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Convexity estimates for mean curvature flow and singularities of mean convex surfaces

by

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1. Introduction

Let $F_0: \mathcal{M} \to \mathbf{R}^{n+1}$ be a smooth immersion of a closed n-dimensional hypersurface of nonnegative mean curvature in Euclidean space, $n \geqslant 2$. The evolution of $\mathcal{M}_0 = F_0(\mathcal{M})$ by mean curvature flow is the one-parameter family of smooth immersions $F: \mathcal{M} \times [0, T[\to \mathbf{R}^{n+1}$ satisfying

$$\frac{\partial F}{\partial t}(p,t) = -H(p,t)\nu(p,t), \quad p \in \mathcal{M},\ t \geqslant 0, \tag{1.1}$$

$$F(\,\cdot\,,0) = F_0, \tag{1.2}$$

where $H(p,t)$ and $\nu(p,t)$ are the mean curvature and the outer normal respectively at the point $F(p,t)$ of the surface $\mathcal{M}_t = F(\,\cdot\,,t)(\mathcal{M})$. The signs are chosen such that $-H\nu = \vec{H}$ is the mean curvature vector and the mean curvature of a convex surface is positive.

For closed surfaces the solution of (1.1)–(1.2) exists on a finite maximal time interval $[0, T[$, $0 < T < \infty$, and the curvature of the surfaces becomes unbounded for $t \to T$. It is important to obtain a detailed description of the singular behaviour for $t \to T$, a future goal being the topologically controlled extension of the flow past singularities.

In the present paper we use the assumption of nonnegative mean curvature to derive new a priori estimates from below for all other elementary symmetric functions of the principal curvatures, strong enough to conclude that any rescaled limit of a singularity is (weakly) convex.

Let $\vec{\lambda} = (\lambda_1, ..., \lambda_n)$ be the principal curvatures of the evolving hypersurfaces $\mathcal{M}_t$, and let

$$S_k(\lambda) = \sum_{1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n} \lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_k}$$

be the elementary symmetric functions with $S_1=H$. Then our main result is

THEOREM 1.1. *Suppose that $F_0: \mathcal{M} \to \mathbf{R}^{n+1}$ is a smooth closed hypersurface immersion with nonnegative mean curvature. For each k, $2 \leqslant k \leqslant n$, and any $\eta>0$ there is a constant $C_{\eta,k}$ depending only on n,k,η and the initial data, such that everywhere on $\mathcal{M} \times [0,T[$ we have the estimate*

$$S_k \geqslant -\eta H^k - C_{\eta,k}. \tag{1.3}$$

The arbitrariness of η breaks the scaling invariance in inequality (1.3) and implies that near a singularity, where $S_1=H$ becomes unbounded, each S_k becomes nonnegative after rescaling:

COROLLARY 1.2. *Let $\mathcal{M}_t$ be a mean convex solution of mean curvature flow on the maximal time interval $[0,T[$ as in Theorem 1.1. Then any smooth rescaling of the singularity for $t \to T$ is convex.*

For a discussion of previous results on the blowup behaviour of mean curvature flow see [12], where we proved a lower bound as in (1.3) for the scalar curvature of $\mathcal{M}_t$. We note that the present result applies to type-II singularities which up to now were understood only in the one-dimensional case. We give a classification of type-II singularities in the mean convex case in Theorem 4.1, complementary to the known classification of type-I singularities in [10], [11].

In view of the results in [14] and [9] respectively, Theorem 1.1 and Corollary 1.2 also hold for star-shaped surfaces in $\mathbf{R}^{n+1}$ and for mean convex surfaces in smooth Riemannian manifolds, see Remarks 3.8 and 3.9.

The proof of Theorem 1.1 proceeds by induction on the degree k of the elementary symmetric polynomial S_k. Assuming that the desired inequalities in (1.3) hold up to some $k \geqslant 2$, we perturb the second fundamental form $A=\{h_{ij}\}$ by adding a small multiple of the metric $g=\{g_{ij}\}$, setting

$$b_{ij} := h_{ij} + \varepsilon H g_{ij} + D g_{ij} \tag{1.4}$$

for small $\varepsilon>0$ and positive D. In view of the induction hypothesis (1.3) we manage to choose $D=D_\varepsilon$ in such a way that the elementary symmetric function $\widetilde{S}_k$ of the eigenvalues of $\{b_{ij}\}$ is strictly positive, allowing us to work with the quotient $\widetilde{Q}_{k+1}=\widetilde{S}_{k+1}/\widetilde{S}_k$ as a test function. The algebraic properties of $S_k, \widetilde{S}_k$ and $Q_{k+1}, \widetilde{Q}_{k+1}$ established in §2, in particular the concavity of Q_{k+1} on the set where $S_k>0$, turn out to be crucial for the proof of the a priori estimates in §3.

2. Symmetric polynomials

As we said in the introduction, in this paper we are concerned with symmetric functions of the principal curvatures on a manifold. We begin by recalling the definition and some basic properties of the elementary symmetric polynomials. In the following $n \geqslant 2$ is a fixed integer.

Definition 2.1. For any $k=1,2,\ldots,n$ we set

$$s_k(\mu) = \sum_{1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n} \mu_{i_1}\mu_{i_2}\ldots\mu_{i_k}, \quad \forall \mu = (\mu_1,\ldots,\mu_n) \in \mathbf{R}^n. \tag{2.1}$$

We also set $s_0 \equiv 1$ and $s_k \equiv 0$ for $k>n$. In addition, we define

$$\Gamma_k = \{\mu \in \mathbf{R}^n : s_1(\mu) > 0, s_2(\mu) > 0, \ldots, s_k(\mu) > 0\}. \tag{2.2}$$

Clearly the sets Γ_k are open cones and satisfy $\Gamma_{k+1} \subset \Gamma_k$ for any $k=1,\ldots,n-1$. In the sequel (Theorem 2.5 and Proposition 2.6) we will also see that these cones are convex and that Γ_n coincides with the positive cone.

Let us denote by $s_{k;i}(\mu)$ the sum of the terms of $s_k(\mu)$ not containing the factor μ_i. Then the following identities hold.

PROPOSITION 2.2. *We have, for any $k=0,\ldots,n$, $i=1,\ldots,n$ and $\mu \in \mathbf{R}^n$,*

$$\frac{\partial s_{k+1}}{\partial \mu_i}(\mu) = s_{k;i}(\mu), \tag{2.3}$$

$$s_{k+1}(\mu) = s_{k+1;i}(\mu) + \mu_i s_{k;i}(\mu), \tag{2.4}$$

$$\sum_{i=1}^n s_{k;i}(\mu) = (n-k)s_k(\mu), \tag{2.5}$$

$$\sum_{i=1}^n \mu_i s_{k;i}(\mu) = (k+1)s_{k+1}(\mu), \tag{2.6}$$

$$\sum_{i=1}^n \mu_i^2 s_{k;i}(\mu) = s_1(\mu)s_{k+1}(\mu) - (k+2)s_{k+2}(\mu). \tag{2.7}$$

Proof. The properties (2.3) and (2.4) follow from the definitions, while (2.6) is a consequence of (2.3) and Euler's theorem on homogeneous functions. Taking sums over i in (2.4) and applying (2.6) we obtain (2.5). By (2.4) we also obtain, for any $\mu \in \mathbf{R}^n$,

$$s_{k+2}(\mu) - s_{k+2;i}(\mu) = \mu_i s_{k+1;i}(\mu) = \mu_i s_{k+1}(\mu) - \mu_i^2 s_{k;i}(\mu).$$

Taking sums over i and using (2.5) we find

$$(k+2)s_{k+2}(\mu)=\sum_{i=1}^{n}(\mu_i s_{k+1}(\mu)-\mu_i^2 s_{k;i}(\mu)),$$

which implies the identity (2.7). □

A less immediate property of these polynomials is the following inequality (for the proof see e.g. [7, pp. 104–105]).

THEOREM 2.3. *For any $k\in\{1,...,n-1\}$ and $\mu\in\mathbf{R}^n$ we have*

$$(n-k+1)(k+1)s_{k-1}(\mu)s_{k+1}(\mu)\leqslant k(n-k)s_k^2(\mu). \tag{2.8}$$

The above result is known as *Newton's inequality.* We observe that it implies the weaker inequality

$$s_{k-1}(\mu)s_{k+1}(\mu)\leqslant s_k^2(\mu). \tag{2.9}$$

The following property of the polynomials $s_{k;i}$ is a consequence of Newton's inequality.

LEMMA 2.4. *Let $\mu\in\Gamma_k$ be given for some $k\in\{1,...,n\}$. Then we have $s_{h;i}(\mu)>0$ for any $h\in\{0,...,k-1\}$ and $i\in\{1,...,n\}$.*

Proof. We proceed by induction on h. The case $h=0$ is trivial. Consider now an arbitrary $h\in\{1,...,k-1\}$. By the assumption that $\mu\in\Gamma_k$ and by (2.4) we have, for any $i=1,...,n$,

$$\mu_i s_{h;i}(\mu)+s_{h+1;i}(\mu)=s_{h+1}(\mu)>0, \quad \mu_i s_{h-1;i}(\mu)+s_{h;i}(\mu)=s_h(\mu)>0.$$

By the induction hypothesis we have $s_{h-1;i}(\mu)>0$. Let us argue by contradiction and suppose that $s_{h;i}(\mu)\leqslant 0$. Then we have $\mu_i s_{h-1;i}(\mu)=s_h(\mu)-s_{h;i}(\mu)>0$, which implies $\mu_i>0$. Therefore

$$s_{h+1;i}(\mu)>-\mu_i s_{h;i}(\mu)\geqslant 0, \quad \mu_i s_{h-1;i}(\mu)>-s_{h;i}(\mu)\geqslant 0.$$

We deduce that $s_{h+1;i}(\mu)s_{h-1;i}(\mu)>s_{h;i}^2(\mu)$. On the other hand, formula (2.9) evaluated for $\mu_i=0$ and $k=h$ yields the opposite inequality. The contradiction proves that $s_{h;i}(\mu)>0$. □

Let us now define, for $k=2,...,n$ and $\mu\in\Gamma_{k-1}$,

$$q_k(\mu)=\frac{s_k(\mu)}{s_{k-1}(\mu)}. \tag{2.10}$$

Similarly we set, for $i=1,...,n$, $q_{k,i}(\mu)=s_{k,i}(\mu)/s_{k-1,i}(\mu)$. Observe that, if $\mu\in\Gamma_k$, then $q_{k,i}(\mu)$ is well defined, by virtue of the previous lemma. The functions q_k are homogeneous of degree one. We now give a concavity result for these functions which is fundamental for our later purposes. Our proof is inspired by the one of [13, Theorem 1], where the concavity of q_k was proved in the positive cone.

THEOREM 2.5. *The cone Γ_k is convex and the function q_{k+1} is concave on Γ_k for any $k=1,...,n-1$. More precisely, given $\mu\in\Gamma_k$ and $\xi\in\mathbf{R}^n$, we have $\partial^2 q_{k+1}/\partial\xi^2(\mu)\leqslant 0$, and equality occurs only in the two following cases*:

(i) *ξ is a scalar multiple of μ,*

(ii) *μ has exactly $n-k$ zero components and these components are zero also for ξ.*

Proof. We proceed by induction on k and consider first the case $k=1$. The convexity of Γ_1 is immediate since Γ_1 is a half-space. In addition we have the identity

$$2q_2(\mu)-q_2(\mu+\xi)-q_2(\mu-\xi)=\frac{\sum_{i=1}^n(\xi_i s_1(\mu)-\mu_i s_1(\xi))^2}{s_1(\mu)s_1(\mu+\xi)s_1(\mu-\xi)},$$

valid for all μ,ξ such that $\mu,\mu\pm\xi\in\Gamma_1$. Using this we obtain, for all $\mu\in\Gamma_1$ and $\xi\in\mathbf{R}^n$,

$$-\frac{\partial^2 q_2}{\partial\xi^2}(\mu)=\lim_{\varepsilon\to 0}\frac{2q_2(\mu)-q_2(\mu+\varepsilon\xi)-q_2(\mu-\varepsilon\xi)}{\varepsilon^2}=\frac{\sum_{i=1}^n(\xi_i s_1(\mu)-\mu_i s_1(\xi))^2}{s_1(\mu)^3}.$$

The right-hand side is strictly positive if ξ is not a multiple of μ, and this proves the assertion in this case.

Let us now consider k arbitrary. The convexity of Γ_k follows from the induction hypothesis, since we have $\Gamma_k=\{\mu\in\Gamma_{k-1}:q_k(\mu)>0\}$. To prove the concavity of q_{k+1}, let us take $\mu\in\Gamma_k$. Then we have, by (2.4) and by Lemma 2.4,

$$\mu_i+q_{k,i}(\mu)=\frac{s_k(\mu)}{s_{k-1;i}(\mu)}>0. \tag{2.11}$$

Applying first (2.7) and then again (2.4) we obtain

$$\begin{aligned}(k+1)q_{k+1}(\mu)&=\sum_{i=1}^n\left(\mu_i-\mu_i^2\frac{s_{k-1;i}(\mu)}{s_k(\mu)}\right)\\&=\sum_{i=1}^n\left(\mu_i-\mu_i^2\frac{s_{k-1;i}(\mu)}{s_{k;i}(\mu)+\mu_i s_{k-1;i}(\mu)}\right)\\&=\sum_{i=1}^n\left(\mu_i-\frac{\mu_i^2}{q_{k;i}(\mu)+\mu_i}\right).\end{aligned} \tag{2.12}$$

Let us now take $\xi\in\mathbf{R}^n$ small enough to have $\mu\pm\xi\in\Gamma_k$. Using (2.12) we obtain

$$
\begin{aligned}
&(k+1)(2q_{k+1}(\mu)-q_{k+1}(\mu+\xi)-q_{k+1}(\mu-\xi))\\
&\quad=\sum_{i=1}^{n}\left(\frac{(\mu_i+\xi_i)^2}{q_{k;i}(\mu+\xi)+\mu_i+\xi_i}+\frac{(\mu_i-\xi_i)^2}{q_{k;i}(\mu-\xi)+\mu_i-\xi_i}-\frac{(2\mu_i)^2}{q_{k;i}(\mu+\xi)+q_{k;i}(\mu-\xi)+2\mu_i}\right)\\
&\qquad+\sum_{i=1}^{n}\left(\frac{(2\mu_i)^2}{q_{k;i}(\mu+\xi)+q_{k;i}(\mu-\xi)+2\mu_i}-\frac{2\mu_i^2}{q_{k;i}(\mu)+\mu_i}\right)\\
&\quad=\sum_{i=1}^{n}\left(\frac{[(\mu_i+\xi_i)q_{k;i}(\mu-\xi)-(\mu_i-\xi_i)q_{k;i}(\mu+\xi)]^2}{[q_{k;i}(\mu+\xi)+\mu_i+\xi_i][q_{k;i}(\mu-\xi)+\mu_i-\xi_i][q_{k;i}(\mu+\xi)+q_{k;i}(\mu-\xi)+2\mu_i]}\right)\\
&\qquad-2\sum_{i=1}^{n}\mu_i^2\frac{q_{k;i}(\mu+\xi)+q_{k;i}(\mu-\xi)-2q_{k;i}(\mu)}{(q_{k;i}(\mu+\xi)+q_{k;i}(\mu-\xi)+2\mu_i)(q_{k;i}(\mu)+\mu_i)}.
\end{aligned}
$$

It follows, for ξ arbitrary,

$$
\begin{aligned}
\frac{\partial^2 q_{k+1}}{\partial\xi^2}(\mu)&=\lim_{\varepsilon\to0}\frac{q_{k+1}(\mu+\varepsilon\xi)+q_{k+1}(\mu-\varepsilon\xi)-2q_{k+1}(\mu)}{\varepsilon^2}\\
&\leqslant\lim_{\varepsilon\to0}\sum_{i=1}^{n}\frac{2\mu_i^2}{k+1}\cdot\frac{q_{k;i}(\mu+\varepsilon\xi)+q_{k;i}(\mu-\varepsilon\xi)-2q_{k;i}(\mu)}{\varepsilon^2(q_{k;i}(\mu+\varepsilon\xi)+q_{k;i}(\mu-\varepsilon\xi)+2\mu_i)(q_{k;i}(\mu)+\mu_i)}\\
&=\sum_{i=1}^{n}\frac{\mu_i^2(\partial^2q_{k;i}/\partial\xi^2)(\mu)}{(k+1)(q_{k;i}(\mu)+\mu_i)^2}.
\end{aligned}
$$

Let us denote by $[\mu]_i$, $[\xi]_i$ the vectors obtained by setting the ith component of μ,ξ equal to zero. Then $[\mu]_i\in\Gamma_{k-1}$ for every i by Lemma 2.4. Thus, by the induction hypothesis, we have

$$
\mu_i^2\frac{\partial^2q_{k;i}}{\partial\xi^2}(\mu)=\mu_i^2\frac{\partial^2q_k}{\partial([\xi]_i)^2}([\mu]_i)\leqslant0. \tag{2.13}
$$

This proves the concavity of q_{k+1}. Let us now analyse the cases when we have equality in (2.13) for every i. We first consider the case when μ has more than k nonzero components, say for instance the first $k+1$. Then, for any $i=1,\dots,k+1$, the vector $[\mu]_i$ has at least k nonzero components and $\mu_i\neq0$. By the induction hypothesis, equality in (2.13) is possible only if $[\xi]_i$ is a multiple of $[\mu]_i$. Since this must hold for every $i=1,\dots,k+1$ and since $k+1>2$, we obtain that ξ is a scalar multiple of μ.

Let us now consider the case when μ has exactly k nonzero components, for instance the first k. By the induction hypothesis, $[\xi]_1$ and $[\mu]_1$ satisfy either property (i) or property (ii). In both cases we find that the last $n-k$ components of $[\xi]_1$ are zero. Then the same holds for ξ. This shows that μ and ξ satisfy property (ii).

The case when μ has less than k nonzero components is excluded, since it would imply that $s_k(\mu)=0$, in contradiction with our assumption that $\mu\in\Gamma_k$. □

We can now obtain a characterization of the cones Γ_k.

PROPOSITION 2.6. *The sets* Γ_k *coincide with the connected component of* $\{\mu\in\mathbf{R}^n: s_k(\mu)>0\}$ *containing the positive cone.* Γ_n *coincides with the positive cone.*

Proof. Let us denote by G_k the connected component of the set $\{\mu\in\mathbf{R}^n: s_k(\mu)>0\}$ containing the positive cone. Since Γ_k is convex and contains the positive cone, we deduce that $\Gamma_k\subset G_k$. On the other hand, it is known (see [3], [2]) that $G_k\subset G_{k-1}$ for any $k=2,...,n$. This implies that $s_1,s_2,...,s_{k-1}$ are positive in G_k, and so $G_k\subset\Gamma_k$. This proves that $G_k=\Gamma_k$. To obtain the last assertion, it suffices to observe that s_n vanishes on the boundary of the positive cone, and so G_n cannot be strictly larger than the positive cone. □

We define now, for any $k=1,...,n$, a function on the space of symmetric $(n\times n)$-matrices in the following way: to each matrix $\Theta=(\theta_{ij})$ we associate the polynomial s_k evaluated in the eigenvalues of Θ. With an abuse of notation, we use the same symbol and denote by $s_k(\mu)$ the function on vectors and by $s_k(\Theta)$ the function on matrices. Similarly, we can consider the functions q_k, $s_{k;i}$, $q_{k;i}$ as functions defined on symmetric matrices.

Let us observe that $s_k(\Theta)$ is a homogeneous polynomial of degree k of the entries θ_{ij}. In fact we have, for $t\in\mathbf{R}$,

$$\det(tI+\Theta)=t^n+s_1(\Theta)t^{n-1}+...+s_{n-1}(\Theta)t+s_n(\Theta).$$

This shows that $s_k(\Theta)$ is the sum of subdeterminants of order $k\times k$ of Θ.

Consider now a smooth n-dimensional immersed manifold $\mathcal{M}$, and let x be a local coordinate on $\mathcal{M}$. We denote as usual by $g=(g_{ij})$ the induced metric and by $A=(h_{ij})$ the second fundamental form A. Then the Weingarten operator $W: T_p\mathcal{M}\to T_p\mathcal{M}$ associated with g and A has coefficients $h^i_j=g^{il}h_{lj}$. We denote by $\lambda_1,...,\lambda_n$ the principal curvatures of $\mathcal{M}$, which are also the eigenvalues of W. They appear in the following formulas in a symmetric way and therefore the order in which we label them has no influence. Then, given $k=1,...,n$, we can consider the function $S_k:\mathcal{M}\to\mathbf{R}$ defined by $S_k=s_k(\lambda_1,...,\lambda_n)$. To simplify the notation, it is convenient sometimes to consider S_k as a function of the λ_i's or of the h^i_j's; in this way we have

$$\frac{\partial S_k}{\partial\lambda_i}=\frac{\partial s_k}{\partial\mu_i}(\lambda),\quad \frac{\partial S_k}{\partial h^i_j}=\frac{\partial s_k}{\partial\theta_{ij}}(W).$$

Thus we will write, for instance,

$$\nabla_l S_k=\sum_{i,j}\frac{\partial S_k}{\partial h^i_j}\nabla_l h^i_j\quad\text{instead of}\quad \nabla_l S_k(p)=\sum_{i,j}\frac{\partial s_k}{\partial\theta^i_j}(W)\nabla_l h^i_j(p).$$

Analogously, we can define on $\mathcal{M}$ functions $Q_k, S_{k;i}, Q_{k;i}$ evaluating $q_k, s_{k;i}, q_{k;i}$ in $(\lambda_1, \ldots, \lambda_n)$. Observe that, according to the previous definition, S_1 coincides with the mean curvature; therefore we will denote this function in the sequel by the usual letter H.

As we mentioned in the introduction, it is convenient for us to define some suitable perturbations of the quantities introduced above. Given $\varepsilon, D \geqslant 0$, let us define

$$\tilde{\lambda}_{i;\varepsilon,D} = \lambda_i + \varepsilon H + D, \quad b_{ij;\varepsilon,D} = h_{ij} + g_{ij}(\varepsilon H + D). \tag{2.14}$$

We call $\tilde{A}_{\varepsilon,D}$ (resp. $\widetilde{W}_{\varepsilon,D}$) the matrix whose entries are $b_{ij;\varepsilon,D}$ (resp. $b^i_{j;\varepsilon,D}$). Then $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n$ are the eigenvalues of $\widetilde{W}_{\varepsilon,D}$. We denote by $\widetilde{S}_{k;\varepsilon,D}$ the functions on $\mathcal{M}$ obtained by evaluating s_k at $\tilde{\lambda}_{\varepsilon,D}$ instead of λ. We will drop the subscript ε, D if there is no risk of confusion. From the definition it follows that

$$\begin{aligned}
\widetilde{H}_{\varepsilon,D} &= \widetilde{S}_{1;\varepsilon,D} = s_1(\tilde{\lambda}_{\varepsilon,D}) = (1+n\varepsilon)H + nD,\\
|\tilde{A}_{\varepsilon,D}|^2 &= \sum_{i=1}^{n} \tilde{\lambda}^2_{i;\varepsilon,D} = |A|^2 + n(\varepsilon H + D)^2 + 2H(\varepsilon H + D),\\
\widetilde{S}_{2;\varepsilon,D} &= s_2(\tilde{\lambda}_{\varepsilon,D}) = S_2 + (n-1)(\varepsilon H + D)H + \tfrac{1}{2}n(n-1)(\varepsilon H + D)^2.
\end{aligned}$$

In general, we find

$$\widetilde{S}_{k;\varepsilon,D} = \sum_{h=0}^{k} \binom{n-k+h}{h} (\varepsilon H + D)^h S_{k-h}. \tag{2.15}$$

If we regard $\widetilde{S}_k$ as a function of $\tilde{\lambda}$ or of $\widetilde{W}$, we have

$$\frac{\partial \widetilde{S}_k}{\partial \tilde{\lambda}_i} = \frac{\partial s_k}{\partial \mu_i}(\tilde{\lambda}), \quad \frac{\partial \widetilde{S}_k}{\partial b^i_j} = \frac{\partial s_k}{\partial \theta_{ij}}(\widetilde{W}).$$

Let us also observe that

$$\nabla_l b_{ij} = \nabla_l h_{ij} + g_{ij} \varepsilon \nabla_l H,$$

and therefore the Codazzi equation does not hold for $\nabla_l b_{ij}$. It is also easy to check that $\nabla A = 0$ if and only if $\nabla \tilde{A} = 0$.

We will denote by $\mathcal{M}_{\Gamma_k}$ the set of all points $x \in \mathcal{M}$ such that $(\lambda_1(x), \ldots, \lambda_n(x)) \in \Gamma_k$. Similarly, for given ε, D, we will denote by $\mathcal{M}_{\tilde{\Gamma}_k}$ the set of all points $x \in \mathcal{M}$ such that $(\tilde{\lambda}_1(x), \ldots, \tilde{\lambda}_n(x)) \in \Gamma_k$. On the set $\mathcal{M}_{\tilde{\Gamma}_k}$ we will consider the function $\widetilde{Q}_{k+1} := \widetilde{S}_{k+1}/\widetilde{S}_k$.

Now we investigate the relation between certain estimates from below for the polynomials S_k and other estimates for the perturbed functions $\widetilde{S}_{k;\varepsilon,D}$. The interest of these results for our later purposes will be explained in Remark 2.10.

LEMMA 2.7. *Let $\mathcal{M}$ have positive mean curvature and let $k\in\{2,...,n-1\}$ be given. Suppose that for any $\theta>0$ there exists C_θ such that*

$$S_l \geqslant -\theta H^l - C_\theta, \quad l=2,...,k, \tag{2.16}$$

everywhere on $\mathcal{M}$. Then for any $\varepsilon\in]0,1/n]$ there exists $D_\varepsilon>0$ such that

$$\widetilde{S}_{k;\varepsilon,D} \geqslant \frac{\varepsilon}{1+n\varepsilon}\cdot\frac{n-k+1}{k}\widetilde{S}_{k-1;\varepsilon,D}\widetilde{H}_{\varepsilon,D} \tag{2.17}$$

for all $D\geqslant D_\varepsilon$.

Proof. Let $\varepsilon\in]0,1/n]$ be given. By (2.15) we have on $\mathcal{M}$, for any $D>0$,

$$\begin{aligned}
\widetilde{S}_{k;\varepsilon,D} &- \frac{\varepsilon}{1+n\varepsilon}\cdot\frac{n-k+1}{k}\widetilde{S}_{k-1;\varepsilon,D}\widetilde{H}_{\varepsilon,D} \\
&= \sum_{h=0}^{k}\binom{n-k+h}{h}(\varepsilon H+D)^h S_{k-h} \\
&\qquad - \frac{\varepsilon}{1+n\varepsilon}\cdot\frac{n-k+1}{k}\sum_{h=0}^{k-1}\binom{n-k+h+1}{h}(\varepsilon H+D)^h S_{k-1-h}\widetilde{H} \\
&= S_k + \sum_{j=1}^{k}\binom{n-k+j}{j}(\varepsilon H+D)^{j-1}S_{k-j}\left(\frac{k-j}{k}\varepsilon H + \frac{k+(k-j)\varepsilon n}{k(1+n\varepsilon)}D\right).
\end{aligned}$$

We can now estimate the right-hand side using assumption (2.16) to obtain, for any $\theta>0$,

$$\begin{aligned}
\widetilde{S}_{k;\varepsilon,D} &- \frac{\varepsilon}{1+n\varepsilon}\cdot\frac{n-k+1}{k}\widetilde{S}_{k-1;\varepsilon,D}\widetilde{H}_{\varepsilon,D} \\
&\geqslant -\theta H^k - C_\theta - \sum_{j=1}^{k-2}\binom{n-k+j}{j}(\varepsilon H+D)^j(\theta H^{k-j}+C_\theta) \\
&\qquad + \binom{n-1}{k-1}(\varepsilon H+D)^{k-2}H\left(\frac{\varepsilon H}{k}+\frac{k+\varepsilon n}{k(1+\varepsilon n)}D\right) + \binom{n}{k}(\varepsilon H+D)^{k-1}\left(\frac{D}{1+\varepsilon n}\right) \\
&\geqslant -\binom{n-1}{k-1}\sum_{j=0}^{k-2}(\varepsilon H+D)^j(\theta H^{k-j}+C_\theta) + \frac{1}{2}\binom{n-1}{k-1}(\varepsilon H+D)^k \\
&= \binom{n-1}{k-1}\sum_{j=0}^{k-2}(\varepsilon H+D)^j\left(\frac{(\varepsilon H+D)^{k-j}}{2(k-1)} - (\theta H^{k-j}+C_\theta)\right).
\end{aligned}$$

Let us now define $\theta_\varepsilon = \varepsilon^k(2(k-1))^{-1}$ and $D_\varepsilon = 1+2(k-1)C_{\theta_\varepsilon}$. Then, since $(\varepsilon H+D)^{k-j}\geqslant(\varepsilon H)^{k-j}+D^{k-j}$, the right-hand side of the above formula is positive for $\theta=\theta_\varepsilon$ and for any $D\geqslant D_\varepsilon$. $\square$

LEMMA 2.8. *Let the assumptions of Lemma 2.7 be satisfied and let* $|A|^2 \leqslant \alpha H^2$ *on* $\mathcal{M}$ *for some* $\alpha>0$. *Suppose also that, for any* $\varepsilon \in]0, 1/n]$ *and* $\eta \in]0,1]$, *there exist* $C>0$, $D \geqslant 0$, $\sigma \in]0,1]$ *such that*

$$\frac{\widetilde{Q}_{k+1;\varepsilon,D}+\eta \widetilde{H}_{\varepsilon,D}}{\widetilde{H}^{1-\sigma}_{\varepsilon,D}} \geqslant -C. \tag{2.18}$$

Then for any $\theta>0$ *there exists* $C_\theta>0$ *such that*

$$S_{k+1} \geqslant -\theta H^{k+1} - C_\theta. \tag{2.19}$$

Proof. Inequality (2.18) is equivalent to $\widetilde{Q}_{k+1} \geqslant -\eta \widetilde{H} - C\widetilde{H}^{1-\sigma}$, which implies that $\widetilde{Q}_{k+1} \geqslant -2\eta \widetilde{H} - c_1$ for some $c_1 = c_1(C, \eta, \sigma) > 0$. We rewrite this inequality as

$$\widetilde{S}_{k+1} \geqslant -2\eta \widetilde{H} \widetilde{S}_k - c_1 \widetilde{S}_k.$$

On the other hand, using the identity (2.15) and the assumption $|A|^2 \leqslant \alpha H^2$ we find

$$\widetilde{S}_{k+1} \leqslant S_{k+1} + \varepsilon c_2 H^{k+1} + c_3(H^k+1),$$

for suitable constants $c_2 = c_2(n, \alpha)$ and $c_3 = c_3(n, D, \alpha)$. Therefore we obtain

$$S_{k+1} \geqslant -2\eta \widetilde{H} \widetilde{S}_k - c_1 \widetilde{S}_k - \varepsilon c_2 H^{k+1} - c_3(H^k+1).$$

Again using (2.15) we can estimate

$$\widetilde{S}_k \leqslant c_4(1+H^k), \quad \widetilde{H}\widetilde{S}_k \leqslant c_5 H^{k+1} + c_6(H^k+1),$$

where $c_4 = c_4(n, D, \alpha)$, $c_5 = c_5(n, \alpha)$ and $c_6 = c_6(n, D, \alpha)$. We conclude that

$$S_{k+1} > -c_7(\varepsilon+\eta) H^{k+1} - c_8(H^k+1) > -2c_7(\varepsilon+\eta) H^{k+1} - c_9,$$

where $c_7 = c_7(n, \alpha)$, while c_8 and c_9 depend on $n, D, \alpha, C, \eta, \sigma, \varepsilon$. This proves the lemma, since ε and η can be taken arbitrarily small. □

Remark 2.9. It is clear from the above proofs that, if we have a family $\{\mathcal{M}_t\}$ of manifolds satisfying the hypotheses of the two previous lemmas with constants independent of t, then the conclusions also hold with a constant independent of t. We recall that, if $\{\mathcal{M}_t\}$ is a family of closed manifolds with positive mean curvature evolving by mean curvature flow, then the inequality $|A|^2 \leqslant \alpha H^2$ is satisfied for some $\alpha>0$ independent of t (see e.g. [12]).

Remark 2.10. We observe that the thesis of Lemma 2.8 allows to apply Lemma 2.7 with k replaced by $k+1$. In addition, we recall that in [12] we proved that a family of hypersurfaces evolving by mean curvature flow fulfils property (2.16) for $k=2$. Therefore, an iterative application of the two previous lemmas yields our main Theorem 1.1, provided we are able to prove at each step that estimate (2.18) holds. This will be our aim in §3; to this purpose we need some properties of the functions $\widetilde{S}_k$ which will be proved in the remainder of this section.

Following [1, Lemmas 2.22 and 7.12] we now derive some estimates exploiting the concavity of q_{k+1}. First of all, we derive a suitable expression for the second derivatives of q_k with respect to the matrix entries θ_{ij} in terms of the derivatives of q_k with respect to the eigenvalues μ_i. Given $i,p\in\{1,...,n\}$, with $i\neq p$, let us denote by $\eta_{ip}\in\mathbf{R}^n$ the vector whose ith component is 1, the pth component is -1, and all other components are 0.

LEMMA 2.11. *Let $\bar\mu\in\Gamma_k$ and let $\overline{\Theta}$ be the diagonal matrix with entries $\bar\mu_1,...,\bar\mu_n$. Then we have*

$$\frac{\partial^2 q_{k+1}}{\partial\theta_{ij}\,\partial\theta_{pq}}(\overline{\Theta})=\frac{\partial^2 q_{k+1}}{\partial\mu_i\,\partial\mu_p}(\bar\mu)\,\delta_{ij}\delta_{pq}+G_{ip}(\bar\mu)\,\delta_{iq}\delta_{jp}, \tag{2.20}$$

where we have set $G_{ip}(\mu)=0$ for $i=p$, while if $i\neq p$,

$$G_{ip}(\mu)=\frac{1}{2}\int_0^1\frac{\partial^2 q_{k+1}}{\partial\eta_{ip}^2}\bigl(\mu+\sigma\cdot\tfrac{1}{2}(\mu_p-\mu_i)\eta_{ip}\bigr)\,d\sigma. \tag{2.21}$$

Proof. By the chain rule

$$\frac{\partial^2 q_{k+1}}{\partial\theta_{ij}\,\partial\theta_{pq}}(\overline{\Theta})=\sum_{l,m}\frac{\partial^2 q_{k+1}}{\partial\mu_l\,\partial\mu_m}(\bar\mu)\,\frac{\partial\mu_l}{\partial\theta_{ij}}(\overline{\Theta})\,\frac{\partial\mu_m}{\partial\theta_{pq}}(\overline{\Theta})+\sum_l\frac{\partial q_{k+1}}{\partial\mu_l}(\bar\mu)\,\frac{\partial^2\mu_l}{\partial\theta_{ij}\,\partial\theta_{pq}}(\overline{\Theta}). \tag{2.22}$$

Actually, this formula cannot be applied in general, since the eigenvalues are not an everywhere differentiable function of the entries of a matrix. However, using the implicit function theorem, it is easily checked that, if the eigenvalues $\bar\mu_1,...,\bar\mu_n$ are distinct, then the above derivatives exist and are equal to

$$\frac{\partial\mu_l}{\partial\theta_{ij}}(\overline{\Theta})=\begin{cases}1 & \text{if } i=j=l,\\ 0 & \text{otherwise,}\end{cases}$$

$$\frac{\partial^2\mu_l}{\partial\theta_{ij}\,\partial\theta_{pq}}(\overline{\Theta})=\begin{cases}1/(\bar\mu_i-\bar\mu_p) & \text{if } i\neq p,\ i=q=l,\ j=p,\\ 1/(\bar\mu_p-\bar\mu_i) & \text{if } i\neq p,\ j=p=l,\ i=q,\\ 0 & \text{otherwise.}\end{cases}$$

It follows that

$$\frac{\partial^2 q_{k+1}}{\partial\theta_{ij}\,\partial\theta_{pq}}(\overline{\Theta})=\frac{\partial^2 q_{k+1}}{\partial\mu_i\,\partial\mu_p}(\bar\mu)\,\delta_{ij}\delta_{pq}+\frac{1-\delta_{ip}}{\bar\mu_i-\bar\mu_p}\left(\frac{\partial q_{k+1}}{\partial\mu_i}(\bar\mu)-\frac{\partial q_{k+1}}{\partial\mu_p}(\bar\mu)\right)\delta_{iq}\delta_{jp}. \tag{2.23}$$

Let us write the second term on the right-hand side in a more convenient way. For fixed $i \neq p$, let us call μ^* the vector in Γ_k with entries

$$\mu_j^* = \begin{cases} \bar\mu_j & \text{if } j \neq i,\ j \neq p, \\ \frac{1}{2}(\bar\mu_i + \bar\mu_p) & \text{if } j = i \text{ or if } j = p. \end{cases}$$

Then $\mu_i^* = \mu_p^*$ and so

$$\frac{\partial q_{k+1}}{\partial \eta_{ip}}(\mu^*) = \frac{\partial q_{k+1}}{\partial \mu_i}(\mu^*) - \frac{\partial q_{k+1}}{\partial \mu_p}(\mu^*) = 0$$

since q_{k+1} is a symmetric function. From the definition of G_{ip} we obtain

$$G_{ip}(\bar\mu) = \frac{\dfrac{\partial q_{k+1}}{\partial \eta_{ip}}(\mu^*) - \dfrac{\partial q_{k+1}}{\partial \eta_{ip}}(\bar\mu)}{\bar\mu_p - \bar\mu_i} = \frac{\dfrac{\partial q_{k+1}}{\partial \eta_{ip}}(\bar\mu)}{\bar\mu_i - \bar\mu_p} = \frac{\dfrac{\partial q_{k+1}}{\partial \mu_i}(\bar\mu) - \dfrac{\partial q_{k+1}}{\partial \mu_p}(\bar\mu)}{\bar\mu_i - \bar\mu_p}.$$

Recalling (2.23), this proves the conclusion in the case when the eigenvalues $\bar\mu_1, ..., \bar\mu_n$ are distinct. Since both sides of (2.20) are well defined and continuous for $\bar\mu \in \Gamma_k$, the general case follows by continuity. □

LEMMA 2.12. *Given $\bar\mu \in \Gamma_k$ and $i \neq p$ we have $G_{ip}(\bar\mu) \leqslant 0$, with equality only if $\bar\mu$ has exactly k nonzero entries and $\bar\mu_i \neq 0$, $\bar\mu_p \neq 0$.*

Proof. We observe that $s_1(\eta_{ip}) = 0$, and so no scalar multiple of η_{ip} belongs to Γ_k. Then the result is a direct consequence of Theorem 2.5. □

LEMMA 2.13. *Let $\bar\mu \in \Gamma_k$ satisfy $s_{k+1}(\bar\mu) < 0$, and let a_{lip} be a nonzero totally symmetric tensor. Given $\varepsilon \geqslant 0$, set*

$$\tilde a_{lip} = a_{lip} + \varepsilon \delta_{ip} \sum_{h=1}^{n} a_{lhh},$$

$$J(\bar\mu, a, \varepsilon) = \sum_{i,p,l} \frac{\partial^2 q_{k+1}}{\partial \mu_i \partial \mu_p}(\bar\mu)\, \tilde a_{lii} \tilde a_{lpp} + \sum_{\substack{i,p,l \\ i \neq p}} G_{ip}(\bar\mu)(\tilde a_{lip})^2,$$

where G_{ip} is defined as in (2.21). *Then $J(\bar\mu, a, \varepsilon) < 0$.*

Proof. From Theorem 2.5 and Lemma 2.12 we immediately obtain that $J(\bar\mu, a, \varepsilon) \leqslant 0$. Let us now prove that the inequality is strict. We have by assumption $s_l(\bar\mu) > 0$ for any $l = 1, ..., k$ and $s_{k+1}(\bar\mu) < 0$. We deduce in particular that at least $k+1$ components of $\bar\mu$ are nonzero. By Lemma 2.12, $G_{ip}(\bar\mu) < 0$ for any $i \neq p$; thus, $J(\bar\mu, a, \varepsilon)$ can vanish only if $\tilde a_{lip} = 0$ for all l and all $i \neq p$. Since $\tilde a_{lip} = a_{lip}$ for $i \neq p$ and a_{lip} is symmetric, we deduce that

$$a_{lip} = 0 \quad \text{unless } l = i = p. \tag{2.24}$$

By assumption a_{lip} is not identically zero; therefore a_{lll} must be nonzero for some l. Let us assume for instance that $a_{111}\neq 0$. We deduce from equation (2.24) that $\sum_{h=1}^{n} a_{1hh}=a_{111}$, and so

$$\tilde{a}_{1ii}=(\delta_{i1}+\varepsilon)a_{111}, \quad 1\leqslant i\leqslant n. \tag{2.25}$$

From Theorem 2.5 we deduce that $J(\bar{\mu},a,\varepsilon)$ can vanish only if the vector $(\tilde{a}_{1ii})_{1\leqslant i\leqslant n}$ is a scalar multiple of $\bar{\mu}$. This implies, by (2.25), that the components of $\bar{\mu}$ have all the same sign. This contradicts the assumption that $s_1(\bar{\mu})>0$, $s_{k+1}(\bar{\mu})<0$. □

To simplify the notation, we assume that all following computations are performed in a coordinate system which is orthonormal at the point under consideration. Then we have $g_{ij}=\delta_{ij}$, $h^i_j=h_{ij}$, $b^i_j=b_{ij}$ at that point, and so we will make no distinction in our formulas between upper and lower indices.

THEOREM 2.14. *At any point $P\in\mathcal{M}_{\widetilde{\Gamma}_k}$ we have*

$$\sum_{i,j,p,q,l}\frac{\partial^2\widetilde{Q}_{k+1}}{\partial b_{ij}\,\partial b_{pq}}\nabla_l b_{ij}\nabla_l b_{pq}\leqslant 0. \tag{2.26}$$

Moreover, given $c>\eta>0$ there exists a constant $C=C(c,\eta,n)$ such that, for any $\varepsilon\in[0,1]$, for any $D\geqslant 0$, and for any point $P\in\mathcal{M}_{\widetilde{\Gamma}_k}$ satisfying $-c\widetilde{H}(P)<\widetilde{Q}_{k+1}(P)<-\eta\widetilde{H}(P)$, we have

$$\sum_{i,j,p,q,l}\frac{\partial^2\widetilde{Q}_{k+1}}{\partial b_{ij}\,\partial b_{pq}}\nabla_l b_{ij}\nabla_l b_{pq}\leqslant -\frac{1}{C}\cdot\frac{|\nabla\tilde{A}|^2}{|\tilde{A}|}.$$

Proof. Let $P\in\mathcal{M}_{\widetilde{\Gamma}_k}$ and let our coordinate system be such that h_{ij} is diagonal at P. Then, by Lemma 2.11, we have at the point P

$$\sum_{i,j,p,q,l}\frac{\partial^2\widetilde{Q}_{k+1}}{\partial b_{ij}\,\partial b_{pq}}\nabla_l b_{ij}\nabla_l b_{pq}=\sum_{i,p,l}\frac{\partial^2 q_{k+1}}{\partial\mu_i\,\partial\mu_p}(\tilde{\lambda})\nabla_l b_{ii}\nabla_l b_{pp}+\sum_{\substack{i,p,l\\ i\neq p}}G_{ip}(\tilde{\lambda})(\nabla_l b_{ip})^2. \tag{2.27}$$

Inequality (2.26) follows then from Theorem 2.5 and Lemma 2.12. To prove the second assertion, we proceed as follows. We denote by E the set of all elements $(\mu,a,\varepsilon)\in\Gamma_k\times(\mathbf{R}^n\otimes\mathbf{R}^n\otimes\mathbf{R}^n)\times[0,1]$ with the following properties.

(i) The vector μ satisfies $-cs_1(\mu)s_k(\mu)\leqslant s_{k+1}(\mu)\leqslant -\eta s_k(\mu)s_1(\mu)$.

(ii) The tensor $a=(a_{lip})$ is totally symmetric.

(iii) $|\mu|=|a|=1$.

Let us define the tensor $\tilde{a}$ and the function $J(\mu,a,\varepsilon)$ as in the previous lemma. Then the function $J(\mu,a,\varepsilon)$ is everywhere negative on E. Since E is compact, we deduce that there exists $C>0$ such that

$$J(\mu,a,\varepsilon)\leqslant -\frac{1}{C}\cdot\frac{|\tilde{a}|^2}{|\mu|}$$

for any $(\mu, a, \varepsilon) \in E$. By homogeneity, such an inequality remains true if we drop property (iii) in the definition of E and assume instead only $\mu \neq 0$. This inequality, with $\mu = \tilde{\lambda}_{\varepsilon,D}$ and $a_{lip} = \nabla_l h_{ip}$, proves the theorem, thanks to formula (2.27). $\square$

LEMMA 2.15. *Given $\varepsilon, D \geqslant 0$ we have*

$$\begin{aligned}
\sum_{i,j} \frac{\partial \widetilde{S}_k}{\partial b_{ij}} \nabla_i \nabla_j \widetilde{S}_{k+1} = {} & \sum_{i,j,p,q,l,m} \frac{\partial \widetilde{S}_k}{\partial b_{ij}} \cdot \frac{\partial^2 \widetilde{S}_{k+1}}{\partial b_{lm} \partial b_{pq}} \nabla_i b_{lm} \nabla_j b_{pq} \\
& + \sum_{i,j,l,m} \frac{\partial \widetilde{S}_k}{\partial b_{ij}} \cdot \frac{\partial \widetilde{S}_{k+1}}{\partial b_{lm}} \nabla_l \nabla_m b_{ij} \\
& + \frac{\varepsilon}{1+n\varepsilon} \sum_{i,j} \left((n-k) \widetilde{S}_k \frac{\partial \widetilde{S}_k}{\partial b_{ij}} - (n-k+1) \widetilde{S}_{k-1} \frac{\partial \widetilde{S}_{k+1}}{\partial b_{ij}} \right) \nabla_i \nabla_j \widetilde{H} \\
& - \widetilde{H} \widetilde{S}_k \widetilde{S}_{k+1} + (k+1) \widetilde{S}_{k+1}^2 + k[(k+1) \widetilde{S}_{k+1}^2 - (k+2) \widetilde{S}_k \widetilde{S}_{k+2}] \\
& + \left(\frac{\varepsilon \widetilde{H} + D}{1+n\varepsilon} \right)^2 [(k+1)(n-k+1) \widetilde{S}_{k+1} \widetilde{S}_{k-1} - k(n-k) \widetilde{S}_k^2] \\
& + \left(\frac{\varepsilon \widetilde{H} + D}{1+n\varepsilon} \right) [(n-k) \widetilde{S}_k (\widetilde{H} \widetilde{S}_k - (k+1) \widetilde{S}_{k+1}) \\
& \qquad + (n-k+1) \widetilde{S}_{k-1} ((k+2) \widetilde{S}_{k+2} - \widetilde{H} \widetilde{S}_{k+1})].
\end{aligned}$$

Proof. Throughout the proof we use the summation convention for repeated indices (except of course for the index k appearing in $\widetilde{S}_k$). We have

$$\frac{\partial \widetilde{S}_k}{\partial b_{ij}} \nabla_i \nabla_j \widetilde{S}_{k+1} = \frac{\partial \widetilde{S}_k}{\partial b_{ij}} \cdot \frac{\partial^2 \widetilde{S}_{k+1}}{\partial b_{lm} \partial b_{pq}} \nabla_i b_{lm} \nabla_j b_{pq} + \frac{\partial \widetilde{S}_k}{\partial b_{ij}} \cdot \frac{\partial \widetilde{S}_{k+1}}{\partial b_{lm}} \nabla_i \nabla_j b_{lm}. \tag{2.28}$$

We recall the commutation identity

$$\nabla_i \nabla_j h_{lm} - \nabla_l \nabla_m h_{ij} = h_{ij} h_{lr} h_{rm} - h_{lm} h_{ir} h_{rj} + h_{im} h_{lr} h_{rj} - h_{lj} h_{mr} h_{ri}.$$

Then we obtain

$$\begin{aligned}
\frac{\partial \widetilde{S}_k}{\partial b_{ij}} \cdot \frac{\partial \widetilde{S}_{k+1}}{\partial b_{lm}} \nabla_i \nabla_j b_{lm} & - \frac{\partial \widetilde{S}_k}{\partial b_{ij}} \cdot \frac{\partial \widetilde{S}_{k+1}}{\partial b_{lm}} \nabla_l \nabla_m b_{ij} \\
& = \varepsilon \frac{\partial \widetilde{S}_k}{\partial b_{ij}} \cdot \frac{\partial \widetilde{S}_{k+1}}{\partial b_{lm}} (\delta_{lm} \nabla_i \nabla_j H - \delta_{ij} \nabla_l \nabla_m H) \\
& \quad + \frac{\partial \widetilde{S}_k}{\partial b_{ij}} \cdot \frac{\partial \widetilde{S}_{k+1}}{\partial b_{lm}} (h_{ij} h_{lr} h_{rm} - h_{lm} h_{ir} h_{rj} + h_{im} h_{lr} h_{rj} - h_{lj} h_{mr} h_{ri}).
\end{aligned} \tag{2.29}$$

From the general identity

$$\frac{\partial \widetilde{S}_k}{\partial b_{ij}} \delta_{ij} = (n-k+1) \widetilde{S}_{k-1}, \quad k \in \{1, ..., n\},$$

we deduce that

$$\begin{aligned}&\varepsilon\frac{\partial\widetilde{S}_k}{\partial b_{ij}}\cdot\frac{\partial\widetilde{S}_{k+1}}{\partial b_{lm}}(\delta_{lm}\nabla_i\nabla_j H-\delta_{ij}\nabla_l\nabla_m H)\\&\qquad=\varepsilon\left((n-k)\widetilde{S}_k\frac{\partial\widetilde{S}_k}{\partial b_{ij}}-(n-k+1)\widetilde{S}_{k-1}\frac{\partial\widetilde{S}_{k+1}}{\partial b_{ij}}\right)\nabla_i\nabla_j H.\end{aligned}\tag{2.30}$$

To evaluate the other term on the right-hand side of (2.29) we choose a coordinate system such that h_{ij} is diagonal at x, that is, $h_{ij}=\delta_{ij}\lambda_i$. We find

$$\begin{aligned}&\frac{\partial\widetilde{S}_k}{\partial b_{ij}}\cdot\frac{\partial\widetilde{S}_{k+1}}{\partial b_{lm}}(h_{ij}h_{lr}h_{rm}-h_{lm}h_{ir}h_{rj}+h_{im}h_{lr}h_{rj}-h_{lj}h_{mr}h_{ri})\\&\qquad=\frac{\partial\widetilde{S}_k}{\partial b_{ii}}\cdot\frac{\partial\widetilde{S}_{k+1}}{\partial b_{mm}}(\lambda_i\lambda_m^2-\lambda_i^2\lambda_m)+\frac{\partial\widetilde{S}_k}{\partial b_{ij}}\cdot\frac{\partial\widetilde{S}_{k+1}}{\partial b_{ij}}(\lambda_i\lambda_j^2-\lambda_i^2\lambda_j)\\&\qquad=\frac{\partial\widetilde{S}_k}{\partial\tilde{\lambda}_i}\cdot\frac{\partial\widetilde{S}_{k+1}}{\partial\tilde{\lambda}_m}(\lambda_i\lambda_m^2-\lambda_i^2\lambda_m)\\&\qquad=\frac{\partial\widetilde{S}_k}{\partial\tilde{\lambda}_i}\cdot\frac{\partial\widetilde{S}_{k+1}}{\partial\tilde{\lambda}_m}(\tilde{\lambda}_i\tilde{\lambda}_m^2-\tilde{\lambda}_i^2\tilde{\lambda}_m)+(\varepsilon H+D)^2\frac{\partial\widetilde{S}_k}{\partial\tilde{\lambda}_i}\cdot\frac{\partial\widetilde{S}_{k+1}}{\partial\tilde{\lambda}_m}(\tilde{\lambda}_m-\tilde{\lambda}_i)\\&\qquad\quad+(\varepsilon H+D)\frac{\partial\widetilde{S}_k}{\partial\tilde{\lambda}_i}\cdot\frac{\partial\widetilde{S}_{k+1}}{\partial\tilde{\lambda}_m}(\tilde{\lambda}_i^2-\tilde{\lambda}_m^2).\end{aligned}\tag{2.31}$$

Now we can use identities (2.3), (2.6) and (2.7) to obtain

$$\begin{aligned}&\frac{\partial\widetilde{S}_k}{\partial\tilde{\lambda}_i}\cdot\frac{\partial\widetilde{S}_{k+1}}{\partial\tilde{\lambda}_m}(\tilde{\lambda}_i\tilde{\lambda}_m^2-\tilde{\lambda}_i^2\tilde{\lambda}_m)\\&\qquad=k\widetilde{S}_k\sum_{m=1}^n\widetilde{S}_{k;m}\tilde{\lambda}_m^2-(k+1)\widetilde{S}_{k+1}\sum_{i=1}^n\tilde{\lambda}_i^2\widetilde{S}_{k-1;i}\\&\qquad=k\widetilde{S}_k(\widetilde{H}\widetilde{S}_{k+1}-(k+2)\widetilde{S}_{k+2})-(k+1)\widetilde{S}_{k+1}(\widetilde{H}\widetilde{S}_k-(k+1)\widetilde{S}_{k+1})\\&\qquad=-\widetilde{H}\widetilde{S}_k\widetilde{S}_{k+1}+(k+1)\widetilde{S}_{k+1}^2+k[(k+1)\widetilde{S}_{k+1}^2-(k+2)\widetilde{S}_k\widetilde{S}_{k+2}].\end{aligned}\tag{2.32}$$

Similarly, using (2.5) and (2.6) we find

$$\begin{aligned}\frac{\partial\widetilde{S}_k}{\partial\tilde{\lambda}_i}\cdot\frac{\partial\widetilde{S}_{k+1}}{\partial\tilde{\lambda}_m}(\tilde{\lambda}_m-\tilde{\lambda}_i)&=(k+1)\widetilde{S}_{k+1}\sum_{i=1}^n\widetilde{S}_{k-1;i}-k\widetilde{S}_k\sum_{m=1}^n\widetilde{S}_{k;m}\\&=(k+1)(n-k+1)\widetilde{S}_{k+1}\widetilde{S}_{k-1}-k(n-k)\widetilde{S}_k^2,\end{aligned}\tag{2.33}$$

while we deduce from (2.5) and (2.7)

$$\begin{aligned}\frac{\partial\widetilde{S}_k}{\partial\tilde{\lambda}_i}\cdot\frac{\partial\widetilde{S}_{k+1}}{\partial\tilde{\lambda}_m}(\tilde{\lambda}_i^2-\tilde{\lambda}_m^2)&=(n-k)\widetilde{S}_k\sum_{i=1}^n\widetilde{S}_{k-1;i}\tilde{\lambda}_i^2-(n-k+1)\widetilde{S}_{k-1}\sum_{m=1}^n\widetilde{S}_{k;m}\tilde{\lambda}_m^2\\&=(n-k)\widetilde{S}_k(\widetilde{H}\widetilde{S}_k-(k+1)\widetilde{S}_{k+1})\\&\qquad-(n-k+1)\widetilde{S}_{k-1}(\widetilde{H}\widetilde{S}_{k+1}-(k+2)\widetilde{S}_{k+2}).\end{aligned}\tag{2.34}$$

From (2.28), (2.29), (2.30), (2.31), (2.32), (2.33) and (2.34) we obtain our conclusion. □

COROLLARY 2.16. *Suppose that the assumptions of Lemma 2.7 are satisfied. Let* $\eta, D, \varepsilon>0$. *Suppose that*

$$\varepsilon \leqslant \min\left\{\frac{1}{n}, \sqrt{\frac{\eta}{2k(n-k)}}\right\}, \quad D \geqslant D_\varepsilon,$$

where D_ε *is given from Lemma 2.7. Then, at any point* $x \in \mathcal{M}$ *such that* $\widetilde{Q}_{k+1}(x) < -\eta \widetilde{H}(x)$, *we have*

$$\begin{aligned}\sum_{i,j} \frac{\partial \widetilde{S}_k}{\partial b_{ij}} \nabla_i \nabla_j \widetilde{S}_{k+1} > &\sum_{i,j,p,q,l,m} \frac{\partial \widetilde{S}_k}{\partial b_{ij}} \cdot \frac{\partial^2 \widetilde{S}_{k+1}}{\partial b_{lm} \partial b_{pq}} \nabla_i b_{lm} \nabla_j b_{pq} + \sum_{i,j,l,m} \frac{\partial \widetilde{S}_k}{\partial b_{ij}} \cdot \frac{\partial \widetilde{S}_{k+1}}{\partial b_{lm}} \nabla_l \nabla_m b_{ij} \\ &+ \frac{\varepsilon}{1+n\varepsilon} \sum_{i,j} \left((n-k)\widetilde{S}_k \frac{\partial \widetilde{S}_k}{\partial b_{ij}} - (n-k+1)\widetilde{S}_{k-1} \frac{\partial \widetilde{S}_{k+1}}{\partial b_{ij}}\right) \nabla_i \nabla_j \widetilde{H} \\ &+ (k+1)\widetilde{S}_{k+1}^2 + \tfrac{1}{2}\eta \widetilde{H}^2 \widetilde{S}_k^2 - CD|\tilde{A}|^{2k}(D+|\tilde{A}|),\end{aligned}$$

where $C=C(n)$.

Proof. Let us consider the identity given by the previous lemma. We have to estimate the polynomials of maximum degree $2k+2$ appearing on the right-hand side, since the other terms are dominated by $CD|\tilde{A}|^{2k}(D+|\tilde{A}|)$ for a suitable $C=C(n)$. By hypothesis we have $\widetilde{S}_l > 0$ at x for $l=1, \ldots, k$ and $\widetilde{S}_{k+1} < -\eta \widetilde{H} \widetilde{S}_k$. This implies

$$-\widetilde{H}\widetilde{S}_k \widetilde{S}_{k+1} \geqslant \eta \widetilde{H}^2 \widetilde{S}_k^2, \quad (\varepsilon \widetilde{H}+D)\widetilde{S}_k(\widetilde{H}\widetilde{S}_k - (k+1)\widetilde{S}_{k+1}) > 0.$$

Grouping the two terms containing the factor $\widetilde{H}^2 \widetilde{S}_{k-1} \widetilde{S}_{k+1}$ we obtain the contribution

$$\frac{\varepsilon(n-k+1)}{(1+n\varepsilon)^2} \widetilde{H}^2 \widetilde{S}_{k-1} \widetilde{S}_{k+1}((k+1)\varepsilon - (1+n\varepsilon)),$$

which is positive, since $\varepsilon \leqslant 1/(k+1)$. In addition we have

$$-\frac{\varepsilon^2 k(n-k)}{(1+n\varepsilon)^2} \widetilde{H}^2 \widetilde{S}_k^2 > -\tfrac{1}{2}\eta \widetilde{H}^2 \widetilde{S}_k^2.$$

It remains to estimate the term

$$k(k+1)\widetilde{S}_{k+1}^2 - k(k+2)\widetilde{S}_k \widetilde{S}_{k+2} + \frac{\varepsilon}{1+n\varepsilon} \widetilde{H}(n-k+1)(k+2)\widetilde{S}_{k-1}\widetilde{S}_{k+2}.$$

Let us first suppose that $\widetilde{S}_{k+2} > 0$. Then the third term is clearly positive, while the sum of the first two is positive by Newton's inequality (2.8). Suppose instead that $\widetilde{S}_{k+2} \leqslant 0$. Then the first term is positive, and the sum of the second and the third is nonnegative by Lemma 2.7. This concludes the proof. $\square$

3. The a priori estimates

Let $F: \mathcal{M}\times[0,T[\to\mathbf{R}^{n+1}$ be a solution of mean curvature flow (1.1)–(1.2) with closed, smoothly immersed evolving surfaces $\mathcal{M}_t=F(\cdot,t)(\mathcal{M})$. The induced metric $g=\{g_{ij}\}$, the surface measure $d\mu$ and the second fundamental form $A=\{h_{ij}\}$ satisfy the evolution equations previously computed in [8]:

LEMMA 3.1. *We have the evolution equations*
(i) $\partial g_{ij}/\partial t=-2Hh_{ij}$,
(ii) $\partial(d\mu)/\partial t=-H^2\,d\mu$,
(iii) $\partial h_{ij}/\partial t=\Delta h_{ij}-2Hh_{il}h^l_j+|A|^2h_{ij}$.

If we consider the Weingarten map $W: T_p\mathcal{M}\to T_p\mathcal{M}$ associated with A and g, given by the matrix $\{h^i_j\}=\{g^{il}h_{lj}\}$, and let $P=P(W)$ be any invariant function of the elementary symmetric polynomials of the principal curvatures as considered in §2, then Lemma 3.1 implies

COROLLARY 3.2. *If $W=\{h^i_j\}$ is the Weingarten map and $P(W)$ is an invariant function of degree α, i.e. $P(\varrho W)=\varrho^\alpha P(W)$, then*
(i) $\partial h^i_j/\partial t=\Delta h^i_j+|A|^2h^i_j$,
(ii) $\partial P/\partial t=\Delta P-(\partial^2P/\partial h^i_j\partial h^p_q)\nabla^l h^i_j\nabla_l h^p_q+\alpha|A|^2P$.

Let us first consider the case of an initial surface $\mathcal{M}_0=F_0(\mathcal{M})$ satisfying $S_k(W)\geqslant 0$ for some k, $1\leqslant k\leqslant n$. The following two propositions, although not needed for the proof of the main Theorem 1.1, are of independent interest and demonstrate that the elementary symmetric functions provide natural curvature conditions for the flow.

PROPOSITION 3.3. (i) *If $S_k\geqslant 0$ on any closed hypersurface $\mathcal{M}\subset\mathbf{R}^{n+1}$ for some k, $1\leqslant k\leqslant n$, then also $S_l\geqslant 0$ for all $1\leqslant l\leqslant k$. The same result holds for the case of strict inequalities.*

(ii) *If $S_k\geqslant 0$ on $\mathcal{M}_0$ for some k, $1\leqslant k\leqslant n$, then the strict inequalities $S_l>0$, $1\leqslant l\leqslant k$, hold on $\mathcal{M}_t$ for each $t\in]0,T[$.*

Proof. (i) For a closed $\mathcal{M}\subset\mathbf{R}^{n+1}$ each connected component admits at least one strictly convex point $p_0\in\mathcal{M}$ where $S_l(p_0)>0$ for all $1\leqslant l\leqslant n$. The claim then follows since the cones Γ_l introduced in §2 are connected and satisfy $\Gamma_l\subset\Gamma_k$ for all $1\leqslant k<l\leqslant n$.

(ii) The argument is an iterative application of the maximum principle: First observe that the mean curvature $S_1=H$ satisfies the evolution equation

$$\frac{\partial H}{\partial t}=\Delta H+|A|^2H, \tag{3.1}$$

implying strict positivity of H for all $t>0$ in view of the strict parabolic maximum principle and the fact that there is at least one strictly convex point on $\mathcal{M}$. Also notice that the evolution equation in Corollary 3.2 (i) preserves any of the closed convex cones $\bar{\Gamma}_l$ by a general result due to Hamilton ([4, §4]). Now suppose that we have already shown $S_{l-1}>0$ for some $l\geqslant 2$, and consider for $t\geqslant t_0>0$ the quotient $Q_l=S_l/S_{l-1}$. By Corollary 3.2 (ii) we have the evolution equation

$$\frac{\partial}{\partial t}Q_l=\Delta Q_l-\frac{\partial^2 Q_l}{\partial h^i_j\,\partial h^p_q}\nabla^l h^i_j\nabla_l h^p_q+|A|^2 Q_l, \tag{3.2}$$

and the second term on the right-hand side is nonnegative since Q_l is concave by Theorem 2.5. Since we already know that Q_l remains nonnegative, it now follows from the strict parabolic maximum principle that $Q_l>0$ on $]0,T[$, and the iteration may be continued. □

Now suppose that S_l, $1\leqslant l\leqslant k$, is already strictly positive on $\mathcal{M}_0$ for some k, $2\leqslant k\leqslant n$. Then there are constants $\varepsilon_l>0$ for $2\leqslant l\leqslant k$ such that on $\mathcal{M}_0$

$$S_l\geqslant \varepsilon_l H S_{l-1}. \tag{3.3}$$

We show that this uniform estimate on the curvature is also preserved by the flow:

PROPOSITION 3.4. *If on the initial surface $\mathcal{M}_0$ the inequalities $S_l\geqslant\varepsilon_l HS_{l-1}$ hold for constants $\varepsilon_l>0$, $2\leqslant l\leqslant k$, then the same inequalities continue to hold on $[0,T[$.*

Proof. In view of (3.1) and (3.2) the quantity $g=Q_l-\varepsilon_l H$ satisfies the inequality

$$\frac{\partial g}{\partial t}\geqslant \Delta g+|A|^2 g, \tag{3.4}$$

where we used the concavity of Q_l. The result is then a direct consequence of the maximum principle. Note that the result can also be deduced from a general result of Hamilton [4], since the inequality describes a convex cone in the space of symmetric 2-tensors. □

To prove the main Theorem 1.1, we will use an iterative procedure employing quotients of consecutive elementary symmetric polynomials similar as in the proof of Proposition 3.3. The first step of the iteration is contained in [12], where we showed that S_2 satisfies the desired estimate if the initial data have nonnegative mean curvature:

For all $\theta>0$ there is C_θ depending only on θ and $\mathcal{M}_0$ such that

$$S_2\geqslant -\theta S_1^2-C_\theta. \tag{3.5}$$

Now suppose that $2\leqslant k\leqslant n-1$ is given and that the desired estimate has already been established for all $2\leqslant l\leqslant k$, i.e. for all $\theta>0$ there is $C_{\theta,l}$ with

$$S_l\geqslant -\theta H^l-C_{\theta,l}. \tag{3.6}$$

To derive the lower bound for S_{k+1}, we have to overcome the difficulty that S_k is not quite positive and therefore Q_{k+1} is not well defined on the evolving surfaces. We study a perturbed second fundamental form $\tilde{A}=\tilde{A}_{\varepsilon,D}=\{b_{ij}\}$ as already discussed in §2,

$$b_{ij}=b_{ij;\varepsilon,D}=h_{ij}+\varepsilon Hg_{ij}+Dg_{ij}, \tag{3.7}$$

where $\varepsilon>0$ and $D>0$ are constants to be chosen appropriately. We will then denote by $\widetilde{H},\widetilde{S}_l,\widetilde{Q}_l$ the invariant functions of $\tilde{A}=\tilde{A}_{\varepsilon,D}$, indicating the dependence on ε, D explicitly only where necessary.

In view of Lemma 2.7 and assumption (3.6) there is $C_n>0$ depending only on n such that for all $\varepsilon\in]0,1/n]$ we may choose $D=D_\varepsilon$ so large that for all $2\leqslant l\leqslant k$

$$\widetilde{S}_l=\widetilde{S}_{l,\varepsilon,D_\varepsilon}\geqslant C_n\varepsilon^{l-1}\widetilde{H}^l \quad\text{and}\quad \widetilde{S}_l\geqslant C_n\varepsilon\widetilde{H}\widetilde{S}_{l-1}. \tag{3.8}$$

Note that D_ε depends on $C_{\theta,l}$ for $2\leqslant l\leqslant k$ and hence on the estimates established in previous steps; in particular, D_ε depends on the initial data.

The quotient $\widetilde{Q}_{k+1}=\widetilde{S}_{k+1}/\widetilde{S}_k$ is now well defined and we infer from the first inequality in (3.8) that for every $\varepsilon>0$ there is a constant $C=C(n,k,\varepsilon)$ such that

$$|\widetilde{Q}_{k+1}|\leqslant C(n,k,\varepsilon)\widetilde{H}. \tag{3.9}$$

Here we also used the fact that in view of (3.8) in case $l=2$ we always have the inequality

$$|A|^2\leqslant|\tilde{A}|^2\leqslant\widetilde{H}^2. \tag{3.10}$$

For future reference we also note that in view of (3.8) and (3.10)

$$\left|\frac{\partial\widetilde{S}_k}{\partial b_{ij}}\xi^i\zeta^j\right|\leqslant C(n,k,\varepsilon)\widetilde{H}^{k-1}|\xi|\,|\zeta|, \qquad \left|\nabla_l\frac{\partial\widetilde{S}_k}{\partial b_{ij}}\right|\leqslant C(n,k,\varepsilon)\widetilde{H}^{k-2}|\nabla\tilde{A}|. \tag{3.11}$$

In the following we will assume that for each $\varepsilon>0$ the constant D_ε in the definition of the perturbed second fundamental form is fixed once and for all such that the inequalities (3.8) hold and therefore (3.9)–(3.11) hold with fixed constants $C(n,k,\varepsilon)$.

In view of Lemma 3.1 and Corollary 3.2 we have the evolution equations

$$\frac{\partial}{\partial t}b_{ij}=\Delta b_{ij}-2Hh_{il}b_j^l+|A|^2b_{ij}-D|A|^2g_{ij},$$
$$\frac{\partial}{\partial t}b_j^i=\Delta b_j^i+|A|^2b_j^i-D|A|^2\delta_j^i.$$

Hence if P is a symmetric homogeneous function of degree α satisfying the evolution equation in Corollary 3.2 (ii), then the corresponding function $\widetilde{P}$ of $\{b_{ij}\}$ satisfies the equation

$$\frac{\partial}{\partial t}\widetilde{P}=\Delta\widetilde{P}-\frac{\partial^2\widetilde{P}}{\partial b_j^i\partial b_q^p}\nabla^l b_j^i\nabla_l b_q^p+\alpha|A|^2\widetilde{P}-D|A|^2\operatorname{tr}\left(\frac{\partial\widetilde{P}}{\partial b_j^i}\right). \tag{3.12}$$

To prove the estimate (3.6) now for $l=k+1$ we first fix an arbitrary $0<\eta\leqslant 1$ and then consider for $0\leqslant\sigma\leqslant\frac{1}{2}$, $\varepsilon\geqslant 0$ the function

$$f=f_{\sigma,\varepsilon,D_\varepsilon}=\frac{-\widetilde{Q}_{k+1}-\eta\widetilde{H}}{\widetilde{H}^{1-\sigma}}. \tag{3.13}$$

Notice that for $k=1$, $\varepsilon=D_\varepsilon=0$ this agrees with the choice of test function in [12]. For simplicity we write from now on $\widetilde{Q}$ for $\widetilde{Q}_{k+1}$ and make no further distinction between upper and lower indices. From (3.12) we derive the following evolution equation in a straightforward calculation:

$$\begin{aligned}\frac{\partial f}{\partial t}&=\Delta f+\frac{2(1-\sigma)}{\widetilde{H}}\langle\nabla\widetilde{H},\nabla f\rangle-\frac{\sigma(1-\sigma)}{\widetilde{H}^2}f|\nabla\widetilde{H}|^2\\&\quad+\frac{1}{\widetilde{H}^{1-\sigma}}\cdot\frac{\partial^2\widetilde{Q}}{\partial b_{ij}\partial b_{pq}}\nabla_m b_{ij}\nabla_m b_{pq}+\sigma|A|^2f+D_\varepsilon|A|^2\operatorname{tr}\left(\frac{\partial}{\partial b_{ij}}\left(\frac{\widetilde{Q}+\eta\widetilde{H}}{\widetilde{H}^{1-\sigma}}\right)\right).\end{aligned} \tag{3.14}$$

In view of the estimates (3.8)–(3.11) the last term on the right-hand side of this equation can be estimated by

$$D_\varepsilon|A|^2\operatorname{tr}\left(\frac{\partial}{\partial b_{ij}}\left(\frac{\widetilde{Q}+\eta\widetilde{H}}{\widetilde{H}^{1-\sigma}}\right)\right)\leqslant C(n,k,\varepsilon)D_\varepsilon\widetilde{H}^{1+\sigma}. \tag{3.15}$$

We now establish L^p-estimates for the quantity $f_+=\max(f,0)$.

LEMMA 3.5. *There are positive constants c_2 and c_3 depending only on n,k,η,ε such that for any $p\geqslant c_2$ there are constants c_4 and c_5 depending only on $n,k,p,\eta,\varepsilon,D_\varepsilon$ and σ, such that we have the inequality*

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{M}_t}f_+^p\,d\mu&\leqslant-\tfrac{1}{2}p(p-1)\int_{\mathcal{M}_t}f_+^{p-2}|\nabla f|^2\,d\mu-\frac{p}{c_3}\int_{\mathcal{M}_t}\frac{f_+^{p-1}}{\widetilde{H}^{2-\sigma}}|\nabla\tilde{A}|^2\,d\mu\\&\quad+2p\sigma\int_{\mathcal{M}_t}|\widetilde{H}|^2f_+^p\,d\mu+c_4\int_{\mathcal{M}_t}f_+^p\,d\mu+c_5\int_{\mathcal{M}_t}d\mu.\end{aligned}$$

Proof. For $p\geqslant 3$ we derive from (3.14), (3.15) and Lemma 3.1 (ii)

$$\begin{aligned}\frac{d}{dt}\int f_+^p\,d\mu&\leqslant-p(p-1)\int f_+^{p-2}|\nabla f|^2\,d\mu+2(1-\sigma)p\int\frac{f_+^{p-1}}{\widetilde{H}}\langle\nabla\widetilde{H},\nabla f\rangle\,d\mu\\&\quad+p\int_{\mathcal{M}_t}\frac{f_+^{p-1}}{\widetilde{H}^{1-\sigma}}\cdot\frac{\partial^2\widetilde{Q}}{\partial b_{ij}\partial b_{qr}}\nabla_m b_{ij}\nabla_m b_{qr}\,d\mu\\&\quad+p\sigma\int|A|^2f_+^p\,d\mu+pC(n,k,\varepsilon)D_\varepsilon\int\widetilde{H}^{1+\sigma}f_+^{p-1}\,d\mu.\end{aligned}$$

Notice that $f \leqslant c_0 \widetilde{H}^\sigma$ for some $c_0 = c_0(n,k,\eta)$. In addition, on the set where $f>0$ we have the inequalities $-C(n,k,\varepsilon)\widetilde{H} \leqslant \widetilde{Q}_{k+1} < -\eta \widetilde{H}$, such that on this set by Theorem 2.14

$$\frac{\partial^2 \widetilde{Q}}{\partial b_{ij}\partial b_{qr}} \nabla_m b_{ij} \nabla_m b_{qr} \leqslant -\frac{1}{c_1} \cdot \frac{|\nabla \tilde{A}|^2}{\widetilde{H}} \tag{3.16}$$

with a constant $c_1 = c_1(n,k,\varepsilon,\eta)$. For $p \geqslant \max\{4, 1+4c_0c_1\}$, $c_3 = 2c_1$, the terms involving derivatives of curvature may then be estimated as desired since $|\nabla \tilde{A}|^2 \geqslant (1/n)|\nabla \widetilde{H}|^2$.

Furthermore we use Young's inequality to estimate

$$f_+^{p-1}\widetilde{H}^{1+\sigma} \leqslant \sigma f_+^p \widetilde{H}^2 + C(\sigma) f_+^{p-2/(1-\sigma)} \leqslant \sigma f_+^p \widetilde{H}^2 + C(\sigma) f_+^p + C(\sigma,p).$$

The conclusion then follows from (3.10). □

To absorb the positive integrals on the right-hand side of the inequality just established, we use the commutator of the second derivatives of $\{h_{ij}\}$ to derive a Poincaré-type estimate:

PROPOSITION 3.6. *There exists a constant c_6 depending only on n,k,ε, such that for any $p>2$, $\beta>0$*

$$\frac{1}{c_6}\int_{\mathcal{M}_t} \widetilde{H}^2 f_+^p \, d\mu \leqslant \left(p+\frac{p}{\beta}\right)\int_{\mathcal{M}_t} f_+^{p-2}|\nabla f|^2 \, d\mu + (1+\beta p)\int_{\mathcal{M}_t} \frac{f_+^{p-1}}{\widetilde{H}^{2-\sigma}}|\nabla \tilde{A}|^2 \, d\mu + \int_{\mathcal{M}_t} f_+^p \, d\mu.$$

Proof. From the definition of f in (3.13) we compute

$$\nabla_j f = -\widetilde{H}^{\sigma-1}\widetilde{S}_k^{-1}\nabla_j \widetilde{S}_{k+1} + \widetilde{H}^{\sigma-1}\widetilde{S}_k^{-2}\widetilde{S}_{k+1}\nabla_j \widetilde{S}_k - \eta \widetilde{H}^{\sigma-1}\nabla_j \widetilde{H} + (\sigma-1)\widetilde{H}^{-1} f \nabla_j \widetilde{H}$$

as well as

$$\begin{aligned}
\nabla_i \nabla_j f = & -\widetilde{H}^{\sigma-1}\widetilde{S}_k^{-1}\nabla_i\nabla_j \widetilde{S}_{k+1} - (\sigma-1)\widetilde{H}^{\sigma-2}\widetilde{S}_k^{-1}\nabla_i \widetilde{H}\nabla_j \widetilde{S}_{k+1} \\
& + \widetilde{H}^{\sigma-1}\widetilde{S}_k^{-2}\nabla_i \widetilde{S}_k \nabla_j \widetilde{S}_{k+1} + (\sigma-1)\widetilde{H}^{\sigma-2}\widetilde{S}_k^{-2}\widetilde{S}_{k+1}\nabla_i \widetilde{H}\nabla_j \widetilde{S}_k \\
& - 2\widetilde{H}^{\sigma-1}\widetilde{S}_k^{-3}\widetilde{S}_{k+1}\nabla_i \widetilde{S}_k \nabla_j \widetilde{S}_k + \widetilde{H}^{\sigma-1}\widetilde{S}_k^{-2}\nabla_i \widetilde{S}_{k+1}\nabla_j \widetilde{S}_k \\
& + \widetilde{H}^{\sigma-1}\widetilde{S}_k^{-2}\widetilde{S}_{k+1}\nabla_i\nabla_j \widetilde{S}_k - \eta(\sigma-1)\widetilde{H}^{\sigma-2}\nabla_i \widetilde{H}\nabla_j \widetilde{H} - \eta \widetilde{H}^{\sigma-1}\nabla_i\nabla_j \widetilde{H} \\
& + (\sigma-1)\widetilde{H}^{-1}\nabla_i f \nabla_j \widetilde{H} - (\sigma-1)\widetilde{H}^{-2} f \nabla_i \widetilde{H}\nabla_j \widetilde{H} + (\sigma-1)\widetilde{H}^{-1} f \nabla_i\nabla_j \widetilde{H}.
\end{aligned}$$

We now take the trace of the Hessian of f with respect to $\partial \widetilde{S}_k/\partial b_{ij}$ and restrict attention to the set where $f>0$, also bearing in mind that D_ε is fixed such that the inequalities (3.8)–(3.11) all hold. We then conclude that

$$\begin{aligned}
\frac{\partial \widetilde{S}_k}{\partial b_{ij}}\nabla_i\nabla_j f \leqslant & -\widetilde{H}^{\sigma-1}\widetilde{S}_k^{-1}\frac{\partial \widetilde{S}_k}{\partial b_{ij}}\nabla_i\nabla_j \widetilde{S}_{k+1} + C\widetilde{H}^{k+\sigma-3}|\nabla \tilde{A}|^2 + C\widetilde{H}^{k-2}|\nabla f|\,|\nabla \tilde{A}| \\
& + \frac{\partial \widetilde{S}_k}{\partial b_{ij}}\{\widetilde{H}^{\sigma-1}\widetilde{S}_k^{-2}\widetilde{S}_{k+1}\nabla_i\nabla_j \widetilde{S}_k - (\eta \widetilde{H}^{\sigma-1} - (\sigma-1)\widetilde{H}^{-1} f)\nabla_i\nabla_j \widetilde{H}\},
\end{aligned}$$

where the constants depend on n, k and ε. Here we also used $|\nabla\widetilde{H}|^2 \leqslant n|\nabla\tilde{A}|^2$ and

$$|\nabla\widetilde{S}_k| \leqslant C(n,k,\varepsilon)\widetilde{H}^{k-1}|\nabla\tilde{A}|, \quad |\nabla\widetilde{S}_{k+1}| \leqslant C(n,k,\varepsilon)\widetilde{H}^{k}|\nabla\tilde{A}|. \tag{3.17}$$

We now multiply this inequality by $f_+^p\widetilde{H}^{-k+1-\sigma}$ and fix $\varepsilon=\varepsilon(\eta,n,k)>0$ such that Corollary 2.16 can be applied to the first term on the right-hand side to yield

$$\begin{aligned}
\tfrac{1}{2}\eta\int \widetilde{H}^{2-k}\widetilde{S}_k f_+^p\,d\mu \leqslant & -\int f_+^p\widetilde{H}^{-k+1-\sigma}\frac{\partial\widetilde{S}_k}{\partial b_{ij}}\nabla_i\nabla_j f\,d\mu \\
& -\int f_+^p\widetilde{H}^{-k}\frac{\partial\widetilde{S}_k}{\partial b_{ij}}\bigg\{-\widetilde{S}_k^{-2}\widetilde{S}_{k+1}\nabla_i\nabla_j\widetilde{S}_k \\
& \qquad +\widetilde{S}_k^{-1}\frac{\partial^2\widetilde{S}_{k+1}}{\partial b_{lm}\partial b_{qr}}\nabla_i b_{lm}\nabla_j b_{qr}+\widetilde{S}_k^{-1}\frac{\partial\widetilde{S}_{k+1}}{\partial b_{lm}}\nabla_l\nabla_m b_{ij}\bigg\}\,d\mu \\
& +\int f_+^p\widetilde{H}^{-k}\frac{\partial\widetilde{S}_k}{\partial b_{ij}}\Big(-\eta-\frac{\varepsilon}{1+n\varepsilon}(n-k)+(\sigma-1)\widetilde{H}^{-\sigma}f\Big)\nabla_i\nabla_j\widetilde{H}\,d\mu \\
& +\frac{\varepsilon}{1+n\varepsilon}(n-k+1)\int f_+^p\widetilde{H}^{-k}\widetilde{S}_{k-1}\widetilde{S}_k^{-1}\frac{\partial\widetilde{S}_{k+1}}{\partial b_{ij}}\nabla_i\nabla_j\widetilde{H}\,d\mu \\
& +C(n,k,\eta)\int f_+^p(\widetilde{H}^{-2}|\nabla\tilde{A}|^2+\widetilde{H}^{-1-\sigma}|\nabla f|\,|\nabla\tilde{A}|)\,d\mu \\
& +C(n,k,\eta)\int f_+^p\widetilde{H}^k\widetilde{S}_k^{-1}(c(n,k,\eta)+\widetilde{H})\,d\mu.
\end{aligned}$$

Now notice that in view of (3.8) we have $\widetilde{S}_k \geqslant C(n)\varepsilon^{k-1}\widetilde{H}^k$, where $\varepsilon=\varepsilon(\eta,n,k)$ was fixed above. Hence the crucial term on the left-hand side is estimated from below by a small fraction of

$$\int \widetilde{H}^2 f_+^p\,d\mu,$$

whereas the last term on the right-hand side is lower order compared to the left-hand side, and can by interpolation be replaced by

$$C(n,k,\eta)\int f_+^p\,d\mu.$$

We now integrate by parts all terms of the right-hand side involving second derivatives of curvature. Using repeatedly the inequalities (3.8)–(3.11), (3.17) and the fact that $f \leqslant C(n,k,\eta)\widetilde{H}^\sigma$ we derive

$$\begin{aligned}
\int \widetilde{H}^2 f_+^p\,d\mu \leqslant{} & C(n,k,\eta)\int f_+^p(\widetilde{H}^{-2}|\nabla\tilde{A}|^2+\widetilde{H}^{-1-\sigma}|\nabla f|\,|\nabla\tilde{A}|)\,d\mu \\
& +pC(n,k,\eta)\int f_+^{p-2}(|\nabla f|^2+f_+\widetilde{H}^{-1}|\nabla f|\,|\nabla\tilde{A}|)\,d\mu+C(n,k,\eta)\int f_+^p\,d\mu,
\end{aligned}$$

and the conclusion of the proposition follows. □

COROLLARY 3.7. *There are constants c_7 and c_8 depending only on n, k, η and the initial data such that for $p \geqslant c_7$, $0 \leqslant \sigma \leqslant c_8 p^{-1/2}$ there is a uniform bound*

$$\int_{\mathcal{M}_t} f_+^p \, d\mu \leqslant c_9 \quad \text{on } [0, T[,$$

depending only on n, k, η, σ, p and $\mathcal{M}_0$.

Proof. From Lemma 3.5 and Proposition 3.6 with $\beta = p^{-1/2}$ we infer that for p, σ as above

$$\frac{d}{dt} \int f_+^p \, d\mu \leqslant C \int f_+^p \, d\mu + C \int d\mu$$

with constants depending on n, k, p, η, σ and $\mathcal{M}_0$. This yields the desired estimate since T is bounded in terms of the diameter of the initial surface. □

Having established the L^p-bound for $\sigma = O(p^{-1/2})$ the proof of the sup-bound for f proceeds exactly as in [12], compare also [14] and [8]. This establishes the next step in the iteration on k according to Lemma 2.8, and completes the proof of the main result in Theorem 1.1. □

Remark 3.8. It was shown by Smoczyk in [14] that a star-shaped surface evolving by mean curvature satisfies the bound on the scalar curvature corresponding to the first step in our iteration. He also proved that, for such surfaces, the uniform bounds $H \geqslant -C_1$ and $|A|^2 \leqslant C_2 H^2 + C_3$ hold for suitable constants C_i. It is easily checked that our Lemmas 2.7 and 2.8 are still valid (with a different choice of the constants) replacing the assumption $H > 0$ with these weaker bounds. Therefore, the iterative procedure of this section can be applied, and the main results in Theorem 1.1 and Corollary 1.2 hold in the class of star-shaped surfaces as well.

Remark 3.9. Let $F: \mathcal{M} \times [0, T[\to (N^{n+1}, h)$ be a closed hypersurface of nonnegative mean curvature evolving by mean curvature flow in a smooth Riemannian manifold (N^{n+1}, h) satisfying bounds $-K_1 \leqslant \sigma_N \leqslant K_2$, $|\nabla^N \mathrm{Riem}| \leqslant L$, $i_N > 0$ on its sectional curvature, gradient of Riemann curvature tensor and injectivity radius respectively, similarly as in [9]. Then the estimates in Theorem 1.1 continue to hold with constants $C_{\eta,k}$ now also depending on the data of (N^{n+1}, h) above, and we again conclude that singularities of mean convex surfaces have convex blowups. To see this, first note that the mean curvature satisfies (compare [9])

$$\frac{\partial H}{\partial t} = \Delta H + H(|A|^2 + \mathrm{Ric}^N(\nu, \nu))$$

and therefore remains nonnegative due to the parabolic maximum principle.

Furthermore, the crucial evolution equation for the full second fundamental form differs from the Euclidean case only by lower-order terms, see [9]:

$$\frac{\partial}{\partial t}A=\Delta A+|A|^2A+\mathrm{Riem}^N*A+\nabla^N\mathrm{Riem}^N.$$

Similar lower-order terms appear in the commutator identity for the second derivatives of the second fundamental form, and hence the additional terms in Lemma 3.5 and Proposition 3.6 can all be estimated by

$$C\int f_+^p\,d\mu,$$

with a constant depending on n, k, p, K_1, K_2 and L. The proof then proceeds as before.

4. Description of singularities

Let $[0,T[$ be the maximal time interval where a smooth solution of the flow exists, such that $\sup_{\mathcal{M}_t}|A|^2$ becomes unbounded for $t\to T$. We assume the reader to be familiar with the rescaling procedure described in [12], see also [6] and [5]. The new estimates in Theorem 1.1 yield a description of type-II singularities of the flow in the case of nonnegative mean curvature.

Let (x_k,t_k) be an essential blowup sequence as in (4.2) and (4.3) of [12], and let $\widetilde{\mathcal{M}}_{k,t}$ be the corresponding rescaled surfaces with limiting mean curvature flow $\widetilde{\mathcal{M}}_\tau$, $\tau\in\mathbf{R}$, as described in [12, Lemma 4.4 and Theorem 4.5]. As in this reference we call a solution of mean curvature flow (1.1) which exists for all time and moves by translation in $\mathbf{R}^{n+1}$ a translating soliton.

THEOREM 4.1. *If $\mathcal{M}_0$ has nonnegative mean curvature, then any limiting flow of a type-*II *singularity has convex surfaces $\widetilde{\mathcal{M}}_\tau$, $\tau\in\mathbf{R}$. Furthermore, either $\widetilde{\mathcal{M}}_\tau$ is a strictly convex translating soliton or (up to rigid motion) $\widetilde{\mathcal{M}}_\tau=\mathbf{R}^{n-k}\times\Sigma_\tau^k$, where Σ_τ^k is a lower-dimensional strictly convex translating soliton in $\mathbf{R}^{k+1}$.*

Proof. The convexity of the limiting flow is for both type-I and type-II singularities an immediate consequence of the a priori estimates in Theorem 1.1. To obtain the splitting result we apply the strict maximum principle for symmetric 2-tensors of Hamilton [4, Lemma 8.2] to the evolution of the nonnegative second fundamental form

$$\frac{\partial}{\partial t}h_j^i=\Delta h_j^i+|A|^2h_j^i \tag{4.1}$$

on the limiting flow $\widetilde{\mathcal{M}}_\tau$, $\tau\in\mathbf{R}$. Thus the rank of A is a constant, and the null space of A is invariant under parallel translation and invariant in time. Using the Frobenius theorem as in [4, §8] or as in [11, Theorem 5.1] we see that each $\widetilde{\mathcal{M}}_\tau$ splits as an isometric product $\mathbf{R}^{n-k}\times\Sigma_\tau^k$, where Σ_τ^k is strictly convex unless $\widetilde{\mathcal{M}}_\tau$ is strictly convex itself. Finally, Σ_τ^k is a translating soliton since it is convex, solves mean curvature flow for all $\tau\in\mathbf{R}$ and admits the maximum of the mean curvature at one point at least, compare [6, Theorem 1.3]. $\square$

We wish to point out that the splitting result can also be obtained by a successive application of the (scalar) maximum principle to the quantities Q_k, $2\leqslant k\leqslant n$, satisfying the parabolic equations (3.2). The splitting dimension k is then characterised by the relations $S_n=S_{n-1}=\ldots=S_{k+1}=0$, $S_l>0$ when $1\leqslant l\leqslant k$.

In the case $k=1$ the only translating soliton is the "grim reaper" curve $x=\log\cos y+t$, and in higher dimensions it is known that there are rotationally symmetric translating solitons. It is an open problem whether in higher dimensions there are other translating solitons, convex or of positive mean curvature.

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