the T^n-action is locally modelled on the standard representation of T^n on $\mathbb{C}^n$ and that μ identifies M^{2n}/T^n with P^n. Thus $\mu\colon M^{2n} \to P^n$ is a toric manifold.

Nonsingular toric varieties fit into this picture. Indeed, the Kähler form on such a toric variety is a symplectic form with respect to which the T^n-action is Hamiltonian.

In [Delzant] the following theorem, reminiscent of our Proposition 1.8, is proved.

THEOREM 7.4 ([DELZANT]). *Suppose that (M_1, ω_1), (M_2, ω_2) are two symplectic manifolds with Hamiltonian T^n-actions such that the images of the momentum maps coincide (perhaps after translation). Then M_1 is equivariantly symplectomorphic to M_2.*

Delzant also gives complete characterization of the convex polytopes arising as images of momentum maps.

THEOREM 7.5 ([DELZANT]). *A convex polytope P in t^* is the image of the momentum map for some symplectic manifold (M^{2n}, ω) with Hamiltonian T^n-action if and only if for each vertex $p \in P$, there are n points q_i, lying on the rays obtained by extending the edges emanating from p, so that n vectors $\{q_i - p\}$ constitute a basis of $(\mathbb{Z}^n)^* \subset t^*$.*

It is a byproduct of Delzant's result, that every symplectic $2n$-manifold with a Hamiltonian T^n-action is equivariantly diffeomorphic (but not *symplectomorphic*) to a toric variety. Thus one sees that the characteristic functions of the Hamiltonian toric manifolds constitute rather restricted family. One may still ask the following topological question.

Problem 7.6. Let P^n be a simple polytope with m codimension-one faces, and $\lambda\colon \mathbb{Z}^m \to \mathbb{Z}^n$ a characteristic function and $M^{2n}(\lambda)$ the resulting toric manifold. Find conditions on λ so that $M^{2n}(\lambda)$ admits a T^n invariant almost complex structure.

7.7. *Tridiagonal isospectral manifolds.* As in Example 1.15, let M^{2n} be an isospectral manifold of tridiagonal hermitian $(n + 1) \times (n + 1)$ matrices, and P^n the pemutohedron. Then $M^{2n} \to P^n$ is a toric manifold. The manifold M^{2n} is naturally a submanifold of $U(n)$-orbit of a diagonal matrix. This orbit is identified with the flag manifold $U(n)/T^n$. The left T^n-action on the flag manifold is Hamiltonian and the moment map $\mu\colon U(n)/T^n \to \mathbb{R}^n$ can be identified with the (restriction of the) projection onto the diagonal matrices. In this case it is a classical theorem of Schur and Horn, that the image of μ is the permutohedron, viewed as the convex hull of all permutations of a diagonal matrix in the orbit. Unfortunately, the restriction of the symplectic form on the flag manifold to M^{2n} is degenerate, and the image of the restriction of the momentum map is not convex.

Recently, it has been shown in [Bloch, et. al.] how to remedy this problem. There is a different imbedding of M^{2n} into the flag manifold so that the restriction of symplectic form is nonsingular and such that the image of M^{2n} coincides with P^n. Hence, the isospectral manifolds fit into the context of 7.3.

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