Internat. J. Math. & Math. Sci.
Vol. 2 (1979) 1-13

CONVEXITY, BOUNDEDNESS, AND ALMOST PERIODICITY FOR DIFFERENTIAL EQUATIONS IN HILBERT SPACE

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(Received October 27, 1978)

ABSTRACT. There are three kinds of results. First we extend and sharpen a convexity inequality of Agmon and Nirenberg for certain differential inequalities in Hilbert space. Next we characterize the bounded solutions of a differential equation in Hilbert space involving and arbitrary unbounded normal operator. Finally, we give a general sufficient condition for a bounded solution of a differential equation in Hilbert space to be almost periodic.

KEY WORDS AND PHRASES. *Differential equations in Hilbert space, Convexity inequality, Self-adjoint operators, Bounded solutions, Almost periodic solutions.*

AMS(MOS) SUBJECT CLASSIFICATION (1970) CODES. *Primary 34G05, 47A50 Secondary 34C25, 47B15.*

1. INTRODUCTION. Let S_1 , S_2 be two commuting self-adjoint operators on a complex Hilbert space H. Let $u : [a,b] \to H$ satisfy the inequality

$$\|du(t)/dt - (S_1 + iS_2)\, u(t)\| \le \phi(t)\, \|u(t)\| , \quad a \le t \le b , \tag{1.1}$$

where $\int_a^b \phi(t)dt \le c < 1/2$. We shall show that this implies the convexity inequality

$$\|u(t)\| \le K_c\, \|u(a)\|^{\frac{b-t}{b-a}}\, \|u(b)\|^{\frac{t-a}{b-a}} ,$$

which holds for some constant K_c and all $t \in [a,b]$. S. Agmon and L. Nirenberg [1] first proved this assuming $c = 2^{-3/2}$; recently S. Zaidman [7] extended it to weak solutions of (1.1). Our results apply to weak solutions and to the range of values $0 < c < 1/2$; moreover, we obtain a smaller constant K_c than did these previous authors. This result is presented in Section 2.

Section 3 is devoted to obtaining the structure of the set of all bounded solutions of

$$du(t)/dt = (S_1 + iS_2)u(t) \qquad (-\infty < t < \infty).$$

The results generalize and improve a recent result of Zaidman [8].

In Section 4 we study almost periodic solutions of the inhomogeneous equation

$$du(t)/dt = Au(t) + f(t) \qquad (-\infty < t < \infty) ;$$

here A is a closed linear operator on H and f is an H-valued function. Under a finite dimensionality assumption we show that bounded solutions are almost periodic. This generalizes the results obtained by Zaidman in [6].

2. A CONVEXITY THEOREM. Let u map the real interval $[a,b]$ into a complex Hilbert space $\mathcal{H}$ with inner product $\langle\cdot,\cdot\rangle$. Let $B : \mathcal{D}(B) \subset \mathcal{H} \to \mathcal{H}$ be a closed, densely defined linear operator. u is a *strong solution* of

$$\|du(t)/dt - Bu(t)\| \leq \phi(t)\ \|u(t)\| \tag{2.1}$$

if u is continuously differentiable on $[a,b]$, takes values in $\mathcal{D}(B)$, and $f(t) \equiv du/dt - Bu$ satisfies $\|f(t)\| \leq \phi(t)\ \|u(t)\|$, $a \leq t \leq b$. u is a *weak solution* of (2.1) if u is continuous and for continuously differentiable functions ψ with compact support in $]a,b[$ and with values in $\mathcal{D}(B^*)$, we have

$$-\int_a^b \langle u(t),\psi'(t)\rangle\, dt = \int_a^b \{\langle u(t),B^*\psi(t)\rangle + \langle f(t),\psi(t)\rangle\}dt\ ,$$

$$\|f(t)\| \leq \phi(t)\ \|u(t)\|\ , \quad a \leq t \leq b.$$

That a strong solution of (2.1) is a weak solution follows from an integration by parts.

THEOREM 2.1. *Let* $u : [a,b] \to \mathcal{H}$ *be a weak solution of* (2.1) *where* B *is symmetric. If*

$$\int_a^b \phi(t)dt \leq c < 1/2\ , \tag{2.2}$$

then the convexity inequality

$$\|u(t)\| \leq K_c\ \|u(a)\|^{\alpha}\ \|u(b)\|^{1-\alpha}\ , \tag{2.3}$$

holds, where

$$\alpha = \frac{b-t}{b-a}\ , \quad K_c = \left(\frac{2}{1-2c}\right)^{1/2} .$$

In particular, when $c = 1/2\sqrt{2}$, we get $K_c = (4 + 2\sqrt{2})^{1/2}$. Agmon and Nirenberg [1] proved this result for strong solutions, taking $c = 1/2\sqrt{2}$ and obtaining the constant $K_c = 2\sqrt{2}$ ($> (4 + 2\sqrt{2})^{1/2}$). This result also appears in Friedman's book [3, p.219]. Zaidman [7] extended the Agmon-Nirenberg result to weak solutions. The new features of Theorem 2.1 are (i) the result is extended to cover the case $\frac{1}{2\sqrt{2}} < c < \frac{1}{2}$, (ii) the constant K_c is sharpened for each value of c (including $c \le 1/2\sqrt{2}$).

By enlarging the Hilbert space $\mathcal{H}$, we can extend B to be a self-adjoint operator (cf. Sz.-Nagy [5]). Also, for S_1 and S_2 commuting self-adjoint operators (i.e., e^{itS_1} and e^{isS_2} commute for all real t and s), we may extend the theorem to the case where B is replaced by the (unbounded) normal operator $S_1 + iS_2$ according to the observation made in [1, p.138].

PROOF OF THEOREM 2.1. The proof follows Zaidman [7, pp. 236-244] with the following changes on pp. 242-244. We use Zaidman's notation. From

$$\|u(t)\|^2 \le \|u_1(b)\|^2 + \|u_2(a)\|^2 + 2M\int_a^b \|f(s)\|\, ds$$

(cf. [7, p.242, line 3]) we get

$$\|u(t)\|^2 \le \|u_1(b)\|^2 + \|u_1(a)\|^2 + \varepsilon M^2 + \varepsilon^{-1}\left(\int_a^b \|f(s)\|\, ds\right)^2$$

for each $\varepsilon > 0$; here $M = \sup\{\|u(s)\| : a \le s \le b\}$. This implies

$$M^2 \le \beta + \varepsilon M^2 + \varepsilon^{-1} N^2$$

where $\beta = \|u(a)\|^2 + \|u(b)\|^2$, $N = \int_a^b \|f(s)\|\, ds$. Consequently

$$M^2 \le (\beta + \varepsilon^{-1} N^2)(1 - \varepsilon)^{-1} \tag{2.4}$$

for $0 < \varepsilon < 1$. (This becomes [7, p.242, eqn. (*)] when $\varepsilon = 1/2$.) Since u is a weak solution of $u' - Bu = f$ (where $\|f(t)\| \le \phi(t)\,\|u(t)\|$), it follows that $\omega_\sigma(t) \equiv e^{\sigma t}\,u(t)$ is a weak solution of $\omega' - B_\sigma\omega_\sigma = e^{\sigma t}\,f(t)$ where $B_\sigma = B - \sigma I$ (cf. [7, Lemma 4, p.242]). Letting

$$M_\sigma = \sup\,\{\|e^{\sigma t}\,u(t)\|^2 : a \le t \le b\}\,,$$

$$B_\sigma = \|e^{\sigma a}\,u(a)\|^2 + \|e^{\sigma b}\,u(b)\|^2\,,$$

$$N_\sigma = \int_a^b \|e^{\sigma t}\,f(t)\|\,dt\,,$$

we have that (2.4) (applied to ω_σ rather than u) yields

$$M_\sigma^{\;2} \le (\beta_\sigma + \varepsilon^{-1}\,N_\sigma^{\;2})(1-\varepsilon)^{-1} \tag{2.5}$$

for all real σ and all ε, $0 < \varepsilon < 1$. But by (2.1) and (2.2),

$$N_\sigma \le \int_a^b e^{\sigma t}\,\phi(t)\,\|u(t)\|\,dt$$

$$\le \sup\,\{\|e^{\sigma s}\,u(s)\| : a \le s \le b\}\int_a^b \phi(t)\,dt$$

$$\le M_\sigma\,c.$$

Squaring this gives

$$N_\sigma^{\;2} \le M_\sigma^{\;2}\,c^2.$$

Plugging into (2.5) yields

$$M_\sigma^{\;2} \le (\beta_\sigma + \varepsilon^{-1}\,c^2\,M_\sigma^{\;2})(1-\varepsilon)^{-1}$$

or

$$M_\sigma^{\;2} \le \frac{\varepsilon\,\beta_\sigma}{\varepsilon(1-\varepsilon) - c^2} \tag{2.6}$$

provided $0 < \varepsilon < 1$ and $\varepsilon(1-\varepsilon) > c^2$, i.e., $0 < c < 1/2$ and $|2\varepsilon - 1| < (1-4c^2)^{1/2}$. As in [7, pp. 243, 244], $u(a) = 0$ or $u(b) = 0$ implies $u \equiv 0$, so to prove the theorem we may suppose $u(a) \neq 0$, $u(b) \neq 0$. Choosing $\sigma = (b-a)^{-1} \log(\|u(a)\| / \|u(b)\|)$ makes $e^{\sigma t} = (\|u(a)\| / \|u(b)\|)^{\frac{t}{b-a}}$ and $\|e^{\sigma a} u(a)\| = \|e^{\sigma b} u(b)\|$. Thus (2.6) becomes, for all $t \in [a,b]$,

$$\left(\frac{\|u(a)\|}{\|u(b)\|}\right)^{\frac{2t}{b-a}} \|u(t)\|^2 \leq L \left\{ \|u(a)\|^2 \left(\frac{\|u(a)\|}{\|u(b)\|}\right)^{\frac{2a}{b-a}} + \|u(b)\|^2 \left(\frac{\|u(a)\|}{\|u(b)\|}\right)^{\frac{2b}{b-a}} \right\}$$

$$= 2L \left(\frac{\|u(a)\|^{2b}}{\|u(b)\|^{2a}}\right)^{\frac{1}{b-a}}$$

where $L = \varepsilon(\varepsilon(1-\varepsilon) - c^2)^{-1}$. Consequently

$$\|u(t)\| \leq (2L)^{1/2} \|u(a)\|^{\frac{b-t}{b-a}} \|u(b)\|^{\frac{t-a}{b-a}}$$

holds for $a \leq t \leq b$. Regard $g(\varepsilon) \equiv (2L)^{1/2} = \left(\frac{2\varepsilon}{\varepsilon(1-\varepsilon) - c^2}\right)^{1/2}$ as a function of ε. It is minimized when $\varepsilon = c^{*}$, in which case $(2L)^{1/2} = \left(\frac{2}{1-2c}\right)^{1/2}$. This is a legitimate choice of ε since $|2\varepsilon - 1| < (1-4c^2)^{1/2}$ holds in this case. The proof of the theorem is now complete.

3. <u>BOUNDED SOLUTIONS</u>. Let S_1, S_2 be commuting self-adjoint operators on H. We study functions $u \in C^1(\mathbb{R}, H)$ $(\mathbb{R} =]-\infty,\infty[)$ which are bounded (strong) solutions of

$$du(t)/dt = (S_1 + iS_2)\, u(t) \ , \quad t \in \mathbb{R} . \tag{3.1}$$

LEMMA 3.1. *Let* u *be a bounded solution of* (3.1). *Then* $u(t) = e^{itS_2} h$ *for all* $t \in \mathbb{R}$ *and some* $h \in \mathrm{Ker}(S_1) = \{f \in \mathcal{H} : S_1 f = 0\}$.

PROOF. Let $h = u(0)$. Then

$$u(t) = e^{tS_1}(e^{itS_2} h) = e^{itS_2}(e^{tS_1} h).$$

(Recall that e^{tS_1}, e^{itS_2} are defined by the operational calculus associated with the spectral theorem.) Since e^{itS_2} is unitary, $\|u(t)\| = \|e^{tS_1} h\|$ follows. But $\|e^{tS_1} h\|$ is bounded for $t \in \mathbb{R}$ if and only if $h \in \mathrm{Ker}(S_1)$, in which case $e^{tS_1} h = h$, and so $u(t) = e^{itS_2} h$, as advertised.

A special case occurs when

$$\mathrm{Ker}(S_1) = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_n ,$$

where S_2 restricted to $\mathcal{M}_j$ is a real constant λ_j times the identity on $\mathcal{M}_j$ for $1 \le j \le n$. Then any bounded solution of (3.1) is of the form

$$u(t) = \sum_{j=1}^{n} e^{it\lambda_j} h_j \tag{3.2}$$

where h_j is the orthogonal projection of $u(0)$ onto $\mathcal{M}_j$, $1 \le j \le n$. This covers the result obtained by Zaidman in [8]. More precisely, let $\{E(\theta) : \theta \in \mathbb{R}\}$ be a resolution of the identity and let

$$S_1 = \int_{-\infty}^{\infty} x(\theta)\, dE(\theta) \ , \quad S_2 = \int_{-\infty}^{\infty} y(\theta)\, dE(\theta)$$

be associated commuting self-adjoint operators, where x and y are continuous real functions on $\mathbb{R}$. If the zero set of x is the finite set $\{\theta_1, \ldots, \theta_n\}$ then S_2 is $\lambda_j = y(\theta_j)$ times the identity on $M_j = (E(\theta_j^+) - E(\theta_j^-))(H)$, $1 \le j \le n$, and so any bounded solution of (3.1) is of the form (3.2) with $h_j \in M_j$, $1 \le j \le n$. This is Zaidman's result [8].

4. ALMOST PERIODIC SOLUTIONS.

THEOREM 4.1. *Let* $A : H \to H$ *be a bounded linear operator and let* $f : \mathbb{R} \to H$ *be almost periodic. Let* $u : \mathbb{R} \to H$ *be a bounded (i.e.* $\sup \{\|u(t)\| : t \in \mathbb{R}\} < \infty$*) strong solution of*

$$du(t)/dt = Au(t) + f(t) \qquad (t \in \mathbb{R}) . \tag{4.1}$$

$$\left.\begin{array}{l}\textit{Suppose there is a finite dimensional subspace } H_1 \textit{ of } H \\ \textit{such that } H_1 \supset \{Af(s) : s \in \mathbb{R}\} \cup \{Au(0)\} \textit{ and} \\ e^{tA}(H_1) \subset H_1 \textit{ for all } t \in \mathbb{R} .\end{array}\right\} \tag{4.2}$$

Then u *is almost periodic.*

When H is finite dimensional, this is the classical Bohr-Neugebauer-Bochner theorem (cf. Amerio-Prouse [2, p.85]). When A is a finite rank operator we can take H_1 to be the range of A, and Theorem 4.1 becomes the theorem of Zaidman [6] in this case.

PROOF OF THEOREM 4.1. Let $H_2 = H \ominus H_1$ be the orthogonal complement of H_1, and let P_j be the orthogonal projection onto H_j, $j = 1,2$. Let $u_j(t) = P_j u(t)$, $j = 1,2$. Note that if L is as upper bound for $\|u(s)\|$ $(s \in \mathbb{R})$, then for all real t,

$$L^2 \ge \|u(t)\|^2 = \|u_1(t)\|^2 + \|u_2(t)\|^2 ,$$

whence u_1 and u_2 are bounded. Also,

$$du/dt = du_1/dt + du_2/dt = Au_1 + Au_2 + P_1f + P_2f .$$

Applying P_1 to both sides gives

$$du_1/dt = P_1Au_1 + P_1Au_2 + P_1f . \tag{4.3}$$

The function u admits the variation of parameters representation

$$u(t) = e^{tA}u(0) + \int_0^t e^{(t-s)A} f(s)\, ds$$

$$= e^{tA}u(0) + \int_0^t f(s)\, ds + \sum_{n=1}^{\infty} \int_0^t \frac{(t-s)^n}{n!} A^n f(s)\, ds .$$

The last (summation) term belongs to H_1 by (4.2). Applying P_2 to this expression gives

$$u_2(t) = P_2\, e^{tA}u(0) + \int_0^t P_2\, f(s)\, ds ;$$

differentiating yields

$$du_2(t)/dt = P_2\, e^{tA}Au(0) + P_2\, f(t) = P_2\, f(t)$$

by (4.2). Since f is almost periodic and P_2 is bounded it follows that du_2/dt is almost periodic. Since u_2 is bounded, u_2 itself is almost periodic (see [2, p.55]).

Next, by (4.3),

$$du_1(t)/dt = P_1Au_1(t) + g(t) , \tag{4.4}$$

where $g(t) \equiv P_1Au_2(t) + P_1f(t)$ is almost periodic. Since u_1 is bounded and $P_1A : H_1 \to H_1$ is linear, (4.4) is a linear system in the finite dimensional Hilbert space H_1 (see (4.2)). It follows from the classical Bohr-Neugebauer-Bochner theorem [2] that u_1 is almost periodic. Consequently $u = u_1 + u_2$ is almost periodic, and the proof is complete.

Theorem 4.1 can be easily extended to the case when A is unbounded, as follows.

THEOREM 4.2. *Let* $A : \mathcal{D}(A) \subset H \to H$ *generate a* (C_0) *group of bounded linear operators* $\{T(t) : t \in \mathbb{R}\}$ *on* H (cf. [4]). *Let* $u : \mathbb{R} \to H$ *be a bounded solution of* (4.1) *where* f *is almost periodic. Suppose there is a finite dimensional subspace* H_1 *of* H *such that*

$$H_1 \supset \{(T(t) - I) f(s) : s \in \mathbb{R}, t \in \mathbb{R}\} \cup \{Au(0)\}$$

and $T(t)(H_1) \subset H_1$ *for all* $t \in \mathbb{R}$. *Then* u *is almost periodic.*

The proof, which differs from the proof of Theorem 4.1 only in inessential ways, is omitted.

COROLLARY 4.3. *Let* $\lambda_1,\ldots,\lambda_n$ *be eigenvalues of the linear operator* $A : \mathcal{D}(A) \subset H \to H$ *and let* $\phi_1,\ldots,\phi_n$ *be corresponding eigenvectors. Let* H_1 *be the span of* $\phi_1,\ldots,\phi_n$. *Then any bounded solution of* (4.1) *is almost periodic, provided* $f : \mathbb{R} \to H_1$ *is almost periodic and* $u(0) \in H_1$.

This follows immediately from Theorem 4.2.

COROLLARY 4.4. *In Corollary* 4.3 *one can omit the hypothesis that* $u(0) \in H_1$ *provided that one assumes that* A *is a compact normal operator.*

PROOF. Let P_1 , P_2 , u_1 , u_2 be as in the proof of Theorem 4.1. Applying P_j to (4.1) and noting that A commutes with P_j in this case gives

$$du_1(t)/dt = Au_1(t) + f(t) \ ,$$
$$du_2(t)/dt = Au_2(t) \qquad (t \in \mathbb{R}) \ . \tag{4.5}$$

u_1 is almost periodic by the Bohr-Neugebauer-Bochner theorem. Thus it only remains to show that u_2 is almost periodic. Let B be the restriction of A to H_2. B is a compact normal operator, hence by the spectral theorem there is an orthonormal basis $\{\psi_m\}$ for H_2 and complex numbers $\mu_m \to 0$ such that

$$B\phi = \sum_{m=1}^{\infty} \mu_m \langle\phi,\psi_m\rangle \psi_m$$

for all $\phi \in H_2$. Let Q_m be the orthogonal projection (in H_2) onto the span of $\psi_1, \ldots, \psi_m$. Let $v_m = Q_m u_2$. Then

$$dv_m/dt = Q_m \, du_2/dt = Q_m \, Au_2 = Bv_m$$

by (4.5). Also, v_m is bounded (since u_2 is) and takes values in a finite dimensional space, whence v_m is almost periodic. We *claim* that $u_2(t) = \lim_{m\to\infty} v_m(t)$, uniformly for $t \in \mathbb{R}$. It then follows that u_2 is almost periodic [2] and the proof is done. So it only remains to prove the *claim*. We have

$$\frac{d}{dt}(u_2(t) - v_m(t)) = B(u_2(t) - v_m(t)) = (B - Q_m B)(u_2(t) - v_m(t)) \ ,$$

therefore

$$u_2(t) - v_m(t) = \sum_{k=m+1}^{\infty} e^{t\mu_k} \langle(u_2 - v_m)(0), \psi_k\rangle \psi_k \ .$$

Consequently

$$\|u_2(t) - v_m(t)\|^2 = \sum_{k=m+1}^{\infty} e^{t \operatorname{Re} \mu_k} |\langle(u_2 - v_m)(0), \psi_k\rangle|^2 \ .$$

Since $\|u_2(t) - v_m(t)\| \le \|u_2(t)\| \le L < \infty$ for some L and all $t \in \mathbb{R}$, it follows that for every k for which $\langle (u_2 - v_m)(0), \psi_k \rangle \neq 0$ for some m, μ_k must be purely imaginary. Therefore

$$\|u_2(t) - v_m(t)\|^2 = \sum_{k=m+1}^{\infty} |\langle (u_2 - v_m)(0), \psi_k \rangle|^2$$
$$= \|(I - Q_m)u_2(0)\|^2 \to 0$$

as $n \to \infty$, uniformly for $t \in \mathbb{R}$. Q.E.D.

ACKNOWLEDGEMENT: This research is an outgrowth of discussions I had with Samuel Zaidman in April, 1978 when I had the pleasure to visit the Université de Montréal. I thank Professor Zaidman for some stimulating dicussions, and I gratefully acknowledge that my trip to Montréal was supported by his NRC grant. Finally, I gratefully acknowledge the support of an NSF grant.

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