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Convex Representation for Lower Semicontinuous Envelopes of Functionals in L^1

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G. Alberti, G. Bouchitté and G. Dal Maso recently found sufficient conditions for the minimizers of the (nonconvex) Mumford-Shah functional. Their method consists in an extension of the calibration method (that is used for the characterization of minimal surfaces), adapted to this functional. The existence of a calibration, given a minimizer of the functional, remains an open problem. We introduce general framework for the study of this problem. We first observe that, roughly, the minimization of any functional of a scalar function can be achieved by minimizing a convex functional, in higher dimension. Although this principle is in general too vague, in some situations, including the Mumford-Shah case in dimension one, it can be made more precise and leads to the conclusion that for every minimizer, the calibration exists — although, still, in a very weak (asymptotical) sense.

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1. Introduction

In a recent paper [1], G. Alberti, G. Bouchitté and G. Dal Maso have introduced a new representation for the Mumford-Shah functional (written here without "forcing term")

$$F(u) \,=\, \int_\Omega |\nabla u(x)|^2\,dx \,+\, \mathcal{H}^{N-1}(S_u)$$

defined for $u \in SBV(\Omega)^1$, where Ω is a bounded open subset of $\mathbb{R}^N$. They express F in the following way

$$F(u) \,=\, \sup_{\varphi\in K} \int_{\Omega\times\mathbb{R}} \left\langle \varphi(x,t), D\mathbf{1}_{\{t<u(x)\}} \right\rangle \tag{1}$$

where $\mathbf{1}_{\{t<u(x)\}}$ is the function on $(x,t) \in \Omega\times\mathbb{R}$ that takes the value 1 when $t < u(x)$ and 0 when $t \geq u(x)$, $D\mathbf{1}_{\{t<u(x)\}}$ is the distributional derivative of $\mathbf{1}_{\{t<u(x)\}}$, which is a bounded Radon measure as soon as $u \in BV(\Omega)$, and K is the convex set of Borel $(N+1)$-dimensional fields $\varphi(x,t) = (\varphi^x(x,t), \varphi^t(x,t)) \in \mathbb{R}^N\times\mathbb{R}$, defined on $\Omega\times\mathbb{R}$ and satisfying

[1] We recall that here S_u is the set of essential discontinuities of u (the complement of the Lebesgue points) and that the function $u \in BV(\Omega)$ (i.e., u of bounded variation) is in $SBV(\Omega)$ if and only if its derivative Du is the sum of an absolutely continuous part $\nabla u \cdot dx$ and a singular part carried by the $(N-1)$-dimensional set S_u. $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure. For the general properties of functions of bounded variations we refer to [13, 14, 16], for a short introduction to SBV functions see for instance [2, 3].

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the inequalities:

$$\begin{cases} \varphi^t(x,t) \geq \dfrac{|\varphi^x(x,t)|^2}{4} & \text{for every } x,t, \text{ and} \\ \left| \displaystyle\int_{t_1}^{t_2} \varphi^x(x,t)\,dt \right| \leq 1 & \text{for every } x, t_1, t_2. \end{cases} \tag{2}$$

This representation for the Mumford-Shah functional is derived from a general representation of "local" functionals in BV due to Guy Bouchitté, that was described in a series of lectures given at the Scuola Normale Superiore di Pisa in 1998 [5].

The most interesting consequence of this representation lies in the fact that, although F is not convex, it is possible to characterize minimizers of F (with, for instance, prescribed boundary conditions) by finding a free-divergence field φ in K that achieves the supremum in (1) (a "calibration"). This is related to Weierstrass' and Carathéodory's approach to sufficient conditions for variational problems [15], although the case of Mumford-Shah functional is not handled by this classical theory. A few examples of calibrations for minimizers of the Mumford-Shah functional are given in [1, 10]. On the other hand, it is not known whether for every minimizer of F, there exists such a calibration. This problem is related to the following: if, for any v with bounded variation on $\Omega \times \mathbb{R}$ such that $v(x,t)$ goes to 1 as $t \to -\infty$ and to 0 as $t \to +\infty$, we define a functional $\mathcal{F}(v)$ by substituting $D\mathbf{1}_{\{t<u(x)\}}$ with Dv in the right-hand side of (1), is the minimum of the convex functional $\mathcal{F}$ equal to the minimum of F? This is clearly true if we can find a calibration for some minimizer u of F. Conversely, if it were true, for every minimizer u of F, $\mathbf{1}_{\{t<u(x)\}}$ would be a minimizer of $\mathcal{F}$, so that it should imply using convex analysis argument the existence in some sense of a calibration for u.

The minima of F and of $\mathcal{F}$ would be the same if, for instance, (1) held and we could prove a coarea formula [13, 14]

$$\mathcal{F}(v) = \int_{-\infty}^{+\infty} \mathcal{F}\left(\mathbf{1}_{\{v>s\}}\right) ds. \tag{3}$$

In the case where $F(u)$ is the total variation $|Du|(\Omega)$ of u (and K is the corresponding convex set $\{\varphi = (\varphi^x, \varphi^t) : \varphi^t \geq 0, \|\varphi^x\|_\infty \leq 1\}$), then such a formula is well known, and actually, there exists a calibration for every minimizer u. However, (3) does not hold if F is the Mumford-Shah functional. Assume indeed, for instance, that $\Omega = (0,1)$, and that we want to minimize F with the boundary conditions $u(0) = 0$, $u(1) = 1$. Then, two minimizers are given by $u_1(x) = x$ and $u_2(x) = 0$ if $x < 1/2$, $u_2(x) = 1$ if $x \geq 1/2$. In particular, if $v = \left(\mathbf{1}_{\{t<u_1(x)\}} + \mathbf{1}_{\{t<u_2(x)\}}\right)/2$, then it is not difficult to show that $\mathcal{F}(v) = \mathcal{F}(u_1) = \mathcal{F}(u_2) = 1$. However, the right-hand side of (3) is $3/2$. Moreover, it is possible to find a calibration for both u_1 and u_2, so that the minimum value of $\mathcal{F}$ is actually 1 and v is also a minimizer of $\mathcal{F}$.

This example suggests that the coarea formula (3) should be replaced with a more general variant that should read (we intentionally remain vague in this introduction)

$$\mathcal{F}(v) = \int_0^1 \mathcal{F}\left(\mathbf{1}_{\{t<u_s(x)\}}\right) ds$$

with $v = \int_0^1 \mathbf{1}_{\{t<u_s(x)\}}\,ds$ (considering only functions v with $0 \leq v \leq 1$), so that every

minimizer of $\mathcal{F}$ should lay inside some convex set whose extremal points are characteristic functions of subgraphs of minimizers of F.

We propose in this paper an approach for the study of this problem. We show, in particular, that in the one-dimensional case the Mumford-Shah functional actually admits an equivalent convex representation (Theorem 8.1, and Corollary 8.2, Remark 8.3 for the Mumford-Shah functional with "forcing term" $\int_\Omega (u-u_0)^2$).

The approach we follow also provides a new point of view on the problem of the relaxation (the computation of the l.s.c. envelope) of a functional in L^1. However, it only applies to the case of functionals $F(u)$ depending on a scalar function u, so that it does not seem, up to now, to provide new relaxation results (with respect to the many papers that address this problem, see [6, 4, 8, 9]). It is related to the method of P. Aviles and Y. Giga [4] who also introduce functionals on graphs of functions in order to derive relaxation results (in both scalar and vectorial cases, though). A generalization of this method, due to G. Bouchitté, will be described in a forthcoming paper.

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2. Notations

In what follows, we will consider an arbitrary functional

$$F : L^1(\Omega;[0,1]) \rightarrow [0,+\infty].$$

The fact that we will be considering only bounded functions u is not restrictive as soon as we are studying the minimization or relaxation of a functional $F(u)$ that decreases by truncation of the function u at some (large enough) level $l>0$ (i.e., if $l>0$ is large enough, $F((-l\vee u)\wedge l)\le F(u)$ for every u).

We will denote by X the set of measurable functions $v(x,t):\Omega\times[0,1]\rightarrow\{0,1\}$, nonincreasing in t, with $v(\cdot,0)\equiv 1$ and $v(\cdot,1)\equiv 0$. X is essentially the set of all functions $\mathbf{1}_{\{t<u(x)\}}$, $u\in L^1(\Omega;[0,1])$, where we adopt the convention, here and in the sequel, that $\mathbf{1}_{\{0<u(x)\}}=1$ even if $u(x)=0$. Notice that this set also contains the functions $\mathbf{1}_{\{t\le u(x)\}}$, and many others, but for u Lebesgue-integrable the set $\{(x,u(x)) : x\in\Omega\}$ is negligible in $\Omega\times[0,1]$ so that $\mathbf{1}_{\{t\le u(x)\}}=\mathbf{1}_{\{t<u(x)\}}$ a.e. in $\Omega\times[0,1]$. As we will consider the $L^1(\Omega\times[0,1])$ topology on X we do not really need to take into account these functions.

The closed convex envelope of X in the $L^1(\Omega\times[0,1])$ topology will be denoted by $\overline{X}$: it is the set of measurable functions $v(x,t):\Omega\times[0,1]\rightarrow[0,1]$, nonincreasing in t, with $v(\cdot,0)\equiv 1$ and $v(\cdot,1)\equiv 0$. Indeed, every v in this latter set is the limit, as n goes to infinity, of the convex combinations of the n characteristic functions of the level sets $\{v>\frac{i}{n}\}$, $i=0,\dots,n-1$, each one with weight $\frac{1}{n}$.

The boundary conditions $v(\cdot,0)\equiv 1$ and $v(\cdot,1)\equiv 0$, which do not really make sense in the $L^1(\Omega\times[0,1])$ topology, will take their full meaning when we consider in a further section the derivative Dv of bounded variation functions $v\in X$ as a measure on $\Omega\times[0,1]$.

As usual, $C_0(\Omega\times[0,1];\mathbb{R}^{N+1})$ is the space of all continuous $(N+1)$-dimensional fields on $\Omega\times[0,1]$ vanishing at the boundary $\partial\Omega\times[0,1]$, endowed with the L^∞-topology, whereas

$\mathcal{M}(\Omega \times [0,1]; \mathbb{R}^{N+1}) = C_0(\Omega \times [0,1]; \mathbb{R}^{N+1})'$ is the space of vector-valued bounded Radon measures on $\Omega \times [0,1]$ (up to the boundary $\Omega \times \{0,1\}$).

We will also denote by $C^1_0(\Omega\times[0,1]; \mathbb{R}^{N+1})$ (respectively, $C^1_c(\Omega\times[0,1]; \mathbb{R}^{N+1})$) the subspace of all C^1-regulars fields of $C_0(\Omega \times [0,1]; \mathbb{R}^{N+1})$ (respectively, of all C^1-regulars fields of $C_0(\Omega \times [0,1]; \mathbb{R}^{N+1})$ with compact support in $\Omega \times [0,1]$).

3. Relaxation via convexification

Define for every v in $\overline{X}$ the functional

$$\mathcal{F}(v) = \begin{cases} F(u) & \text{if } v \in X, v = \mathbf{1}_{\{t<u(x)\}} \text{ a.e.,} \\ +\infty & \text{if } v \in \overline{X} \setminus X. \end{cases} \tag{4}$$

We define then on $\overline{X}$ the function $\overline{\mathcal{F}}$ as the *convex lower semicontinuous envelope* of $\mathcal{F}$ in the $L^1(\Omega \times [0,1])$ topology, i.e.,

$$\overline{\mathcal{F}}(v) = \inf \liminf_{n\to\infty} \sum_{i=1}^{k^n} \theta_i^n F(u_i^n)$$

where the infimum is taken over all convex combinations ($k^n \geq 1$, $u_i^n \in L^1(\Omega; [0,1])$, $\theta_i^n \geq 0$ for every $i = 1, \dots, k^n$, and $\sum_{i=1}^{k^n} \theta_i^n = 1$ for all n) $\sum_{i=1}^{k^n} \theta_i^n \mathbf{1}_{\{t<u_i^n(x)\}}$ that converge to v in $L^1(\Omega \times [0,1])$ as $n\to\infty$.

Let then

$$\overline{F}(u) = \overline{\mathcal{F}}\left(\mathbf{1}_{\{t<u(x)\}}\right) \tag{5}$$

for every $u \in L^1(\Omega; [0,1])$. We have the following result.

Proposition 3.1. *The function $\overline{F}$ is the relaxed (i.e., l.s.c.) envelope of F in the $L^1(\Omega; [0,1])$ topology.*

This, in fact, is related to the Young measure approach to relaxation, since for every $v \in \overline{X}$ the opposite of the derivative $v_t = D_t v$ (which is correctly defined since v is nondecreasing in t) is a Young measure, and it is the measure associated to u whenever $v = \mathbf{1}_{\{t<u(x)\}} \in X$.

Proof of Proposition 3.1. Since by construction of $\overline{\mathcal{F}}$ and $\overline{F}$ it is obvious that, if $u_n\to u$, $\overline{F}(u) \leq \liminf_{n\to\infty} F(u_n)$, we must show that given any $u \in L^1(\Omega; [0,1])$, we can find a sequence u_n such that

$$\limsup_{n\to\infty} F(u_n) \leq \overline{F}(u). \tag{6}$$

From the definition of $\overline{\mathcal{F}}$, we know that there exists a sequence of convex combinations $v^n = \sum_{i=1}^{k^n} \theta_i^n \mathbf{1}_{\{t<u_i^n(x)\}}$, $n \geq 1$, converging to $\mathbf{1}_{\{t<u(x)\}}$ as $n\to\infty$, with

$$\lim_{n\to\infty} \sum_{i=1}^{k^n} \theta_i^n F(u_i^n) = \overline{\mathcal{F}}\left(\mathbf{1}_{\{t<u(x)\}}\right) = \overline{F}(u).$$

Define $A^+ = \{(x,t) : \mathbf{1}_{\{t<u(x)\}} = 1\}$ and $A^- = \{(x,t) : \mathbf{1}_{\{t<u(x)\}} = 0\}$. Then,

$$\int_{\Omega\times[0,1]} \left|v^n - \mathbf{1}_{\{t<u(x)\}}\right| \, dxdt \;=\; \int_{A^+} 1 - v^n \, dxdt \;+\; \int_{A^-} v^n \, dxdt \;=$$

$$= \sum_{i=1}^{k^n} \theta_i^n \int_{A^+} 1 - \mathbf{1}_{\{t<u_i^n(x)\}} \, dxdt \;+\; \sum_{i=1}^{k^n} \theta_i^n \int_{A^-} \mathbf{1}_{\{t<u_i^n(x)\}} \, dxdt$$

$$= \sum_{i=1}^{k^n} \theta_i^n \int_{\Omega\times[0,1]} \left|\mathbf{1}_{\{t<u_i^n(x)\}} - \mathbf{1}_{\{t<u(x)\}}\right| \, dxdt \;=\; \sum_{i=1}^{k^n} \theta_i^n \int_\Omega |u - u_i^n| \, dx.$$

We therefore have that

$$\lim_{n\to\infty} \sum_{i=1}^{k^n} \theta_i^n \int_\Omega |u - u_i^n| \, dx \;=\; 0$$

and

$$\lim_{n\to\infty} \sum_{i=1}^{k^n} \theta_i^n F(u_i^n) \;=\; \overline{F}(u).$$

Set for every n and $i = 1, \dots, k^n$, $\lambda_i^n = \int_\Omega |u - u_i^n| \, dx$ and $\varphi_i^n = F(u_i^n)$. We can always rearrange the indices i in order to have $\lambda_i^n \le \lambda_{i+1}^n$ for every $i = 1, \dots, k_n - 1$. We define then for $s \in (0,1)$ the functions $\Lambda^n(s)$ and $\Phi^n(s)$ by $\Lambda^n(s) = \lambda_i^n$ and $\Phi^n(s) = \varphi_i^n$ when $\sum_{j<i} \theta_i^n \le s < \sum_{j\le i} \theta_i^n$.

Λ^n is nondecreasing, nonnegative, and $\int_0^1 \Lambda^n(s) \, ds$ goes to 0 as $n\to\infty$. Φ^n is nonnegative and $\int_0^1 \Phi^n(s) \, ds$ goes to $\overline{F}(u)$ as $n\to\infty$.

In particular, Λ^n goes uniformly to 0 as $n\to\infty$ on every interval $(0, 1-\delta)$, for $1 > \delta > 0$. We fix a small $\delta > 0$, and write

$$\fint_0^{1-\delta} \Phi^n(s) \, ds \;=\; \frac{1}{1-\delta} \int_0^{1-\delta} \Phi^n(s) \, ds \;\le\; \frac{1}{1-\delta} \int_0^1 \Phi^n(s) \, ds \;\overset{n\to\infty}{\longrightarrow}\; \frac{\overline{F}(u)}{1-\delta}$$

so that for n large enough (and $\delta < \frac{1}{2}$),

$$\fint_0^{1-\delta} \Phi^n(s) \, ds \;\le\; \overline{F}(u)(1+2\delta).$$

In this case, $|\{s \in (0, 1-\delta) : \Phi^n(s) \le \overline{F}(u)(1+2\delta)\}| > 0$ so that there exists $s(n) < 1-\delta$ and $i(n)$ with $\Phi^n(s(n)) = \varphi_{i(n)}^n \le \overline{F}(u)(1+2\delta)$. Set $u_n = u_{i(n)}^n$ for every (large) n: since $\lambda_{i(n)}^n = \Lambda^n(s(n))\to 0$, u_n goes to u as $n\to\infty$, and

$$\limsup_{n\to\infty} \varphi_{i(n)} \;=\; \limsup_{n\to\infty} F(u_n) \;\le\; \overline{F}(u)(1+2\delta).$$

Sending δ to zero, we can build a diagonal sequence (still denoted by $(u_n)_{n\ge 1}$) such that (6) holds, and this achieves the proof of Proposition 3.1. □

4. BV coercivity

We will now make an additional coercivity assumption on F in order to ensure that we only deal with bounded variation functions. We assume that there exists $c_1, c_2 \geq 0$ with

$$|Du|(\Omega) \leq c_1 + c_2 F(u) \tag{7}$$

for every $u \in L^1(\Omega; [0,1])$.

Example 4.1. For instance, if

$$F(u) = \begin{cases} \int_\Omega |\nabla u(x)|^2 + |u(x) - u_0(x)|^2 \, dx + \mathcal{H}^{N-1}(S_u) & \text{if } u \in SBV(\Omega; [0,1]) \\ +\infty & \text{otherwise} \end{cases} \tag{8}$$

is the Mumford-Shah functional, we have, for every $u \in SBV(\Omega; [0,1])$,

$$\begin{aligned} |Du|(\Omega) &= \int |\nabla u(x)| \, dx + \int_{S_u} (u^+(x) - u^-(x)) \, d\mathcal{H}^{N-1}(x) \\ &\leq |\Omega|^{\frac{1}{2}} \left(\int_\Omega |\nabla u(x)|^2 \, dx \right)^{\frac{1}{2}} + \mathcal{H}^{N-1}(S_u) \\ &\leq |\Omega|^{\frac{1}{2}} \sqrt{F(u)} + F(u) \leq |\Omega|^{\frac{1}{2}} + (1 + |\Omega|^{\frac{1}{2}}) F(u). \end{aligned}$$

Assumption (7) implies the following result.

Lemma 4.2. *If (7) holds, then for every $v \in \overline{X}$,*

$$|Dv|(\Omega \times [0,1]) \leq |\Omega| + c_1 + c_2 \overline{\mathcal{F}}(v). \tag{9}$$

Here, the variation $|Dv|(\Omega \times [0,1])$ takes into account the boundary values $v(\cdot, 0) \equiv 1$ and $v(\cdot, 1) \equiv 0$.

Proof of Lemma 4.2. If $v \in X \cap BV(\Omega \times [0,1])$, $v = \mathbf{1}_{\{t<u(x)\}}$, then

$$|Dv|(\Omega \times [0,1]) \leq |\Omega| + |Du|(\Omega) \leq |\Omega| + c_1 + c_2 F(u).$$

(Remember that we assume $v \equiv 1$ on $\Omega \times \{0\}$ and $v \equiv 0$ on $\Omega \times \{1\}$, i.e., $\mathbf{1}_{\{0<u(\cdot)\}} \equiv 1$ and $\mathbf{1}_{\{1<u(\cdot)\}} \equiv 0$.) Thus,

$$|Dv|(\Omega \times [0,1]) \leq |\Omega| + c_1 + c_2 \mathcal{F}(v)$$

where $\mathcal{F}$ is defined by (4).

This is valid for every $v \in \overline{X}$, since when $v \notin X$ the right-hand side of the last expression is $+\infty$. Since $\overline{\mathcal{F}}$ is the convex l.s.c. envelope of $\mathcal{F}$, and since the total variation is also convex and l.s.c. in $L^1(\Omega \times [0,1])$, we deduce (9). □

5. A representation formula for $\overline{\mathcal{F}}$

We now denote by X_b and $\overline{X}_b$, respectively, the intersections $X \cap BV(\Omega \times [0,1])$ and $\overline{X} \cap BV(\Omega \times [0,1])$.

We want to show that if (7) holds, then for every $v \in \overline{X}_b$,

$$\overline{\mathcal{F}}(v) = \sup_{\varphi \in K} \int_{\Omega \times [0,1]} \langle \varphi, Dv \rangle$$

for some convex closed set $K \subset C_0(\Omega \times [0,1]; \mathbb{R}^{N+1})$. By standard results in convex analysis [7, 12], this is equivalent to finding a convex, w-$*$ l.s.c., and 1-homogeneous extension to $\mathcal{M}(\Omega \times [0,1]; \mathbb{R}^{N+1})$ of $\overline{\mathcal{F}}$, seen as a function of the measure Dv.

In order to simplify the notations we will always denote in the sequel the scalar product $\langle \varphi, Dv \rangle$ simply by φDv.

Let us introduce the set

$$\overline{Y} = \left\{ \nu \in \mathcal{M}(\Omega \times [0,1]; \mathbb{R}^{N+1}) \, : \, \exists v \in \overline{X}_b, \nu = Dv \right\}.$$

If we consider an arbitrary function $\psi \in C_0(\Omega)$ with $\int_\Omega \psi(x)\,dx = 1$ and, for every $\nu = (\nu^x, \nu^t) \in \mathcal{M}(\Omega \times [0,1]; \mathbb{R}^{N+1})$, we let $\Lambda(\nu) = \int_{\Omega \times [0,1]} \psi(x)\,d\nu^t(x,t)$, then, $\mathbb{R}_+\overline{Y} \cap \{\Lambda = 1\} = \overline{Y}$ and $\nu \in \mathbb{R}_+\overline{Y}$ if and only if $\Lambda(\nu) \geq 0$ and there exists $v \in \overline{X}_b$, $\nu = \Lambda(\nu) Dv$.

We introduce the function on $\mathcal{M}(\Omega \times [0,1]; \mathbb{R}^{N+1})$

$$\mathcal{H}(\nu) = \begin{cases} +\infty & \text{if } \nu \notin \mathbb{R}_+\overline{Y} \\ \lambda\overline{\mathcal{F}}(v) & \text{if } \nu = \lambda Dv \in \mathbb{R}_+\overline{Y}, \lambda \geq 0, v \in \overline{X}_b. \end{cases} \tag{10}$$

Lemma 5.1. *$\mathcal{H}$ is convex, positively 1-homogeneous, and w-$*$ lower semicontinuous on $\mathcal{M}(\Omega \times [0,1]; \mathbb{R}^{N+1})$.*

Proof. $\mathcal{H}$ is 1-homogeneous by construction and it is straightforward to check that it is convex (using the convexity of $\overline{\mathcal{F}}$ and the fact that if $\nu_1, \nu_2 \in \mathbb{R}_+\overline{Y}$, $\nu_1 = \lambda_1 Dv_1$, $\nu_2 = \lambda_2 Dv_2$, and $\theta \in [0,1]$, then $\theta\nu_1 + (1-\theta)\nu_2 = \lambda_3 Dv_3$ with $\lambda_3 = \theta\lambda_1 + (1-\theta)\lambda_2$ and $v_3 = (\theta\lambda_1 v_1 + (1-\theta)\lambda_2 v_2)/\lambda_3$).

The weakly-$*$ lower semicontinuity of $\mathcal{H}$ is a consequence of the Krein-Šmulian theorem [11, Thm. V.5.7]. Actually, $\mathcal{H}$ is sequentially w-$*$ l.s.c., and thus its epigraph is w-$*$ relatively closed in every bounded subset of $\mathcal{M}(\Omega \times [0,1]; \mathbb{R}^{N+1}) \times \mathbb{R}$, since it is convex we deduce it is w-$*$ closed in $\mathcal{M}(\Omega \times [0,1]; \mathbb{R}^{N+1}) \times \mathbb{R}$. □

The main consequence of Lemma 5.1 is [7, 12] the existence of a closed convex set $K \subset C_0(\Omega \times [0,1]; \mathbb{R}^{N+1})$ such that $\mathcal{H}$ is the Legendre-Fenchel conjugate of the characteristic function of K

$$\chi_K(\varphi) = \begin{cases} 0 & \text{if } \varphi \in K, \\ +\infty & \text{if } \varphi \notin K, \end{cases}$$

i.e.,

$$\mathcal{H}(\nu) = {\chi_K}^*(\nu) = \sup_{\varphi} \int_{\Omega \times [0,1]} \varphi d\nu - \chi_K(\varphi) = \sup_{\varphi \in K} \int_{\Omega \times [0,1]} \varphi d\nu.$$

Therefore, for all $v \in \overline{X}_b$,

$$\overline{\mathcal{F}}(v) \;=\; \sup_{\varphi\in K} \int_{\Omega\times[0,1]} \varphi Dv. \tag{11}$$

6. Some properties of the convex K

The set K is defined by

$$\chi_K(\varphi) \;=\; \mathcal{H}^*(\varphi) \;=\; \sup_{\nu} \int_{\Omega\times[0,1]} \varphi d\nu \;-\; \mathcal{H}(\nu).$$

Since $\mathcal{H}(\nu) = +\infty$ when $\nu \notin \mathbb{R}_+\overline{Y}$, we can consider in the supremum only measures of the form λDv with $\lambda \geq 0$ and $v \in \overline{X}_b$, so that

$$K \;=\; \left\{ \varphi \in C_0(\Omega\times[0,1];\mathbb{R}^{N+1}) \;:\; \int_{\Omega\times[0,1]} \varphi Dv \;\leq\; \overline{\mathcal{F}}(v)\ \forall v \in \overline{X}_b \right\}.$$

In particular, if $\varphi \in K$, then

$$\int_{\Omega\times[0,1]} \varphi D\mathbf{1}_{\{t<u(x)\}} \;\leq\; \overline{\mathcal{F}}\left(\mathbf{1}_{\{t<u(x)\}}\right) \;=\; \overline{F}(u) \;\leq\; F(u)$$

for every $u \in BV(\Omega;[0,1])$. On the other hand, by the construction of $\overline{\mathcal{F}}$, it is not difficult to check that if $\int_{\Omega\times[0,1]} \varphi D\mathbf{1}_{\{t<u(x)\}} \leq F(u)$ for every $u \in BV(\Omega;[0,1])$, then $\varphi \in K$: thus

$$K \;=\; \left\{ \varphi \in C_0(\Omega\times[0,1];\mathbb{R}^{N+1}) \;:\; \int_{\Omega\,\times\,[0,1]} \varphi D\mathbf{1}_{\{t<u(x)\}} \;\leq\; F(u)\ \forall u \in BV(\Omega;[0,1]) \right\}. \tag{12}$$

Lemma 6.1. *The set K contains an open set, so that K is the closure of its interior ($K = \overline{\mathit{Int}\,K}$).*

We recall that a closed convex set is the closure of its interior if and only if it contains an open set. An important consequence of this lemma is that the smooth functions with compact support in $\Omega\times[0,1]$ are dense in K, so that the supremum in (11) can be taken only on these functions.

Proof of Lemma 6.1. If $\varphi \in C_0(\Omega\times[0,1];\mathbb{R}^{N+1})$ and $u \in BV(\Omega;[0,1])$, we have

$$\begin{aligned}\int_{\Omega\times[0,1]} \varphi D\mathbf{1}_{\{t<u(x)\}} \;&\leq\; \|\varphi\|_\infty \left|D\mathbf{1}_{\{t<u(x)\}}\right|(\Omega\times[0,1]) \\ &\leq\; \|\varphi\|_\infty(|\Omega| \;+\; |Du|(\Omega)).\end{aligned}$$

If ψ is chosen as before ($\psi \in C_0(\Omega)$, $\int_\Omega \psi(x)\,dx = 1$) and $\lambda > 0$, we find that

$$\int_{\Omega\times[0,1]} (\varphi^x(x,t), \varphi^t(x,t) + \lambda\psi(x))D\mathbf{1}_{\{t<u(x)\}} \;\leq\; (-\lambda + \|\varphi\|_\infty)|\Omega| + \|\varphi\|_\infty(c_1 + c_2F(u)). \tag{13}$$

In particular, if $\|\varphi\|_\infty < 1/c_2$ and $\lambda = (1 + c_1/|\Omega|)/c_2$, then the right hand side of (13) is less than $F(u)$. Since this is true for every u, we deduce that for this value of λ, the ball of radius $1/c_2$ and center $\varphi_\lambda(x,t) = (0, \lambda\psi(x))$ is in K. $\square$

7. Reduction of the set K

We have shown up to now that given any functional F on $L^1(\Omega;[0,1])$, with the coercivity property (7), the functional $\overline{\mathcal{F}}$ on $\overline{X}_b$ defined by (11), with K given by (12), is the largest convex l.s.c. functional on $\overline{X}_b$ whose trace on X_b coincides with the l.s.c. envelope of F, in the sense given by (5). In particular,

$$\inf_{u\in BV(\Omega;[0,1])} F(u) \;=\; \min_{v\in\overline{X}_b} \overline{\mathcal{F}}(v)$$

and every minimizer of $\overline{\mathcal{F}}$ is the convex combination of functions $\mathbf{1}_{\{t<u(x)\}}$ with u minimizing $\overline{F}$.

However, this representation is totally useless. Indeed, in the one-dimensional case where u is not a function, but just a scalar in $[0,1]$, it is not difficult to check that this result is no more than saying that $\overline{F}$ is the supremum of all continuous functions below F. What happens is that the definition (12) of the set K is too global, and makes usually impossible to say whether a particular field $\varphi\in C_0(\Omega\times[0,1];\mathbb{R}^{N+1})$ is in K or not. The further step is therefore to understand under which additional assumptions on F we can expect that there exists a "simpler" convex set K' such that (11) still holds if we replace K with K', and that we can better describe. In particular, we should have, letting $\mathcal{H}'=\chi_{K'}{}^*$,

(i) $\mathcal{H}'\le\mathcal{H}$, since $\mathcal{H}=+\infty$ outside of $\mathbb{R}_+\overline{Y}$, and we want $\mathcal{H}'$ to be equal to $\mathcal{H}$ on $\mathbb{R}_+\overline{Y}$,
(ii) $K'\subset K$, as a consequence of (i),

We will first study the subset of K of "useless" fields, which have no effect in the integral in (11).

Let us define the sets

$$\begin{aligned} N &= \left\{\varphi\in C^1_0(\Omega\times[0,1];\mathbb{R}^{N+1}) \,:\, \int_{\Omega\times[0,1]}\varphi D\mathbf{1}_{\{t<u(x)\}} = 0\ \forall u\in BV(\Omega;[0,1])\right\}\\ &= \left\{\varphi\in C^1_0(\Omega\times[0,1];\mathbb{R}^{N+1}) \,:\, \int_\Omega \varphi^t(x,0)\,dx = 0, \operatorname{div}\varphi = 0\right\}, \end{aligned}$$

and

$$P \;=\; \left\{\varphi\in C_0(\Omega\times[0,1];\mathbb{R}^{N+1}) \,:\, \varphi^x = 0, \varphi^t\ge 0 \text{ on } \Omega\times[0,1]\right\}.$$

Like previously, we choose an arbitrary function $\psi\in C_0(\Omega)$ with $\int_\Omega\psi(x)\,dx=1$ and Λ is still defined by $\Lambda(\nu)=\int_{\Omega\times[0,1]}\psi(x)\,d\nu^t(x,t)$ for every $\nu=(\nu^x,\nu^t)\in\mathcal{M}(\Omega\times[0,1];\mathbb{R}^{N+1})$.

If $A\subset C_0(\Omega\times[0,1];\mathbb{R}^{N+1})$ and $B\subset\mathcal{M}(\Omega\times[0,1];\mathbb{R}^{N+1})$, we define the *dual cones* of A and B, respectively denoted by A° and B°, as the closed convex cones

$$A^\circ \;=\; \left\{\nu\in\mathcal{M}(\Omega\times[0,1];\mathbb{R}^{N+1}) \,:\, \int_{\Omega\times[0,1]}\varphi d\nu\le 0\ \forall\varphi\in A\right\}$$

and

$$B^\circ \;=\; \left\{\varphi\in C_0(\Omega\times[0,1];\mathbb{R}^{N+1}) \,:\, \int_{\Omega\times[0,1]}\varphi d\nu\le 0\ \forall\nu\in B\right\}$$

From the Hahn-Banach Theorem we know that $(A^\circ)^\circ$ (resp., $(B^\circ)^\circ$) is the closed convex envelope of $\mathbb{R}_+A$ (resp., $\mathbb{R}_+B$).

Lemma 7.1. *The cone $\mathbb{R}_+\overline{Y}$ is the dual cone $(N+P)^\circ$ of $N+P$, and $\overline{Y} = (N+P)^\circ \cap \{\Lambda = 1\}$.*

Proof. Let us first compute the dual cone of N. Since N is a vector space, we have $N^\circ = \{\nu : \int_{\Omega\times[0,1]} \varphi d\nu = 0\ \forall \varphi \in N\}$. In particular, the fact that $\nu \in N^\circ$ is orthogonal to every vector field in $N \cap \mathcal{D}(\Omega\times(0,1), \mathbb{R}^{N+1})$ implies by standard results that ν is a gradient in $\Omega\times(0,1)$ and there exists $v \in BV(\Omega\times(0,1))$ with $\nu = Dv$ inside $\Omega\times(0,1)$. Thus $\nu = \nu \llcorner \Omega\times\{0\} + Dv + \nu \llcorner \Omega\times\{1\}$.

Now, if we denote by $\tilde{v}$ the trace of v on $\partial(\Omega\times(0,1))$, using Green formula we find that for every $\varphi \in N$,

$$0 = \int_{\Omega\times[0,1]} \varphi\, d\nu = \int_{\Omega\times\{0\}} \varphi\, d\nu - \tilde{v}\varphi^t\, dx + \int_{\Omega\times\{1\}} \varphi\, d\nu + \tilde{v}\varphi^t\, dx \tag{14}$$

where dx is used to denote $d\mathcal{H}^N(x,0)$ or $d\mathcal{H}^N(x,1)$ on, respectively, $\Omega\times\{0\}$ or $\Omega\times\{1\}$. If we consider a field φ where $\varphi^t = \frac{\partial\rho}{\partial x_i}(x)\sigma(t)$, $1 \le i \le N$, and the ith component of φ^x is $-\rho(x)\sigma'(t)$, all other being zero, with $\rho \in C^1_c(\Omega)$ and $\sigma \in C^1([0,1])$, then $\varphi \in N$ and (14) becomes

$$\begin{aligned} 0 = {} & \sigma(0)\int_{\Omega\times\{0\}} \frac{\partial\rho}{\partial x_i}\,(d\nu^t - \tilde{v}\,dx) + \sigma(1)\int_{\Omega\times\{1\}} \frac{\partial\rho}{\partial x_i}\,(d\nu^t + \tilde{v}\,dx) \\ & - \sigma'(0)\int_{\Omega\times\{0\}} \rho\, d\nu^{x_i} - \sigma'(1)\int_{\Omega\times\{1\}} \rho\, d\nu^{x_i} \end{aligned}$$

Since $\sigma(0)$, $\sigma'(0)$, $\sigma(1)$ and $\sigma'(1)$ can be chosen independently, we deduce that $\nu^x \llcorner \Omega\times\{0,1\} = 0$ and that there exist constants λ_0, λ_1 such that

$$\begin{cases} \nu^t \llcorner \Omega\times\{0\} = (\lambda_0 + \tilde{v})\,dx \\ \nu^t \llcorner \Omega\times\{1\} = (\lambda_1 - \tilde{v})\,dx \end{cases}$$

Since v is determined up to a constant, we can always assume that $\lambda_1 = 0$ Then, extending v to $\Omega\times[0,1]$ by setting

$$\begin{cases} v(\cdot,0) \equiv -\lambda_0 & \text{on } \Omega\times\{0\} \text{ and} \\ v(\cdot,1) \equiv 0 & \text{on } \Omega\times\{1\} \end{cases}$$

we find that $\nu = Dv$. Now, if $\nu \in N^\circ \cap P^\circ = (N+P)^\circ$, it is easy to check that we also must have that $\nu^t \le 0$, i.e., $v(x,t)$ is nonincreasing in t. In particular, $-\lambda_0 \ge 0$ and $\nu \in -\lambda_0\overline{Y}$.

On the other hand, it is obvious to check that $\mathbb{R}_+\overline{Y} \subseteq (N+P)^\circ$. This achieves the proof of Lemma 7.1. $\square$

Now, if we want to find $\mathcal{H}'$, K' satisfying (i) and (ii) above, we should have

(iii) $K = \overline{(K' + N + P)}$,

since by Lemma 7.1, $\chi_{N+P}{}^* = \chi_{\mathbb{R}_+\overline{Y}}$.

We will require in general that $K' + P = K'$, since we would like the supremum of $\int_{\Omega\times[0,1]} \varphi Dv$ over $\varphi \in K'$ to be equal to $+\infty$ whenever $v(x,t) \in BV(\Omega \times [0,1])$ is not nonincreasing in t. In this case, (iii) may be replaced by $K = \overline{(K' + N)}$.

In the sequel, we will consider "local" functionals, in the following sense.

Definition 7.2. We say that the functional $F : L^1(\Omega; [0,1]) \to [0, +\infty]$ is *local* if for every $u \in L^1(\Omega; [0,1])$, there exists a positive Borel measure $F(u, \cdot)$ such that $F(u, \Omega) = F(u)$ and that satisfies $F(u, A) = F(v, A)$ for every open set A and every u, v with $u \equiv v$ on A.

We will try to identify in what cases the set K' defined by

$$K' = \left\{ \varphi \in C_0(\Omega \times [0,1]; \mathbb{R}^{N+1}) : \int_{A\times[0,1]} \varphi D\mathbf{1}_{\{t<u(x)\}} \le F(u, A) \ \ \forall u, A \right\}, \tag{15}$$

where $u \in BV(\Omega; [0,1])$ and $A \in \mathcal{A}(\Omega)$, the set of all open subsets of Ω, enjoys property (iii).

The simplest example is the case where F is already lower semicontinuous, we already now that $F(u) = \sup_{\varphi\in K'} \int_{\Omega\times[0,1]} \varphi D\mathbf{1}_{\{t<u(x)\}}$ for K' defined as above, and the functional $\overline{\mathcal{F}}'(v) = \sup_{\varphi\in K'} \int_{\Omega\times[0,1]} \varphi Dv$ satisfies the coarea formula

$$\overline{\mathcal{F}}'(v) = \int_0^1 \overline{\mathcal{F}}' \left(\mathbf{1}_{\{v>s\}}\right) ds.$$

Then, it is easy to show that $\overline{\mathcal{F}} = \overline{\mathcal{F}}'$, so that (iii) must hold. For instance, this is true if $F(u) = |\Omega| + |Du|(\Omega)$. In this case,

$$K' = \left\{ \varphi \in C_0(\Omega \times [0,1]; \mathbb{R}^{N+1}) : \varphi^t(x,t) \ge -1, \|\varphi^x(x,t)\| \le 1 \ \forall (x,t) \in \Omega \times [0,1] \right\}.$$

On the other hand, when the relaxed functional $\overline{F}$ is not convex (for instance, if it is the Mumford-Shah functional (8)), it can be shown that neither $\overline{\mathcal{F}}$, nor the function $\overline{\mathcal{F}}'$ defined as above satisfy the coarea formula. In this case, it seems much harder to find out whether these two functionals are equal or not. In the next section we will concentrate on the one-dimensional case, and establish a list of assumptions under which we can show how to "reduce" (or "localize") the set K

8. The localization of K

8.1. Local functionals on $BV(\Omega; [0,1])$

In general, we would like to consider functionals of the form

$$F(u) = \int_\Omega f(x, u(x), \nabla u(x))\, dx + \int_{S_u} g(x, u^-(x), u^+(x), \nu_u(x))\, d\mathcal{H}^{N-1}(x)$$

for $u \in SBV(\Omega, [0,1])$, and $F(u) = +\infty$ otherwise, with $f(x,t,p) \ge \gamma(|p| \wedge |p|^q)$ $(q \ge 1)$ and $g(x, \alpha, \beta, \nu) = g(x, \beta, \alpha, -\nu)$, $g(x, \alpha, \beta, \nu) \ge \gamma(|\beta - \alpha| \wedge 1)$ for some $\gamma > 0$ to ensure some coercivity (7).

These functionals enjoy the fundamental property that if an open set $A \subseteq \Omega$ can be decomposed in the disjoint union of two open sets B, C and a $(N-1)$-dimensional closed set $L = \partial B \cap \partial C \cap A$, then $F(u,A) = F(u,B) + F(u,C)$ as soon as $\mathcal{H}^{N-1}(S_u \triangle L) = 0$ (i.e., u is essentially approximately continuous at the interface L), where $S_u \triangle L$ denotes the symmetric difference of the sets S_u and L.

In this section we will investigate the one-dimensional case, under further assumptions on f and g. We assume that $\Omega \subset \mathbb{R}$, that for every $A \in \mathcal{A}(\Omega)$ and $u \in BV(\Omega;[0,1])$,

$$F(u,A) = \begin{cases} \int_A f(x,u(x),u'(x))\,dx + \sum_{x\in S_u\cap A} g(x,u(x-0),u(x+0)) & \text{if } u \in SBV(A,[0,1]), \\ +\infty & \text{otherwise,} \end{cases}$$

with $F(u) = F(u,\Omega)$, and f and g satisfy the following assumptions (for some $\gamma > 0$ and $q \geq 1$):

(A1) $f(x,t,p) \geq \gamma(|p| \wedge |p|^q)$,
(A2) $g(x,\alpha,\beta) \geq \gamma(|\beta-\alpha| \wedge 1)$,[1]
(A3) there exists a modulus of continuity $\omega(\xi,\tau)$ ($\lim_{|\xi|+|\tau|\to 0}\omega(\xi,\tau) = 0$) such that

$$|f(x,t,p) - f(x',t',p)| \leq \omega(x'-x,t'-t)(1+f(x,t,p)),$$

(A4) g is continuous on $\Omega \times [0,1] \times [0,1]$,

for every $x, x' \in \Omega$, $t, t', \alpha, \beta \in [0,1]$ and $p \in \mathbb{R}$, and

(A5) there exist two continuous and nondecreasing functions $\gamma^+, \gamma^- : (0,1] \to (0,+\infty)$, with $g(x,\alpha,\alpha \pm t) \geq \gamma^\pm(t) \geq \gamma \cdot (t \wedge 1)$, such that the functions

$$k^+(x,t) = \lim_{\substack{\alpha,\beta\to t \\ \alpha<t<\beta \\ x'\to x}} \frac{g(x',\alpha,\beta)}{\gamma^+(\beta-\alpha)} \quad \text{and} \quad k^-(x,t) = \lim_{\substack{\alpha,\beta\to t \\ \alpha<t<\beta \\ x'\to x}} \frac{g(x',\beta,\alpha)}{\gamma^-(\beta-\alpha)}$$

are well defined and continuous on $\Omega \times (0,1)$, with a continuous extension to $\Omega \times [0,1]$. (In other words, the behaviour of $g(x,\alpha,\beta)$ as β goes to α from one side is, up to a continuous perturbation, uniform.)

Notice that the case $g \geq \underline{\gamma} > 0$ is a special case of (A5).

In this case, it is not hard to check that

$$K' = \left\{ \varphi \in C_0(\Omega \times [0,1];\mathbb{R}^2) : \varphi^t(x,t) \geq f^*(x,t,\varphi^x(x,t)), \int_{t_1}^{t_2} \varphi^x(x,s)\,ds \leq g(x,t_1,t_2) \right\} \tag{16}$$

where the inequalities hold pointwise, and we adopt the standard convention that when $t_1 > t_2$, $\int_{t_1}^{t_2} \varphi^x(x,s)\,ds = -\int_{t_2}^{t_1} \varphi^x(x,s)\,ds$. The function $f^*(x,t,\cdot)$ is the Legendre-Fenchel conjugate of $f(x,t,\cdot)$. We will show the following result:

[1]Since α, β are bounded we can in fact assume that $g(x,\alpha,\beta) \geq \gamma|\beta-\alpha|$, but we prefer to write (A2) in this form since usually F is the restriction to $BV(\Omega;[0,1])$ of a functional defined in the same way on $BV(\Omega)$, with g satisfying (A2) for every α, β.

Theorem 8.1. *Under assumptions* (A1)–(A5), $K = \overline{K' + N}$, *so that*

$$\overline{\mathcal{F}}(v) \;=\; \sup_{\varphi \in K'} \int_{\Omega\times[0,1]} \varphi Dv.$$

As a consequence, for every $u \in BV(\Omega;[0,1])$,

$$\begin{aligned}\overline{F}(u) \;&=\; \sup_{\varphi\in K} \;\int_\Omega u'(x)\varphi^x(x,u(x)) - f^*(x,u(x),\varphi^x(x,u(x)))\,dx \\ &\qquad + \int_\Omega \varphi^x(x,\tilde u(x))Cu \;+\; \sum_{x\in S_u}\int_{u(x-0)}^{u(x+0)} \varphi^x(x,s)\,ds.\end{aligned}$$

We can deduce the standard relaxation results of coercive functionals in BV (see for instance [6]). For instance, in the case where the functions $\gamma^\pm$ in (A5) satisfy $\lim_{t\to 0}\gamma^\pm(t)/t = +\infty$, if we also assume that $f(x,t,p)/|p| \to +\infty$ as $|p|\to\infty$, we can check that $\varphi^x(x,t)$ is (pointwise) unbounded as φ describes K, so that $\sup_{\varphi\in K}\int_\Omega \varphi^x(x,\tilde u(x))Cu = +\infty$ in case $Cu \neq 0$ ($u \notin SBV(\Omega)$), which yields that the l.s.c. envelope of F is $+\infty$ outside of $SBV(\Omega)$, and

$$\overline{F}(u) \;=\; \int_\Omega f^{**}(x,u(x),u'(x))\,dx \;+\; \sup_{\varphi\in K}\sum_{x\in S_u}\int_{u(x-0)}^{u(x+0)} \varphi^x(x,s)\,ds$$

for every $u \in SBV(\Omega,[0,1])$. Here, $\sup_{\varphi\in K}\int_\alpha^\beta \varphi^x(x,s)\,ds$ is in some sense a subadditive (in $\beta-\alpha$) envelope of $g(x,\alpha,\beta)$.

Another consequence is that if a function u minimizes $\overline{F}$, then $\mathbf{1}_{\{t<u(x)\}}$ minimizes $\overline{\mathcal{F}}$ on $\overline{X}_b$, i.e., $D\mathbf{1}_{\{t<u(x)\}}$ minimizes $\mathcal{H}+\lambda\Lambda$, for a given Lagrange multiplier λ. This means that $-\lambda\psi(x) \in \partial\mathcal{H}(D\mathbf{1}_{\{t<u(x)\}})$, in particular $\mathcal{H}(D\mathbf{1}_{\{t<u(x)\}}) = \int_{\Omega\times[0,1]} -\lambda\psi D\mathbf{1}_{\{t<u(x)\}} = \lambda$, so that the multiplier is $\lambda = \min_u \overline{F} = \inf_u F$. There exists φ'_n in K' and φ^0_n in N such that $\varphi'_n + \varphi^0_n \to -\lambda\psi$ as n goes to infinity, thus $\operatorname{div}\varphi'_n \rightharpoonup 0$ in the distributional sense, whereas $\int_{\Omega\times[0,1]} \varphi'_n D\mathbf{1}_{\{t<u(x)\}} \to F(u)$. If φ'_n has a limit, this limit is a calibration for u, moreover, it is easy to check that it must be a calibration for every other minimizer of $\overline{F}$. However, in general, φ'_n will not have a limit in K'. We will investigate this problem in a forthcoming paper with G. Bouchitté and explain in what sense there exists a calibration for the minimizers of $\overline{F}$.

Let us now consider again the Mumford-Shah functional, which motivated this work. Still, $\Omega \subset \mathbb{R}$ is a bounded interval. Let $u_0 \in L^\infty(\Omega,[0,1])$, and we consider the Mumford-Shah functional without “forcing term”

$$F_0(u) \;=\; \begin{cases} \int_\Omega |u'(x)|^2\,dx \;+\; \mathcal{H}^0(S_u) & \text{if } u \in SBV(\Omega), \\ +\infty & \text{otherwise.}\end{cases}$$

and the Mumford-Shah functional with “forcing term”

$$F(u) \;=\; F_0(u) \;+\; \int_\Omega (u(x)-u_0(x))^2\,dx.$$

We define $\overline{\mathcal{F}}$, K, K' as in the previous section, and similarly $\overline{\mathcal{F}}_0$, K_0, K_0' for the functional F_0. Clearly,

$$K' = \left\{ \varphi \in C_0(\Omega \times [0,1]; \mathbb{R}^2) : \varphi^t(x,t) \geq \frac{\varphi^x(x,t)^2}{4} - (t - u_0(x))^2, \left| \int_{t_1}^{t_2} \varphi^x(x,s)\, ds \right| \leq 1 \right\},$$

and

$$K_0' = \left\{ \varphi \in C_0(\Omega \times [0,1]; \mathbb{R}^2) : \varphi^t(x,t) \geq \frac{\varphi^x(x,t)^2}{4}, \left| \int_{t_1}^{t_2} \varphi^x(x,s)\, ds \right| \leq 1 \right\}$$

where in the definitions above the inequalities must hold for a.e. $x \in \Omega$ and every t, t_1, t_2. From Theorem 8.1, we know that

$$\overline{\mathcal{F}}_0(v) \;=\; \sup_{\varphi \in K_0'} \int_{\Omega \times [0,1]} \varphi D v, \tag{17}$$

and

$$\overline{\mathcal{F}}(v) \;=\; \sup_{\varphi \in K'} \int_{\Omega \times [0,1]} \varphi D v \tag{18}$$

as soon as u_0 is continuous. We can show that (18) still holds under slightly weaker assumptions on u_0 (for instance, if $u_0 \in BV(\Omega; [0,1])$). However, (18) is not true in general, and we will give a counterexample. Let us first state the following corollary of Theorem 8.1.

Corollary 8.2. *Assume that u_0 has a l.s.c. and an u.s.c. representative in $L^\infty(\Omega, [0,1])$. Then, $K = \overline{K' + N}$, and (18) holds.*

Proof. Defining $\mathcal{F}_0$ and $\mathcal{F}$ as before, i.e., by $\mathcal{F}_0(v) = \mathcal{F}(v) = +\infty$ if $v \notin X$, and $\mathcal{F}_0(v) = F_0(u)$, $\mathcal{F}(v) = F(u)$ if $v = \mathbf{1}_{\{t < u(x)\}} \in X$, we know that by construction $\overline{\mathcal{F}}_0$ and $\overline{\mathcal{F}}$ are respectively the convex l.s.c. envelopes of $\mathcal{F}_0$ and $\mathcal{F}$ in $\overline{X}$ (with the L^1 topology). But since for $v = \mathbf{1}_{\{t < u(x)\}}$,

$$\mathcal{F}(v) \;=\; \mathcal{F}_0(v) \;+\; \int_{\Omega \times [0,1]} 2(t - u_0(x)) v(x,t)\, dx dt \;+\; \int_\Omega u_0(x)^2\, dx,$$

we deduce that for every $v \in \overline{X}_b$,

$$\overline{\mathcal{F}}(v) \;=\; \overline{\mathcal{F}}_0(v) \;+\; \int_{\Omega \times [0,1]} 2(t - u_0(x)) v(x,t)\, dx dt \;+\; \int_\Omega u_0(x)^2\, dx. \tag{19}$$

Now, since by assumption u_0 is a.e. equal to the supremum (respectively, the infimum) of the continuous functions below it (resp., above), then for every $v \in \overline{X}_b$,

$$\int_{\Omega \times [0,1]} 2(t - u_0(x)) v(x,t)\, dx dt + \int_\Omega u_0(x)^2\, dx \;=\; \sup_{\mu, \mu_0} \int_{\Omega \times [0,1]} \mu(x,t) v(x,t)\, dx dt + \int_\Omega \mu_0(x)\, dx,$$

where the supremum is taken over all functions $\mu \in C_0(\Omega \times [0,1])$, $\mu_0 \in C_0(\Omega)$, such that $\mu(x,t) \leq 2(t - u_0(x))$ and $\mu_0(x) \leq u_0(x)^2$ for every t and a.e. x.

Noticing that

$$\int_{\Omega\times[0,1]} \mu(x,t)v(x,t)\,dxdt \;+\; \int_\Omega \mu_0(x)\,dx \;=\; -\int_{\Omega\times[0,1]} \left(\mu_0(x) + \int_0^t \mu(x,s)\,ds\right) D_t v,$$

and that $\mu_0(x) + \int_0^t \mu(x,s)\,ds \le u_0(x)^2 + \int_0^t 2(s-u_0(x))\,dx = (t-u_0(x))^2$ a.e., we deduce the inequality

$$\int_{\Omega\times[0,1]} 2(t-u_0(x))v(x,t)\,dxdt + \int_\Omega u_0(x)^2\,dx \;\le\; \sup_{\sigma\in K_{u_0}} -\int_{\Omega\times[0,1]} \sigma(x,t)D_t v(x,t) \qquad (20)$$

where we have let $K_{u_0} = \{\sigma \in C_0(\Omega\times[0,1]) : \sigma(x,t) \le (t-u_0(x))^2 \text{ a.e.}\}$. We deduce from (19), (20), and using (17), that

$$\begin{aligned}
\overline{\mathcal{F}}(v) \;&\le\; \overline{\mathcal{F}}_0(v) + \sup_{\sigma\in K_{u_0}} -\int_{\Omega\times[0,1]} \sigma D_t v \\
&=\; \sup_{\varphi\in K_0'} \int_{\Omega\times[0,1]} \varphi Dv \;+\; \sup_{\sigma\in K_{u_0}} -\int_{\Omega\times[0,1]} \sigma D_t v \\
&=\; \sup_{\varphi\in K_0',\sigma\in K_{u_0}} \int_{\Omega\times[0,1]} (\varphi-(0,\sigma))Dv \\
&\le\; \sup_{\varphi\in K'} \int_{\Omega\times[0,1]} \varphi Dv,
\end{aligned}$$

since $(\varphi-(0,\sigma)) \in K'$ for every $\varphi\in K_0'$ and $\sigma\in K_{u_0}$. Since we already know the reverse inequality, the thesis of Corollary 8.2 is proven. □

Remark 8.3. Now, if u_0 does not satisfy the assumption of Corollary 8.2, then the thesis may be false, as shows the following example. Let $(r_n)_{n\ge1}$ be the sequence of all rational numbers in $\Omega\cap\mathbb{Q}$. Let $\varepsilon>0$ be small and $u_0(x) = \min\{1, 2^n|x-r_n|/\varepsilon : n\ge1\}$. Then the supremum of the continuous functions that remain below u_0 is 0, and is almost nowhere equal to u_0, in other words, u_0 has no l.s.c. representative. If $\varphi\in K'$, $\varphi^t(x,t) \ge \frac{\varphi^x(x,t)}{4}^2 - t^2$ for every $x\in\Omega\cap\mathbb{Q}$, and thus for every x. In particular, for every $u\in SBV(\Omega)$,

$$\sup_{\varphi\in K'} \int_{\Omega\times[0,1]} \varphi D\mathbf{1}_{\{t<u(x)\}} \;\le\; \int_\Omega u'(x)^2 + u(x)^2\,dx \;+\; \mathcal{H}^0(S_u).$$

If $u=0$, the right-hand side is 0, whereas $F(0) = \int_\Omega u_0(x)^2\,dx \ge |\Omega| - 4\varepsilon/3$. It means that the topological setting in which we have considered this approach is too restrictive to study very general functionals. Notice though that if we introduce the set $\mathcal{K}'$ of Borel fields $\varphi\in L^\infty(\Omega\times[0,1];\mathbb{R}^{N+1})$ (possibly continuous in t) satisfying <u>pointwise</u> $\varphi^t(x,t) \ge \frac{\varphi^x(x,t)}{4}^2 - (u_0(x)-t)^2$ and $\left|\int_{t_1}^{t_2} \varphi^x(x,s)\,ds\right| \le 1$ for every x,t,t_1,t_2, then, again, we can define (following [1]) the convex functional

$$\mathcal{G}(v) \;=\; \sup_{\varphi\in\mathcal{K}'} \int_{\Omega\times[0,1]} \varphi Dv$$

for every $v \in \overline{X}_b$. If we consider in the proof of Corollary 8.2, this time, functions μ and μ_0 that are not continuous (just Borel) in x, and replace K_{u_0} by a set $\mathcal{K}_{u_0}$ of functions σ that are just Borel in x and satisfy the same inequality pointwise, we deduce that $\overline{\mathcal{F}} \le \mathcal{G}$. Since $\min_{BV(\Omega;[0,1])} F = \min_{\overline{X}} \overline{\mathcal{F}}$ and $\mathcal{G}(\mathbf{1}_{\{t<u(x)\}}) = F(u)$ for every u [1], we deduce, once again, that $\min_{\overline{X}} \mathcal{G} = \min_{BV(\Omega;[0,1])} F$. It is not clear whether $\overline{\mathcal{F}} = \mathcal{G}$, or $\overline{\mathcal{F}}$ is only the L^1-l.s.c. envelope of $\mathcal{G}$ in $\overline{X}_b$.

It remains to give the proof of Theorem 8.1.

8.2. Proof of Theorem 8.1

Without loss of generality we can assume that $\Omega = (0,1)$. In order to prove Theorem 8.1, we must find, given any $\varphi \in K$, two fields $\varphi' \in K'$ and $\varphi_0 \in N$ such that $\|\varphi - (\varphi' + \varphi_0)\|_\infty$ is arbitrarily close to zero. Notice first that by density it is not restrictive to assume that $\varphi \in C^1_c((0,1)\times[0,1];\mathbb{R}^2) \cap \mathrm{Int}\, K$, and we will fix $\varepsilon > 0$ with $\mathrm{supp}\,\varphi \subset (\varepsilon, 1-\varepsilon)\times[0,1]$. By a small translation $\varphi \to (\varphi^x, \varphi^t + \kappa\psi(x))$, for $\kappa > 0$ small and $\psi \in C^1_c(\Omega)$ with $\int_\Omega \psi(x)\,dx = 1$, we can furthermore assume that for every $u \in BV(\Omega;[0,1])$,

$$\int_{\Omega\times[0,1]} \varphi D\mathbf{1}_{\{t<u(x)\}} \;\le\; F(u) - \kappa \tag{21}$$

and we choose ε small enough to have, letting $f_\infty = \sup_{x,t} f(x,t,0)$,

$$2\varepsilon f_\infty \;\le\; \kappa. \tag{22}$$

Now, suppose that $\varphi_0 \in N$ is such that $\varphi - \varphi_0 \in K'$. Since $\mathrm{div}\,\varphi_0 = \partial_x\varphi_0^x + \partial_t\varphi_0^t = 0$, there exists $\Lambda \in C^1_0(\Omega\times[0,1])$ such that $\nabla\Lambda = (-\varphi_0^t, \varphi_0^x)$. This functions satisfies

$$\Lambda(1-0,0) - \Lambda(0+0,0) \;=\; -\int_0^1 \varphi_0^t(x,0)\,dx \;=\; 0,$$

and we should have for every a, b, $0 < a < b < 1$, and every $u \in BV(\Omega;[0,1])$,

$$\int_{(a,b)\times[0,1]} \varphi D\mathbf{1}_{\{t<u(x)\}} \;-\; F(u,(a,b)) \;\le\; \int_{(a,b)\times[0,1]} \varphi_0 D\mathbf{1}_{\{t<u(x)\}} \;=\; \Lambda(b,\beta) - \Lambda(a,\alpha),$$

with $\alpha = u(a+0)$ and $\beta = u(b-0)$. On the other hand, if a scalar function Λ satisfies these two properties, then, setting $(\varphi_0^x, \varphi_0^t) = (\partial_t\Lambda, -\partial_x\Lambda)$ will give the correct φ_0.

These considerations suggest us to define the following quantities, for every $a, b \in (\varepsilon, 1-\varepsilon)$ and $\alpha, \beta \in [0,1]$:

$$\Lambda^-(b,\beta) \;=\; \sup\left\{ \int_{(\varepsilon,\,b)\,\times\,[0,1]} \varphi D\mathbf{1}_{\{t<u(x)\}} - F(u,(\varepsilon,b)) \;:\; u \in BV(\Omega;[0,1]), u(b-0) = \beta \right\}$$

and

$$-\Lambda^+(a,\alpha) \;=\; \sup\left\{ \int_{(a,\,1-\varepsilon)\,\times\,[0,1]} \varphi D\mathbf{1}_{\{t<u(x)\}} - F(u,(a,1-\varepsilon)) \;:\; u \in BV(\Omega;[0,1]), u(a+0) = \alpha \right\}.$$

Lemma 8.4. *The functions $\Lambda^\pm$ satisfy $-\infty < \Lambda^- \le \Lambda^+ < +\infty$, and for every $a, b \in (\varepsilon, 1-\varepsilon)$, $\alpha, \beta \in [0,1]$ and u with $a < b$, $u(a+0) = \alpha$ and $u(b-0) = \beta$,*

$$\Lambda^\pm(b,\beta) - \Lambda^\pm(a,\alpha) \ \ge\ \int_{(a,b)\times[0,1]} \varphi D\mathbf{1}_{\{t<u(x)\}} \ -\ F(u,(a,b)). \tag{23}$$

Proof. Clearly, $\Lambda^- > -\infty$ and $\Lambda^+ < +\infty$. Let $(a,\alpha) \in (\varepsilon, 1-\varepsilon)\times[0,1]$. If $(u_n^-)_{n\ge1}$ is a maximizing sequence for the sup defining $\Lambda^-(a,\alpha)$, and $(u_n^+)_{n\ge1}$ a maximizing sequence for the sup defining $-\Lambda^+(a,\alpha)$, then, letting for every n and $x \in \Omega$

$$u_n(x) \ =\ \begin{cases} u_n^-(\varepsilon+0) & \text{if } x \le \varepsilon, \\ u_n^-(x) & \text{if } \varepsilon < x \le a, \\ u_n^+(x) & \text{if } a < x < 1-\varepsilon, \text{ and} \\ u_n^+(1-\varepsilon-0) & \text{if } x \ge 1-\varepsilon, \end{cases}$$

we build a function $u_n \in BV(\Omega;[0,1])$ with $\varepsilon, a, 1-\varepsilon \notin S_{u_n}$. In particular, for every n, we have $|D\mathbf{1}_{\{t<u_n(x)\}}|(\{a\}\times[0,1]) = 0$, and (21) yields

$$\begin{aligned}
-\kappa \ &\ge\ \int_{\Omega\times[0,1]} \varphi D\mathbf{1}_{\{t<u_n(x)\}} - F(u_n) \\
&=\ \int_{(0,\,a)\,\times\,[0,\,1]} \varphi D\mathbf{1}_{\{t<u_n(x)\}} - F(u_n,(0,a)) \ +\ \int_{(a,\,1)\,\times\,[0,\,1]} \varphi D\mathbf{1}_{\{t<u_n(x)\}} - F(u_n,(a,1)) \\
&=\ \Bigg(\int_{(\varepsilon,a)\times[0,1]} \varphi D\mathbf{1}_{\{t<u_n(x)\}} - F(u_n,(\varepsilon,a)) \\
&\qquad + \int_{(a,1-\varepsilon)\times[0,1]} \varphi D\mathbf{1}_{\{t<u_n(x)\}} - F(u_n,(a,1-\varepsilon))\Bigg) \ -\ F(u_n,(0,\varepsilon]\cup[1-\varepsilon,1)).
\end{aligned}$$

Since

$$F(u_n,(0,\varepsilon]) \ =\ \int_0^\varepsilon f(x,u_n(\varepsilon+0),0)\,dx \ \le\ \varepsilon f_\infty$$

and

$$F(u_n,[1-\varepsilon,1)) \ =\ \int_{1-\varepsilon}^1 f(x,u_n(1-\varepsilon-0),0)\,dx \ \le\ \varepsilon f_\infty,$$

by (22) we deduce that

$$\int_{(\varepsilon,\,a)\,\times\,[0,\,1]} \varphi D\mathbf{1}_{\{t<u_n^-(x)\}} - F(u_n^-,(\varepsilon,a)) \ +\ \int_{(a,\,1-\varepsilon)\,\times\,[0,\,1]} \varphi D\mathbf{1}_{\{t<u_n^+(x)\}} - F(u_n^+,(a,1-\varepsilon)) \le 0.$$

Sending n to infinity, we deduce that $\Lambda^-(a,\alpha) - \Lambda^+(a,\alpha) \le 0$, which proves the first assertion of the Lemma.

Now choose a, b with $\varepsilon < a < b < 1-\varepsilon$ and $\alpha, \beta \in [0,1]$, and let $u \in BV(\Omega;[0,1])$ with $u(a+0) = \alpha$ and $u(b-0) = \beta$. Let $(u_n^-)_{n\ge1}$ be a maximizing sequence for the sup defining $\Lambda^-(a,\alpha)$ and define u_n in the following way:

$$u_n(x) \ =\ \begin{cases} u_n^-(x) & \text{if } x \le a, \\ u(x) & \text{if } x > a. \end{cases}$$

Since $a \notin S_{u_n}$ and thus $|D\mathbf{1}_{\{t<u_n(x)\}}|(\{a\}\times[0,1]) = 0$, by definition of $\Lambda^-(b,\beta)$ we have for every n:

$$\begin{aligned}
\Lambda^-(b,\beta) &\geq \int_{(\varepsilon,b)\times[0,1]} \varphi D\mathbf{1}_{\{t<u_n(x)\}} - F(u_n,(\varepsilon,b)) \\
&= \int_{(\varepsilon,a)\times[0,1]} \varphi D\mathbf{1}_{\{t<u_n(x)\}} - F(u_n,(\varepsilon,a)) + \int_{(a,b)\times[0,1]} \varphi D\mathbf{1}_{\{t<u_n(x)\}} - F(u_n,(a,b)) \\
&= \int_{(\varepsilon,a)\times[0,1]} \varphi D\mathbf{1}_{\{t<u_n^-(x)\}} - F(u_n^-,(\varepsilon,a)) + \int_{(a,b)\times[0,1]} \varphi D\mathbf{1}_{\{t<u(x)\}} - F(u,(a,b)) \\
&\longrightarrow \Lambda^-(a,\alpha) + \int_{(a,b)\times[0,1]} \varphi D\mathbf{1}_{\{t<u(x)\}} - F(u,(a,b))
\end{aligned}$$

as n goes to infinity. Therefore (23) holds for Λ^-. The proof of (23) for Λ^+ is identical. $\square$

Let us now return to the proof of Theorem 8.1. We extend Λ^- to $\{\varepsilon\}\times[0,1]$ and Λ^+ to $\{1-\varepsilon\}\times[0,1]$ by the value 0, clearly, this extension is consistent with the definition of $\Lambda^\pm$, and (23) holds up to $a=\varepsilon$ for Λ^- and $b = 1-\varepsilon$ for Λ^+. Let now, for $(x,t)\in[\varepsilon,1-\varepsilon]\times[0,1]$,

$$\Lambda(x,t) = \left(1-\frac{x-\varepsilon}{1-2\varepsilon}\right)\Lambda^-(x,t) + \frac{x-\varepsilon}{1-2\varepsilon}\Lambda^+(x,t)$$

(for $x=\varepsilon$ or $x=1-\varepsilon$, we have thus $\Lambda(x,t)=0$). If $\varepsilon\le a<b\le 1-\varepsilon$,

$$\begin{aligned}
\Lambda(b,\beta)-\Lambda(a,\alpha) &= \left(1-\frac{b-\varepsilon}{1-2\varepsilon}\right)\Lambda^-(b,\beta) + \\
&\quad + \frac{b-\varepsilon}{1-2\varepsilon}\Lambda^+(b,\beta) - \left(1-\frac{a-\varepsilon}{1-2\varepsilon}\right)\Lambda^-(a,\alpha) - \frac{a-\varepsilon}{1-2\varepsilon}\Lambda^+(a,\alpha) \\
&= \left(1-\frac{a-\varepsilon}{1-2\varepsilon}\right)\big(\Lambda^-(b,\beta)-\Lambda^-(a,\alpha)\big) + \\
&\quad + \frac{a-\varepsilon}{1-2\varepsilon}\big(\Lambda^+(b,\beta)-\Lambda^+(a,\alpha)\big) + \frac{b-a}{1-2\varepsilon}\big(\Lambda^+(b,\beta)-\Lambda^-(b,\beta)\big)
\end{aligned}$$

Since $\Lambda^+\ge\Lambda^-$ and $b>a$, it yields using (23) both for Λ^- and Λ^+

$$\Lambda(b,\beta)-\Lambda(a,\alpha) \geq \int_{(a,b)\times[0,1]} \varphi D\mathbf{1}_{\{t<u(x)\}} - F(u,(a,b)) \tag{24}$$

as soon as $u(a+0)=\alpha$ and $u(b-0)=\beta$. We now extend Λ to $((0,\varepsilon)\cup(1-\varepsilon,1))\times[0,1]$ by the value 0, it is easy to check that (24) still holds if $a<\varepsilon$ or $b>1-\varepsilon$, indeed, if for instance $a<\varepsilon<b\le 1-\varepsilon$, we have

$$\Lambda(b,\beta)-\Lambda(a,\alpha) = \Lambda(b,\beta) = \Lambda(b,\beta)-\Lambda(\varepsilon,u(\varepsilon+0))$$

and

$$\begin{aligned}
\Lambda(b,\beta)-\Lambda(\varepsilon,u(\varepsilon+0)) &\geq \int_{(\varepsilon,b)\times[0,1]} \varphi D\mathbf{1}_{\{t<u(x)\}} - F(u,(\varepsilon,b)) \\
&\geq \int_{(a,b)\times[0,1]} \varphi D\mathbf{1}_{\{t<u(x)\}} - F(u,(a,b)).
\end{aligned}$$

If we could prove that Λ is of class C^1 then the proof of Theorem (8.1) would be achieved. Unfortunately this is far from being clear (is is possible to prove, quite easily, that Λ is continuous). In order to achieve the proof we will therefore mollify Λ and φ.

Let $k(x,t)$ be a symmetric mollifier with support in $\{x^2+t^2\le 1\}$ and for every $\eta>0$ let $k_\eta(x,t)=(1/\eta^2)k(x/\eta,t/\eta)$.

Fix first $x_0\in\Omega=(0,1)$, choose $\alpha,\beta\in[0,1]$ and let $u(x)=\alpha$ if $x<x_0$, $u(x)=\beta$ otherwise. If $\delta<x_0\wedge(1-x_0)$, then by (24)

$$\begin{aligned}\Lambda(x_0+\delta,\beta)-\Lambda(x_0-\delta,\alpha) &\ge \int_\alpha^\beta \varphi^x(x_0,s)\,ds-g(x_0,\alpha,\beta)\\ &\quad+\int_{x_0-\delta}^{x_0+\delta}-\varphi^t(x,u(x))-f(x,u(x),0)\,dx\\ &\ge \int_\alpha^\beta \varphi^x(x_0,s)\,ds-g(x_0,\alpha,\beta)\;-\;2c\,\delta\end{aligned}$$

where $c=\sup_{x,t}\left(\varphi^t(x,t)+f(x,t,0)\right)$. Now, if η is small (in particular, $\eta<<\varepsilon$) and $x_0\wedge(1-x_0)>\eta$, if δ is small enough we deduce that for every ξ,τ, assuming also $\eta\le\alpha,\beta\le 1-\eta$,

$$k_\eta(\xi,\tau)\left(\Lambda(x_0+\delta-\xi,\beta-\tau)-\Lambda(x_0-\delta-\xi,\alpha-\tau)\right)\ge$$

$$\ge\int_\alpha^\beta k_\eta(\xi,\tau)\varphi^x(x_0-\xi,s-\tau)\,ds-k_\eta(\xi,\tau)g(x_0-\xi,\alpha-\tau,\beta-\tau)\;-\;2ck_\eta(\xi,\tau)\,\delta$$

We define $g_\eta(x,\alpha,\beta)=\int_{\mathbb{R}^2}k_\eta(\xi,\tau)g(x-\xi,\alpha-\tau,\beta-\tau)\,d\xi\,d\tau$ for every $x\in(\eta,1-\eta)$ and $\eta\le\alpha,\beta\le 1-\eta$, $\Lambda_\eta=k_\eta*\Lambda$, $\varphi_\eta=k_\eta*\varphi$ (on $(0,1)\times[\eta,1-\eta]$, since clearly we can extend by the value 0 the functions Λ and φ for $x\le 0$ or $x\ge 1$). Integrating the last equation over ξ,τ yields

$$\Lambda_\eta(x_0+\delta,\beta)-\Lambda_\eta(x_0-\delta,\alpha)\ge\int_\alpha^\beta\varphi^x_\eta(x_0,s)\,ds-g_\eta(x_0,\alpha,\beta)\;-\;2c\,\delta.$$

Sending δ to 0, we get

$$\int_\alpha^\beta\varphi^x_\eta(x_0,s)\,ds-g_\eta(x_0,\alpha,\beta)\;\le\;\Lambda_\eta(x_0,\beta)-\Lambda_\eta(x_0,\alpha)\tag{25}$$

for every x_0 (actually, in $[0,1]$), and every $\alpha,\beta\in[\eta,1-\eta]$.

Choose now some $p\in\mathbb{R}$, $x_0\in(0,1)$, $t_0\in(0,1)$ and define $u(x)=t_0+p(x-x_0)$. Now, (24) yields for small δ

$$\Lambda(x_0+\delta,t_0+p\delta)-\Lambda(x_0-\delta,t_0-p\delta)\ge$$

$$\int_{x_0-\delta}^{x_0+\delta}\varphi^x(x,t_0+p(x-x_0))p-\varphi^t(x,t_0+p(x-x_0))\;-\;f(x,t_0+p(x-x_0),p)\,dx.$$

Evaluating this expression at $(x_0-\xi,t_0-\tau)$, multiplying again by $k_\eta(\xi,\tau)$ and integrating over ξ and τ, we get

$$\Lambda_\eta(x_0+\delta,t_0+p\delta)-\Lambda_\eta(x_0-\delta,t_0-p\delta)\;\geq$$
$$\int_{x_0-\delta}^{x_0+\delta}\varphi^x_\eta(x,t_0+p(x-x_0))p-\varphi^t_\eta(x,t_0+p(x-x_0))\;-\;f_\eta(x,t_0+p(x-x_0),p)\,dx,$$

where now $f_\eta(x,t,p)=\int_{\mathbb{R}^2}k_\eta(\xi,\tau)f(x-\xi,t-\tau,p)\,d\xi\,d\tau$. Dividing by 2δ and sending δ to 0, we deduce

$$\partial_x\Lambda_\eta(x_0,t_0)+p\partial_t\Lambda_\eta(x_0,t_0)\;\geq\;\varphi^x_\eta(x_0,t_0)p-\varphi^t_\eta(x_0,t_0)\;-\;f_\eta(x_0,t_0,p),\tag{26}$$

this for every $x_0\in(0,1)$, $t_0\in(\eta,1-\eta)$ (in fact, $[\eta,1-\eta]$ by continuity), and $p\in\mathbb{R}$.

Now, we need to find inequalities such as (25) and (26) satisfied up to $\alpha,\beta,t_0\in[0,1]$. To this end, we just "stretch" the variable t. We introduce $T_\eta:[0,1]\to[\eta,1-\eta]$, $T_\eta t=\eta+(1-2\eta)t$, and let

$$\tilde\Lambda_\eta(x,t)\;=\;\Lambda_\eta(x,T_\eta t)\qquad\text{and}\qquad\tilde\varphi_\eta(x,t)\;=\;((1-2\eta)\varphi^x_\eta(x,T_\eta t),\varphi^t_\eta(x,T_\eta t)).$$

Equations (25) and (26) yield

$$\int_\alpha^\beta\tilde\varphi^x_\eta(x,s)\,ds-g_\eta(x,T_\eta\alpha,T_\eta\beta)\;\leq\;\tilde\Lambda_\eta(x,\beta)-\tilde\Lambda_\eta(x,\alpha)\tag{27}$$

and

$$\partial_x\tilde\Lambda_\eta(x,t)+\frac{p}{1-2\eta}\partial_t\tilde\Lambda_\eta(x,t)\;\geq\;\tilde\varphi^x_\eta(x,t)\frac{p}{1-2\eta}-\tilde\varphi^t_\eta(x,t)\;-\;f_\eta(x,T_\eta t,p),\tag{28}$$

for every $x\in\Omega$, $\alpha,\beta,t\in[0,1]$, and every $p\in\mathbb{R}$. Notice that for every $A,B\in\mathbb{R}$, if

$$A\frac{p}{1-2\eta}+B\;\leq\;f_\eta(x,T_\eta t,p)$$

for every $p\in\mathbb{R}$, then, choosing $p=0$ yields $B\;\leq\;f_\eta(x,T_\eta t,0)\leq f_\infty$, and multiplying then by $1-2\eta$ yields

$$Ap+B\;\leq\;(1-2\eta)f_\eta(x,T_\eta t,p)+2\eta B\leq\;(1-2\eta)f_\eta(x,T_\eta t,p)+2\eta f_\infty$$

for every p. Applying this remark to equation (28), we get

$$\partial_x\tilde\Lambda_\eta(x,t)+p\partial_t\tilde\Lambda_\eta(x,t)\;\geq\;\tilde\varphi^x_\eta(x,t)p-\tilde\varphi^t_\eta(x,t)-2\eta f_\infty\;-\;(1-2\eta)f_\eta(x,T_\eta t,p)$$

for every $x\in\Omega$, $t\in[0,1]$, and $p\in\mathbb{R}$. Since $\tilde\Lambda_\eta$ and $\tilde\varphi_\eta$ have compact support, in $[\varepsilon-\eta,1-\varepsilon+\eta]\times[0,1]$, if we choose a smooth function $\tilde\psi:\Omega\to[0,1]$ with compact support such that $\tilde\psi\equiv1$ in $[\varepsilon/2,1-\varepsilon/2]$, the last equation can be written, for η small enough,

$$\partial_x\tilde\Lambda_\eta(x,t)+p\partial_t\tilde\Lambda_\eta(x,t)\;\geq\;\tilde\varphi^x_\eta(x,t)p-\tilde\varphi^t_\eta(x,t)-2\eta f_\infty\tilde\psi(x)\;-\;(1-2\eta)f_\eta(x,T_\eta t,p)\tag{29}$$

for every $x\in\Omega$, $t\in[0,1]$, and $p\in\mathbb{R}$.

To conclude, we first notice that by assumption (A3) there exists a constant c_η going to zero as η goes to zero such that for every x,t,

$$f_\eta(x,T_\eta t,p)\;\leq\;(1+c_\eta)f(x,t,p)\;+\;c_\eta.\tag{30}$$

Then, using now assumption (A5), we introduce the function $h^+(x,t,\sigma)$, $x \in \Omega$, $t \in [0,1]$, $0 \le \sigma \le 2t \wedge 2(1-t)$, defined by

$$h^+\left(x, \frac{\alpha+\beta}{2}, \beta-\alpha\right) = \begin{cases} \dfrac{g(x,\alpha,\beta)}{\gamma^+(\beta-\alpha)} & \text{if } \beta > \alpha, \\ k^+(x,\alpha) & \text{if } \beta = \alpha. \end{cases}$$

The function h^+ is uniformly continuous on $\{(x,t,\sigma) : \varepsilon/4 \le x \le 1-\varepsilon/4, t \in [0,1], 0 \le \sigma \le 2t \wedge 2(1-t)\}$, so that, introducing $h^+_\eta(x,t,\sigma) = \int_{\mathbb{R}^2} k_\eta(\xi,\tau) h^+(x-\xi, t-\tau, \sigma)\, d\xi\, d\tau$, there exists a constant c'_η going to zero with η such that

$$|h^+_\eta(x, T_\eta t, (1-2\eta)\sigma) - h^+(x,t,\sigma)| \le c'_\eta$$

for every $x \in [\varepsilon/2, 1-\varepsilon/2]$ (remember $\eta << \varepsilon$), $t \in [0,1], \sigma \ge 0$. By (A5) we also have that $h^+ \ge 1$, therefore

$$h^+_\eta(x, T_\eta t, (1-2\eta)\sigma) \le h^+(x,t,\sigma) + c'_\eta \le \left(1 + c'_\eta\right) h^+(x,t,\sigma).$$

Since $g_\eta(x, T_\eta\alpha, T_\eta\beta) = \gamma^+((1-2\eta)(\beta-\alpha)) h^+_\eta(x, T_\eta((\alpha+\beta)/2), (1-2\eta)(\beta-\alpha))$ and γ^+ is nondecreasing, we deduce that

$$g_\eta(x, T_\eta\alpha, T_\eta\beta) \le \left(1 + c'_\eta\right) g(x,\alpha,\beta) \tag{31}$$

for every $\beta > \alpha$, and x in $[\varepsilon/2, 1-\varepsilon/2]$. On the other hand, using now $\overline{\gamma}^-$, we get that the same inequality holds for $\alpha > \beta$.

We now let $\theta_\eta^{-1} = \max\{(1-2\eta)(1+c_\eta), 1+c'_\eta\}$. We define

$$\hat{\varphi}_\eta(x,t) = \theta_\eta \left(\tilde{\varphi}^x_\eta(x,t), \tilde{\varphi}^t_\eta(x,t) + (2\eta f_\infty + (1-2\eta)c_\eta)\tilde{\psi}(x)\right)$$

for every $(x,t) \in \Omega \times [0,1]$. As $\eta \to 0$, $\hat{\varphi}_\eta \to \varphi$ in $C_0(\Omega \times [0,1]; \mathbb{R}^2)$, and since $\varphi \in \operatorname{Int} K$ it implies that $\hat{\varphi}_\eta \in K$ for small η. For every $(x,t) \in \Omega \times [0,1]$, we also define

$$\varphi_{0\eta}(x,t) = \theta_\eta(\partial_t \tilde{\Lambda}_\eta(x,t), -\partial_x \tilde{\Lambda}_\eta(x,t))$$

and we let $\varphi'_\eta = \hat{\varphi}_\eta - \varphi_{0\eta}$. Clearly, by construction, $\varphi_{0\eta} \in N$, and by (27), (31), and (29), (30), we get

$$\int_\alpha^\beta {\varphi'_\eta}^x(x,s)\, ds \le g(x,\alpha,\beta)$$

and

$${\varphi'_\eta}^t(x,t) \ge {\varphi'_\eta}^x(x,t) p - f(x,t,p)$$

for every $x \in \Omega$, $\alpha, \beta, t \in [0,1]$ and $p \in \mathbb{R}$. (The last inequality has to be checked independently on the sets $\{\tilde{\psi}(x) = 1\}$ and $\{\tilde{\psi}(x) < 1\}$, on the latter set it follows from the fact that $\tilde{\varphi}_\eta$ and $\tilde{\Lambda}_\eta$ are identically zero, if η is small enough.) It is not difficult to check that these two conditions characterize the convex K': therefore $\varphi'_\eta \in K'$, and the proof of Theorem 8.1 is achieved. Notice that taking the supremum over p in the last equation yields

$${\varphi'_\eta}^t(x,t) \ge f^*(x,t,{\varphi'_\eta}^x(x,t)),$$

which is the condition that appears in equation (16). □

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