

CONVEXITY OF SMOOTH YIELD SURFACE OF FRICTIONAL MATERIAL

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INTRODUCTION

The yield condition, loading surface or inelastic potential of an inelastic material may be described by a surface in the space of principal stresses σ_1 , σ_2 and σ_3 . In the deviatoric cross sections, which form equal angles with the principal stress axes, the Tresca criterion is represented by a regular hexagon, which possesses six axes of symmetry, as does the von Mises criterion, which is represented by a circle. For frictional materials, however, only three axes of symmetry are to be expected, as is exemplified by the Mohr-Coulomb criterion which represents an irregular hexagon. This criterion, however, is not very convenient for numerical finite element analysis because the corners impair convergence of numerical computation. At the same time, material test data are never precise enough to indicate whether the loading surface has corners or is smooth. Therefore, smooth yield conditions should be used in finite element analysis.

Various known smooth yield conditions which have a reduced degree of symmetry in the deviatoric cross section, similar to the Mohr-Coulomb criterion, have been recently reviewed by Zienkiewicz and Pande (1977). Yet, the question of convexity of the yield surface, which is important for theoretical reasons as well as numerical applications, has apparently, not been examined. This question, will, therefore, be addressed here.

EXISTING FORMULAS

The oldest as well as the simplest formula for a smooth deviatoric cross section which possesses three rather than six axes of symmetry was proposed simultaneously by Gudehus (1973) and by Argyris, et al. (1974). It may be described by the formula

$$r(\theta) = r_c \frac{2k}{(1+k) + (1-k) \cos 3\theta} \quad (1)$$

in which r = distance to the yield surface from its axis (i.e., the straight line forming equal angles with the principal stress axes); k = empirical constant and θ = similarity angle, which is given by the expression $\cos 3\theta = \frac{1}{3} J_2^{-3/2} \sqrt{3} J_3 / 2$, in which J_2 and J_3 = second and third invariants of

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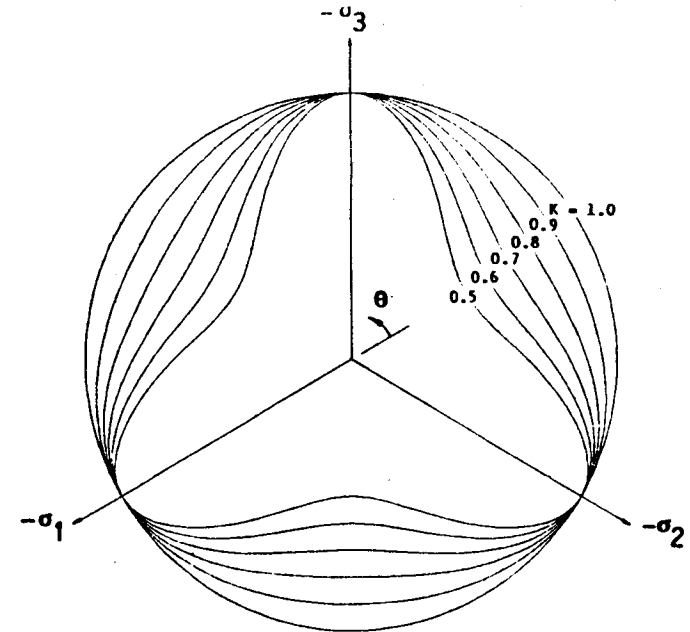


FIG. 1.—Deviatoric Cross Sections According to the Criterion Proposed by Gudehus and by Argyris, et al.

the stress deviator. Angle θ is measured from the positive σ_1 direction (see Fig. 1). r_c is the value of $r(\theta)$ on the compression meridian, $\theta = 60^\circ$.

Another well known expression was proposed by Willam and Warnke (1975) as part of their five-parameter failure criterion for concrete. This expression has two parameters r_c and r_t and reads

$$r = r_c \frac{R + (2r_t - r_c)(2R \cos \theta + 5r_t^2 - 4r_t r_c)^{1/2}}{2R \cos \theta + (2r_t - r_c)^2} \quad (2)$$

in which $R = 2(r_c^2 - r_t^2) \cos \theta$ (3)

and r_t = value of $r(\theta)$ on the tension meridian, $\theta = 0^\circ$. Eq. 2 represents a section of an ellipse spanning from the tensile direction of one principal stress axis to the compressive stress direction of another principal stress axis; see Fig. 2 (where $k = r_t/r_c$). The ellipse intersects both axes normally, so that the entire deviatoric section, patched up from six identical ellipse segments, has a continuous slope at the axes.

CONVEXITY

Since the segment of an ellipse is always convex, the yield condition of Willam and Warnke need not be examined. However, the yield condition of Gudehus and Argyris, et al. needs to be examined. We do so graphically in Fig. 1, in which the shapes of the deviatoric cross section of this yield criterion are plotted according to Eq. 1. We see from this

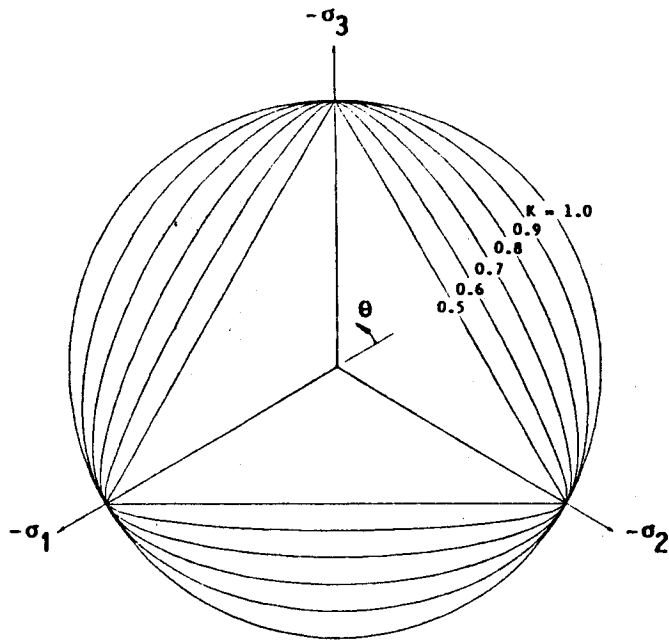


FIG. 2.—Deviatoric Cross Sections According to Willam and Warnke's Criterion

figure that convexity is violated for small values of k . Evaluating the minimum curvature numerically for various values of k , and iterating to find the case of zero minimum curvature, we find that the condition of convexity of this criterion is

$$k \geq 0.777 \dots \dots \dots (4)$$

Therefore, if the k -values which are less than the foregoing limit cannot be ruled out in advance, the criterion of Gudehus and Argyris, et al. should be replaced by the criterion of Willam and Warnke.

A generalization of the formula of Gudehus and Argyris, et al., which may be written as

$$r(\theta) = r_c \frac{2k(c_1 + c_2 \cos 3\theta)}{(c_3 + k) + (c_3 - k) \cos 3\theta} \quad \text{with } c_1 - c_2 = 1, c_1 + c_2 = c_3 \dots \dots (5)$$

has been also examined. It is found that this type of yield criterion generally does not assure convexity for the entire range of parameter k , up to the limiting case of an equilateral triangle (Fig. 2).

CONCLUSION

While the yield criterion of Willam and Warnke is always convex, the yield criteria of Gudehus and of Argyris, et al., are convex only for a certain range of material parameters. However, parameter values which fall outside this range might not be excluded for applications to frictional

materials, and in such a case the criterion of Willam and Warnke should be preferred despite the fact that it is given by a more complex expression.

APPENDIX.—REFERENCES

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