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CONVEXITY ACCORDING TO THE GEOMETRIC MEAN

CONSTANTIN P. NICULESCU

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Abstract. We develop a parallel theory to the classical theory of convex functions, based on a change of variable formula, by replacing the arithmetic mean by the geometric one. It is shown that many interesting functions such as exp, sinh, cosh, sec, csc, arc sin, Γ etc illustrate the multiplicative version of convexity when restricted to appropriate subintervals of $(0, \infty)$. As a consequence, we are not only able to improve on a number of classical elementary inequalities but also to discover new ones.

1. Introduction

The usual definition of a convex function (of one real variable) depends on the structure of $\mathbb{R}$ as an ordered vector space. As $\mathbb{R}$ is actually an ordered field, it is natural to ask what happens when addition is replaced by multiplication and the arithmetic mean is replaced by the geometric mean. A moment's reflection reveals an entire new world of beautiful inequalities, involving a broad range of functions from the elementary ones, such as sin, cos, exp, to the special ones, such as Γ, Psi, L (the Lobacevski's function), Si (the integral sine) etc.

Depending on which type of mean, arithmetic (A), or geometric (G), we consider respectively on the domain and the codomain of definition, we shall encounter one of the following four classes of functions:

$AA-$ *convex functions,* the usual convex functions

$AG-$ *convex functions*

$GA-$ *convex functions*

$GG-$ *convex functions.*

It is worth noticing that while (A) makes no restriction about the interval I where it applies (it is so because $x, y \in I,\ \lambda \in [0, 1]$ implies $(1-\lambda)x + \lambda y \in I$), the use of (G) forces us to restrict to the subintervals J of $(0, \infty)$ in order to assure that

$$x, y \in J,\ \lambda \in [0, 1] \Rightarrow x^{1-\lambda}y^{\lambda} \in J.$$

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To be more specific, the $AG-$ *convex functions* (usually known as $\log -$ *convex functions*) are those functions $f : I \to (0, \infty)$ for which

$$x, y \in I \text{ and } \lambda \in [0, 1] \Rightarrow f((1-\lambda)x + \lambda y) \leqslant f(x)^{1-\lambda} f(y)^{\lambda}, \tag{AG}$$

i.e., for which $\log f$ is convex.

The $GG-$ *convex functions* (called in what follows *multiplicatively convex functions*) are those functions $f : I \to J$ (acting on subintervals of $(0, \infty)$) such that

$$x, y \in I \text{ and } \lambda \in [0, 1] \Rightarrow f(x^{1-\lambda} y^{\lambda}) \leqslant f(x)^{1-\lambda} f(y)^{\lambda}. \tag{GG}$$

Due to the following form of the $AM - GM$ Inequality,

$$a, b \in (0, \infty),\ \lambda \in [0, 1] \Rightarrow a^{1-\lambda} b^{\lambda} \leqslant (1-\lambda)a + \lambda b, \tag{$*$}$$

every $\log -$ convex function is also convex. The most notable example of such a function is Euler's *gamma function,*

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\, dt, \quad x > 0.$$

In fact,

$$\frac{d^2}{dx^2} \log \Gamma(x) = \sum_{n=0}^{\infty} \frac{1}{(x+n)^2} \quad \text{for } x > 0.$$

See [15]. As noticed by H. Bohr and J. Mollerup [2], [1], the gamma function is the only function $f : (0, \infty) \to (0, \infty)$ with the following three properties:

(Γ1) f is $\log -$ convex;
(Γ2) $f(x+1) = x f(x)$ for every $x > 0$;
(Γ3) $f(n+1) = n!$ for every $n \in \mathbb{N}$.

The class of all $GA-$ *convex* functions is constituted by all functions $f : I \to \mathbb{R}$ (defined on subintervals of $(0, \infty)$) for which

$$x, y \in I \text{ and } \lambda \in [0, 1] \Rightarrow f(x^{1-\lambda} y^{\lambda}) \leqslant (1-\lambda) f(x) + \lambda f(y). \tag{GA}$$

In the context of twice differentiable functions $f : I \to \mathbb{R}$, $GA-$ convexity means $x^2 f'' + x f' \geqslant 0$, so that all twice differentiable nondecreasing convex functions are also $GA-$ convex. Notice that the inequality $(*)$ above is of this nature.

The aim of this paper is to investigate the class of multiplicatively convex functions as a source of inequalities. We shall develop a parallel to the classical theory of convex functions based on the following remark, which relates the two classes of functions: *Suppose that I is a subinterval of $(0, \infty)$ and $f : I \to (0, \infty)$ is a multiplicatively convex function. Then*

$$F = \log \circ f \circ \exp : \log(I) \to \mathbb{R}$$

is a convex function. Conversely, if J is an interval (for which $\exp(J)$ is a subinterval of $(0, \infty)$) and $F : J \to \mathbb{R}$ is a convex function, then

$$f = \exp \circ F \circ \log : \exp(J) \to (0, \infty)$$

is a convex function.

Equivalently, f is multiplicatively convex if, and only if, $\log f(x)$ is a convex function of $\log x$. See Lemma 2.1 below. Modulo this characterization, the class of all multiplicatively convex functions was first considered by P. Montel [10], in a beautiful paper discussing the analogues of the notion of convex function in n variables. However, the roots of the research in this area can be traced long before him. Let us mention two such results here:

HADAMARD'S THREE CIRCLES THEOREM. *Let f be an analytical function in the annulus $a < |z| < b$. Then $\log M(r)$ is a convex function of $\log r$, where*

$$M(r) = \sup_{|z|=r} |f(z)|.$$

G. H. HARDY'S MEAN VALUE THEOREM. *Let f be an analytical function in the annulus $a < |z| < b$ and let $p \in [1, \infty)$. Then $\log M_p(r)$ is a convex function of $\log r$, where*

$$M_p(r) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{1/p}.$$

As $\lim_{n\to\infty} M_n(r) = M(r)$, Hardy's aforementioned result implies Hadamard's. As is well known, Hadamard's result is instrumental in deriving the celebrating Riesz-Thorin Interpolation Theorem (see [5]).

Books like those of Hardy, Littlewood and Polya [5] and A. W. Roberts and D. E. Varberg [12] make some peripheric references to the functions f for which $\log f(x)$ is a convex function of $\log x$. Nowadays, the subject of multiplicative convexity seems to be even forgotten, which is a pity because of its richness. What we try to do in this paper is not only to call the attention to the beautiful zoo of inequalities falling in the realm of multiplicative convexity, but also to prove that many classical inequalities such as the $AM - GM$ Inequality can benefit of a better understanding via the multiplicative approach of convexity.

2. Generalities on multiplicatively convex functions

The class of multiplicatively convex functions can be easily described as being constituted by those functions f (acting on subintervals of $(0, \infty)$) such that $\log f(x)$ is a convex function of $\log x$:

LEMMA 2.1. *Suppose that I is a subinterval of $(0, \infty)$. A function $f : I \to (0, \infty)$ is multiplicatively convex if, and only if,*

$$\begin{vmatrix} 1 & \log x_1 & \log f(x_1) \\ 1 & \log x_2 & \log f(x_2) \\ 1 & \log x_3 & \log f(x_3) \end{vmatrix} \geqslant 0$$

for every $x_1 \leqslant x_2 \leqslant x_3$ in I; equivalently, if and only if,

$$f(x_1)^{\log x_3} f(x_2)^{\log x_1} f(x_3)^{\log x_2} \geqslant f(x_1)^{\log x_2} f(x_2)^{\log x_3} f(x_3)^{\log x_1}$$

for every $x_1 \leqslant x_2 \leqslant x_3$ *in* I.

Proof. That follows directly from the definition of multiplicative convexity, taking logarithms and noticing that any point between x_1 and x_3 is of the form $x_1^{1-\lambda}x_3^{\lambda}$, for some $\lambda \in (0,1)$. $\square$

COROLLARY 2.2. *Every multiplicatively convex function* $f : I \to (0,\infty)$ *has finite lateral derivatives at each interior point of* I. *Moreover, the set of all points where* f *is not differentiable is at most countable.*

An example of a multiplicatively convex function which is not differentiable at countably many points is

$$\exp\left(\sum_{n=0}^{\infty} \frac{|\log x - n|}{2^n}\right).$$

By Corollary 2.2, every multiplicatively convex function is continuous in the interior of its domain of definition. Under the presence of continuity, the multiplicative convexity can be restated in terms of geometric mean:

THEOREM 2.3. *Suppose that* I *is a subinterval of* $(0,\infty)$. *A continuous function* $f : I \to [0,\infty)$ *is multiplicatively convex if, and only if,*

$$x, y \in I \Rightarrow f(\sqrt{xy}) \leqslant \sqrt{f(x)f(y)}.$$

Proof. The necessity is clear. The sufficiency part follows from the connection between the multiplicative convexity and the usual convexity (as noticed in the Introduction) and the well known fact that mid-convexity (i.e., Jensen convexity) is equivalent to convexity under the presence of continuity. See [5]. $\square$

Theorem 2.3 above reveals the essence of multiplicative convexity as being the *convexity according to the geometric mean*; in fact, under the presence of continuity, the multiplicatively convex functions are precisely those functions $f : I \to [0,\infty)$ for which

$$x_1, \ldots, x_n \in I \Rightarrow f(\sqrt[n]{x_1 \ldots x_n}) \leqslant \sqrt[n]{f(x_1)\ldots f(x_n)}.$$

In this respect, it is natural to call a function $f : I \to (0,\infty)$ *multiplicatively concave* if $1/f$ is multiplicatively convex and *multiplicatively affine* if f is of the form Cx^{α} for some $C > 0$ and some $\alpha \in \mathbb{R}$.

A refinement of the notion of multiplicative convexity is that of *strict multiplicative convexity,* which in the context of continuity will mean

$$f(\sqrt[n]{x_1 \ldots x_n}) < \sqrt[n]{f(x_1)\ldots f(x_n)}$$

unless $x_1 = \ldots = x_n$. Clearly, our remark concerning the connection between the multiplicatively convex functions and the usual convex functions has a "strict" counterpart.

A large class of strictly multiplicatively convex functions, is indicated by the following result, which developed from [5], Theorem 177, page 125:

PROPOSITION 2.4. *Every polynomial* $P(x)$ *with non-negative coefficients is a multiplicatively convex function on* $[0,\infty)$. *More generally, every real analytic function*

$f(x) = \sum_{n=0}^{\infty} c_n x^n$ with non-negative coefficients is a multiplicatively convex function on $(0, R)$, where R denotes the radius of convergence.

Moreover, except for the case of functions Cx^n (with $C > 0$ and $n \in \mathbb{N}$), the above examples are strictly multiplicatively convex functions.

Examples of such real analytic functions are:

$$\begin{aligned}
&\exp,\ \sinh,\ \cosh \quad \text{on } (0, \infty)\\
&\tan,\ \sec,\ \csc,\ \frac{1}{x} - \cot x \quad \text{on } (0, \pi/2)\\
&\arcsin \quad \text{on } (0, 1]\\
&-\log(1-x),\ \frac{1+x}{1-x} \quad \text{on } (0, 1).
\end{aligned}$$

See the table of series of I. S. Gradshteyn and I. M. Ryzhik [4].

Proof. By continuity, it suffices to prove only the first assertion. For, suppose that $P(x) = \sum_{n=0}^{N} c_n x^n$. According to Theorem 2.3, we have to prove that

$$x, y > 0 \Rightarrow (P(\sqrt{xy}))^2 \leqslant P(x)P(y),$$

equivalently,

$$x, y > 0 \Rightarrow (P(xy))^2 \leqslant P(x^2)P(y^2).$$

Or, the latter is an easy consequence of the Cauchy-Schwarz Inequality. □

REMARK 2.1. $i)$ If a function f is multiplicatively convex, then so is $x^{\alpha} f^{\beta}(x)$ (for all $\alpha \in \mathbb{R}$ and all $\beta > 0$).

$ii)$ If f is continuous, and one of the functions $f(x)^x$ and $f(e^{1/\log x})$ is multiplicatively convex, then so is the other.

REMARK 2.2. S. Saks [13] noticed that for a continuous function $f : I \to (0, \infty)$, $\log f(x)$ is a convex function of $\log x$ if, and only if, for every $\alpha > 0$ and every compact subinterval J of I, $x^{\alpha} f(x)$ should attain its maximum in J at one of the ends of J.

APPLICATIONS. Proposition 2.4 is the source of many interesting inequalities. Here are several elementary examples, obtained via Theorem 2.3:

$a)$ (See D. Mihet [9]). *If P is a polynomial with non-negative coefficients then*

$$P(x_1)\dots P(x_n) \geqslant \left(P(\sqrt[n]{x_1 \dots x_n})\right)^n \quad \text{for every } x_1, \dots, x_n \geqslant 0.$$

This inequality extends the classical inequality of Huygens (which corresponds to the case where $P(x) = 1 + x$) and complements a remark made by C. H. Kimberling [7] to Chebyshev's Inequality, namely,

$$(P(1))^{n-1} P(x_1 \dots x_n) \geqslant P(x_1) \dots P(x_n)$$

if all x_k are either in $[0, 1]$ or in $[1, \infty)$.

A similar conclusion is valid for every real analytic function as in Proposition 2.4 above.

$b)$ The $AM - GM$ Inequality is an easy consequence of the strict multiplicative convexity of e^x on $[0, \infty)$. A strengthened version of this will be presented in Section 5 below.

$c)$ Because $\dfrac{1+x}{1-x}$ is strictly multiplicatively convex on $(0, 1)$,

$$\prod_{k=1}^{n} \frac{1+x_k}{1-x_k} > \left(\frac{1+(\prod x_k)^{1/n}}{1-(\prod x_k)^{1/n}} \right)^n \qquad \text{for every } x_1, ..., x_n \in [0, 1)$$

unless $x_1 = ... = x_n$.

$d)$ Because $\arcsin$ is a strictly multiplicatively convex function on $(0, 1]$, in any triangle (excepts for the equilateral ones) the following inequality

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} < \left(\sin(\frac{1}{2}\sqrt[3]{ABC}\,) \right)^3$$

holds. That improves on a well known fact namely,

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} < \frac{1}{8}$$

unless $A = B = C$ (which is a consequence of the strict $\log -$ concavity of the function $\sin$). In a similar way one can argue that

$$\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} < \left(\sin(\frac{1}{2}\sqrt[3]{(\pi - A)\,(\pi - B)\,(\pi - C)}\,) \right)^3$$

unless $A = B = C.$

$e)$ As $\tan$ is a strictly multiplicatively convex function on $(0, \pi/2)$, in any triangle we have

$$\tan\frac{A}{2}\,\tan\frac{B}{2}\,\tan\frac{C}{2} > \left(\tan(\frac{1}{2}\sqrt[3]{ABC}) \right)^3$$

unless $A = B = C.$

The next example provides an application of Proposition 2.4 via Lemma 2.1:

$f)$ If $0 < a < b < c$ (or $0 < b < c < a$, or $0 < c < a < b$), then

$$P(a)^{\log c} P(b)^{\log a} P(c)^{\log b} > P(a)^{\log b} P(b)^{\log c} P(c)^{\log a}$$

for every polynomial P with non-negative coefficients and positive degree (and, more generally, for every strictly multiplicatively convex function). That complements the conclusion of the standard rearrangement inequalities (cf. [3], page 167): *If* $0 < a < b < c$, *and* $^\circ P > 0$, *then*

$$\begin{aligned} P(a)^{\log c} P(b)^{\log b} P(c)^{\log a} &= \inf_{\sigma} P(a)^{\log \sigma(a)} P(b)^{\log \sigma(b)} P(c)^{\log \sigma(c)} \\ P(a)^{\log a} P(b)^{\log b} P(c)^{\log c} &= \sup_{\sigma} P(a)^{\log \sigma(a)} P(b)^{\log \sigma(b)} P(c)^{\log \sigma(c)} \end{aligned}$$

where σ runs the set of all permutations of $\{a, b, c\}$.

The integral characterization of multiplicatively convex functions is another source of inequalities. We leave the (straightforward) details to the interested reader.

3. The analogue of Popoviciu's Inequality

The technique of majorization, which dominates the classical study of convex functions, can be easily adapted in the context of multiplicatively convex functions via the correspondence between the two classes of functions (as mentioned in the Introduction). We shall restrict here to the multiplicative analogue of a famous inequality due to Hardy, Littlewood and Polya [5]:

PROPOSITION 3.1. *Suppose that* $x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n$ *and* $y_1 \geqslant y_2 \geqslant \ldots \geqslant y_n$ *are two families of numbers in a subinterval* I *of* $(0,\infty)$ *such that*

$$\begin{aligned} x_1 &\geqslant y_1 \\ x_1x_2 &\geqslant y_1y_2 \\ &\ldots \\ x_1x_2\ldots x_{n-1} &\geqslant y_1y_2\ldots y_{n-1} \\ x_1x_2\ldots x_n &= y_1y_2\ldots y_n. \end{aligned}$$

Then

$$f(x_1)f(x_2)\ldots f(x_n) \geqslant f(y_1)f(y_2)\ldots f(y_n)$$

for every multiplicatively convex function $f : I \to (0,\infty)$.

A result due to H. Weyl [14] (see also [8], p. 231) gives us the basic example of a pair of sequences satisfying the hypothesis of Proposition 3.1: *Given any matrix* $A \in M_n(\mathbb{C})$ *having the eigenvalues* $\lambda_1, \ldots, \lambda_n$ *and the singular values* $s_1, \ldots, s_n$, *they can be rearranged such that*

$$\begin{aligned} |\lambda_1| &\geqslant \ldots \geqslant |\lambda_n|, \quad s_1 \geqslant \ldots \geqslant s_n \\ \left|\prod_{k=1}^{m} \lambda_k\right| &\leqslant \prod_{k=1}^{m} s_k \quad \text{for } k = 1, \ldots, n-1 \quad \text{and} \quad \left|\prod_{k=1}^{n} \lambda_k\right| = \prod_{k=1}^{n} s_k\,. \end{aligned}$$

Recall that the *singular values* of A are precisely the eigenvalues of its modulus, $|A| = (A^\star A)^{1/2}$. The spectral mapping theorem assures that $s_k = |\lambda_k|$ when A is self-adjoint. One could suppose that for an arbitrary matrix, $|\lambda_k| \leqslant s_k$ for all k. However, this is not true. A counter example is given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$$

whose eigenvalues are $\lambda_1 = 2 > \lambda_2 = -2$ and the singular values are $s_1 = 4 > s_2 = 1$.

As noticed A. Horn [6] (see also [8], p. 233), the converse of Weyl's aforementioned result is also true, i.e., all the families of numbers which fulfil the hypotheses of Proposition 3.1 come that way.

According to the above discussion, the following result holds:

PROPOSITION 3.2. *Let* $A \in M_n(\mathbb{C})$ *be any matrix having the eigenvalues* $\lambda_1, \ldots, \lambda_n$ *and the singular values* $s_1, \ldots, s_n$, *listed such that* $|\lambda_1| \geqslant \ldots \geqslant |\lambda_n|$ *and* $s_1 \geqslant \ldots \geqslant s_n$. *Then*

$$\prod_{k=1}^{n} f(s_k) \geqslant \prod_{k=1}^{n} f(|\lambda_k|)$$

for every multiplicatively convex function f *which is continuous on* $[0, \infty)$.

We shall give another application of Proposition 3.1, which seems to be new even for polynomials with non-negative coefficients:

THEOREM 3.3. (The multiplicative analogue of Popoviciu's Inequality [11]) . *Suppose that* $f : I \to (0, \infty)$ *is a multiplicatively convex function. Then*

$$f(x)\, f(y)\, f(z)\, f^3\left(\sqrt[3]{xyz}\right) \geqslant f^2\left(\sqrt{xy}\right) f^2\left(\sqrt{yz}\right) f^2\left(\sqrt{zx}\right)$$

for every $x, y, z \in I$. *Moreover, for the strictly multiplicatively convex functions the equality occurs only when* $x = y = z$.

Proof. Without loss of generality we may assume that $x \geqslant y \geqslant z$. Then

$$\sqrt{xy} \geqslant \sqrt{zx} \geqslant \sqrt{yz} \quad \text{and} \quad x \geqslant \sqrt[3]{xyz} \geqslant z.$$

If $x \geqslant \sqrt[3]{xyz} \geqslant y \geqslant z$, the desired conclusion follows from Proposition 3.1 applied to

$$\begin{aligned} x_1 &= x, \quad x_2 = x_3 = x_4 = \sqrt[3]{xyz}, \quad x_5 = y, \quad x_6 = z \\ y_1 &= y_2 = \sqrt{xy}, \quad y_3 = y_4 = \sqrt{xz}, \quad y_5 = y_6 = \sqrt{yz} \end{aligned}$$

while in the case $x \geqslant y \geqslant \sqrt[3]{xyz} \geqslant z$, we have to consider

$$\begin{aligned} x_1 &= x, \quad x_2 = y, \quad x_3 = x_4 = x_5 = \sqrt[3]{xyz}, \quad x_6 = z \\ y_1 &= y_2 = \sqrt{xy}, \quad y_3 = y_4 = \sqrt{xz}, \quad y_5 = y_6 = \sqrt{yz}\,. \end{aligned} \qquad \square$$

According to Theorem 3.3 (applied to $f(x) = e^x$), for every $x, y, z > 0$ we have

$$\frac{x+y+z}{3} + \sqrt[3]{xyz} > \frac{2}{3}\left(\sqrt{xy} + \sqrt{yz} + \sqrt{zx}\right)$$

unless $x = y = z$.

Other homogeneous inequalities can be obtained by extending Proposition 3.3 to longer sequences and/or to more general convex combinations.

4. Multiplicative convexity of special functions

We start this section by recalling the following result:

PROPOSITION 4.1. (P. Montel [10]) *Let* $f : [0, a) \to [0, \infty)$ *be a continuous function, which is multiplicatively convex on* $(0, a)$. *Then*

$$F(x) = \int_0^x f(t)\, dt$$

is also continuous on $[0, a)$ *and multiplicatively convex on* $(0, a)$.

Proof. Montel's original argument was based on the fact that under the presence of continuity, f is multiplicatively convex if, and only if,

$$2f(x) \leqslant k^{\alpha} f(kx) + k^{-\alpha} f(x/k),$$

for every $x \in I$ and every $k > 0$ such that kx and x/k both belong to I.

Actually, due to the continuity of F, it suffices to show that

$$(F(\sqrt{xy}))^2 \leqslant F(x)\, F(y) \text{ for every } x, y \in [0, a),$$

which is a consequence of the corresponding inequality at the level of integral sums,

$$\left[\frac{\sqrt{xy}}{n} \sum_{k=0}^{n-1} f\left(k\,\frac{\sqrt{xy}}{n}\right)\right]^2 \leqslant \left[\frac{x}{n} \sum_{k=0}^{n-1} f\left(k\,\frac{x}{n}\right)\right] \left[\frac{y}{n} \sum_{k=0}^{n-1} f\left(k\,\frac{y}{n}\right)\right]$$

i.e., of

$$\left[\sum_{k=0}^{n-1} f\left(k\,\frac{\sqrt{xy}}{n}\right)\right]^2 \leqslant \left[\sum_{k=0}^{n-1} f\left(k\,\frac{x}{n}\right)\right] \left[\sum_{k=0}^{n-1} f\left(k\,\frac{y}{n}\right)\right].$$

To see that the latter inequality holds, notice that

$$\left[\, f\left(k\,\frac{\sqrt{xy}}{n}\right)\right]^2 \leqslant \left[\, f\left(k\,\frac{x}{n}\right)\right] \left[f\left(k\,\frac{y}{n}\right)\right]$$

and then apply the Cauchy-Schwarz Inequality. $\square$

As tan is continuous on $[0, \pi/2)$ and multiplicatively convex on $(0, \pi/2)$, a repeated application of Proposition 4.1 shows us that the *Lobacevski's function*,

$$\mathrm{L}(x) = -\int_0^x \log\cos t \; dt$$

is multiplicatively convex on $(0, \pi/2)$.

Starting with $\frac{t}{\sin t}$ and then switching to $\frac{\sin t}{t}$, which is multiplicatively concave, a similar argument leads us to the fact that the *integral sine*,

$$\mathrm{Si}(x) = \int_0^x \frac{\sin\, t}{t}\; dt\,,$$

is multiplicatively concave on $(0, \pi/2)$.

Another striking example is the following:

PROPOSITION 4.2. Γ *is a strictly multiplicatively convex function on* $[1, \infty)$.

Proof. In fact, $\log\Gamma(1 + x)$ is strictly convex and increasing on $(1, \infty)$. Or, an increasing strictly convex function of a strictly convex function is strictly convex too. So, $F(x) = \log\Gamma(1 + e^x)$ is strictly convex on $(0, \infty)$ and thus

$$\Gamma(1 + x) = e^{F(\log x)}$$

is strictly multiplicatively convex on $[1, \infty)$. As $\Gamma(1+x) = x\Gamma(x)$, we conclude that Γ itself is strictly multiplicatively convex on $[1, \infty)$. $\square$

According to Proposition 4.2,

$$\Gamma^3(\sqrt[3]{xyz}) < \Gamma(x)\,\Gamma(y)\,\Gamma(z) \quad \text{for every } x,\ y,\ z \geqslant 1$$

except the case where $x = y = z$.

On the other hand, by Theorem 3.3, we infer that

$$\Gamma(x)\,\Gamma(y)\,\Gamma(z)\,\Gamma^3\left(\sqrt[3]{xyz}\right) \geqslant \Gamma^2\left(\sqrt{xy}\right)\Gamma^2\left(\sqrt{yz}\right)\Gamma^2\left(\sqrt{zx}\right)$$

for every $x,\ y,\ z \geqslant 1$; the equality occurs only for $x = y = z$.

Probably, the last two inequalities work in the reversed form when $x,\ y,\ z \in (0, 1]$, but at the moment we are unable to prove that.

Another application of Proposition 4.2 is the fact that the function $\frac{\Gamma(2x+1)}{\Gamma(x+1)}$ *is strictly multiplicatively convex on* $[1, \infty)$. In fact, it suffices to recall the *Gauss-Legendre duplication formula,*

$$\frac{\Gamma(2x+1)}{\Gamma(x+1)} = \frac{2^{2x}\,\Gamma(x+1/2)}{\sqrt{\pi}}\,.$$

In order to present further inequalities involving the gamma function we shall need the following criteria of multiplicative convexity for differentiable functions:

PROPOSITION 4.3. *Let* $f : I \to (0, \infty)$ *be a differentiable function defined on a subinterval of* $(0, \infty)$. *Then the following assertions are equivalent:*

$i)$ f *is multiplicatively convex*;

$ii)$ *The function* $\frac{xf'(x)}{f(x)}$ *is nondecreasing*;

$iii)$ f *verifies the inequality*

$$\frac{f(x)}{f(y)} \geqslant \left(\frac{x}{y}\right)^{y\cdot f'(y)\,/\,f(y)} \qquad \text{for every} \quad x, y \in I.$$

If moreover f *is twice differentiable, then* f *is multiplicatively convex if, and only if,*

$$x[f(x)f''(x) - f'^2(x)] + f(x)f'(x) \geqslant 0 \quad \text{for every} \quad x > 0.$$

The corresponding variants for the strictly multiplicatively convex functions also work.

Proof. In fact, according to a remark in the Introduction, a function $f : I \to (0, \infty)$ is multiplicatively convex if, and only if, the function $F : \log(I) \to \mathbb{R}$, $F(x) = \log f(e^x)$, is convex. Taking into account that the differentiability is preserved under the above correspondence, the statement to be proved is just a translation of the usual criteria of convexity (as known in the differentiability framework) into criteria of multiplicative convexity. $\square$

Directly related to the gamma function is the *psi function,*

$$\operatorname{Psi}(x) = \frac{d}{dx}\,\log\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}\,, \quad x > 0$$

also known as the digamma function. It satisfies the functional equation $\psi(x+1) = \psi(x) + \frac{1}{x}$ and can be also be represented as

$$\operatorname{Psi}(x) = -\gamma - \int_0^1 \frac{t^{x-1}-1}{1-t}\,dt,$$

where $\gamma = .57722$ is Euler's constant. See $[4]$.

By combining Propositions 4.2 and 4.3 above, we obtain the inequality

$$\frac{\Gamma(x)}{\Gamma(y)} \geqslant \left(\frac{x}{y}\right)^{y\cdot\operatorname{Psi}(y)} \quad \text{for every} \quad x, y \geqslant 1, \tag{Psi}$$

as well as the fact that $x\operatorname{Psi}(x)$ is increasing for $x \geqslant 1$.

The latter inequality can be used to estimate Γ from below on $[1,2]$. The interest comes from the fact that Γ is convex and attains its global minimum in that interval because $\Gamma(1) = \Gamma(2)$; more precisely, the minimum is attained near 1.46. Taking $y = 1$ and then $y = 3/2$ in (Psi), we get

$$\Gamma(x) \geqslant \max\left\{x^{-\gamma},\ \frac{1}{2}\sqrt{\pi}\,\left(\frac{2x}{3}\right)^{3/2\,(2-\gamma-2\ln 2)}\right\} \quad \text{for every } x \in [1,2].$$

5. An estimate of the AM-GM Inequality

Suppose that I is a subinterval of $(0,\infty)$ and that $f : I \to (0,\infty)$ is a twice differentiable function. We are interested to determine for what values $\alpha \in \mathbb{R}$ the function

$$\varphi(x) = f(x) \cdot x^{(-\alpha/2)\ \log x}$$

is multiplicatively convex on I, equivalently, for what values $\alpha \in \mathbb{R}$ the function

$$\Phi(x) = \log\varphi(e^x) = \log f(e^x) - \frac{\alpha x^2}{2},$$

is convex on $\log(I)$. By using the fact that the convexity of a twice differentiable function Φ is equivalent to $\Phi'' \geqslant 0$, we get a quick answer to the aforementioned problem:

$$\alpha \leqslant A(f),$$

where

$$\begin{aligned} A(f) &= \inf_{x\,\in\,\log(I)} \frac{d^2}{dx^2}\log f(e^x) = \\ &= \inf_{x\,\in\,\log(I)} \frac{x^2(f(x)f''(x) - (f'(x))^2) + xf(x)\,f'(x)}{f(x)^2}. \end{aligned}$$

By considering also

$$B(f) = \sup_{x\,\in\,\log(I)} \frac{d^2}{dx^2}\log f(e^x),$$

we arrive at the following result: *Under the above hypotheses,*

$$\exp\left(\frac{A(f)}{2n^2}\sum\nolimits_{j<k}(\log x_j-\log x_k)^2\right) \leqslant \left(\prod_{k=1}^{n} f(x_k)\right)^{1/n} \Big/ f\left(\left(\prod_{k=1}^{n} x_k\right)^{1/n}\right)$$
$$\leqslant \exp\left(\frac{B(f)}{2n^2}\sum\nolimits_{j<k}(\log x_j-\log x_k)^2\right)$$

for every $x_1,\ldots,x_n\in I$.

Particularly, for $f(x)=e^x$, $x\in[A,B]$ (where $0<A\leqslant B$), we have $A(f)=A$ and $B(f)=B$ and we are led to the following improvement upon the $AM-GM$ Inequality:

THEOREM 5.1. *Suppose that* $0<A\leqslant B$. *Then*

$$\frac{A}{2n^2}\sum\nolimits_{j<k}(\log x_j-\log x_k)^2 \leqslant \frac{1}{n}\sum x_k-\left(\prod_{k=1}^{n} x_k\right)^{1/n}$$
$$\leqslant \frac{B}{2n^2}\sum\nolimits_{j<k}(\log x_j-\log x_k)^2$$

for every $x_1,\ldots,x_n\in[A,B]$.

As
$$\frac{1}{2n^2}\sum\nolimits_{j<k}(\log x_j-\log x_k)^2$$
represents the variance of the random variable whose distribution is

$$\begin{pmatrix} \log x_1 & \log x_2 & \ldots & \log x_k \\ 1/n & 1/n & \ldots & 1/n \end{pmatrix},$$

Theorem 5.1 reveals the probabilistic character of the $AM-GM$ Inequality. Using the technique of approximating the integrable functions by step functions, one can immediately derive from Theorem 5.1 the following more general result:

THEOREM 5.2. *Let* (Ω,Σ,P) *be a probability space and let* X *be a random variable on this space, taking values in the interval* $[A,B]$, *where* $0<A\leqslant B$. *Then*

$$A\leqslant\frac{M(X)-e^{M(\log X)}}{D^2(\log X)}\leqslant B.$$

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Constantin P. Niculescu
University of Craiova
Department of Mathematics
Street A. I. Cuza 13
Craiova 1100
Romania e-mail: `tempus@oltenia.ro`