Convexity and Steinitz's Exchange Property*

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Abstract

"Convex analysis" is developed for functions defined on integer lattice points. We investigate the class of functions which enjoy a variant of Steinitz's exchange property. It includes linear functions on matroids, valuations on matroids (in the sense of Dress and Wenzel), and separable concave functions on the integral base polytope of submodular systems. It is shown that a function ω has the Steinitz exchange property if and only if it can be extended to a concave function $\overline{\omega}$ such that the maximizers of ($\overline{\omega}$+any linear function) form an integral base polytope. A Fenchel-type min-max theorem and discrete separation theorems are established which imply, as immediate consequences, Frank's discrete separation theorem for submodular functions, Edmonds' intersection theorem, Fujishige's Fenchel-type min-max theorem for submodular functions, and also Frank's weight splitting theorem for weighted matroid intersection.

Keywords: Combinatorial optimization, Steinitz's exchange property, convex analysis, valuated matroid.

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1 Introduction

The theory of convex/concave functions (Rockafellar [36], Stoer-Witzgall [40]) has played the core role in the field of nonlinear optimization as well as in other fields of mathematical sciences. Convex/concave functions are computationally tractable by virtue of the following two facts:

- Global optimality is guaranteed by local optimality. Hence myopic (or greedy) strategies work for minimizing/maximizing the function value.
- Strong duality holds for a pair of convex and concave functions. Computationally, this guarantees the existence of a certificate (evidence) for the optimality in terms of the dual variable.

The theory of matroids (Welsh [42], White [44]) has played a similar role in the field of combinatorial optimization. It has successfully captured the combinatorial essence underlying the well-solved class of optimization problems such as those on graphs and networks (cf., e.g., Lawler [24]). Efficient algorithms are known for the optimization problems on matroids such as (i) the problem of optimizing a linear objective function over a single matroid and (ii) the problem of optimizing a linear objective function over the intersection of two matroids ("weighted matroid intersection problem"). The tractability of these problems relies on the following facts:

- Global optimality is guaranteed by local optimality, and moreover, the so-called greedy algorithm works for the problem (i).
- A duality theorem, Edmonds' intersection theorem, guarantees the existence of a certificate for the optimality for the problem (ii) in terms of the dual variable.

The polyhedral approach of Edmonds [9] recognizes a combinatorial optimization problem as a linear programming problem with an extra constraint of integrality. With the combinatorial optimization problem is associated a polyhedron, the convex hull of the relevant incidence vectors, over which the linear objective function is maximized. The polyhedron (convex hull) is described by a system of linear inequalities, that is, it is expressed as the intersection of halfspaces rather than as the convex combinations of the vertices.

The polyhedral approach to matroid optimization, emphasizing faces rather than vertices of the polyhedron, has evolved to the theory of submodular/supermodular functions (Edmonds-Giles [11], Frank [14], Fujishige [19], Schrijver [38]), where a set function $f : 2^V \to \mathbf{R}$ is called submodular if

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y) \qquad (X, Y \subseteq V),$$

and supermodular if $-f$ is submodular. In particular, the matroid intersection problem has been extended to the polymatroid intersection problem (Welsh [42]) and further to the independent flow problem (Fujishige [17]) and the submodular flow problem (Edmonds-Giles [11], Frank [15], Frank-Tardos [16]).

The analogy between convex/concave functions and submodular/supermodular functions has attracted research interest. Fujishige [18] formulates Edmonds' intersection theorem into a Fenchel-type min-max duality theorem and considers further analogy such as subgradients. Frank [14] shows a separation theorem for a pair of submodular/supermodular functions, with integrality assertion for the separating hyperplane in the case of integer-valued functions. This theorem can also be regarded as being equivalent to Edmonds' intersection theorem. A precise statement, beyond analogy, about the relationship between convex functions and submodular functions is made by Lovász [25]. Namely, a set function is submodular if and only if the so-called Lovász extension of that function is convex (see also [19]). This penetrating remark also establishes a direct link between duality for convex/concave functions and duality for submodular/supermodular functions. The essence of the duality principle for submodular/supermodular functions is now recognized as the discreteness (integrality) assertion in addition to the duality for convex/concave functions.

In spite of the developments outlined above, our understanding of the relationship between convexity and submodularity seems to be only partial. In convex analysis, a convex function is minimized over a convex domain of definition which can be described by a system of inequalities in (other) convex functions. In the polyhedral approach to matroid optimization, a linear function is optimized over a (discrete) domain of definition which is described by a system of inequalities involving submodular functions. The relationship between convexity and submodularity we have understood so far is concerned only with the domain of definitions and not with the objective functions. In the literature, however, we can find a number of results on the optimization of nonlinear functions over the base polytope of a submodular system. In particular, the minimization of a separable convex function over such a base polytope has been considered by Fujishige [19] and Groenevelt [20], and the submodular flow problem with a separable convex objective function has been treated by Fujishige [19]. Our present knowledge does not help us understand this result in relation to convex analysis.

Remark 1.1 It may be in order here to mention that the minimization of a submodular function is of primary importance in combinatorial optimization (see Grötschel-Lovász-Schrijver [21]), but this does not seem relevant in the present context. □

Quite independently of these developments in the theory of submodular functions, Dress and Wenzel [5], [8] have recently introduced *valuated matroids*, a quantitative generalization of matroids. A matroid $(V, \mathcal{B})$, defined in terms of its family of bases $\mathcal{B} \subseteq 2^V$, is characterized by the Steinitz exchange property:

> For $X, Y \in \mathcal{B}$ and $u \in X - Y$, there exists $v \in Y - X$ such that $X - u + v \in \mathcal{B}$.

It is well known that this implies a simultaneous exchange property:

> For $X, Y \in \mathcal{B}$ and $u \in X - Y$, there exists $v \in Y - X$ such that $X - u + v \in \mathcal{B}$ and $Y + u - v \in \mathcal{B}$.

A valuation of $(V, \mathcal{B})$ is a function $\omega : \mathcal{B} \to \mathbf{R}$ which enjoys the quantitative extension of the Steinitz exchange property:

(MV) For $X, Y \in \mathcal{B}$ and $u \in X - Y$, there exists $v \in Y - X$ such that $X - u + v \in \mathcal{B}$, $Y + u - v \in \mathcal{B}$ and

$$\omega(X) + \omega(Y) \leq \omega(X - u + v) + \omega(Y + u - v).$$

A matroid equipped with a valuation is called a valuated matroid.

It has turned out recently that valuated matroids afford a nice combinatorial framework to which the optimization algorithms established for matroids generalize naturally. Variants of greedy algorithms work for maximizing a matroid valuation, as has been shown by Dress-Wenzel [5] as well as by Dress-Terhalle [2], [3], [4] and Murota [27]. (These greedy-type algorithms are similar in the vein to, but not the same as, those in Korte-Lovász-Schrader [23].) The weighted matroid intersection problem has been extended by Murota [28], [29] to the valuated matroid intersection problem. The optimality criteria and algorithms for the weighted matroid intersection problem have been generalized for the valuated matroid intersection problem.

This direction of research can be extended further as follows [32]. Let us say that $B \subseteq \mathbf{Z}^V$ is an integral base set if it is a nonempty set that satisfies:

(B1) For $x, y \in B$ and for $u \in \text{supp}^+(x-y)$, there exists $v \in \text{supp}^-(x-y)$ such that $x - \chi_u + \chi_v \in B$,

where $\text{supp}^+(x-y) := \{u \in V \mid x(u) > y(u)\}$, $\text{supp}^-(x-y) := \{v \in V \mid x(v) < y(v)\}$ and χ_u denotes the characteristic vector of $u \in V$. We then consider a function $\omega : B \to \mathbf{R}$ on a finite integral base set B such that:

(EXC) For $x, y \in B$ and $u \in \text{supp}^+(x-y)$, there exists $v \in \text{supp}^-(x-y)$ such that $x - \chi_u + \chi_v \in B$, $y + \chi_u - \chi_v \in B$ and

$$\omega(x) + \omega(y) \leq \omega(x - \chi_u + \chi_v) + \omega(y + \chi_u - \chi_v).$$

We call such ω an M-concave function, where M stands for Matroid. As will be illustrated in Section 2, M-concave functions arise naturally in the context of combinatorial optimization.

In a sense to be made precise later in Theorem 2.1, the exchange property (B1) is equivalent to submodularity. With the correspondence between convexity and submodularity in mind, we may then say that (B1) prescribes a certain "convexity" of the domain of definition of the function ω. The main theme of this paper is to demonstrate that the exchange property (EXC) can be interpreted as "concavity" of the objective function in the context of combinatorial optimization. The three central questions considered in this paper are the following:

- We know a pair of "conjugate" characterizations of the base polytope of a submodular system, namely, the exchange property (B1) for the points in the polytope and the submodularity for (the inequalities describing) the faces of the polytope. The property (EXC) is a quantitative generalization of (B1). Then what is the generalization of submodularity that corresponds to (EXC)?

$$\begin{array}{ccc} \text{[Domain]} & & \text{[Function]} \\ \text{(B1)} & \Longrightarrow & \text{(EXC)} \\ \Updownarrow & & \Updownarrow \\ \text{Submodularity} & \Longrightarrow & \textbf{?} \end{array} \tag{1.1}$$

  An answer is given in Theorem 5.3.

- Can an M-concave function be extended to a concave function in the usual sense, just as a submodular function can be extended to a convex function through the Lovász extension? Theorem 4.6 answers this question affirmatively.

- Is there any duality for M-convex/M-concave functions that corresponds to the duality for convex/concave functions? The main concern here will be the discreteness (integrality) assertion for a pair of such functions which are integer-valued. We answer this in the affirmative in Section 6 by extending the approach of Murota [30] for matroid valuations. To be specific, this amounts to a generalization of the optimality criteria for the weighted matroid intersection problem and its variants and extensions such as the potential characterization of the optimality due to Iri-Tomizawa [22] and Fujishige [17], and the weight splitting theorem of Frank [13].

2 Functions with the Exchange Property

2.1 Definitions

Let V be a finite nonempty set and $\mathbf{R}$ be the set of real numbers. For $u \in V$ we denote by χ_u its characteristic vector, i.e., $\chi_u = (\chi_u(v) \mid v \in V) \in \mathbf{Z}^V$ with $\chi_u(v) := 1$ if $v = u$ and $\chi_u(v) := 0$ otherwise. For $x = (x(v) \mid v \in V) \in \mathbf{R}^V$, $y = (y(v) \mid v \in V) \in \mathbf{R}^V$ we define

$$\mathrm{supp}^+(x) := \{v \in V \mid x(v) > 0\}, \qquad \mathrm{supp}^-(x) := \{v \in V \mid x(v) < 0\},$$

$$x(X) := \sum\{x(v) \mid v \in X\} \qquad (X \subseteq V),$$

$$||x|| := \sum\{|x(v)| \mid v \in V\}, \qquad \langle x, y\rangle := \sum\{x(v)y(v) \mid v \in V\}.$$

Let $B \subseteq \mathbf{Z}^V$ be a finite integral base set, i.e., a finite nonempty set such that

(B1) For $x, y \in B$ and for $u \in \mathrm{supp}^+(x-y)$, there exists $v \in \mathrm{supp}^-(x-y)$ such that $x - \chi_u + \chi_v \in B$.

As is well known, (B1) is equivalent to the simultaneous exchange property

(B2) For $x, y \in B$ and for $u \in \mathrm{supp}^+(x-y)$, there exists $v \in \mathrm{supp}^-(x-y)$ such that $x - \chi_u + \chi_v \in B$ and $y + \chi_u - \chi_v \in B$.

Note that — in view of $||x|| = ||x - \chi_u + \chi_v||$ and $||(x - \chi_u + \chi_v) - y|| < ||x - y||$ for $u \in \mathrm{supp}^+(x-y)$ and $v \in \mathrm{supp}^-(x-y)$ — (B1) implies $x(V) = y(V)$ for $x, y \in B$.

The following theorem is known as a folklore (according to private communications from W. Cunningham and S. Fujishige; see also [1], [41], [42] in this connection). Recall that a function $f : 2^V \to \mathbf{R}$ is said to be submodular if

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y) \qquad (X, Y \subseteq V), \tag{2.1}$$

and $g : 2^V \to \mathbf{R}$ is supermodular if

$$g(X) + g(Y) \leq g(X \cup Y) + g(X \cap Y) \qquad (X, Y \subseteq V). \tag{2.2}$$

Theorem 2.1 *For a finite nonempty set $B \subseteq \mathbf{Z}^V$, the following three conditions are equivalent.*

(a) *B satisfies* (B1).

(b) *There exists an integer-valued submodular function $f : 2^V \to \mathbf{Z}$ with $f(\emptyset) = 0$ such that*

$$B = \mathbf{Z}^V \cap \{x \in \mathbf{R}^V \mid x(X) \leq f(X)\ (\forall X \subset V), x(V) = f(V)\}.$$

(c) *There exists an integer-valued supermodular function* $g : 2^V \to \mathbf{Z}$ *with* $g(\emptyset) = 0$ *such that*

$$B = \mathbf{Z}^V \cap \{x \in \mathbf{R}^V \mid x(X) \geq g(X) \ (\forall X \subset V), x(V) = g(V)\}.$$

Moreover, the functions f *and* g *are given by*

$$f(X) = \max\{x(X) \mid x \in B\}, \qquad g(X) = \min\{x(X) \mid x \in B\}.$$

□

This theorem allows us to say that we assume B to be the integral points of an integral base polytope, where an integral base polytope means the base polytope of an integral submodular/supermodular system [19]. Note that we have

$$B = \mathbf{Z}^V \cap \overline{B}, \tag{2.3}$$

where $\overline{B}$ denotes the convex hull of B.

In this paper we are concerned with a function $\omega : B \to \mathbf{R}$ that satisfies the following variant of Steinitz's exchange property:

(EXC) For $x, y \in B$ and $u \in \mathrm{supp}^+(x-y)$, there exists $v \in \mathrm{supp}^-(x-y)$ such that $x - \chi_u + \chi_v \in B$, $y + \chi_u - \chi_v \in B$ and

$$\omega(x) + \omega(y) \leq \omega(x - \chi_u + \chi_v) + \omega(y + \chi_u - \chi_v). \tag{2.4}$$

Using the notation

$$\omega(x, u, v) := \omega(x - \chi_u + \chi_v) - \omega(x), \tag{2.5}$$

which represents the local behavior of ω around x, we can rewrite (2.4) to

$$\omega(x, u, v) + \omega(y, v, u) \geq 0. \tag{2.6}$$

We often use the convention $\omega(x) := -\infty$ for $x \notin B$. Note that a pair (B, ω) of a nonempty set $B \subseteq \mathbf{Z}^V$ and a function $\omega : B \to \mathbf{R}$ satisfies (B2) and (EXC) if and only if the associated extended function $\omega : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ satisfies

(EXC′) For $x, y \in \mathbf{Z}^V$ with $\omega(x) \neq -\infty$ and $\omega(y) \neq -\infty$, and for $u \in \mathrm{supp}^+(x - y)$, there exists $v \in \mathrm{supp}^-(x-y)$ such that

$$\omega(x) + \omega(y) \leq \omega(x - \chi_u + \chi_v) + \omega(y + \chi_u - \chi_v).$$

2.2 Examples

We discuss a number of natural classes of M-concave functions.

Example 2.1 (Affine function) For $\eta : V \to \mathbf{R}$ and $\alpha \in \mathbf{R}$, the function $\omega : B \to \mathbf{R}$ defined by

$$\omega(x) := \alpha + \langle \eta, x \rangle \qquad (x \in B)$$

satisfies the exchange property (EXC) with equality in (2.4). This is an immediate consequence of the simultaneous exchange property (B2). □

Example 2.2 (Separable concave function) We call $g : \mathbf{Z} \to \mathbf{R}$ concave if its piecewise linear extension $\hat{g} : \mathbf{R} \to \mathbf{R}$ is a concave function, that is, if $g(t-1) + g(t+1) \leq 2g(t)$ holds for all $t \in \mathbf{Z}$. For a family of concave functions $g_v : \mathbf{Z} \to \mathbf{R}$ indexed by $v \in V$, the (separable concave) function $\omega : B \to \mathbf{R}$ defined by

$$\omega(x) := \sum\{g_v(x(v)) \mid v \in V\} \qquad (x \in B)$$

satisfies the exchange property (EXC). See [32] for a proof. □

Example 2.3 (Min-cost flow) Let $G = (V, A)$ be a directed graph with vertex set V and arc set A. Assume further that we are given an upper capacity function $\overline{c} : A \to \mathbf{Z}$ and a lower capacity function $\underline{c} : A \to \mathbf{Z}$. A feasible (integral) flow φ is a function $\varphi : A \to \mathbf{Z}$ such that $\underline{c}(a) \leq \varphi(a) \leq \overline{c}(a)$ for each $a \in A$. Its boundary $\partial\varphi : V \to \mathbf{Z}$ is defined by

$$\partial\varphi(v) := \sum\{\varphi(a) \mid a \in \delta^+ v\} - \sum\{\varphi(a) \mid a \in \delta^- v\},$$

where $\delta^+ v$ and $\delta^- v$ denote the sets of the out-going and in-coming arcs incident to v, respectively. Then

$$B := \{\partial\varphi \mid \varphi : \text{ feasible flow }\}$$

is known to satisfy (B1). See, e.g., [19].

Suppose further that we are given a family of convex functions $f_a : \mathbf{Z} \to \mathbf{R}$ indexed by $a \in A$, where we call $f : \mathbf{Z} \to \mathbf{R}$ convex if its piecewise linear extension $\check{f} : \mathbf{R} \to \mathbf{R}$ is a convex function, that is, if $f(t-1) + f(t+1) \geq 2f(t)$ holds for all $t \in \mathbf{Z}$. Define $\Gamma(\varphi) := \sum\{f_a(\varphi(a)) \mid a \in A\}$. Then the function $\omega : B \to \mathbf{R}$ defined by

$$\omega(x) := -\min\{\Gamma(\varphi) \mid \varphi : \text{ feasible flow with } \partial\varphi = x\} \qquad (x \in B)$$

satisfies the exchange property (EXC) (see [32] for a proof). In general, this construction yields a nonseparable function ω (see [28, Example 3.3] for a concrete instance). □

Example 2.4 (Determinant) Let $A(t)$ be an $m \times n$ matrix of rank m with each entry being a polynomial in a variable t, and let $\mathbf{M} = (V, \mathcal{B})$ denote the (linear) matroid defined on the column set V of $A(t)$ by linear independence of the column vectors, where $J \subseteq V$ belongs to $\mathcal{B}$ if and only if $|J| = m$ and the column vectors with indices in J are linearly independent. Let B be the set of the incidence vectors of the bases (the members of $\mathcal{B}$). Then $\omega : B \to \mathbf{Z}$ defined by

$$\omega(\chi_J) := \deg_t \det A[J] \qquad (J \in \mathcal{B})$$

satisfies (EXC), where χ_J is the incidence vector of J and $A[J]$ denotes the $m \times m$ submatrix with column indices in J. In fact [5], [8], the Grassmann-Plücker identity implies the exchange property of ω. □

Remark 2.1 In connection with the construction in Example 2.2, it is worth while to mention that a general concave function on $\mathbf{R}^V$ (or on a base polytope over $\mathbf{R}$) does not necessarily satisfy (EXC) when restricted to $\mathbf{Z}^V$. See Remark 2.2 of [32] for a concrete instance. In fact, it is one of the main objectives of this paper to identify the precise relationship between Steinitz's exchange property and concavity, which will be stated as Theorem 4.6. □

2.3 Fundamental properties

In this section we mention some consequences of (EXC) that have been used in [32]. We emphasize the analogy to concave functions. We assume that $\omega : B \to \mathbf{R}$ satisfies (EXC).

For $p : V \to \mathbf{R}$ we define $\omega[p] : B \to \mathbf{R}$ by

$$\omega[p](x) := \omega(x) + \langle p, x \rangle. \tag{2.7}$$

Just as a concave function remains concave when a linear function is added, we have the following theorem. The proof is easy.

Theorem 2.2 *$\omega[p]$ satisfies* (EXC)*.* □

For a concave function g, we have the subgradient inequality [36]:

$$g(y) \leq g(x) + \langle \nabla g(x), y - x \rangle,$$

where $\nabla g(x) \in \mathbf{R}^V$ denotes a subgradient of g at x. As a counterpart for ω, we have the "upper-bound lemma" described as follows. For $x, y \in B$ we consider a bipartite graph $G(x,y)$, which has $(V^+, V^-) := (\mathrm{supp}^+(x-y), \mathrm{supp}^-(x-y))$ as its vertex bipartition and

$$\hat{A} := \{(u,v) \mid u \in V^+, v \in V^-, x - \chi_u + \chi_v \in B\}$$

as its arc set. Each arc (u,v) is associated with "arc weight" $\omega(x,u,v)$ of (2.5). We define

$$\begin{aligned} \widehat{\omega}(x,y) \ := \ \max \Bigg\{ & \sum_{(u,v)\in\hat{A}} \omega(x,u,v)\lambda(u,v) \ \Bigg| \\ & \lambda(u,v) \geq 0 \qquad ((u,v) \in \hat{A}), \\ & \sum_{v:(u,v)\in\hat{A}} \lambda(u,v) = x(u) - y(u) \qquad (u \in V^+), \\ & \sum_{u:(u,v)\in\hat{A}} \lambda(u,v) = y(v) - x(v) \qquad (v \in V^-) \Bigg\}. \end{aligned} \tag{2.8}$$

It is known [19, Theorem 3.28] that such $\lambda : \hat{A} \to \mathbf{R}$ exist and that $\widehat{\omega}(x,y)$ is a well-defined finite real number. It may be mentioned that the maximization in (2.8) can be identified as a transportation problem [24]. The "upper-bound lemma" reads as follows.

Theorem 2.3 ([32, Lemma 2.4]) *For $x, y \in B$ we have*

$$\omega(y) \leq \omega(x) + \widehat{\omega}(x,y). \tag{2.9}$$

(Proof) See [32] or [28, Lemma 3.4]. □

This yields the following theorem, stating that local optimality implies global optimality. This is a straightforward extension of a similar result of [5], [8] for a matroid valuation. The analogy to concave functions should be obvious.

Theorem 2.4 ([32]) *Let $x \in B$. Then $\omega(x) \geq \omega(y)$ for all $y \in B$ if and only if*

$$\omega(x,u,v) \leq 0 \qquad (\forall\ u, v \in V). \tag{2.10}$$

(By convention, $\omega(x,u,v) := -\infty$ if $x - \chi_u + \chi_v \notin B$.) □

For $y, z \in \mathbf{Z}^V$ we define

$$B^y := \{x \in B \mid x \leq y\}, \qquad B_z := \{x \in B \mid x \geq z\}, \tag{2.11}$$

called the reduction of B by y and the contraction of B by z respectively. They satisfy (B1). Let $\omega^y : B^y \to \mathbf{R}$ (resp. $\omega_z : B_z \to \mathbf{R}$) be the restriction of ω to B^y (resp. B_z), provided that $B^y \neq \emptyset$ (resp. $B_z \neq \emptyset$).

Lemma 2.5 ω^y *and* ω_z *satisfy* (EXC). □

The above lemma shows that M-concave functions defined on B naturally induce such functions on the reduction (resp. contraction) by a superbase y (resp. subbase z). In Section 6.4, we will see in Theorem 6.10 that an M-concave function can be induced on the sum of two integral base polytopes through the "convolution" operation.

3 Local Exchange Property

We will show that the exchangeability condition (EXC) is in fact a local property, though its definition refers globally to all pairs (x, y). Namely, we may impose exchangeability only on neighboring pairs (x, y). This may be compared to a similar phenomenon for concavity, which is generally defined by a global property but which can also be characterized in local terms (e.g., in terms of the second order derivative).

The following theorem claims that — assuming (B1) for B — the exchange property (EXC) is equivalent to a seemingly weaker local exchange property

(EXC$_{\rm loc}$) For $x, y \in B$ with $||x - y|| = 4$ there exist $u \in \mathrm{supp}^+(x - y)$ and $v \in \mathrm{supp}^-(x - y)$ such that $x - \chi_u + \chi_v \in B$, $y + \chi_u - \chi_v \in B$ and

$$\omega(x) + \omega(y) \leq \omega(x - \chi_u + \chi_v) + \omega(y + \chi_u - \chi_v) \tag{3.1}$$

(see [7, Theorem 3.4] for a similar statement relating to matroid valuations).

Theorem 3.1 *Let $\omega : B \to \mathbf{R}$ be a function defined on a finite integral base set $B \subseteq \mathbf{Z}^V$. Then ω satisfies* (EXC) *if and only if it satisfies* (EXC$_{\rm loc}$). □

We prove (EXC$_{\rm loc}$) $\Longrightarrow$ (EXC). For $p : V \to \mathbf{R}$ we abbreviate $\omega[p]$ of (2.7) to ω_p and define $\omega_p(x, u, v) := \omega_p(x - \chi_u + \chi_v) - \omega_p(x)$ as in (2.5). For $x, y \in B$, we have

$$\omega(x, u, v) + \omega(y, v, u) = \omega_p(x, u, v) + \omega_p(y, v, u). \tag{3.2}$$

Lemma 3.2 *Let $x \in B$, $y := x - \chi_{u_0} - \chi_{u_1} + \chi_{v_0} + \chi_{v_1} \in B$ with $u_0, u_1, v_0, v_1 \in V$ and $\{u_0, u_1\} \cap \{v_0, v_1\} = \emptyset$, and let $p : V \to \mathbf{R}$. If* (EXC$_{\rm loc}$) *is satisfied, then*

$$\omega_p(y) - \omega_p(x) \leq \max(\pi_{00} + \pi_{11}, \pi_{01} + \pi_{10}), \tag{3.3}$$

where $\pi_{ij} := \omega_p(x, u_i, v_j)$ for $i, j = 0, 1$.

(Proof) By (EXC$_{\rm loc}$) we have

$$\omega(y) - \omega(x) \leq \max(\omega(x, u_0, v_0) + \omega(x, u_1, v_1), \omega(x, u_0, v_1) + \omega(x, u_1, v_0)).$$

This shows (3.3) with $p = 0$, which immediately implies the general case. □

Define

$$\begin{aligned} \mathcal{D} \ := \ & \{(x, y) \mid x, y \in B,\ \exists u_* \in \mathrm{supp}^+(x - y), \forall v \in \mathrm{supp}^-(x - y) : \\ & \omega(x, u_*, v) + \omega(y, v, u_*) < 0\}, \end{aligned}$$

which denotes the set of pairs (x, y) for which the exchangeability (EXC) fails. We want to show $\mathcal{D} = \emptyset$.

Suppose to the contrary that $\mathcal{D} \neq \emptyset$, take $(x, y) \in \mathcal{D}$ such that $||x - y||$ is minimum, and let $u_* \in \mathrm{supp}^+(x - y)$ be as in the definition of $\mathcal{D}$. We have $||x - y|| > 4$. Define $p : V \to \mathbf{R}$ by

$$p(v) := \begin{cases} -\omega(x, u_*, v) & (v \in \mathrm{supp}^-(x-y), x - \chi_{u_*} + \chi_v \in B) \\ \omega(y, v, u_*) + \varepsilon & (v \in \mathrm{supp}^-(x-y), x - \chi_{u_*} + \chi_v \notin B, y + \chi_{u_*} - \chi_v \in B) \\ 0 & (\text{otherwise}) \end{cases}$$

with some $\varepsilon > 0$ and consider ω_p.

Claim 1:

$$\omega_p(x, u_*, v) = 0 \quad \text{if } v \in \mathrm{supp}^-(x-y), x - \chi_{u_*} + \chi_v \in B, \tag{3.4}$$

$$\omega_p(y, v, u_*) < 0 \quad \text{for } v \in \mathrm{supp}^-(x-y). \tag{3.5}$$

The equality (3.4) follows from the definition of p, whereas the inequality (3.5) can be shown as follows. If $x - \chi_{u_*} + \chi_v \in B$, we have $\omega_p(x, u_*, v) = 0$ by (3.4) and

$$\omega_p(x, u_*, v) + \omega_p(y, v, u_*) = \omega(x, u_*, v) + \omega(y, v, u_*) < 0,$$

which in turn follows from (3.2) and the definition of u_*. Otherwise, we have $\omega_p(y, v, u_*) = -\varepsilon$ or $-\infty$ according to whether $y + \chi_{u_*} - \chi_v \in B$ or not.

Claim 2: There exist $u_0 \in \mathrm{supp}^+(x - y)$ and $v_0 \in \mathrm{supp}^-(x - y)$ such that $y + \chi_{u_0} - \chi_{v_0} \in B$, and

$$\omega_p(y, v_0, u_0) \geq \omega_p(y, v, u_0) \qquad (v \in \mathrm{supp}^-(x-y)). \tag{3.6}$$

In fact, by (B1) we have $y + \chi_{u_0} - \chi_{v_0} \in B$ for some $u_0 \in \mathrm{supp}^+(x - y)$ and $v_0 \in \mathrm{supp}^-(x - y)$. We can further assume (3.6) by fixing u_0 and redefining v_0 to be the element $v \in \mathrm{supp}^-(x - y)$ that maximizes $\omega_p(y, v, u_0)$.

Claim 3: $(x, y') \in \mathcal{D}$ with $y' := y + \chi_{u_0} - \chi_{v_0}$.

To prove this it suffices to show

$$\omega_p(x, u_*, v) + \omega_p(y', v, u_*) < 0 \qquad (v \in \mathrm{supp}^-(x - y')).$$

We may restrict ourselves to v with $x - \chi_{u_*} + \chi_v \in B$ since otherwise the first term $\omega_p(x, u_*, v)$ is equal to $-\infty$. For such v, the first term is equal to zero by (3.4). For the second term, it follows from Lemma 3.2, (3.5) and (3.6) that

$$\omega_p(y', v, u_*) = \omega_p(y + \chi_{u_0} + \chi_{u_*} - \chi_{v_0} - \chi_v) - \omega_p(y + \chi_{u_0} - \chi_{v_0})$$

$$
\begin{aligned}
&\leq \max\left[\omega_p(y,v_0,u_0)+\omega_p(y,v,u_*),\omega_p(y,v,u_0)+\omega_p(y,v_0,u_*)\right] \\
&\quad -\omega_p(y,v_0,u_0) \\
&< \max\left[\omega_p(y,v_0,u_0),\omega_p(y,v,u_0)\right]-\omega_p(y,v_0,u_0) \\
&= 0.
\end{aligned}
$$

Since $||x-y'|| = ||x-y||-2$, Claim 3 contradicts our choice of $(x,y) \in \mathcal{D}$. Therefore we conclude $\mathcal{D} = \emptyset$, completing the proof of Theorem 3.1.

4 Conjugate Functions and Concave Extensions

4.1 Concave conjugate functions

In line with the standard method in convex analysis [36], [40], we introduce the concept of conjugate functions.

For any nonempty finite set $B \subseteq \mathbf{Z}^V$ and any function $g : B \to \mathbf{R}$, we define $g^\circ : \mathbf{R}^V \to \mathbf{R}$ by

$$g^\circ(p) := \min\{\langle p, x\rangle - g(x) \mid x \in B\}. \tag{4.1}$$

We call g° the concave conjugate function of g. Since B is finite, g° is a polyhedral concave function [36], [40], taking finite values for all p. Furthermore we define $\hat{g} : \mathbf{R}^V \to \mathbf{R}$ by

$$\hat{g}(b) := \inf\{\langle p, b\rangle - g^\circ(p) \mid p \in \mathbf{R}^V\}. \tag{4.2}$$

Obviously, $\hat{g}$ is a concave function, which we call the concave closure of g. By a standard result from convex analysis (cf. [40, §4.8]) (or equivalently by linear programming duality) we have

$$\hat{g}(b) = \begin{cases} \max\{\sum_{y\in B} \lambda_y g(y) \mid b = \sum_{y\in B} \lambda_y y,\ \lambda \in \Lambda(B)\} & (b \in \overline{B}) \\ -\infty & (b \notin \overline{B}) \end{cases} \tag{4.3}$$

where

$$\Lambda(B) := \{\lambda \in \mathbf{R}^B \mid \sum_{y\in B} \lambda_y = 1,\ \lambda_y \geq 0\ (y \in B)\}$$

and $\overline{B}$ denotes the convex hull of B, that is,

$$\overline{B} := \{b \in \mathbf{R}^V \mid b = \sum_{y\in B} \lambda_y y,\ \lambda \in \Lambda(B)\}. \tag{4.4}$$

Also, in general, we denote by $\overline{X}$ the convex hull of a subset $X \subseteq \mathbf{R}^V$.

Define

$$\begin{aligned} \operatorname{argmax}(g) &:= \{x \in B \mid g(x) \geq g(y),\ \forall y \in B\}, & (4.5)\\ \operatorname{argmax}(\hat{g}) &:= \{b \in \overline{B} \mid \hat{g}(b) \geq \hat{g}(c),\ \forall c \in \overline{B}\}, & (4.6) \end{aligned}$$

where we regard $\hat{g}$ as $\hat{g} : \overline{B} \to \mathbf{R}$.

Lemma 4.1

(1) $\hat{g}(x) \geq g(x)$ *for* $x \in B$.

(2) $\max\{\hat{g}(b) \mid b \in \overline{B}\} = \max\{g(x) \mid x \in B\}$.

(3) $\operatorname{argmax}(\hat{g}) = \overline{\operatorname{argmax}(g)}$.

(Proof) These claims follow easily from (4.3). □

For $p : V \to \mathbf{R}$ (that is, $p \in \mathbf{R}^V$), we define $g[p] : B \to \mathbf{R}$ and $\hat{g}[p] : \overline{B} \to \mathbf{R}$ by

$$g[p](x) := g(x) + \langle p, x\rangle, \qquad \hat{g}[p](b) := \hat{g}(b) + \langle p, b\rangle \tag{4.7}$$

as in (2.7). The following relations are easy to see, where $(g[p_0])\hat{}$ denotes the concave closure of $g[p_0]$.

Lemma 4.2

(1) $(g[p_0])^\circ(p) = g^\circ(p - p_0)$.

(2) $(g[p_0])\hat{}(b) = \hat{g}[p_0](b)$. □

4.2 Characterization of M-concavity by the maximizers

Just as the maximizers of a concave function form a convex set, the family of the maximizers of an M-concave function ω enjoys a nice property. In the following we assume that B is a finite integral base set. Recall (4.5) for the notation $\operatorname{argmax}(\omega)$.

Lemma 4.3 *If* $\omega : B \to \mathbf{R}$ *has the exchange property* (EXC)*, then* $\operatorname{argmax}(\omega)$ *is an integral base set, that is,* $\overline{\operatorname{argmax}(\omega)}$ *is an integral base polytope.* □

(Proof) Put $\omega_{\max} := \max\{\omega(x) \mid x \in B\}$. In (EXC) we must have $\omega(x-\chi_u+\chi_v) = \omega(y + \chi_u - \chi_v) = \omega_{\max}$ if $\omega(x) = \omega(y) = \omega_{\max}$. □

The above lemma implies furthermore that $\overline{\operatorname{argmax}(\omega[p])}$ is an integral base polytope for each $p : V \to \mathbf{R}$, since $\omega[p]$ also satisfies (EXC) by Theorem 2.2. This turns out to be a key property for M-concavity as follows.

Theorem 4.4 *Let* $\omega : B \to \mathbf{R}$ *be a function defined on a finite integral base set* $B \subseteq \mathbf{Z}^V$*. Then* ω *satisfies* (EXC) *if and only if* $\overline{\operatorname{argmax}(\omega[p])}$ *is an integral base polytope for each* $p : V \to \mathbf{R}$.

(Proof) The "only if" part has already been shown. For the "if" part, we will show that ω satisfies the local exchange property (EXC$_{\rm loc}$). Then Theorem 3.1 establishes the claim.

Take $x, y \in B$ with $||x - y|| = 4$ and put $c := (x + y)/2 \in \mathbf{R}^V$. By considering the concave closure $\hat{\omega}$ of (4.2) and its supporting hyperplane at c, we see that $c \in \operatorname{argmax}(\hat{\omega}[p])$ for some $p : V \to \mathbf{R}$. On the other hand, putting $B_p := \operatorname{argmax}(\omega[p])$, we have

$$\operatorname{argmax}(\hat{\omega}[p]) = \operatorname{argmax}((\omega[p])\hat{}) = \overline{\operatorname{argmax}(\omega[p])} = \overline{B_p} \tag{4.8}$$

from Lemma 4.2(2) and Lemma 4.1(3). Therefore, we have $c \in \overline{B_p}$. Here $B_p \subseteq B \subseteq \mathbf{Z}^V$ and $\overline{B_p}$ is an integral base polytope by the assumption. [Remark: It is not claimed — and not even true in general — that $\{x, y\} \subseteq B_p$.]

Consider an "interval" I defined by

$$I := \{b \in \mathbf{R}^V \mid x \wedge y \leq b \leq x \vee y\},$$

where $x \wedge y \in \mathbf{Z}^V$ and $x \vee y \in \mathbf{Z}^V$ are given by

$$(x \wedge y)(v) := \min(x(v), y(v)), \qquad (x \vee y)(v) := \max(x(v), y(v)) \qquad (v \in V).$$

We have $I \cap \overline{B_p} \neq \emptyset$ since $c \in I \cap \overline{B_p}$. Hence (cf. [19, Theorem 3.8]), $I \cap \overline{B_p}$ is an integral base polytope that contains c. Therefore, c can be represented as a convex combination of some integral vectors, say $z_1, \cdots, z_m$, in $I \cap \overline{B_p}$. Since $\mathbf{Z}^V \cap (I \cap \overline{B_p}) = I \cap B_p$ by (2.3), we see

$$c = \sum_{k=1}^{m} \lambda_k z_k, \qquad z_k \in I \cap B_p \quad (k = 1, \cdots, m), \tag{4.9}$$

with $\sum_{k=1}^{m} \lambda_k = 1$ and $\lambda_k > 0$ $(k = 1, \cdots, m)$.

Since $||x - y|| = 4$, we can find $v_1, v_2, v_3, v_4 \in V$ such that $\{v_1, v_2\} \cap \{v_3, v_4\} = \emptyset$ and $y = x - \chi_{v_1} - \chi_{v_2} + \chi_{v_3} + \chi_{v_4}$. In the following, we consider the case where $v_1 \neq v_2$ and $v_3 \neq v_4$, since the other cases can be treated similarly (and more easily).

When v_1, v_2, v_3, v_4 are distinct, a vector $z \in I \cap B_p$, which is integral, can be identified with a 2-element subset $\{v_i, v_j\}$ of $V_0 = \{v_1, v_2, v_3, v_4\}$ according to the correspondence

$$z = (x \wedge y) + \chi_{v_i} + \chi_{v_j} \qquad (i \neq j).$$

Denoting this correspondence $z \mapsto \{v_i, v_j\}$ by φ and referring to (4.9), we define an undirected graph $G = (V_0, E_0)$ with vertex set V_0 and edge set $E_0 = \{\varphi(z_k) \mid k = 1, \cdots, m\}$.

Claim: G has a perfect matching (of size 2).

(Proof of Claim) For each i $(1 \leq i \leq 4)$, we have $c(v_i) - (x \wedge y)(v_i) = 1/2$, whereas $z_k(v_i) - (x \wedge y)(v_i) \in \{0, 1\}$ for all k $(1 \leq k \leq m)$ in (4.9). Hence, for each i, there exist k_1 and k_0 such that

$$z_{k_1}(v_i) - (x \wedge y)(v_i) = 1, \qquad z_{k_0}(v_i) - (x \wedge y)(v_i) = 0.$$

Translating this into G, we see that for each vertex v_i there is an edge which covers (is incident to) v_i and also there is another edge which avoids (is not incident to)

v_i. Then it is not difficult to see that this condition implies the existence of a perfect matching in G (either by a straightforward enumeration of all possible configurations or by invoking Tutte's theorem [26]). Thus the claim has been proven.

Finally we derive ($\mathrm{EXC}_{\mathrm{loc}}$) from the above claim. We divide into two cases.

Case 1: In case $\{\{v_1, v_2\}, \{v_3, v_4\}\} \subseteq E_0$, both x and y appear among the z_k's. This means in particular that $\{x, y\} \subseteq B_p$. Since $\overline{B_p}$ is an integral base polytope by assumption, we can apply (B2) to obtain $x - \chi_{v_i} + \chi_{v_j} \in B_p$ and $y + \chi_{v_i} - \chi_{v_j} \in B_p$ for some $i \in \{1, 2\}$ and $j \in \{3, 4\}$. This shows

$$\omega[p](x) = \omega[p](y) = \omega[p](x - \chi_{v_i} + \chi_{v_j}) = \omega[p](y + \chi_{v_i} - \chi_{v_j}) = \max(\omega[p]),$$

which implies (3.1).

Case 2: If $\{\{v_1, v_2\}, \{v_3, v_4\}\} \not\subseteq E_0$, it follows from the above claim that $\{\{v_1, v_i\}, \{v_2, v_j\}\} \subseteq E_0$ for some i, j with $\{i, j\} = \{3, 4\}$. This means both $(x \wedge y) + \chi_{v_1} + \chi_{v_i}$ and $(x \wedge y) + \chi_{v_2} + \chi_{v_j}$ appear among the z_k's, which belong to $B_p = \mathrm{argmax}\,(\omega[p])$. Noting

$$(x \wedge y) + \chi_{v_1} + \chi_{v_i} = x - \chi_{v_2} + \chi_{v_i}, \qquad (x \wedge y) + \chi_{v_2} + \chi_{v_j} = y + \chi_{v_2} - \chi_{v_i},$$

we see

$$\omega[p](x - \chi_{v_2} + \chi_{v_i}) = \omega[p](y + \chi_{v_2} - \chi_{v_i}) = \max(\omega[p]).$$

This implies

$$\omega[p](x) + \omega[p](y) \leq \omega[p](x - \chi_{v_2} + \chi_{v_i}) + \omega[p](y + \chi_{v_2} - \chi_{v_i}),$$

which establishes (3.1). □

4.3 Concave extensions

In this section we reveal a precise relationship between exchangeability (EXC) and concavity. By Lemma 4.1(1), which is independent of (EXC), we know that $\hat{\omega} : \overline{B} \to \mathbf{R}$ is a concave function such that $\hat{\omega}(x) \geq \omega(x)$ for $x \in B$. The exchangeability condition (EXC) guarantees the equality here as follows.

Lemma 4.5 *If $\omega : B \to \mathbf{R}$ has the exchange property* (EXC)*, then $\hat{\omega}(x) = \omega(x)$ for all $x \in B$.*

(Proof) Fix $x \in B$. Since $\hat{\omega}$ is concave, there exists $p : V \to \mathbf{R}$ such that $x \in \mathrm{argmax}\,(\hat{\omega}[p])$, i.e.,

$$\hat{\omega}[p](x) = \max(\hat{\omega}[p]). \tag{4.10}$$

Put $B_p := \mathrm{argmax}\,(\omega[p])\ (\subseteq \mathbf{Z}^V)$. Then by (4.8), we have $x \in \overline{B_p}$, which implies $x \in \mathbf{Z}^V \cap \overline{B_p} = B_p$ by (2.3). That is,

$$\omega[p](x) = \max(\omega[p]). \tag{4.11}$$

Since $\max(\hat{\omega}[p]) = \max((\omega[p])\hat{\ }) = \max(\omega[p])$ by Lemma 4.2(2) and Lemma 4.1(2), we see from (4.10) and (4.11) that $\hat{\omega}[p](x) = \omega[p](x)$, i.e., $\hat{\omega}(x) = \omega(x)$. □

We say that $\overline{\omega} : \overline{B} \to \mathbf{R}$ is an extension of $\omega : B \to \mathbf{R}$ if $\overline{\omega}(x) = \omega(x)$ for $x \in B$.

Theorem 4.6 (Extension Theorem) *Let $\omega : B \to \mathbf{R}$ be a function defined on a finite integral base set $B \subseteq \mathbf{Z}^V$. Then ω satisfies* (EXC) *if and only if it can be extended to a concave function $\overline{\omega} : \overline{B} \to \mathbf{R}$ such that* $\mathrm{argmax}\,(\overline{\omega}[p])$ *is an integral base polytope for each $p : V \to \mathbf{R}$.*

(Proof) To show the "only if" part, we can take $\overline{\omega} = \hat{\omega}$, which is an extension of ω by Lemma 4.5 and meets the requirement by (4.8) and Theorem 4.4.

The "if" part can be shown as follows. Obviously we have

$$\max(\overline{\omega}[p]) := \max\{\overline{\omega}[p](b) \mid b \in \overline{B}\} \geq \max\{\omega[p](x) \mid x \in B\} =: \max(\omega[p])$$

since $\overline{\omega}[p](x) = \omega[p](x)$ for $x \in B$. On the other hand, $\mathrm{argmax}\,(\overline{\omega}[p])$ contains an integral point, which belongs to $\mathbf{Z}^V \cap \overline{B} = B$ (cf. (2.3)). Therefore we have $\max(\overline{\omega}[p]) = \max(\omega[p])$ and

$$\mathbf{Z}^V \cap \mathrm{argmax}\,(\overline{\omega}[p]) = \mathrm{argmax}\,(\omega[p]).$$

Since $\mathrm{argmax}\,(\overline{\omega}[p])$ is an integral base polytope by assumption, it follows from Theorem 4.4 that ω satisfies (EXC). □

5 Supermodularity in Conjugate Function

In Theorem 2.1 we have seen that the exchange property (B1) (or (B2)) of B is equivalent to the sub/supermodularity of the function (f or g) describing the face of the polytope $\overline{B}$. As the exchange property (EXC) for ω can be regarded as a quantitative extension of the simultaneous exchange property (B2) for B, it is quite natural to seek for an extension of the above correspondence between the exchangeability and the sub/supermodularity (see (1.1)). We answer this question in Theorem 5.3 below, which says that (EXC) for ω is equivalent to "local supermodularity" of the concave conjugate function ω°.

5.1 Exchangeability (B1) and supermodularity

We reformulate known facts (cf. Theorem 2.1) about the relationship between (B2) and supermodularity in a form that is suitable for our subsequent discussion. We assume $B \subseteq \mathbf{Z}^V$ is a finite nonempty set such that $B = \mathbf{Z}^V \cap \overline{B}$.

We define $\psi^\circ : \mathbf{R}^V \to \mathbf{R}$ by

$$\psi^\circ(p) := \min\{\langle p, x\rangle \mid x \in B\}. \tag{5.1}$$

Note that ψ° is the concave conjugate function of $\psi \equiv 0$ (on B) in the sense of (4.1), and also that

$$-\psi^\circ(-p) = \max\{\langle p, x\rangle \mid x \in B\} \tag{5.2}$$

agrees with the support function of $\overline{B}$ as defined in [36], [40]. Obviously, $\psi^\circ(p)$ is concave, $\psi^\circ(0) = 0$, and positively homogeneous, i.e., $\psi^\circ(\lambda p) = \lambda\psi^\circ(p)$ for $\lambda > 0$. Hence the hypograph

$$\mathrm{Hyp}(\psi^\circ) := \{(p, q) \in \mathbf{R}^V \times \mathbf{R} \mid q \le \psi^\circ(p)\} \tag{5.3}$$

is a convex cone.

Suppose B satisfies (B1). We first observe that the function $g : 2^V \to \mathbf{R}$ defined by $g(X) := \psi^\circ(\chi_X)$ ($X \subseteq V$) is supermodular. In fact, we have

$$g(X) = \min\{\langle \chi_X, x\rangle \mid x \in B\} = \min\{x(X) \mid x \in B\}$$

and this is how the supermodular function g in Theorem 2.1 is constructed. Secondly, the value of $\psi^\circ(p)$ at arbitrary p can be expressed as a linear combination of $\psi^\circ(\chi_X)$ ($X \subseteq V$). In fact, the greedy algorithm (cf. [19]) for minimizing a linear function over the base polytope, say $B(g)$, of the supermodular system $(2^V, g)$ shows

$$\min\{\langle p, x\rangle \mid x \in B(g)\} = \sum_{j=1}^{n}(p_j - p_{j+1})g(V_j), \tag{5.4}$$

where, for given $p \in \mathbf{R}^V$, the elements of V are indexed as $\{v_1, v_2, \cdots, v_n\}$ (with $n = |V|$) in such a way that

$$p(v_1) \geq p(v_2) \geq \cdots \geq p(v_n);$$

$p_j := p(v_j)$, $V_j := \{v_1, v_2, \cdots, v_j\}$ for $j = 1, \cdots, n$, and $p_{n+1} := 0$. Noting $\overline{B} = B(g)$ we obtain

$$\psi^{\circ}(p) = \sum_{j=1}^{n}(p_j - p_{j+1})\psi^{\circ}(\chi_{V_j}). \tag{5.5}$$

Conversely, suppose $\psi^{\circ}(p)$ defined from B by (5.1) satisfies the two conditions:

(C1) [supermodularity] $g(X) := \psi^{\circ}(\chi_X)$ is supermodular.

(C2) [greediness] $\psi^{\circ}(p) = \sum_{j=1}^{n}(p_j - p_{j+1})\psi^{\circ}(\chi_{V_j})$,

where, for given $p \in \mathbf{R}^V$, the elements of V are indexed as $\{v_1, v_2, \cdots, v_n\}$ in such a way that $p(v_1) \geq p(v_2) \geq \cdots \geq p(v_n)$; $p_j := p(v_j)$, $V_j := \{v_1, v_2, \cdots, v_j\}$ for $j = 1, \cdots, n$, and $p_{n+1} := 0$.

The condition (C1) implies (5.4). Combining this with (C2) and (5.1) we see that

$$\min\{\langle p, x\rangle \mid x \in B(g)\} = \min\{\langle p, x\rangle \mid x \in B\} \qquad (p \in \mathbf{R}^V).$$

This means $B(g) = \overline{B}$, from which follows $B = \mathbf{Z}^V \cap \overline{B} = \mathbf{Z}^V \cap B(g)$. Then Theorem 2.1 shows that B satisfies (B1).

We say that a positively homogeneous function $h : \mathbf{R}^V \to \mathbf{R}$ is "matroidal" if it satisfies (C1) and (C2) with ψ° replaced by h. By a result of Lovász [25] (see also [19, Theorem 6.13]) such h is necessarily concave. We also say that a cone is "matroidal" if it is a hypograph of a "matroidal" h.

With this terminology the above observations are summarized in the following theorem, which characterizes the exchange property of B in the language of ψ° (or the support function of $\overline{B}$).

Theorem 5.1 *Let $B \subseteq \mathbf{Z}^V$ be a finite nonempty set with $B = \mathbf{Z}^V \cap \overline{B}$. Then B satisfies* (B1) *if and only if ψ° is "matroidal" (satisfying* (C1) *and* (C2)*).* □

The following fact will be used later. The proof is easy from (C1) and (C2).

Lemma 5.2 *Let $h_1, h_2 : \mathbf{R}^V \to \mathbf{R}$ be positively homogeneous functions. If h_1 and h_2 are "matroidal", then $h_1 + h_2$ is also "matroidal".* □

5.2 M-concavity (EXC) and supermodularity

We now consider the concave conjugate function

$$\omega^\circ(p) := \min\{\langle p, x\rangle - \omega(x) \mid x \in B\} \tag{5.6}$$

of $\omega : B \to \mathbf{R}$ defined on a finite integral base set $B \subseteq \mathbf{Z}^V$. As opposed to ψ°, ω° is not a positively homogeneous function though it is concave. Accordingly, the hypograph

$$\mathrm{Hyp}(\omega^\circ) := \{(p, q) \in \mathbf{R}^V \times \mathbf{R} \mid q \leq \omega^\circ(p)\} \tag{5.7}$$

is not a cone but a polyhedron. Its characteristic cone (or recession cone) [36], [39], [40] is given by $\mathrm{Hyp}(\psi^\circ)$ of (5.3), and hence it is "matroidal" by Theorem 5.1.

Since $\omega^\circ(p)$ is a concave function, we can think of its subdifferential in the ordinary sense of convex analysis. Namely, the subdifferential of ω° at $p_0 \in \mathbf{R}^V$, denoted by $\partial\omega^\circ(p_0)$, is defined by

$$\partial\omega^\circ(p_0) := \{b \in \mathbf{R}^V \mid \omega^\circ(p) - \omega^\circ(p_0) \leq \langle p - p_0, b\rangle,\ \forall p \in \mathbf{R}^V\}. \tag{5.8}$$

Using this notion, we define a positively homogeneous concave function $\hat{L}(\omega^\circ, p_0) : \mathbf{R}^V \to \mathbf{R}$ by

$$\hat{L}(\omega^\circ, p_0)(p) := \inf\{\langle p, b\rangle \mid b \in \partial\omega^\circ(p_0)\}, \tag{5.9}$$

which we call the localization of ω° at p_0 (provided $\partial\omega^\circ(p_0) \neq \emptyset$). Note that

$$\omega^\circ(p) \leq \omega^\circ(p_0) + \hat{L}(\omega^\circ, p_0)(p - p_0) \tag{5.10}$$

and that $\omega^\circ(p)$ is equal to the right-hand side in the neighborhood of p_0. Also note that

$$\mathrm{Hyp}(\hat{L}(\omega^\circ, p_0)) = \{(p, q) \in \mathbf{R}^V \times \mathbf{R} \mid q \leq \langle p, b\rangle,\ b \in \partial\omega^\circ(p_0)\}. \tag{5.11}$$

The following theorem establishes a link between (EXC) and supermodularity, showing that (EXC) for ω is equivalent to the localization of ω° being "matroidal" at each point. Recalling that the first condition (C1) for being "matroidal" refers to supermodularity, while (C2) is related to greediness, we may say that the exchange property (EXC) is nothing but "a collection of local supermodularity", just as the exchange property (B1) corresponds to supermodularity.

Theorem 5.3 (Local Supermodularity Theorem) *Let $\omega : B \to \mathbf{R}$ be a function defined on a finite integral base set $B \subseteq \mathbf{Z}^V$. Then ω satisfies* (EXC) *if and only if the localization $\hat{L}(\omega^\circ, p_0)$ of ω° is "matroidal" (satisfying* (C1) *and* (C2)*) at each point p_0.*

(Proof) The hyperplane $H_x := \{(p,q) \in \mathbf{R}^V \times \mathbf{R} \mid q = \langle p, x\rangle - \omega(x)\}$ in $\mathbf{R}^V \times \mathbf{R}$, indexed by $x \in B$, contains $(p,q) := (p_0, \omega^\circ(p_0))$ if and only if

$$\langle p_0, x\rangle - \omega(x) = \omega^\circ(p_0) = \min\{\langle p_0, y\rangle - \omega(y) \mid y \in B\},$$

which means $x \in \mathrm{argmax}\,(\omega[-p_0])$ and $\langle p, x\rangle - \omega(x) = \langle p - p_0, x\rangle + \omega^\circ(p_0)$ for such x. Therefore, in the neighborhood of p_0, $\omega^\circ(p)$ is equal to

$$\begin{aligned} &\min\{\langle p, x\rangle - \omega(x) \mid x \in \mathrm{argmax}\,(\omega[-p_0])\} \\ &= \min\{\langle p - p_0, x\rangle \mid x \in \mathrm{argmax}\,(\omega[-p_0])\} + \omega^\circ(p_0). \end{aligned}$$

This shows

$$\hat{L}(\omega^\circ, p_0)(p) = \min\{\langle p, x\rangle \mid x \in \mathrm{argmax}\,(\omega[-p_0])\}. \tag{5.12}$$

By Theorem 5.1, this is "matroidal" if and only if $\mathrm{argmax}\,(\omega[-p_0])$ satisfies (B1), whereas the latter condition for all p_0 is equivalent to (EXC) by Theorem 4.4. □

Remark 5.1 It follows from Theorem 5.3 (with $\omega = 0$) that the localization of a "matroidal" function is again "matroidal". Therefore, it is sufficient in Theorem 5.3 (for a general ω) to consider the localization of ω° at points p_0 such that $(p_0, \omega^\circ(p_0))$ lies in the minimal faces of $\mathrm{Hyp}(\omega^\circ)$. □

Remark 5.2 For an affine function $\omega(x) = \alpha + \langle \eta, x\rangle$ on B, we have

$$\omega^\circ(p) = \psi^\circ(p - \eta) - \alpha, \qquad \hat{L}(\omega^\circ, \eta) = \psi^\circ,$$

where ψ° is defined by (5.1). □

Remark 5.3 Just as in Theorem 5.3, a valuated Δ-matroid [6], [43] can be characterized in terms of local bisupermodularity. See [33] for details. □

Finally, note that a combination of Theorem 5.1 and Theorem 5.3 yields the following variant of Theorem 5.3.

Theorem 5.4 *Let $\omega : B \to \mathbf{R}$ be a function defined on a finite nonempty set $B \subseteq \mathbf{Z}^V$ with $B = \mathbf{Z}^V \cap \overline{B}$. Then ω satisfies* (EXC) *if and only if* (i) *the characteristic cone of the hypograph* $\mathrm{Hyp}(\omega^\circ)$ *of ω° is "matroidal" and* (ii) *the localization $\hat{L}(\omega^\circ, p_0)$ of ω° is "matroidal" at each point p_0.* □

6 Duality

Using the standard Fenchel duality framework of convex analysis [36], [40], we derive a min-max duality formula for a pair of an M-convex and an M-concave function. Its content lies in the integrality assertion that both the primal (maximization) problem and the dual (minimization) problem have integral optimum solutions when the given functions satisfying (EXC) are integer-valued. This min-max formula is a succinct unification of two groups of more or less equivalent theorems, (i) Edmonds' polymatroid intersection theorem [9], Fujishige's Fenchel-type duality theorem [18], and Frank's discrete separation theorem for a pair of sub/supermodular functions [14], and (ii) (an extension of) Iri-Tomizawa's potential characterization of optimality for the independent assignment problem [22], Fujishige's generalization thereof to the independent flow problem [17] and Frank's weight splitting theorem for the matroid intersection problem [13]. The min-max formula can also be reformulated as discrete separation theorems, which are distinct from Frank's.

6.1 Convex conjugate function

Dually to (4.1), for an arbitrary function $f : B \to \mathbf{R}$, we define the convex conjugate function $f^\bullet : \mathbf{R}^V \to \mathbf{R}$ by

$$f^\bullet(p) := \max\{\langle p, x\rangle - f(x) \mid x \in B\}. \tag{6.1}$$

We also define, dually to (4.2), the convex closure $\check{f} : \mathbf{R}^V \to \mathbf{R}$ of f by

$$\check{f}(b) \ := \ \sup\{\langle p, b\rangle - f^\bullet(p) \mid p \in \mathbf{R}^V\}. \tag{6.2}$$

The following relations are immediate from the definitions, where $(-f)\hat{}$ denotes the concave closure of $-f$.

Lemma 6.1

(1) $(-f)^\circ(p) = -f^\bullet(-p)$.

(2) $(-f)\hat{}(b) = -\check{f}(b)$. □

This lemma allows us to translate the results for $g^\circ(p)$ and $\hat{g}(b)$ into corresponding ones for $f^\bullet(p)$ and $\check{f}(b)$. For example, from (4.3) we obtain

$$\check{f}(b) = \begin{cases} \min\{\sum_{y\in B} \lambda_y f(y) \mid b = \sum_{y\in B} \lambda_y y, \ \lambda \in \Lambda(B)\} & (b \in \overline{B}) \\ +\infty & (b \notin \overline{B}) \end{cases} \tag{6.3}$$

Accordingly, we may regard $\check{f}$ as $\check{f} : \overline{B} \to \mathbf{R}$. Lemma 4.1 translates as follows:

Lemma 6.2

(1) $\check{f}(x) \leq f(x)$ *for* $x \in B$.

(2) $\min\{\check{f}(b) \mid b \in \overline{B}\} = \min\{f(x) \mid x \in B\}$.

(3) $\mathrm{argmin}\,(\check{f}) = \overline{\mathrm{argmin}\,(f)}$. □

6.2 Duality theorems

Let B_1 and B_2 be finite integral base sets ($\subseteq \mathbf{Z}^V$). For $\omega : B_1 \to \mathbf{R}$ and $\zeta : B_2 \to \mathbf{R}$, we define the conjugate functions ω° and $\zeta^\bullet$ by (4.1) and (6.1) with reference to B_1 and B_2, respectively, and also the concave/convex closure functions $\hat{\omega}$ and $\check{\zeta}$ by (4.2) and (6.2), respectively. We sometimes use the following convention:

$$\omega(x) = -\infty \quad (x \notin B_1), \qquad \zeta(x) = +\infty \quad (x \notin B_2). \tag{6.4}$$

Note that $\omega^\circ(p) \in \mathbf{Z}$ and $\zeta^\bullet(p) \in \mathbf{Z}$ for $p \in \mathbf{Z}^V$ if ω and ζ are integer-valued.

We define a primal-dual pair of problems as follows.

[**Primal problem**]
Maximize $\Phi(x) := \omega(x) - \zeta(x) \quad (x \in B_1 \cap B_2)$.

[**Dual problem**]
Minimize $\Psi(p) := \zeta^\bullet(p) - \omega^\circ(p) \quad (p \in \mathbf{R}^V)$.

Using the concave/convex closures, we also introduce a relaxation of the primal problem:

[**Relaxed primal problem**]
Maximize $\tilde{\Phi}(b) := \hat{\omega}(b) - \check{\zeta}(b) \quad (b \in \overline{B_1} \cap \overline{B_2})$.

The following identity is known as the Fenchel duality [36], [40]:

$$\max\{\hat{\omega}(b) - \check{\zeta}(b) \mid b \in \overline{B_1} \cap \overline{B_2}\} = \inf\{\zeta^\bullet(p) - \omega^\circ(p) \mid p \in \mathbf{R}^V\}, \tag{6.5}$$

which holds true independently of (EXC). Here we assume the convention that the maximum taken over an empty family is equal to $-\infty$. With this convention, the above formula implies in particular that $\overline{B_1} \cap \overline{B_2} \neq \emptyset$ if the infimum on the right-hand side is finite.

Combining (6.5) with the obvious inequalities (cf. Lemma 4.1(1) and Lemma 6.2(1)):

$$\omega(x) \leq \hat{\omega}(x) \quad (x \in B_1), \qquad \zeta(x) \geq \check{\zeta}(x) \quad (x \in B_2),$$

we obtain the following weak duality.

Lemma 6.3 *For any functions* $\omega : B_1 \to \mathbf{R}$ *and* $\zeta : B_2 \to \mathbf{R}$,

$$\max\{\omega(x) - \zeta(x) \mid x \in B_1 \cap B_2\}$$
$$\leq \max\{\hat{\omega}(b) - \check{\zeta}(b) \mid b \in \overline{B_1} \cap \overline{B_2}\} = \inf\{\zeta^\bullet(p) - \omega^\circ(p) \mid p \in \mathbf{R}^V\}.$$

(This is even independent of the property (B1) *for* B_1 *and* B_2*.)* □

Naturally, we are interested in whether the equality holds in the weak duality above. The next theorem shows that this is indeed the case if ω and $-\zeta$ enjoy the exchange property (EXC).

Theorem 6.4 *Let* $\omega : B_1 \to \mathbf{R}$ *and* $\zeta : B_2 \to \mathbf{R}$ *be such that* ω *and* $-\zeta$ *satisfy* (EXC).

(1) [Primal integrality]

$$\max\{\omega(x) - \zeta(x) \mid x \in B_1 \cap B_2\}$$
$$= \max\{\hat{\omega}(b) - \check{\zeta}(b) \mid b \in \overline{B_1} \cap \overline{B_2}\} = \inf\{\zeta^\bullet(p) - \omega^\circ(p) \mid p \in \mathbf{R}^V\}.$$

To be more precise,

(P1) *If* $\inf\{\zeta^\bullet(p) - \omega^\circ(p) \mid p \in \mathbf{R}^V\} \neq -\infty$, *then* $B_1 \cap B_2 \neq \emptyset$,

(P2) *If* $B_1 \cap B_2 \neq \emptyset$, *all these values are finite and equal, and the infimum is attained by some* $p \in \mathbf{R}^V$.

(2) [Dual integrality] If ω *and* ζ *are integer-valued, the infimum can be taken over integral vectors, i.e.,*

$$\max\{\omega(x) - \zeta(x) \mid x \in B_1 \cap B_2\} = \inf\{\zeta^\bullet(p) - \omega^\circ(p) \mid p \in \mathbf{Z}^V\},$$

and the infimum is attained by some $p \in \mathbf{Z}^V$ *if it is finite.* □

Before giving the proof, we observe that the essence of the first half of Theorem 6.4 lies in the integrality of the relaxed primal problem. Since $B_i = \mathbf{Z}^V \cap \overline{B_i}$ $(i = 1, 2)$, we have

$$B_1 \cap B_2 = \mathbf{Z}^V \cap (\overline{B_1} \cap \overline{B_2}).$$

Hence, if the relaxed primal problem has an integral optimal solution, say b, then b belongs to $B_1 \cap B_2$. Furthermore, $\omega(b) = \hat{\omega}(b)$ and $\zeta(b) = \check{\zeta}(b)$ by Lemma 4.5 and Lemma 6.1. So, Theorem 6.4(1) would follow.

The proof of Theorem 6.4 relies on Frank's discrete separation theorem for a pair of sub/supermodular functions and a recent theorem of the present author.

Theorem 6.5 (Discrete Separation Theorem [14]) *Let* $f : 2^V \to \mathbf{R}$ *and* $g : 2^V \to \mathbf{R}$ *be submodular and supermodular functions, respectively, with* $f(\emptyset) = g(\emptyset) = 0$*. If* $g(X) \leq f(X)$ $(X \subseteq V)$*, there exists* $x^* \in \mathbf{R}^V$ *such that*

$$g(X) \leq x^*(X) \leq f(X) \qquad (X \subseteq V). \tag{6.6}$$

Moreover, if f *and* g *are integer-valued, there exists such an* x^* *in* $\mathbf{Z}^V$*.* □

Remark 6.1 The original statement of the discrete separation theorem covers the more general class of sub/supermodular functions on crossing-families. Note also that Frank's discrete separation theorem, Edmonds' intersection theorem [9], [10], and Fujishige's Fenchel-type min-max theorem [18] can be regarded as, essentially, equivalent assertions (see [19, §6.1(b)]). □

Theorem 6.6 ([32, Theorem 4.1]) *Assume that* $\omega_1 : B_1 \to \mathbf{R}$ *and* $\omega_2 : B_2 \to \mathbf{R}$ *satisfy* (EXC) *and let* $x^* \in B_1 \cap B_2$*. Then*

$$\omega_1(x^*) + \omega_2(x^*) \geq \omega_1(x) + \omega_2(x), \ \forall\, x \in B_1 \cap B_2$$

if and only if there exists some $p^* \in \mathbf{R}^V$ *such that*

$$\omega_1[-p^*](x^*) \geq \omega_1[-p^*](x), \ \forall\, x \in B_1; \qquad \omega_2[p^*](x^*) \geq \omega_2[p^*](x), \ \forall\, x \in B_2.$$

Moreover, if ω_1 *and* ω_2 *are integer-valued, there exists such an* p^* *in* $\mathbf{Z}^V$*.*

(Proof) See [32], [28], or [35]. □

Remark 6.2 When ω_1 and ω_2 are affine functions, the above theorem coincides with the optimality criterion (Fujishige's potential characterization [17]) for the weighted intersection problem for a pair of submodular systems (see also [19]). On the other hand, when $B_1, B_2 \subseteq \{0,1\}^V$ representing a pair of matroids, the above theorem reduces to the optimality criterion [28, Theorem 4.2] for the valuated matroid intersection problem. If, in addition, ω_1 is affine and $\omega_2 = 0$, this criterion recovers Frank's weight splitting theorem [13] for the weighted matroid intersection problem, which is in turn equivalent to Iri-Tomizawa's potential characterization of the optimality for the independent assignment problem [22]. □

We now prove the assertion (P1) in Theorem 6.4(1). Recall Theorem 2.1 and let g_1 be the supermodular function describing B_1 and f_2 be the submodular function describing B_2. We have $g_1(\emptyset) = f_2(\emptyset) = 0$. We also introduce (cf. (5.1))

$$\psi_1{}^\circ(p) := \min\{\langle p, x\rangle \mid x \in B_1\}, \qquad \psi_2{}^\bullet(p) := \max\{\langle p, x\rangle \mid x \in B_2\}.$$

The following fact is fundamental.

Lemma 6.7

$$
\begin{aligned}
&\inf\{\zeta^\bullet(p) - \omega^\circ(p) \mid p \in \mathbf{Z}^V\} \neq -\infty & (6.7)\\
\iff\ &\inf\{\zeta^\bullet(p) - \omega^\circ(p) \mid p \in \mathbf{R}^V\} \neq -\infty & (6.8)\\
\iff\ &{\psi_2}^\bullet(p) \geq {\psi_1}^\circ(p) \qquad (p \in \mathbf{R}^V) & (6.9)\\
\iff\ &f_2(X) \geq g_1(X) \quad (X \subseteq V), \qquad f_2(V) = g_1(V). & (6.10)
\end{aligned}
$$

□

(Proof) Since

$$
|\omega^\circ(p) - {\psi_1}^\circ(p)| \leq \max_{x\in B_1} |\omega(x)|, \qquad |\zeta^\bullet(p) - {\psi_2}^\bullet(p)| \leq \max_{x\in B_2} |\zeta(x)|,
$$

and ${\psi_1}^\circ(p)$ and ${\psi_2}^\bullet(p)$ are positively homogeneous, we have

$$
\begin{aligned}
&\inf\{\zeta^\bullet(p) - \omega^\circ(p) \mid p \in \mathbf{R}^V\} \neq -\infty\\
\iff\ &\inf\{{\psi_2}^\bullet(p) - {\psi_1}^\circ(p) \mid p \in \mathbf{R}^V\} \neq -\infty\\
\iff\ &{\psi_2}^\bullet(p) \geq {\psi_1}^\circ(p) \qquad (p \in \mathbf{R}^V).
\end{aligned}
$$

By Theorem 5.1 and (5.2), it suffices to consider the last inequality for $p = \chi_X$ ($X \subseteq V$). A straightforward calculation using (5.5) shows that this in turn is equivalent to (6.10). The above argument is also valid when p is restricted to an integral vector, and therefore (6.7) is equivalent to the other conditions. □

If (6.10) is true, we can apply Theorem 6.5 to obtain $x^* \in B_1 \cap B_2$. [End of proof of (P1)]

Remark 6.3 In view of the above proof, we may say that (P1) in Theorem 6.4(1) is equivalent, modulo Lemma 6.7, to Theorem 6.5. □

Next, we prove the assertion (P2) in Theorem 6.4(1). By Lemma 6.3, we see that (P2) is equivalent to the existence of $x^* \in B_1 \cap B_2$ and $p^* \in \mathbf{R}^V$ such that

$$
\omega(x^*) - \zeta(x^*) = \zeta^\bullet(p^*) - \omega^\circ(p^*). \tag{6.11}
$$

Put $\omega_1 := \omega$ and $\omega_2 := -\zeta$ and denote by x^* a common base that maximizes $\omega_1(x) + \omega_2(x)$. By Theorem 6.6, we have

$$
\begin{aligned}
\omega_1[-p^*](x^*) &= \max\{\omega_1[-p^*](x) \mid x \in B_1\},\\
\omega_2[p^*](x^*) &= \max\{\omega_2[p^*](x) \mid x \in B_2\}
\end{aligned}
$$

for some $p^* \in \mathbf{R}^V$. This implies

$$\begin{aligned}
\omega(x^*) - \zeta(x^*) &= \omega_1(x^*) + \omega_2(x^*) \\
&= \omega_1[-p^*](x^*) + \omega_2[p^*](x^*) \\
&= \max_{x \in B_1} \omega_1[-p^*](x) + \max_{x \in B_2} \omega_2[p^*](x) \\
&= \max_{x \in B_1}(-\langle p^*, x\rangle + \omega(x)) + \max_{x \in B_2}(\langle p^*, x\rangle - \zeta(x)) \\
&= \zeta^\bullet(p^*) - \omega^\circ(p^*).
\end{aligned}$$

The second half of Theorem 6.4 follows from the second half of Theorem 6.6 which guarantees the existence of an integral vector p^*. [End of proof of Theorem 6.4]

Remark 6.4 The above proof shows that (P2) with the integrality assertion in (2) is in fact equivalent to Theorem 6.6. □

The min-max identity of Theorem 6.4 yields a pair of separation theorems, one for the primal pair (ω, ζ) and the other for the dual (conjugate) pair $(\omega^\circ, \zeta^\bullet)$. It is emphasized that these separation theorems do not exclude the case of $B_1 \cap B_2 = \emptyset$.

Theorem 6.8 (Primal Separation Theorem) *Let $\omega : B_1 \to \mathbf{R}$ and $\zeta : B_2 \to \mathbf{R}$ be such that ω and $-\zeta$ satisfy* (EXC). *If $\omega(x) \le \zeta(x)$ $(x \in B_1 \cap B_2)$, there exist $\alpha^* \in \mathbf{R}$ and $p^* \in \mathbf{R}^V$ such that*

$$\omega(x) \le \alpha^* + \langle p^*, x\rangle \le \zeta(x) \qquad (x \in \mathbf{Z}^V). \tag{6.12}$$

[This is a short-hand expression for

$$\omega(x) \le \alpha^* + \langle p^*, x\rangle \quad (x \in B_1), \qquad \alpha^* + \langle p^*, x\rangle \le \zeta(x) \quad (x \in B_2)$$

relying on our convention (6.4)*.]*

Moreover, if ω and ζ are integer-valued, there exist such α^* in $\mathbf{Z}$ and p^* in $\mathbf{Z}^V$.

(Proof) First note that

$$(6.12) \iff \zeta^\bullet(p^*) \le -\alpha^* \le \omega^\circ(p^*).$$

In case $B_1 \cap B_2 \ne \emptyset$, we see from (6.11) and Theorem 6.4, (P2), that there exist $x^* \in B_1 \cap B_2$ and $p^* \in \mathbf{R}^V$ such that

$$[\langle p^*, x^*\rangle - \zeta(x^*)] - [\langle p^*, x^*\rangle - \omega(x^*)] = \zeta^\bullet(p^*) - \omega^\circ(p^*).$$

Hence we have

$$\omega^{\circ}(p^*) = \min\{\langle p^*, x\rangle - \omega(x) \mid x \in B_1\} = \langle p^*, x^*\rangle - \omega(x^*), \tag{6.13}$$
$$\zeta^{\bullet}(p^*) = \max\{\langle p^*, x\rangle - \zeta(x) \mid x \in B_2\} = \langle p^*, x^*\rangle - \zeta(x^*). \tag{6.14}$$

Since $\omega(x^*) \leq \zeta(x^*)$ by assumption, there exists $\alpha^* \in \mathbf{R}$ with

$$\zeta^{\bullet}(p^*) = \langle p^*, x^*\rangle - \zeta(x^*) \leq -\alpha^* \leq \langle p^*, x^*\rangle - \omega(x^*) = \omega^{\circ}(p^*).$$

Next we consider the case of $B_1 \cap B_2 = \emptyset$. By Theorem 6.4, (P1), this implies $\zeta^{\bullet}(p^*) \leq \omega^{\circ}(p^*)$ for some $p^* \in \mathbf{R}^V$. By choosing $\alpha^* \in \mathbf{R}$ with $\zeta^{\bullet}(p^*) \leq -\alpha^* \leq \omega^{\circ}(p^*)$, we obtain (6.12).

The integrality assertion for α^* and p^* follows from the integrality assertions in Theorem 6.4 and Lemma 6.7. □

Remark 6.5 Conversely, the min-max formula of Theorem 6.4 can be derived from the primal separation theorem. Note in this connection that the primal separation theorem implies the following: If $B_1 \cap B_2 = \emptyset$, then for any $M \in \mathbf{R}$ there exist $\alpha^* \in \mathbf{R}$ and $p^* \in \mathbf{R}^V$ such that

$$0 \leq \alpha^* + \langle p^*, x\rangle \quad (x \in B_1), \qquad \alpha^* + \langle p^*, x\rangle \leq -M \quad (x \in B_2);$$

which implies ${\psi_2}^{\bullet}(p^*) - {\psi_1}^{\circ}(p^*) \leq (-\alpha^* - M) - (-\alpha^*) = -M$. The assertion (P1) in Theorem 6.4(1) is immediate from this. For (P2) we apply the primal separation theorem to $(\tilde{\omega}, \zeta)$ with $\tilde{\omega}(x) := \omega(x) - \max\{\omega(y) - \zeta(y) \mid y \in B_1 \cap B_2\}$. □

Remark 6.6 The primal separation theorem in case $B_1 \cap B_2 \neq \emptyset$ has been established in [32]. □

Theorem 6.9 (Dual Separation Theorem) *Let $\omega : B_1 \to \mathbf{R}$ and $\zeta : B_2 \to \mathbf{R}$ be such that ω and $-\zeta$ satisfy* (EXC). *If $\omega^{\circ}(p) \leq \zeta^{\bullet}(p)$ $(p \in \mathbf{R}^V)$, there exist $\beta^* \in \mathbf{R}$ and $x^* \in B_1 \cap B_2$ such that*

$$\omega^{\circ}(p) \leq \beta^* + \langle p, x^*\rangle \leq \zeta^{\bullet}(p) \qquad (p \in \mathbf{R}^V). \tag{6.15}$$

Moreover, if ω and ζ are integer-valued, there exists such an β^* in $\mathbf{Z}$.

(Proof) First note that

$$(6.15) \iff \check{\zeta}(x^*) \leq -\beta^* \leq \hat{\omega}(x^*).$$

The assumption means $\inf_p(\zeta^\bullet(p) - \omega^\circ(p)) \geq 0$, which in turn implies $B_1 \cap B_2 \neq \emptyset$ by Theorem 6.4, (P1). Then by Theorem 6.4, (P2), as well as (6.11), there exist $x^* \in B_1 \cap B_2$ and $p^* \in \mathbf{R}^V$ such that

$$\omega(x^*) - \zeta(x^*) = [\langle p^*, x^* \rangle - \omega^\circ(p^*)] - [\langle p^*, x^* \rangle - \zeta^\bullet(p^*)].$$

The left-hand side is equal to $\hat{\omega}(x^*) - \check{\zeta}(x^*)$ since $\hat{\omega}(x^*) = \omega(x^*)$ and $\check{\zeta}(x^*) = \zeta(x^*)$ by Lemma 4.5. Hence we have

$$\begin{aligned} \hat{\omega}(x^*) &= \inf\{\langle p, x^* \rangle - \omega^\circ(p) \mid p \in \mathbf{R}^V\} = \langle p^*, x^* \rangle - \omega^\circ(p^*), \\ \check{\zeta}(x^*) &= \sup\{\langle p, x^* \rangle - \zeta^\bullet(p) \mid p \in \mathbf{R}^V\} = \langle p^*, x^* \rangle - \zeta^\bullet(p^*). \end{aligned}$$

Since $\omega^\circ(p^*) \leq \zeta^\bullet(p^*)$ by assumption, there exists $\beta^* \in \mathbf{R}$ with

$$\check{\zeta}(x^*) = \langle p^*, x^* \rangle - \zeta^\bullet(p^*) \leq -\beta^* \leq \langle p^*, x^* \rangle - \omega^\circ(p^*) = \hat{\omega}(x^*).$$

The integrality assertion for β^* and p^* follows from the integrality assertion in Theorem 6.4. □

Remark 6.7 The dual separation theorem for $\omega = 0$ and $\zeta = 0$ reduces to the discrete separation theorem (Theorem 6.5) for sub/supermodular functions. In fact, the assumption reduces to (6.9), which is equivalent to (6.10), and we have $\beta^* = 0$ in the conclusion. □

Remark 6.8 The dual separation theorem, with the aid of Lemma 4.5 and Lemma 6.7, implies the min-max formula of Theorem 6.4. Apply the dual separation theorem to $(\tilde{\omega}, \zeta)$ with $\tilde{\omega}(x) := \omega(x) - \inf_p(\zeta^\bullet(p) - \omega^\circ(p))$ to obtain (P2). □

Finally we schematically summarize the relationship among the min-max duality (Theorem 6.4), the discrete separation theorem (Theorem 6.5), the optimality criterion for the weighted intersection problem (Theorem 6.6), the primal separation theorem (Theorem 6.8), and the dual separation theorem (Theorem 6.9). It is emphasized that the "equivalence" relies on Lemma 4.5 and Lemma 6.7.

Primal separation
(Theorem 6.8)

$\Updownarrow$

Min-max duality (Theorem 6.4) $\left\{ \begin{array}{lcl} \text{(P1)} & \Longleftrightarrow & \text{Frank's discrete separation (Theorem 6.5)} \\ \text{(P2)} & \Longleftrightarrow & \text{M-concave weighted intersection (Theorem 6.6)} \end{array} \right.$

$\Updownarrow$

Dual separation
(Theorem 6.9)

6.3 Reduction to affine cases

Based on Theorem 5.3 (Local Supermodularity Theorem), we can derive the dual separation theorem (Theorem 6.9) for general pairs (ω, ζ) from its special case for affine functions.

Let ω and ζ be as in Theorem 6.9. Recall (5.9) and (5.12):

$$\begin{aligned} \hat{L}(\omega^\circ, p_1)(p) &= \inf\{\langle p, b\rangle \mid b \in \partial\omega^\circ(p_1)\} \\ &= \min\{\langle p, x\rangle \mid x \in \mathrm{argmax}\,(\omega[-p_1])\}. \end{aligned}$$

Dually, we define

$$\begin{aligned} \partial\zeta^\bullet(p_2) &:= \{b \in \mathbf{R}^V \mid \zeta^\bullet(p) - \zeta^\bullet(p_2) \geq \langle p - p_2, b\rangle \ (\forall p \in \mathbf{R}^V)\}, \\ \check{L}(\zeta^\bullet, p_2)(p) &:= \sup\{\langle p, b\rangle \mid b \in \partial\zeta^\bullet(p_2)\} \\ &= \max\{\langle p, x\rangle \mid x \in \mathrm{argmin}\,(\zeta[-p_2])\}. \end{aligned}$$

Putting

$$\begin{aligned} \omega_L^\circ(p) &:= \omega^\circ(p_1) + \hat{L}(\omega^\circ, p_1)(p - p_1), \\ \zeta_L^\bullet(p) &:= \zeta^\bullet(p_2) + \check{L}(\zeta^\bullet, p_2)(p - p_2), \end{aligned}$$

we see (cf. (5.10))

$$\omega^\circ(p) \leq \omega_L^\circ(p), \qquad \zeta_L^\bullet(p) \leq \zeta^\bullet(p). \tag{6.16}$$

From the assumption, $\omega^\circ(p) \leq \zeta^\bullet(p)$ $(p \in \mathbf{R}^V)$, we see there exist $p_1, p_2 \in \mathbf{Z}^V$ such that

$$\omega_L^\circ(p) \leq \zeta_L^\bullet(p) \qquad (p \in \mathbf{R}^V).$$

(This is geometrically obvious, and also easy to prove.)

Theorem 5.3 shows that ω_L° is the concave conjugate of some function with the property (EXC), which we denote by $\omega_L : B_1' \to \mathbf{R}$. In fact, $B_1' = \mathrm{argmax}\,(\omega[-p_1])$ and $\omega_L(x) = -\omega^\circ(p_1) + \langle p_1, x\rangle$, which is an affine function on B_1' (cf. Remark 5.2). Similarly, $\zeta_L^\bullet$ is the convex conjugate of $\zeta_L : B_2' \to \mathbf{R}$, where $B_2' = \mathrm{argmin}\,(\zeta[-p_2])$ and $\zeta_L(x) = -\zeta^\bullet(p_2) + \langle p_2, x\rangle$.

Applying the dual separation theorem to the pair (ω_L, ζ_L) of affine functions, we obtain

$$\omega_L^\circ(p) \le \beta^* + \langle p, x^*\rangle \le \zeta_L^\bullet(p) \qquad (p \in \mathbf{R}^V)$$

for some $\beta^* \in \mathbf{R}$ and $x^* \in B_1' \cap B_2' \subseteq B_1 \cap B_2$. From (6.16) this means

$$\omega^\circ(p) \le \beta^* + \langle p, x^*\rangle \le \zeta^\bullet(p) \qquad (p \in \mathbf{R}^V).$$

Finally, if ω and ζ are integer-valued, so are ω_L and ζ_L, and we can take $\beta^* \in \mathbf{Z}$. Thus we have derived the dual separation theorem for the general pair (ω, ζ).

6.4 Convolution

We show that the supremum convolution operation of two functions preserves the property (EXC). This means as a corollary that the union operation can be defined for a pair of valuated matroids.

Assume that $\omega_1 : B_1 \to \mathbf{R}$ and $\omega_2 : B_2 \to \mathbf{R}$ satisfy (EXC). We define their (supremum) convolution, $\omega_1 \Box \omega_2 : B_1 + B_2 \to \mathbf{R}$, by

$$(\omega_1 \Box \omega_2)(x) := \sup\{\omega_1(x_1) + \omega_2(x_2) \mid x_1 + x_2 = x, x_1 \in B_1, x_2 \in B_2\}.$$

Here

$$B_1 + B_2 := \{x_1 + x_2 \mid x_1 \in B_1, x_2 \in B_2\},$$

which is known [19] to satisfy (B1).

We first observe

$$(\omega_1 \Box \omega_2)^\circ = {\omega_1}^\circ + {\omega_2}^\circ, \tag{6.17}$$

which follows immediately from the definitions.

Theorem 6.10 *If $\omega_1 : B_1 \to \mathbf{R}$ and $\omega_2 : B_2 \to \mathbf{R}$ satisfy* (EXC)*, then $\omega_1 \Box \omega_2 : B_1 + B_2 \to \mathbf{R}$ satisfies* (EXC).

(Proof) It follows from Theorem 5.3 (Local Supermodularity Theorem) that both $\hat{L}({\omega_1}^{\circ}, p_0)$ and $\hat{L}({\omega_2}^{\circ}, p_0)$ are "matroidal" for each p_0. This implies by Lemma 5.2 that $\hat{L}({\omega_1}^{\circ}, p_0) + \hat{L}({\omega_2}^{\circ}, p_0)$ is also "matroidal" for each p_0. Noting the relation

$$\hat{L}({\omega_1}^{\circ}, p_0) + \hat{L}({\omega_2}^{\circ}, p_0) = \hat{L}({\omega_1}^{\circ} + {\omega_2}^{\circ}, p_0) = \hat{L}((\omega_1 \Box \omega_2)^{\circ}, p_0),$$

where the first equality is due to the definition of localization and the second to (6.17), we see that $\hat{L}((\omega_1 \Box \omega_2)^{\circ}, p_0)$ is "matroidal" for each p_0. Finally we use the other direction of Theorem 5.3 to conclude that $\omega_1 \Box \omega_2$ satisfies (EXC). □

Whereas (6.17) is a trivial identity, its dual counterpart, (6.18) below, relies on the duality theorem. For ${\omega_1}^{\circ}$ and ${\omega_2}^{\circ}$, we distinguish the convolutions over $\mathbf{R}$ and over $\mathbf{Z}$. Namely, we define ${\omega_1}^{\circ} \Box_{\mathbf{R}} {\omega_2}^{\circ} : \mathbf{R}^V \to \mathbf{R}$ by

$$({\omega_1}^{\circ} \Box_{\mathbf{R}} {\omega_2}^{\circ})(p) := \sup\{{\omega_1}^{\circ}(p_1) + {\omega_2}^{\circ}(p_2) \mid p_1 + p_2 = p, p_1 \in \mathbf{R}, p_2 \in \mathbf{R}\}$$

and ${\omega_1}^{\circ} \Box_{\mathbf{Z}} {\omega_2}^{\circ} : \mathbf{Z}^V \to \mathbf{R}$ by

$$({\omega_1}^{\circ} \Box_{\mathbf{Z}} {\omega_2}^{\circ})(p) := \sup\{{\omega_1}^{\circ}(p_1) + {\omega_2}^{\circ}(p_2) \mid p_1 + p_2 = p, p_1 \in \mathbf{Z}, p_2 \in \mathbf{Z}\}.$$

We define $\omega_1 + \omega_2 : B_1 \cap B_2 \to \mathbf{R}$ by

$$(\omega_1 + \omega_2)(x) := \omega_1(x) + \omega_2(x) \qquad (x \in B_1 \cap B_2),$$

provided $B_1 \cap B_2 \neq \emptyset$. Note that $\omega_1 + \omega_2$ does not necessarily satisfy (EXC).

Theorem 6.11 *If $\omega_1 : B_1 \to \mathbf{R}$ and $\omega_2 : B_2 \to \mathbf{R}$ satisfy* (EXC) *and $B_1 \cap B_2 \neq \emptyset$, then*

$$(\omega_1 + \omega_2)^{\circ} = {\omega_1}^{\circ} \Box_{\mathbf{R}} {\omega_2}^{\circ}. \tag{6.18}$$

If, in addition, ω_1 and ω_2 are integer-valued, then

$$(\omega_1 + \omega_2)^{\circ} = {\omega_1}^{\circ} \Box_{\mathbf{Z}} {\omega_2}^{\circ}, \tag{6.19}$$

where it is understood that the left-hand side denotes the restriction of $(\omega_1 + \omega_2)^{\circ}$ to $\mathbf{Z}^V$.

(Proof)

$$\begin{aligned}
(\omega_1 + \omega_2)^{\circ}(p) &= \min_{x \in B_1 \cap B_2} (\langle p, x \rangle - \omega_1(x) - \omega_2(x)) \\
&= -\max_{x \in B_1 \cap B_2} (\omega(x) - \zeta(x)) \qquad (\text{where } \omega := \omega_1[-p], \zeta := -\omega_2) \\
&= -\min_{p'} (\zeta^{\bullet}(p') - \omega^{\circ}(p')) \qquad (\text{by Theorem 6.4}) \\
&= \max_{p'} ((\omega_1[-p])^{\circ}(p') - (-\omega_2)^{\bullet}(p')) \\
&= \max_{p'} ({\omega_1}^{\circ}(p + p') + {\omega_2}^{\circ}(-p')) \qquad (\text{by Lemmas 4.2, 6.1}).
\end{aligned}$$

The integral case follows from the integrality assertion in Theorem 6.4. □

Remark 6.9 Theorem 6.11 implies

$$(\omega_1 + \omega_2)\hat{}\,(b) = \hat{\omega}_1(b) + \hat{\omega}_2(b) \qquad (b \in \overline{B_1} \cap \overline{B_2}).$$

The proof is the same as in ordinary convex analysis. See [30, Theorem 3.8]. □

7 Induction through Networks

7.1 Theorems

We show that an M-concave function can be transformed into another M-concave function through a network. This is an extension of the well known fact in matroid theory that a matroid can be transformed through a bipartite graph into another matroid.

Let $G = (V, A; V^+, V^-)$ be a (directed) graph with a vertex set V, an arc set A, a set V^+ of entrances and a set V^- of exits such that $V^+, V^- \subseteq V$ and $V^+ \cap V^- = \emptyset$. Also let $\overline{c} : A \to \mathbf{Z}$ be an upper capacity function, $\underline{c} : A \to \mathbf{Z}$ be a lower capacity function, and $w : A \to \mathbf{R}$ be a weight function. Suppose further that we are given a finite nonempty set $B^+ \subseteq \mathbf{Z}^{V^+}$ and a function $\omega^+ : B^+ \to \mathbf{R}$.

A flow is a function $\varphi : A \to \mathbf{Z}$. Its boundary $\partial\varphi : V \to \mathbf{Z}$ is defined by

$$\partial\varphi(v) := \sum\{\varphi(a) \mid a \in \delta^+ v\} - \sum\{\varphi(a) \mid a \in \delta^- v\} \qquad (v \in V), \tag{7.1}$$

where $\delta^+ v$ and $\delta^- v$ denote the sets of the out-going and in-coming arcs incident to v, respectively. We denote by $(\partial\varphi)^+$ (resp. $(\partial\varphi)^-$) the restriction of $\partial\varphi$ to V^+ (resp. V^-). A flow φ is called feasible if

$$\underline{c}(a) \le \varphi(a) \le \overline{c}(a) \qquad (a \in A), \tag{7.2}$$

$$\partial\varphi(v) = 0 \qquad (v \in V - (V^+ \cup V^-)), \tag{7.3}$$

$$(\partial\varphi)^+ \in B^+. \tag{7.4}$$

We assume throughout that a feasible flow exists.

Define $\tilde{B} \subseteq \mathbf{Z}^{V^-}$ and $\tilde{\omega} : \tilde{B} \to \mathbf{R}$ by

$$\tilde{B} := \{(\partial\varphi)^- \mid \varphi : \text{ feasible flow}\}, \tag{7.5}$$

$$\tilde{\omega}(x) := \max\{\langle w, \varphi\rangle_A + \omega^+((\partial\varphi)^+) \mid \varphi : \text{ feasible flow with } (\partial\varphi)^- = x\} \qquad (x \in \tilde{B}), \tag{7.6}$$

where

$$\langle w, \varphi\rangle_A := \sum_{a \in A} w(a)\varphi(a).$$

The following fact is easy to see from the similar results for matroids and polymatroids (cf. [19], [37]).

Lemma 7.1 *If B^+ satisfies* (B1)*, then $\tilde{B}$ of* (7.5) *satisfies* (B1). □

The following is the main result of this section.

Theorem 7.2 *If* ω^+ *satisfies* (EXC), *then* $\tilde{\omega}$ *of* (7.6) *satisfies* (EXC). □

Remark 7.1 This theorem affords an alternative proof to Theorem 6.10: Let V_1 and V_2 be disjoint copies of V (in the notation of Section 6) and consider a bipartite graph $G = (V^+, V^-, A)$ with $V^+ := V_1 \cup V_2$, $V^- := V$ and $A := \{(v_1, v) \mid v \in V\} \cup \{(v_2, v) \mid v \in V\}$, where $v_i \in V_i$ is the copy of $v \in V$ $(i = 1, 2)$. Take $\overline{c}$ sufficiently large, $\underline{c}$ sufficiently small, $w \equiv 0$, $B^+ := B_1 \times B_2$, and $\omega^+(x_1, x_2) := \omega_1(x_1) + \omega_2(x_2)$. Then we have $\omega_1 \Box \omega_2 = \tilde{\omega}$, where $\tilde{\omega}$ is the M-concave function induced from ω^+. □

As a special case of Theorem 7.2, a valuated matroid can be induced by matchings in a bipartite graph. Let $G = (V^+, V^-, A)$ be a bipartite graph, $w : A \to \mathbf{R}$ a weight function, and $\mathbf{M}^+ = (V^+, \mathcal{B}^+)$ a matroid with valuation $\omega^+ : \mathcal{B}^+ \to \mathbf{R}$. Then

$$\tilde{\mathcal{B}} := \{\partial^- M \mid M \text{ is a matching with } \partial^+ M \in \mathcal{B}^+\}$$

is known to form the base family of a matroid, provided $\tilde{\mathcal{B}} \neq \emptyset$. Here $\partial^+ M \subseteq V^+$ and $\partial^- M \subseteq V^-$ denote the sets of vertices incident to M. Define $\tilde{\omega} : \tilde{\mathcal{B}} \to \mathbf{R}$ by

$$\tilde{\omega}(X) := \max\{w(M) + \omega^+(\partial^+ M) \mid M : \text{matching}, \ \partial^+ M \in \mathcal{B}^+, \partial^- M = X\} \qquad (X \in \tilde{\mathcal{B}}). \tag{7.7}$$

Theorem 7.3 $\tilde{\omega}$ *of* (7.7) *is a valuation of* $(V^-, \tilde{\mathcal{B}})$.

This theorem has important consequences. Let $\mathbf{M}_1 = (V, \mathcal{B}_1)$ and $\mathbf{M}_2 = (V, \mathcal{B}_2)$ be matroids with valuations $\omega_1 : \mathcal{B}_1 \to \mathbf{R}$ and $\omega_2 : \mathcal{B}_2 \to \mathbf{R}$. Let $\mathbf{M}_1 \vee \mathbf{M}_2 = (V, \mathcal{B}_1 \vee \mathcal{B}_2)$ denote the union of $\mathbf{M}_1$ and $\mathbf{M}_2$, where $\mathcal{B}_1 \vee \mathcal{B}_2$ is defined to be the family of the maximal elements of $\{X_1 \cup X_2 \mid X_1 \in \mathcal{B}_1, X_2 \in \mathcal{B}_2\}$. Define $\omega_1 \vee \omega_2 : \mathcal{B}_1 \vee \mathcal{B}_2 \to \mathbf{R}$ by

$$(\omega_1 \vee \omega_2)(X) := \max\{\omega_1(X_1) + \omega_2(X_2) \mid X_1 \cup X_2 = X, X_1 \in \mathcal{B}_1, X_2 \in \mathcal{B}_2\} \qquad (X \in \mathcal{B}_1 \vee \mathcal{B}_2).$$

Theorem 7.4 $\omega_1 \vee \omega_2$ *is a valuation of the union* $\mathbf{M}_1 \vee \mathbf{M}_2$.

(Proof) Let V_1 and V_2 be disjoint copies of V, and U be a set of size equal to

$$\mathrm{rank}\,\mathbf{M}_1 + \mathrm{rank}\,\mathbf{M}_2 - \mathrm{rank}\,(\mathbf{M}_1 \vee \mathbf{M}_2).$$

Consider a bipartite graph $G = (V^+, V^-, A)$ with $V^+ := V_1 \cup V_2$, $V^- := V \cup U$ and

$$A := \{(v_1, v) \mid v \in V\} \cup \{(v_2, v) \mid v \in V\} \cup \{(v_2, u) \mid v \in V, u \in U\},$$

where $v_i \in V_i$ is the copy of $v \in V$ $(i = 1, 2)$. Let $\tilde{\omega}$ be the valuation induced on V^- from the valuation ω^+ on V^+ defined by $\omega^+(X_1 \cup X_2) := \omega_1(X_1) + \omega_2(X_2)$ $(X_i \in \mathcal{B}_i \ (i = 1, 2))$. Then $(\omega_1 \vee \omega_2)(X) = \tilde{\omega}(X \cup U)$ for $X \subseteq V$. □

For a matroid $\mathbf{M} = (V, \mathcal{B})$, its truncation to rank k is given by $\mathbf{M}_k = (V, \mathcal{B}_k)$ with

$$\mathcal{B}_k := \{X \subseteq V \mid |X| = k, \ \exists B : X \subseteq B \in \mathcal{B}\}.$$

For a valuation $\omega : \mathcal{B} \to \mathbf{R}$ of $\mathbf{M}$, define $\omega_k : \mathcal{B}_k \to \mathbf{R}$ by

$$\omega_k(X) := \max\{\omega(B) \mid X \subseteq B \in \mathcal{B}\} \qquad (X \in \mathcal{B}_k).$$

The following theorem has been established in [31]; here is an alternative proof by means of the induction through a bipartite graph.

Theorem 7.5 ([31]) *ω_k is a valuation of the truncation $\mathbf{M}_k$, where $k \leq \operatorname{rank} \mathbf{M}$.*

(Proof) Let V' be a copy of V, and U be a set of size $= \operatorname{rank} \mathbf{M} - k$. Consider a bipartite graph $G = (V^+, V^-, A)$ with $V^+ := V'$, $V^- := V \cup U$ and

$$A := \{(v', v) \mid v \in V\} \cup \{(v', u) \mid v \in V, u \in U\},$$

where $v' \in V'$ is the copy of $v \in V$. Let $\tilde{\omega}$ be the valuation induced on V^- from $\omega^+ := \omega$ on V^+. Then $\omega_k(X) = \tilde{\omega}(X \cup U)$ for $X \subseteq V$. □

7.2 Proof of Theorem 7.2

The proof of Theorem 7.2 relies on the optimality criterion of [32] for the submodular flow problem with an objective function satisfying (EXC). First we reformulate Theorem 3.1 of [32] into a form convenient for us. Suppose we are given, in addition to the network $(G = (V, A; V^+, V^-), \underline{c}, \overline{c}, w)$, a pair of M-concave functions $\omega^+ : B^+ \to \mathbf{R}$ and $\omega^- : B^- \to \mathbf{R}$, where $B^+ \subseteq \mathbf{Z}^{V^+}$ and $B^- \subseteq \mathbf{Z}^{V^-}$ are nonempty finite sets. [Do not confuse B^- with $\tilde{B}$, and ω^- with $\tilde{\omega}$.]

[Problem P]

Maximize

$$\langle w, \varphi \rangle_A + \omega^+((\partial\varphi)^+) + \omega^-((\partial\varphi)^-)$$

subject to (7.2), (7.3), (7.4) and

$$(\partial\varphi)^- \in B^-. \tag{7.8}$$

The following theorem involves a "potential" function $q : V \to \mathbf{R}$. We denote by $q^+ : V^+ \to \mathbf{R}$ and $q^- : V^- \to \mathbf{R}$ the restrictions of q and define

$$\begin{aligned} \omega^+[q^+](x) &:= \omega^+(x) + \langle q^+, x\rangle \qquad (x \in B^+), \\ \omega^-[q^-](x) &:= \omega^-(x) + \langle q^-, x\rangle \qquad (x \in B^-), \end{aligned}$$

as in (2.7). We also use the notation

$$w_q(a) := w(a) - q(\partial^+ a) + q(\partial^- a) \qquad (a \in A), \tag{7.9}$$

where $\partial^+ a$ and $\partial^- a$ denote the initial vertex and the terminal vertex of $a \in A$, respectively.

Theorem 7.6 ([32, Theorem 3.1])

(1) A flow $\varphi : A \to \mathbf{Z}$ with (7.2), (7.3), (7.4) *and* (7.8) *is optimal for* P *if and only if there exists a "potential" function $q : V \to \mathbf{R}$ such that* (i)–(iii) *below hold true.*

(i) *For each $a \in A$,*

$$w_q(a) < 0 \implies \varphi(a) = \underline{c}(a), \tag{7.10}$$

$$w_q(a) > 0 \implies \varphi(a) = \overline{c}(a). \tag{7.11}$$

(ii) $(\partial\varphi)^+$ *maximizes* $\omega^+[q^+]$.

(iii) $(\partial\varphi)^-$ *maximizes* $\omega^-[q^-]$.

Moreover, if ω^+ and ω^- are integer-valued, then q can be chosen to be also integer-valued.

(2) Let q be a potential that satisfies (i)–(iii) *above for some (optimal) flow φ. A flow φ' with* (7.2), (7.3), (7.4) *and* (7.8) *is optimal if and only if it satisfies* (i)–(iii) *(with φ replaced by φ').* □

We now start proving Theorem 7.2. By Theorem 4.4, it suffices to show that $\mathrm{argmax}\,(\tilde{\omega}[p])$ satisfies (B1) for each $p : V^- \to \mathbf{R}$.

Since $\tilde{B}$ of (7.5) is a finite set, we can find a finite set $B^- \subseteq \mathbf{Z}^{V^-}$ such that B^- satisfies (B1) and contains $\tilde{B}$ in the relative interior [36], [40] of its convex hull:

$$\tilde{B} \subseteq \mathrm{ri}\ \overline{B^-}. \tag{7.12}$$

Fix $p : V^- \to \mathbf{R}$ and define $\omega^- : B^- \to \mathbf{R}$ by $\omega^-(x) := \langle p, x\rangle$ $(x \in B^-)$. For $x \in \tilde{B}$ we have

$$\begin{aligned}\tilde{\omega}[p](x) &= \max\{\langle w, \varphi\rangle_A + \omega^+((\partial\varphi)^+) \\ &\qquad \mid (7.2),\ (7.3),\ (7.4) \text{ and } (\partial\varphi)^- = x\} + \langle p, x\rangle \\ &= \max\{\langle w, \varphi\rangle_A + \omega^+((\partial\varphi)^+) + \omega^-((\partial\varphi)^-) \\ &\qquad \mid (7.2),\ (7.3),\ (7.4) \text{ and } (\partial\varphi)^- = x\}.\end{aligned}$$

Recalling the definition of $\tilde{B}$ and the relation $\tilde{B} \subseteq B^-$, we see from this expression that

$$\operatorname{argmax}(\tilde{\omega}[p]) = \{(\partial\varphi)^- \mid \varphi : \text{ optimal for } \mathrm{P}(p)\}, \tag{7.13}$$

where $\mathrm{P}(p)$ means the problem P with ω^- defined by $\omega^-(x) := \langle p, x\rangle$.

Let $q : V \to \mathbf{R}$ be the potential function in Theorem 7.6 for the problem $\mathrm{P}(p)$. With reference to q, we define a capacitated network $(G = (V, A; V^+, V^-), \underline{c}_q, \overline{c}_q)$, where the capacity functions $\underline{c}_q$ and $\overline{c}_q$ are given by

$$\begin{aligned} w_q(a) < 0 &\quad\Longrightarrow\quad \underline{c}_q(a) := \overline{c}_q(a) := \underline{c}(a), \\ w_q(a) > 0 &\quad\Longrightarrow\quad \underline{c}_q(a) := \overline{c}_q(a) := \overline{c}(a), \\ w_q(a) = 0 &\quad\Longrightarrow\quad \underline{c}_q(a) := \underline{c}(a), \overline{c}_q(a) := \overline{c}(a). \end{aligned}$$

Recall from Theorem 4.4 that

$$B_q^+ := \operatorname{argmax}(\omega^+[q^+]) \ \subseteq \mathbf{Z}^{V^+}$$

satisfies (B1). Let B_q^- be the subset of $\mathbf{Z}^{V^-}$ induced from B_q^+ by the network $(G = (V, A; V^+, V^-), \underline{c}_q, \overline{c}_q)$ as (7.5). Note that $B_q^- \subseteq \tilde{B}$ and that B_q^- satisfies (B1) by Lemma 7.1. The proof of Theorem 7.2 can now be completed by the following claim.

Claim: $\operatorname{argmax}(\tilde{\omega}[p]) = B_q^-$.

(Proof of Claim) By Theorem 7.6, a flow φ satisfying (7.2), (7.3), (7.4) and (7.8) is optimal to $\mathrm{P}(p)$ if and only if it is a feasible flow for the network $(G = (V, A; V^+, V^-), \underline{c}_q, \overline{c}_q)$ with $B_q^+ \subseteq \mathbf{Z}^{V^+}$ such that $(\partial\varphi)^- \in \operatorname{argmax}(\omega^-[q^-])$. (It should be clear that, by definition, φ is feasible for the network $(G, \underline{c}_q, \overline{c}_q)$ with B_q^+ if φ satisfies (7.2), (7.3), and (7.4) — with $(\underline{c}, \overline{c})$ replaced by $(\underline{c}_q, \overline{c}_q)$ in (7.2) and with B^+ replaced by B_q^+ in (7.4).) Hence, in view of (7.13), we obtain

$$\operatorname{argmax}(\tilde{\omega}[p]) = B_q^- \cap \operatorname{argmax}(\omega^-[q^-]).$$

Since $\omega^-[q^-]$ is a linear function on B^-, $\overline{\operatorname{argmax}(\omega^-[q^-])}$ is a face of $\overline{B^-}$. On the other hand, a proper face of $\overline{B^-}$ is disjoint from $\tilde{B}$ by (7.12). This means that

$\overline{\text{argmax}\,(\omega^-[q^-])} = \overline{B^-}$ since $(\partial\varphi)^- \in \tilde{B} \cap \text{argmax}\,(\omega^-[q^-])$ for an optimal flow φ for P(p). Hence the claim follows. □

8 Conclusion

In this paper, we have restricted ourselves to functions ω defined on the integral points in base polytopes (bounded base polyhedra). The boundedness assumption is not essential: all the results can be extended *mutatis mutandis* to the unbounded case [35]. For instance, the first part of the Fenchel-type duality (Theorem 6.4) in the general case reads as follows:

Theorem 8.1 *Let $\omega : B_1 \to \mathbf{R}$ and $\zeta : B_2 \to \mathbf{R}$ be such that ω and $-\zeta$ satisfy* (EXC)*, where B_1 and B_2 are nonempty (possibly unbounded) subsets of $\mathbf{Z}^V$ satisfying* (B1)*. If $B_1 \cap B_2 \neq \emptyset$ or $\zeta^\bullet(p) - \omega^\circ(p) \neq +\infty$ for some $p \in \mathbf{R}^V$, then*

$$\sup\{\omega(x) - \zeta(x) \mid x \in B_1 \cap B_2\} = \inf\{\zeta^\bullet(p) - \omega^\circ(p) \mid p \in \mathbf{R}^V\}.$$

□

In the Local Supermodularity Theorem, we have characterized the exchangeability (EXC) in terms of the supermodularity of the localization of the conjugate function. A further investigation into the conjugacy between the exchangeability and the sub/supermodularity can be found in [35].

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References

[1] A. Bouchet and W. H. Cunningham: Delta-matroids, jump systems, and bisubmodular polyhedra, *SIAM Journal on Discrete Mathematics* **8** (1995), 17–32.

[2] A. W. M. Dress and W. Terhalle, Well-layered maps — A class of greedily optimizable set functions, *Applied Mathematics Letters*, **8** (1995), 77–80.

[3] A. W. M. Dress and W. Terhalle, Well-layered maps and the maximum-degree $k \times k$-subdeterminant of a matrix of rational functions, *Applied Mathematics Letters* **8** (1995), 19–23.

[4] A. W. M. Dress and W. Terhalle, Rewarding maps — On greedy optimization of set functions, *Advances in Mathematics* **16** (1995), 464–483.

[5] A. W. M. Dress and W. Wenzel, Valuated matroid: A new look at the greedy algorithm, *Applied Mathematics Letters* **3** (1990), 33–35.

[6] A. W. M. Dress and W. Wenzel, A greedy-algorithm characterization of valuated Δ-matroids, *Applied Mathematics Letters* **4** (1991), 55–58.

[7] A. W. M. Dress and W. Wenzel, Perfect matroids, *Advances in Mathematics* **91** (1992), 158–208.

[8] A. W. M. Dress and W. Wenzel, Valuated matroids, *Advances in Mathematics* **93** (1992), 214–250.

[9] J. Edmonds, Submodular functions, matroids and certain polyhedra, in "Combinatorial Structures and Their Applications" (R. Guy, H. Hanai, N. Sauer and J. Schönsheim, eds.), Gordon and Breach, New York, pp. 69–87, 1970.

[10] J. Edmonds, Matroid intersection, *Annals of Discrete Mathematics* **14** (1979), 39–49.

[11] J. Edmonds and R. Giles, A min-max relation for submodular functions on graphs, *Annals of Discrete Mathematics* **1** (1977), 185–204.

[12] U. Faigle, Matroids in combinatorial optimization, in "Combinatorial Geometries" (N. White, ed.), Cambridge University Press, London, pp. 161–210, 1987.

[13] A. Frank, A weighted matroid intersection algorithm, *Journal of Algorithms* **2** (1981), 328–336.

[14] A. Frank, An algorithm for submodular functions on graphs, *Annals of Discrete Mathematics* **16** (1982), 97–120.

[15] A. Frank, Submodular flows, in “Progress in Combinatorial Optimization” (W. R. Pulleyblank, ed.), Academic Press, New York, pp. 147–165, 1984.

[16] A. Frank and E. Tardos, Generalized polymatroids and submodular flows, *Mathematical Programming* **42** (1988), 489–563.

[17] S. Fujishige, Algorithms for solving the independent-flow problems, *Journal of the Operations Research Society of Japan* **21** (1978), 189–204.

[18] S. Fujishige, Theory of submodular programs: A Fenchel-type min-max theorem and subgradients of submodular functions, *Mathematical Programming* **29** (1984), 142-155.

[19] S. Fujishige, “Submodular Functions and Optimization,” Annals of Discrete Mathematics **47**, North-Holland, Amsterdam, 1991.

[20] H. Groenevelt, Two algorithms for maximizing a separable concave function over a polymatroid feasible region, Working Paper, The Graduate School of Management, The University of Rochester, August, 1995.

[21] M. Grötschel, L. Lovász and A. Schrijver, “Geometric Algorithms and Combinatorial Optimization (2nd ed.),” Springer-Verlag, Berlin, 1993.

[22] M. Iri and N. Tomizawa, An algorithm for finding an optimal “independent assignment”, *Journal of the Operations Research Society of Japan* **19** (1976), 32–57.

[23] B. Korte, L. Lovász and R. Schrader, “Greedoids ,” Springer-Verlag, Berlin, 1991.

[24] E. L. Lawler, “Combinatorial Optimization: Networks and Matroids,” Holt, Rinehart and Winston, New York, 1976.

[25] L. Lovász, Submodular functions and convexity, in “Mathematical Programming – The State of the Art” (A. Bachem, M. Grötschel and B. Korte, eds.), Springer-Verlag, Berlin, pp. 235–257, 1983.

[26] L. Lovász and M. Plummer, "Matching Theory," North-Holland, Amsterdam, 1986.

[27] K. Murota, Finding optimal minors of valuated bimatroids, *Applied Mathematics Letters* **8** (1995), 37–42.

[28] K. Murota, Valuated matroid intersection, I: optimality criteria, to appear in *SIAM Journal on Discrete Mathematics* **9** (1996), No. 3.

[29] K. Murota, Valuated matroid intersection, II: algorithms, to appear in *SIAM Journal on Discrete Mathematics* **9** (1996), No. 3.

[30] K. Murota, Fenchel-type duality for matroid valuations, Report No. 95839-OR, Forschungsinstitut für Diskrete Mathematik, Universität Bonn, 1995.

[31] K. Murota, Matroid valuation on independent sets, Report 95842-OR, Forschungsinstitut für Diskrete Mathematik, Universität Bonn, 1995.

[32] K. Murota, Submodular flow problem with a nonseparable cost function, Report No. 95843-OR, Forschungsinstitut für Diskrete Mathematik, Universität Bonn, 1995.

[33] K. Murota, Characterizing a valuated delta-matroid as a family of delta-matroids, Report No. 95849-OR, Forschungsinstitut für Diskrete Mathematik, Universität Bonn, 1995.

[34] K. Murota, Convexity and Steinitz's exchange property (Extended abstract), *Proceedings of Integer Programming and Combinatorial Optimization, V*, June 1996, Lecture Notes in Computer Science, 1084, Springer-Verlag, 1996, pp. 260–274.

[35] K. Murota, Discrete convex analysis, RIMS Preprint No. 1065, Research Institute for Mathematical Sciences, Kyoto University, 1996.

[36] R. T. Rockafellar, "Convex Analysis," Princeton University Press, Princeton, NJ, 1970.

[37] A. Schrijver, "Matroids and Linking Systems," Mathematics Centre Tracts **88**, Amsterdam, 1978.

[38] A. Schrijver, Total dual integrality from directed graphs, crossing families, and sub- and supermodular functions, in "Progress in Combinatorial Optimization" (W. R. Pulleyblank, ed.), Academic Press, New York, pp. 315–361, 1984.

[39] A. Schrijver, "Theory of Linear and Integer Programming," Wiley, Chichester, 1986.

[40] J. Stoer and C. Witzgall, "Convexity and Optimization in Finite Dimensions I," Springer-Verlag, Berlin, 1970.

[41] N. Tomizawa: Theory of hyperspaces (I) – supermodular functions and generalization of concept of 'bases', Papers of the Technical Group on Circuit and System Theory, Institute of Electronics and Communication Engineers of Japan, CAS80-72 (1980). (In Japanese.)

[42] D. J. A. Welsh, "Matroid Theory,"Academic Press, London, 1976.

[43] W. Wenzel, Pfaffian forms and Δ-matroids, *Discrete Mathematics* **115** (1993), 253–266.

[44] N. White (ed.), "Theory of Matroids," Cambridge University Press, London, 1986.