J. Math. Soc. Japan
Vol. 14, No. 2, 1962

# Convexity theorems for Fourier series

By Kenji YANO

(Received March 16, 1959)
(Revised July 5, 1961)

In this paper we shall investigate some convexity theorems for Fourier series. This paper consists with three parts, each of which contains two main theorems (Theorems 1-6). These theorems together with Riesz's theorem (Lemma 5 in § 6) and Dixon-Ferrar's theorem (Lemma 2 in § 3) will constitute a complete system of convexity theorems in this direction, while the last two theorems are independent of Fourier series.

Let $\varphi(t)$ be an even function, integrable in $(0,\pi)$ in Lebesgue sense, periodic of period $2\pi$, and let

$$\varphi(t)\sim\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos nt\,,$$

(0.1) $$\Phi_0(t)=\varphi(t)\,,\qquad \Phi_\alpha(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-u)^{\alpha-1}\varphi(u)du\qquad(\alpha>0)\,,$$

and more generally, for any integer $k\geqq 0$ and $0<t\leqq\pi$,

(0.2) $$\Phi_0^k(t)=t^k\varphi(t)\,,\qquad \Phi_\alpha^k(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-u)^{\alpha-1}u^k\varphi(u)du\qquad(\alpha>0)\,.$$

The Fourier series of $\varphi(t)$ at $t=0$ is $a_0/2+a_1+\cdots+a_n+\cdots$. The $n$-th $(\mathrm{C},\beta)$ sum of this series is

$$s_n^\beta=A_n^\beta\frac{1}{2}a_0+\sum_{\nu=1}^{n}A_{n-\nu}^\beta a_\nu=\sum_{\nu=0}^{n}A_{n-\nu}^{\beta-1}s_\nu\qquad(-\infty<\beta<\infty)\,,$$

where $s_n=s_n^0$, and $A_n^\beta$ is defined by the identity

$$(1-x)^{-\beta-1}=\sum_{n=0}^{\infty}A_n^\beta x^n\qquad(|x|<1)\,.$$

In particular, $s_n^{-1}=a_n\to 0$ as $n\to\infty$.

We understand that $t\to 0$ means $t>0$ and $t\to 0$.

These notations will be used throughout this paper, except when it is stated otherwise.

## Part I.

### 1. Theorems (1).

THEOREM 1. *Let* $0\leqq\beta$, $-1\leqq c$, $0<\beta-b\leqq\gamma-c$ *and* $c-b<1$. (I) *If*

$$\int_0 |\Phi_\beta(u)|\, du = o(t^{\gamma+1}) \qquad as \quad t \to 0\,, \tag{1.1}$$

$$s_n^c = O(n^b) \qquad as \quad n \to \infty\,, \tag{1.2}$$

*then we have*

$$s_n^r = o(n^q)\,, \qquad q = b + (r-c)\frac{\beta-b}{\gamma-c}\,, \tag{1.3}$$

*as* $n \to \infty$, *for*

$$c < r < \gamma'\,, \qquad \gamma' = \inf\left(\gamma, \frac{(b+1)\gamma-(\beta+1)c}{\gamma-c+b-\beta}\right). \tag{1.4}$$

(II) *If* (1.1) *holds, and* $s_n^c = O_L(n^b)$, *then* (1.3) *holds for* $r$ *with* $c+1 \leqq r < \gamma'$, *provided that* $c+1 < \gamma'$.

THEOREM 2. *Let* $0 \leqq b$, $-1 \leqq c$, $0 < \beta - b \leqq \gamma - c$ *and* $c - b < 1$. *If* (1.1) *holds, and*

$$\Phi_b(t) = O(t^c) \qquad as \quad t \to 0\,, \tag{1.5}$$

*then we have* (1.3) *with* (1.4).

(N. B. 1) The range of $r$ in the theorems, i. e. $c < r < \gamma'$ is essentially equivalent to the common range of $c < r < \gamma$ and $r - q < 1$. Consequently, $c < r < \gamma'$ coincides with

$$\begin{cases} c < r < ((b+1)\gamma-(\beta+1)c)/(\gamma-c+b-\beta) & \text{if} \quad \gamma-\beta > 1\,, \\ c < r < \gamma & \text{if} \quad \gamma-\beta \leqq 1\,. \end{cases}$$

(N. B. 2) Since $a_n = o(1)$, the condition (1.2) is a fortiori satisfied when $c - b \leqq -1 \leqq c$. Hence, in Theorem 1 we may assume that $c - b > -1$ with no loss of generality. An analogous notice may be made for Theorem 2, since as a matter of fact the same argument as in the proof of Lemma 1 in § 2 will verify that Theorem 2 is still true when (1.5) is replaced by

$$\int_0^t |\Phi_b(u)|\, du = O(t^{c+1}) \qquad \text{as} \quad t \to 0\,, \tag{1.5$'$}$$

while (1.5)$'$ is actually true when $c - b \leqq -1 \leqq c$. cf. Lemma 3 in § 4.

(N. B. 3) On the contrary, if $c - b \leqq -1$ then, letting $c = -1$ and $b = 0$, we have the following corollary in place of Theorems 1, 2, the condition (1.1) being replaced by a slightly less stringent one, as it will be shown later.

COROLLARY 1.1. *If* $0 < \beta < \gamma + 1$, *and*

$$\int_0^t |\Phi_\beta(u)|\, du = O(t^{\gamma+1})\,, \tag{1.1$'$}$$

*then we have*

$$s_n^r = o(n^{(r+1)\beta/(\gamma+1)})\,,$$

*for* $-1 \leqq r < \inf(\gamma, (\gamma+1+\beta)/(\gamma+1-\beta))$.

Now, letting $r = q = \alpha$ in Theorems 1, 2, we have

(1.6) $$(\gamma-c+b-\beta)\alpha=b\gamma-\beta c\,,$$

and the condition $r-q<1$ is then clearly satisfied. Hence, for the Fourier series of $\varphi(t)$ to be summable $(\mathrm{C},\alpha)$ at $t=0$ for a unique value of $\alpha$, it is necessary that $\gamma-c+b-\beta\neq 0$, i.e.

(1.7) $$0<\beta-b<\gamma-c\,.$$

In these circumstances we have

$$(\gamma-c+b-\beta)(\alpha-c)=(b-c)(\gamma-c)\,,$$

$$(\gamma-c+b-\beta)[\alpha-(c+1)]=(b-c-1)(\gamma-c)+(\beta-b)\,,$$

$$(\gamma-c+b-\beta)(\gamma'-\alpha)=\inf\,[(\gamma-\beta)(\gamma-c),(\gamma-c)]\,.$$

Therefore by (1.7) we see that, from the last three relations,

(1.8) if $b>c$ and $\gamma>\beta$ then $c<\alpha<\gamma'$,

(1.9) if $(b-c-1)(\gamma-c)+(\beta-b)\geqq 0$ and $\gamma>\beta$ then $c+1\leqq\alpha<\gamma'$,

and conversely.

Taking into account these facts, we may deduce the summability theorems from the above theorems. Letting thus $c=-1$ and $b=-(1-\delta)$ in Theorem 1, the condition (1.7) together with those in (1.8) yields $0\leqq\beta<\gamma$ and $0<\delta<1$. And, the first condition in (1.9) then becomes $\gamma\delta\geqq\gamma-\beta$. Hence, using the expression of $\alpha$ in (1.6) we have the following corollary from Theorem 1.

COROLLARY 1.2. *Let* $0\leqq\beta<\gamma$ *and* $0<\delta<1$. *If* (1.1) *holds, and if either of the two conditions*

$$\begin{cases} a_n=O(n^{-(1-\delta)})\,, & \\ a_n=O_L(n^{-(1-\delta)})\,, & \gamma\delta\geqq\gamma-\beta\,, \end{cases}$$

*is satisfied, then*

$$s_n^\alpha=o(n^\alpha)\,,\qquad \alpha=(\gamma\delta-(\gamma-\beta))/(\gamma-\beta+\delta)\,.$$

This proves a conjecture of Sunouchi [**9**].

Similarly, from Theorem 2 we get the following

COROLLARY 2.1. *Let* $0<\beta<\gamma$ *and* $0<\delta<1$. *If* (1.1) *holds, and* $\varphi(t)=O(t^{-\delta})$, *then,*

$$s_n^\alpha=o(n^\alpha)\,,\qquad \alpha=\beta\delta/(\gamma-\beta+\delta)\,.$$

This is a theorem due to Kanno [**10**].

By Corollary 1.1, the last two corollaries can be improved as follows when $\delta=1$.

COROLLARY 1.3. *If* $0<\beta<\gamma$ *and* (1.1)′ *holds, then*

$$s_n^\alpha=o(n^\alpha)\,,\qquad \alpha=\beta/(\gamma-\beta+1)\,.$$

This is a theorem of Yano [**11**].

COROLLARY 2.2. *If* $0\leqq\beta$, $-1<\gamma<\beta+1$, *and*

(1.10) $$\Phi_\beta(t)=o(t^\gamma) \qquad as \quad t\to 0\,,$$
*then we have*
(1.11) $$s_n^{\gamma+\eta}=o(n^{\beta+\eta}) \qquad as \quad n\to\infty\,,$$
*for every* $\eta>0$.

This is due to Obrechkoff [3] when $0\leqq\beta\leqq\gamma<\beta+1$ is satisfied. This corollary is immediately deduced by applying Theorem 2 to (1.10) and $\Phi_{\beta+k}(t)=o(t^{\gamma+k})$, $k>0$, the latter of which follows from the former.

This corollary has a meaning only when $\beta<\gamma+1$, in the sense that when $\beta\geqq\gamma+1>0$ the conclusion (1.11) is a result from $a_n=o(1)$.

REMARK 1. Hyslop [7] has remarked that the proposition "$\gamma>\beta\geqq 0$ and (1.10) imply (1.11) for every $\eta>0$" may be proved by the argument used in Wang [6]. This is true when $\gamma-\beta<1$ (see Corollary 2.2), but generally it is false when $\gamma-\beta\geqq 1$. Indeed, if we put

$$\varphi(t)=\frac{\pi}{2}\cdot\frac{|t|}{\log(2\pi/|t|)} \qquad (|t|\leqq\pi)\,,$$

then it follows that

$$\begin{aligned} s_n^1&=\int_0^\pi\frac{t}{\log(2\pi/t)}\cdot\frac{1-\cos(n+1)t}{(2\sin(t/2))^2}\,dt\\ &=\int_{n^{-1}}^\pi\frac{t}{\log(2\pi/t)}\cdot\frac{dt}{(2\sin(t/2))^2}+O(1)\\ &=\int_{n^{-1}}^\pi\frac{dt}{t\log(2\pi/t)}+O(1)=\log\log n+O(1)\,,\end{aligned}$$

from which we have

$$s_n^{1+\eta}=A_n^\eta\log\log n+O(n^\eta)\,, \qquad \eta>0\,.$$

This gives a negative example for the case $\gamma-\beta=1$, since $s_n^{1+\eta}=o(n^\eta)$, $\eta>0$, does not hold while $\Phi_0(t)=o(t)$.

An example in the case $\gamma-\beta>1$ can be obtained as follows. Let

$$\varphi(t)=\frac{\pi}{2}t^2 \qquad (|t|\leqq\pi)\,.$$

Then, we obtain

$$\begin{aligned} s_n^2&=\int_0^\pi t^2\Big[\frac{n+3/2}{(2\sin(t/2))^2}-\frac{\sin(n+3/2)t}{(2\sin(t/2))^3}\Big]dt\\ &=n\int_0^\pi\frac{t^2dt}{(2\sin(t/2))^2}+O(1)=(2\pi\log 2)n+O(1)\,.\end{aligned}$$

This shows that $\Phi_0(t)=o(t^{2-\eta})$, $0<\eta\leqq 1$, does not imply $s_n^2=o(n^\eta)$.

## 2. Fundamental lemma (1).

LEMMA 1 (Fundamental lemma). *Let* $0\leqq\beta$, $-1\leqq r$, $0<\beta-q\leqq\gamma-r$ *and* $r-q<1$. *If*

$$\int_0^t |\Phi_\beta(u)|\, du = o(t^{\gamma+1}) \qquad as \quad t \to 0\,, \tag{2.1}$$

*and if for any assigned positive number* $\varepsilon$ *there exist an* $\varepsilon' = \varepsilon'(\varepsilon) > 0$ *tending to zero with* $\varepsilon$ *and an* $n_\varepsilon$ *so that*

$$s_{n+\nu}^r - s_n^r > -\varepsilon' n^q\,, \qquad \nu = 1, 2, \cdots, m\,, \tag{2.2}$$

*holds for* $m = [\varepsilon n^{(\beta-q)/(\gamma-r)}]$ *and* $n > n_\varepsilon$, *then one obtains*

$$s_n^r = o(n^q) \qquad as \quad n \to \infty\,. \tag{2.3}$$

REMARK 2. Arguing similarly as in the proof of Lemma 1 we can easily prove the following proposition.

If, in Lemma 1, $o$ in (2.1) is replaced by $O$, and the third assumption by "if $s_{n+\nu}^r - s_n^r > -An^q$, $A > 0$ a constant, holds for $0 < \nu < [n^{(\beta-q)/(\gamma-r)}]$ and $n > 1$", then we have $s_n^r = O(n^q)$ in place of (2.3).

For the proof of this proposition, it is sufficient to take $m = [(2k)^{-1} n^{(\beta-q)/(\gamma-r)}]$, $k$ being an integer greater than $\gamma - r$.

We need some further lemmas. We write

$$D_n^r(t) = \sum_{\nu=0}^{n} A_{n-\nu}^{r-1} D_\nu(t)\,, \tag{2.4}$$

where $D_n(t)$ is the $n$-th Dirichlet kernel, and

$$X_n(t) = \begin{cases} X_n^+(t) = \dfrac{1}{n^q m^k} \displaystyle\sum_{\nu_1=1}^{m} \sum_{\nu_2=1}^{m} \cdots \sum_{\nu_k=1}^{m} D_{n+\nu_1+\nu_2+\cdots+\nu_k}^r(t) \\[2ex] X_n^-(t) = \dfrac{1}{n^q m^k} \displaystyle\sum_{\nu_1=1}^{m} \sum_{\nu_2=1}^{m} \cdots \sum_{\nu_k=1}^{m} D_{n-\nu_1-\nu_2-\cdots-\nu_k}^r(t)\,, \end{cases} \tag{2.5}$$

where $k$ is a fixed positive integer, and $m$ is taken such as $m \leqq (2k)^{-1} n$.

LEMMA 1.1. *Let* $m = m(n, \varepsilon) \leqq (2k)^{-1} n$ *tend to infinity with* $n$ *in as same order as or lower order than* $n$, *and let* $r \geqq -1$, $q$ *be arbitrary. In these circumstances, if*

$$\int_0^\pi \varphi(t) X_n(t)\, dt = o(1) \qquad as \quad n \to \infty\,,$$

*where* $X_n(t)$ *is defined by* (2.5), *and if* (2.2) *holds for* $n > n_\varepsilon$, *then we have* (2.3).

This is Lemma 1 in the paper [**12**] slightly modified. Also cf. the proof of Lemma 8.1 in § 7.

LEMMA 1.2. *If* $r \geqq -1$, *and* $q$ *is arbitrary, and* $m$ *has the same meaning as in Lemma* 1.1, *then* $X_n(t)$ *defined by* (2.5) *has the following properties.*

$$X_n(t) = \Lambda_n(t) + R_n(t)\,,$$

*where, for* $\mu = 0, 1, \cdots$,

$$X_n^{(\mu)}(t) = \left(\frac{\partial}{\partial t}\right)^{\mu} X_n(t) = O(n^{r-q+\mu+1}) \qquad (0 \leqq t \leqq \pi)\,, \tag{2.6}$$

$$(2.7)\qquad A_n^{(\mu)}(t)=\left(\frac{\partial}{\partial t}\right)^{\mu}A_n(t)=\begin{cases} O(n^{\mu-q}/t^{r+1}) & (nt\geqq 1)\\ O(n^{\mu-q}/m^k t^{k+r+1}) & (mt\geqq 1),\end{cases}$$

$$(2.8)\qquad R_n^{(\mu)}(t)=\left(\frac{\partial}{\partial t}\right)^{\mu}R_n(t)=O(n^{r-q-1}/t^{\mu+2})\qquad (nt\geqq 1).$$

PROOF OF LEMMA 1.2. (2.4) yields for $\mu=0,1,\cdots,$

$$\left(\frac{\partial}{\partial t}\right)^{\mu}D_n^r(t)=O(n^{r+\mu+1})\qquad \text{for}\quad 0\leqq t\leqq\pi,$$

which together with (2.5) gives (2.6). Next, $D_n^r(t)$ is written as, as it is well known,

$$D_n^r(t)=A_n^r(t)+R_n^r(t),$$

where $R_n^r(t)$ vanishes when $r=0$ or $r=-1$, and generally

$$(2.9)\qquad A_n^r(t)=\left(2\sin\frac{1}{2}t\right)^{-(r+1)}\sin\left(\left(n+\frac{1}{2}\right)t+\frac{1}{2}r(t-\pi)\right),$$

$$(2.10)\qquad \begin{aligned} R_n^r(t)=&\sum_{j=1}^{p}A_n^{r-j}\left(2\sin\frac{1}{2}t\right)^{-(j+1)}\sin\frac{1}{2}(t-j(t-\pi))\\ &+\left(2\sin\frac{1}{2}t\right)^{-(p+1)}\sum_{\nu=n+1}^{\infty}A_\nu^{r-p-1}\sin\left(\left(\nu-n-\frac{1}{2}\right)t-\frac{1}{2}p(t-\pi)\right),\end{aligned}$$

$p$ being an arbitrary integer greater than $r$. For details, see e.g. Zygmund [**16**, p. 259], and the paper [**14**]. The relation (2.10) implies

$$(2.11)\qquad \left(\frac{\partial}{\partial t}\right)^{\mu}R_n^r(t)=O(n^{r-1}/t^{\mu+2}),\qquad (nt\geqq 1),$$

for $\mu=0,1,\cdots,\mu_0$; $\mu_0$ being as large as we wish with $p$.

Now, dividing $X_n^+(t)$ into two parts,

$$\begin{aligned} X_n^+(t)=&\frac{1}{n^q m^k}\sum_{\nu_1=1}^{m}\sum_{\nu_2=1}^{m}\cdots\sum_{\nu_k=1}^{m}A_{n+\nu_1+\nu_2+\cdots+\nu_k}^r(t)\\ &+\frac{1}{n^q m^k}\sum_{\nu_1=1}^{m}\sum_{\nu_2=1}^{m}\cdots\sum_{\nu_k=1}^{m}R_{n+\nu_1+\nu_2+\cdots+\nu_k}^r(t)=A_n(t)+R_n(t),\end{aligned}$$

say. Then $R_n(t)$ satisfies the condition (2.8) by (2.11). Substituting (2.9), $n$ being replaced by $n+\nu_1+\nu_2+\cdots+\nu_k$, into the expression of $A_n(t)$ and then adding them successively with respect to $\nu$'s we have

$$A_n(t)=\frac{(2\sin(mt/2))^k}{n^q m^k(2\sin(t/2))^{k+r+1}}\sin\left(\left(n+\frac{1}{2}\right)t+\frac{1}{2}k(m+1)t+\frac{1}{2}r(t-\pi)\right),$$

from which we get (2.7) similarly as in Lemma 2 in loc. cit. [**13**].

Similar results will be obtained for $X_n^-(t)$, and we get the lemma.

PROOF OF LEMMA 1. It is sufficient to show that, by Lemma 1.1,

$$(2.12)\qquad \int_0^{\pi}\varphi(t)X_n(t)dt=o(1)\qquad \text{as}\quad n\to\infty,$$

where $X_n(t)$ is defined by (2.5). Let us take the integer $l=[\beta+1]$. Then,

(2.13) $$l-1\leqq\beta<l,$$

and $l\geqq 1$ since $\beta\geqq 0$. By (2.7), (2.8), $r-q<1$, and taking $k>\gamma-r$, we have for $\mu=0,1,\cdots,l-1$, as $n\to\infty$,

$$X_n^{(\mu)}(t)=o(1)\qquad(o<t_0\leqq t\leqq\pi)$$

and then

$$[\Phi_{\mu+1}(t)X_n^{(\mu)}(t)]_{t=0}^{\pi}=o(1).$$

Hence, applying integration by parts $l$-times to the left hand side integral in (2.12), it is seen that (2.12) is equivalent to

(2.14) $$\int_0^{\pi}\Phi_l(t)X_n^{(l)}(t)dt=o(1)\qquad\text{as}\quad n\to\infty.$$

On the other hand, the assumption (2.1) together with $l>\beta$ implies

$$\int_0^t|\Phi_l(u)|\,du=o(t^{\gamma+1+l-\beta}),$$

from which and from (2.6) with $\mu=l$, i.e. $X_n^{(l)}(t)=O(n^{r-q+l+1})$, it follows that

$$\begin{aligned}\int_0^{n^{-1}}\Phi_l(t)X_n^{(l)}(t)dt&=O\left(n^{r-q+l+1}\int_0^{n^{-1}}|\Phi_l(t)|\,dt\right)\\&=o(n^{r-q+l+1}(n^{-1})^{\gamma+1+l-\beta})\\&=o(n^{r-q-(\gamma-\beta)})=o(1),\end{aligned}$$

because $r-q\leqq\gamma-\beta$ by the assumption. Next, by (2.8) with $\mu=l$, i.e. $R_n^{(l)}(t)=O(n^{r-q-1}/t^{l+2})$ for $nt\geqq 1$, we have

$$\int_{n^{-1}}^{\pi}\Phi_l(t)R_n^{(l)}(t)dt=O\left(n^{r-q-1}\int_{n^{-1}}^{\pi}|\Phi_l(t)|\,t^{-(l+2)}dt\right),$$

which is, integrating by parts, and taking into account the fact $r-q<1$,

$$\begin{cases}o(n^{r-q-1}[t^{\gamma-\beta-1}]_{n^{-1}}^{\pi})=o(n^{r-q-(\gamma-\beta)})=o(1) & (\gamma-\beta<1)\\ o(n^{r-q-1}[t^{\gamma-\beta-1}]_{n^{-1}}^{\pi})=o(n^{r-q-1})=o(1) & (\gamma-\beta>1)\\ o\left(n^{r-q-1}\left[\log\dfrac{1}{t}\right]_{n^{-1}}^{\pi}\right)=o(n^{r-q-1}\log n)=o(1) & (\gamma-\beta=1).\end{cases}$$

Hence, observing that $X_n(t)=A_n(t)+R_n(t)$, to get (2.14) it is sufficient to show

(2.15) $$I\equiv\int_{n^{-1}}^{\pi}\Phi_l(t)A_n^{(l)}(t)dt=o(1)\qquad\text{as}\quad n\to\infty.$$

(2.15) may be proved by the same argument as in the proof of Theorem 1 in loc. cit. [**13**], but for the sake of completeness we reproduce it. By the identity

$$\Gamma(l-\beta)\Phi_l(t)=\int_0^t(t-u)^{l-\beta-1}\Phi_\beta(u)du,$$

and neglecting the numerical factor $1/\Gamma(l-\beta)$, it follows that

$$I=\int_{n^{-1}}^{\pi}A_n^{(l)}(t)dt\int_0^t(t-u)^{l-\beta-1}\Phi_\beta(u)du$$
$$=\int_{n^{-1}}^{\pi}dt\int_0^{t-n^{-1}}du+\int_{n^{-1}}^{\pi}dt\int_{t-n^{-1}}^{t}du=I_1+I_2\,.$$

Using the number

$$m=[\varepsilon n^{(\beta-q)/(\gamma-r)}]\,, \tag{2.16}$$

which is clearly less than $n$, and exchanging the order of integration, we have

$$I_1=\int_0^{m^{-1}-n^{-1}}\Phi_\beta(u)du\int_{u+n^{-1}}^{m^{-1}}(t-u)^{l-\beta-1}A_n^{(l)}(t)dt$$
$$+\int_0^{m^{-1}-n^{-1}}du\int_{m^{-1}}^{\pi}dt+\int_{m^{-1}-n^{-1}}^{\pi-n^{-1}}du\int_{u+n^{-1}}^{\pi}dt=J_1+J_2+J_3.$$

Here, for the sake of convienience we write

$$U(t,u)=(t-u)^{l-\beta-1}A_n^{(l)}(t)\,,$$

where $-1<l-\beta-1\leqq 0$ by (2.13). Then, when $n^{-1}\leqq u<u_1<u_2\leqq\pi$, by the second mean value theorem we obtain

$$\left|\int_{u_1}^{u_2}U(t,u)dt\right|\leqq(u_1-u)^{l-\beta-1}\sup_{u_1<t<u_2}|A_n^{l-1}(t)|\,.$$

Thus, (2.7) with $\mu=l-1$ gives

$$\int_{u_1}^{u_2}U(t,u)dt=(u_1-u)^{l-\beta-1}\cdot O\Big(\frac{n^{l-1-q}}{u_1^{r+1}}\Big)\qquad\Big(\frac{1}{n}\leqq u_1<u_2\leqq\frac{1}{m}\Big), \tag{2.17}$$

$$=(u_1-u)^{l-\beta-1}\cdot O\Big(\frac{n^{l-1-q}}{m^k u_1^{k+r+1}}\Big)\qquad\Big(\frac{1}{m}\leqq u_1<u_2\leqq\pi\Big). \tag{2.18}$$

Using (2.17) with $u_1=u+n^{-1}$, i.e. $\int_{u+n^{-1}}^{m^{-1}}U(t,u)dt=O(n^{\beta-q}u^{-(r+1)})$,

$$J_1=\int_0^{m^{-1}-n^{-1}}\Phi_\beta(u)du\int_{u+n^{-1}}^{m^{-1}}U(t,u)dt$$
$$=O\Big(n^{\beta-q}\int_0^{m^{-1}}|\Phi_\beta(u)|\,u^{-(r+1)}du\Big)\,.$$

On account of $\gamma-r>0$, (2.1) and (2.16), integration by parts shows that the last expression is

$$o(n^{\beta-q}[u^{\gamma-r}]_0^{m^{-1}})=o\Big(\frac{n^{\beta-q}}{m^{\gamma-r}}\Big)=o\Big(\frac{1}{\varepsilon^{\gamma-r}}\Big)\,.$$

Next, by (2.17) with $u_1=m^{-1}\geqq u+n^{-1}$, i.e. $\int_{m^{-1}}^{\pi}U(t,u)dt=O(n^{\beta-q}m^{r+1})$, we as above obtain

$$J_2=\int_0^{m^{-1}-n^{-1}}\Phi_\beta(u)du\int_{m^{-1}}^{\pi}U(t,u)dt$$
$$=O\left(n^{\beta-q}m^{r+1}\int_0^{m^{-1}}|\Phi_\beta(u)|\,du\right)$$
$$=o\left(\frac{n^{\beta-q}m^{r+1}}{m^{\gamma+1}}\right)=o\left(\frac{n^{\beta-q}}{m^{\gamma-r}}\right)=o\left(\frac{1}{\varepsilon^{\gamma-r}}\right).$$

Furthermore, by (2.18) with $u_1=u+n^{-1}\geqq m^{-1}$, i.e. $\int_{u+n^{-1}}^{\pi}U(t,u)=O(n^{\beta-q}/m^k u^{k+r+1})$, and since $k>\gamma-r$, one has

$$J_3=\int_{m^{-1}-n^{-1}}^{\pi-n^{-1}}\Phi_\beta(u)du\int_{u+n^{-1}}^{\pi}U(t,u)dt$$
$$=O\left(\frac{n^{\beta-q}}{m^k}\int_{m^{-1}}^{\pi}|\Phi_\beta(u)|\,u^{-(k+r+1)}du\right)$$
$$=o\left(\frac{n^{\beta-q}}{m^k}[u^{\gamma-k-r}]_{m^{-1}}^{\pi}\right)=o\left(\frac{n^{\beta-q}}{m^{\gamma-r}}\right)=o\left(\frac{1}{\varepsilon^{\gamma-r}}\right).$$

The above estimation gives $I_1=o(1)$ as $n\to\infty$.

Concerning $I_2$ we write

$$I_2=\int_{n^{-1}}^{\pi}A_n^{(l)}(t)dt\int_{t-n^{-1}}^{t}(t-u)^{l-\beta-1}\Phi_\beta(u)du$$
$$=\int_0^{n^{-1}}du\int_{n^{-1}}^{u+n^{-1}}dt+\int_{n^{-1}}^{m^{-1}}du\int_u^{u+n^{-1}}dt+\int_{m^{-1}}^{\pi-n^{-1}}du\int_u^{u+n^{-1}}dt$$
$$+\int_{\pi-n^{-1}}^{\pi}du\int_u^{\pi}dt=K_1+K_2+K_3+K_4,$$

say. Using (2.7) with $\mu=l$ we see that all $K$'s are $o(\varepsilon^{-(\gamma-r)})$ by the same argument as above. Hence, $I_2=o(1)$ which together with $I_1=o(1)$ yields (2.15), and the lemma is completely established.

## 3. Proofs of Theorem 1 and Corollary 1.1.

We need a further lemma, which is independent of Fourier series.

LEMMA 2 (Dixson-Ferrar's theorem). *Let $\{s_n\}$ be any sequence of real terms and $s_n^\alpha$ $(-\infty<\alpha<\infty)$ be its n-th* (C, $\alpha$) *sum, and let $-1<\gamma$ and $0<\gamma-c\leqq\beta-b$.*
(I) *If as $n\to\infty$*

(3.1) $$s_n^\beta=o(n^\gamma)$$

(3.2) $$s_n^b=O(n^c),$$

*then we have*

(3.3) $$s_n^r=o(n^q),\qquad q=c+(r-b)\frac{\gamma-c}{\beta-b},\qquad (n\to\infty),$$

*for $b<r<\beta$.*

(II) *If* (3.1) *holds and* $s_n^b = O_L(n^c)$, *then we have* (3.3) *for* $r$ *with* $b+1 \leqq r < \beta$, *provided that* $b+1 < \beta$.

In the part (I), the case $b=0$, $c \geqq 0$ and $\gamma \geqq 0$ is due to Dixson-Ferrar [**2**], and the modified case $b=0$ and $\gamma > -1$ is found in Sunouchi [**9**]. In the part (II), the case $b=-1$ is a corollary of Bosanquet [**5**, Theorem 6]. The above general form is derived from these results by translation.

PROOF OF THEOREM 1. We write

$$\rho = (\beta - b)/(\gamma - c). \tag{3.4}$$

We clearly have

$$(\beta - q)/(\gamma - r) = \rho,$$

and the integer $m$ used in Lemma 1 is written as

$$m = [\varepsilon n^\rho]. \tag{3.5}$$

Also we have, by the assumptions,

$$0 < \rho \leqq 1 \quad \text{and} \quad c - b < 1, \tag{3.6}$$

$$s_n^c = O(n^b), \qquad c \geqq -1, \qquad (b > -2). \tag{3.7}$$

First, we suppose that $b > -1$, and let

$$r = c + \delta, \qquad q = b + \delta\rho \quad \text{and} \quad 0 < \delta < 1. \tag{3.8}$$

We write

$$\begin{aligned} s_{n+m}^{c+\delta} - s_n^{c+\delta} &= \sum_{\nu=0}^{n+m} A_{n+m-\nu}^{\delta-1} s_\nu^c - \sum_{\nu=0}^{n} A_{n-\nu}^{\delta-1} s_\nu^c, \qquad\qquad n' = \left[\frac{n}{2}\right], \\ &= \sum_{\nu=0}^{n'} (A_{n+m-\nu}^{\delta-1} - A_{n-\nu}^{\delta-1}) s_\nu^c + \sum_{n'+1}^{n-m} (A_{n+m-\nu}^{\delta-1} - A_{n-\nu}^{\delta-1}) s_\nu^c \\ &\quad - \sum_{n-m+1}^{n} A_{n-\nu}^{\delta-1} s_\nu^c + \sum_{n-m+1}^{n+m} A_{n+m-\nu}^{\delta-1} s_\nu^c = S_1 + S_2 - S_3 + S_4, \end{aligned}$$

say. Then, by (3.5), (3.7) and $b > -1$, one obtains in turn

$$\begin{aligned} S_1 &= \sum_{\nu=0}^{n'} O(mn^{\delta-2}) \cdot O(\nu+1)^b = O(mn^{\delta-2} n^{b+1}) \\ &= O(\varepsilon n^{\rho+\delta-1+b}) = O(\varepsilon n^q), \end{aligned}$$

since $(\rho+\delta-1+b) - q = -(1-\delta)(1-\rho) \leqq 0$,

$$\begin{aligned} S_2 &= \sum_{n'+1}^{n-m} O(m(n-\nu)^{\delta-2}) \cdot O(n^b) \\ &= O(mn^b \sum_{\nu=0}^{n-m} (n-\nu)^{\delta-2}), \qquad 0 < \delta < 1, \\ &= O(mn^b m^{\delta-1}) = O(m^\delta n^b) \\ &= O(\varepsilon^\delta n^{\delta\rho} n^b) = O(\varepsilon^\delta n^q), \end{aligned}$$

and

$$S_3=\sum_{n-m+1}^{n} A_{n-\nu}^{\delta-1} s_\nu^c = O(n^b \sum_{n-m+1}^{n} A_{n-\nu}^{\delta-1})$$
$$=O(n^b m^\delta)=O(\varepsilon^\delta n^q).$$

Similarly, $S_4=O(\varepsilon^\delta n^q)$. Hence, replacing $c+\delta$ by $r$ we have

$$s_{n+m}^r-s_n^r=O(\varepsilon^\delta n^q).$$

Clearly, by the same argument as above we obtain, for all $n>1$,

(3.9) $$s_{n+\nu}^r-s_n^r=O(\varepsilon^\delta n^q), \qquad \nu=1,2,\cdots,m.$$

Here, it will be easily verified that $O$ depends only on $\delta$ and $O$ in (3.2).

On the other hand, we see that by (3.8) and (3.6)

$$r-q=c-b+\delta(1-\rho)<1$$

for a sufficiently small $\delta>0$. From $r-q<1$, (3.9) and the assumption (1.1), i. e. $\int_0^t |\Phi_\beta(u)|\,du=o(t^{\gamma+1})$, it follows, by Lemma 1 with $\varepsilon'=\varepsilon^\delta$, that $s_n^r=o(n^q)$ holds for every $r$ such that $c<r\leqq c+\delta_0$, $\delta_0$ being small enough.

Thus, when $b>-1$, starting from $r=c+\delta_0$ and repeating the above argument, we have (1.3), i. e.

$$s_n^r=o(n^q), \qquad q=b+(r-c)\rho,$$

for all values of $r$ and $q$ as far as $c<r<\gamma$ and $r-q<1$.

Further, we suppose that $b\leqq -1$. The condition (1.1) implies $\Phi_{\beta+1}(t)=o(t^{\gamma+1})$, and then $\Phi_{\beta+1}(t)=o(t^{\beta+1+c-b})$ since $\gamma\geqq\beta+c-b$. Observing that $c-b<1$, we have

(3.10) $$s_n^{\beta+2+c-b}=o(n^{\beta+2}),$$

by a Obrechkoff's theorem, i. e. Corollary 2.2. (3.10) together with $s_n^c=O(n^b)$ yields $s_n^{c+1}=o(n^{b+1})$ by Lemma 2. Hence, on account of $b>-2$ we obtain

$$S_1=\sum_{\nu=0}^{n'-1}(A_{n+m-\nu}^{\delta-2}-A_{n-\nu}^{\delta-2})s_\nu^{c+1}+(A_{n+m-n'}^{\delta-1}-A_{n-n'}^{\delta-1})s_{n'}^{c+1}, \qquad n'=\left[\frac{n}{2}\right],$$
$$=\sum_{\nu=0}^{n'} O(mn^{\delta-3})\cdot o((\nu+1)^{b+1})+O(mn^{\delta-2})\cdot o(n^{b+1})$$
$$=o(mn^{\delta-1+b})=o(\varepsilon n^{\rho+\delta-1+b})=o(\varepsilon n^q).$$

The rest of the proof is unchanged, and we get the part (I) of the theorem.

Next, from the one-sided condition $s_n^c=O_L(n^b)$, it follows that

$$s_{n+m}^{c+1}-s_n^{c+1}=\sum_{\nu=1}^{m} s_{n+\nu}^c=O_L(mn^b)=O_L(\varepsilon n^{\rho+b}).$$

Besides, we see that $(c+1)-(\rho+b)<1$ by the assumption $c+1<\gamma'$. Hence, we have $s_n^{c+1}=o(n^{b+\rho})$ by Lemma 1. The part (II) of the theorem, thus, follows from the part (I).

PROOF OF COROLLARY 1.1. If we put $c=-1$ and $b=0$, then the conditions in Theorem 1, i.e.

$$0 \leqq \beta, \quad -1 \leqq c, \quad 0<\beta-b \leqq \gamma-c \quad \text{and} \quad c-b<1$$

are reduced to a single one $0<\beta \leqq \gamma+1$ where the case $\beta=\gamma+1$ is trivial. Hence, disregarding the trivial case we obtain, by the part (I) of Theorem 1 and Remark 2 following Lemma 1, that the two conditions

$$\int_0^t |\Phi_\beta(u)|\, du = O(t^{\gamma+1}), \qquad o<\beta<\gamma+1,$$

and $a_n = s_n^{-1} = O(1)$ imply

$$s_n^r = O(n^{(r+1)\beta/(\gamma+1)}), \qquad -1 \leqq r < \inf\left(\gamma, \frac{\gamma+1+\beta}{\gamma+1-\beta}\right). \tag{3.11}$$

On the other hand, $a_n$ is actually $o(1)$ of course. So, Dixson-Ferrar's theorem, by the conditions $a_n = o(1)$ and (3.11), will give

$$s_n^{r'} = o(n^{(r'+1)\beta/(\gamma+1)}), \qquad -1 \leqq r' < r,$$

which proves the theorem.

## 4. Proof of Theorem 2.

The kernel $X_n(t) = X_n^+(t)$ in (2.5) is implicitely defined by the identity

$$\frac{1}{n^q m^k} \sum_{\nu_1=1}^{m} \sum_{\nu_2=1}^{m} \cdots \sum_{\nu_k=1}^{m} s_{n+\nu_1+\nu_2+\cdots+\nu_k}^r = \frac{2}{\pi}\int_0^\pi \varphi(t) X_n^+(t)\, dt.$$

And, a slight modification of the proof shows that Lemma 1 holds for $r \geqq -2$ in place of $r \geqq -1$. So, the principal part of Lemma 1 should be expressed as follows, if $k$ is allowed to be unity, i.e. if $\gamma-r<1$.

Let $0 \leqq \beta$, $-2 \leqq r$, $0<\beta-q \leqq \gamma-r$, $\gamma-r<1$ and $r-q<1$. Then

$$\int_0^t |\Phi_\beta(u)|\, du = o(t^{\gamma+1}) \qquad \text{as} \quad t \to 0 \tag{4.1}$$

implies

$$s_{n+m}^{r+1} - s_n^{r+1} = o(\varepsilon^{-(\gamma-r)} n^q m) \qquad \text{as} \quad n \to \infty, \tag{4.2}$$

for $m = [\varepsilon n^{(\beta-q)/(\gamma-r)}]$, $\varepsilon > 0$.

Here, examining the proof of Lemma 1, we see that $o$ in (4.2) will depend only on $o$ in (4.1) and on the value of $(\gamma-r)$, $r$ being considered as a variable parameter.

Now, letting

$$r' = r+1 \quad \text{and} \quad q' = q+(\beta-q)/(\gamma-r),$$

one obtains $(q'-\beta)/(r'-\gamma) = (\beta-q)/(\gamma-r)$, $r'-\gamma > 0$, and (4.2) then becomes

$$s_{n+m}^{r'} - s_n^{r'} = o(\varepsilon^{r'-\gamma} n^{q'}) \qquad \text{as} \quad n \to \infty \tag{4.2$'$}$$

Thus, the above proposition is expressed again as follows.

Let $0\leqq\beta$, $-1\leqq r'$, $0<q'-\beta\leqq r'-\gamma$, $r'-\gamma<1$ and

$$(r'-1)-\left(q'-\frac{q'-\beta}{r'-\gamma}\right)<1\,.$$

Then, (4.1) implies (4.2)$'$ for $m=[\varepsilon n^{(q'-\beta)/(r'-\gamma)}]$, $\varepsilon>0$.

Replacing $\beta$, $\gamma$, $q'$, $r'$ by $b$, $c$, $q$, $r$ respectively, and letting

$$\frac{q-b}{r-c}=\rho \text{ (clearly then } 0<\rho\leqq 1), \tag{4.3}$$

and $r-c=\delta$, and taking into account the above remark, we have the following

LEMMA 3. *Let* $0\leqq b$, $-1\leqq c$ *and* $c-b<1$. *If*

$$\int_0^t |\Phi_b(u)|\,du=O(t^{c+1}) \qquad as \quad t\to 0\,, \tag{4.4}$$

*and if* $0<\delta<1$ *and* $0<\rho\leqq 1$, *then*

$$|s_{n+\nu}^{c+\delta}-s_n^{c+\delta}|\leqq A\varepsilon^{\delta}n^{b+\delta\rho}\,, \qquad \nu=1,2,\cdots,m\,, \tag{4.5}$$

*holds for* $m=[\varepsilon n^{\rho}]$, $\varepsilon>0$, *and* $n>1$, *A being a constant depending only on* $\delta$ *and* $O$ *in* (4.4).

When $b\geqq c+1$, (4.4) is trivial since $\varphi(t)\in L$ in $(0,\pi)$.

We now prove Theorem 2. Taking $\rho=(\beta-b)/(\gamma-c)$, as we may, the assumptions give (4.5) for every positive $\delta<1$. The condition (4.5) is (3.9) with $r=c+\delta$. So, the theorem follows from the part (I) of Theorem 1.

## Part II.

### 5. Theorems (2).

THEOREM 3. *Let* $-1\leqq\beta$, $0\leqq c$ *and* $0<\gamma+1-c\leqq\beta+1-b$. (I) *If*

$$\sum_{\nu=0}^{n}|s_\nu^{\beta}|=o(n^{\gamma+1}) \qquad as \quad n\to\infty\,, \tag{5.1}$$

$$\Phi_c(t)=O(t^b) \qquad as \quad t\to 0\,, \tag{5.2}$$

*then we have*

$$\Phi_r(t)=o(t^q)\,, \qquad q=b+(r-c)\frac{\beta+1-b}{\gamma+1-c}\,, \qquad (t\to 0)\,, \tag{5.3}$$

*for* $c<r<\gamma+1$.

(II) *If* (5.1) *holds, and* $\Phi_c(t)=O_L(t^b)$ *as* $t\to 0$, *then we have* (5.3) *for r with* $c+1\leqq r<\gamma+1$, *provided that* $c<\gamma$.

THEOREM 4. *Let* $-1\leqq\beta$, $0\leqq c$ *and* $0<\gamma+1-c\leqq\beta+1-b$. *If* (5.1) *holds, and*

$$s_n^{b-1}=O(n^{c-1}) \qquad as \quad n\to\infty. \tag{5.4}$$

*then we have* (5.3) *for* $c<r<\gamma+1$.

(N. B. 4) In Theorem 4, we may suppose that $b-c>-1$ when $b\geqq 0$, as was noticed in (N. B. 2) in § 1. An analogous notice may be made for Theorem 3 when $c\geqq 1$.

(N. B. 5) On the contrary, if $b-c\leqq -1$, letting $c=1$ and $b=0$ in Theorems 3, 4, then we have the following corollary, the condition (5.1) being replaced by a less stringent one.

COROLLARY 3.1. *If* $0<\gamma<\beta+1$, *and*

$$(5.1)' \qquad \sum_{\nu=0}^{n}|s_\nu^\beta|=O(n^{\gamma+1}) \qquad as \quad n\to\infty,$$

*then we have*

$$(5.5) \qquad \Phi_r(t)=o(t^q), \qquad q=(r-1)\frac{\beta+1}{\gamma}, \qquad (t\to 0),$$

*for* $1\leqq r<\gamma+1$.

REMARK 3. In view of Corollary 3.1, we see that if $c\geqq 1$ then Theorems 3, 4 have meanings only when

$$b+(r-c)\frac{\beta+1-b}{\gamma+1-c}>(r-1)\cdot\frac{\beta+1}{\gamma},$$

which is equivalent to

$$(5.6) \qquad (\beta+1)(1-c)+b\gamma>0.$$

And, if $c<1$ and we take into account $\Phi_1(t)=o(1)$ together with either (5.2) or (5.4), then the truth of the above conclusion can be readily verified.

Now, letting $r=q=\alpha$ and repeating the same discussion as in § 1, we have the following corollaries from the above theorems.

COROLLARY 3.2. *Let* $-1<\gamma<\beta$ *and* $0<\delta$. *If* (5.1) *holds, and if either of the two conditions*

$$\begin{cases} \varphi(t)=O(t^{-\delta}) \\ \varphi(t)=O_L(t^{-\delta}), \qquad \beta-\gamma\leqq\delta\gamma, \end{cases}$$

*is satisfied, then*

$$\Phi_\alpha(t)=o(t^\alpha), \qquad \alpha=\delta(\gamma+1)/(\beta-\gamma+\delta).$$

As was noticed in Remark 3, this corollary has a meaning only when $\delta\gamma<\beta+1$ by (5.6) with $c=0$ and $b=-\delta$, which proves a conjecture of Sunouchi [**9**].

When $\delta\gamma=\beta+1$, we have the following

COROLLARY 3.3. *If* $0<\gamma<\beta$, *and the condition* (5.1)′ *in Corollary* 3.1 *holds, then*

$$\Phi_\alpha(t)=o(t^\alpha), \qquad \alpha=(\beta+1)/(\beta+1-\gamma).$$

COROLLARY 4.1. *Let* $-1\leqq c$ *and* $b<c<\gamma<\beta$, *If* (5.1) *holds, and* $s_n^b=O(n^c)$, *then we have*

$$\Phi_\alpha(t)=o(t^\alpha), \qquad \alpha=1+\frac{c\beta-b\gamma}{\beta-b+c-\gamma}.$$

COROLLARY 4.1.° *Let* $0<\delta<1$ *and* $-(1-\delta)<\gamma<\beta$. *If* (5.1) *holds, and* $a_n=O(n^{-(1-\delta)})$, *then*

$$\Phi_\alpha(t)=o(t^\alpha), \qquad \alpha=\delta(\beta+1)/(\beta-\gamma+\delta).$$

In the case $\delta=1$, we have Corollary 3.3 in place of this corollary.

COROLLARY 4.2. *If* $-1<\gamma$ *and* $s_n^\beta=o(n^\gamma)$, *then we have*

$$\Phi_{\gamma+1+\eta}(t)=o(t^{\beta+1+\eta}),$$

*for every* $\eta>0$.

This is immediately deduced by applying Theorem 4 to $s_n^\beta=o(n^\gamma)$ and $s_n^{\beta+k}=o(n^{\gamma+k})$ where $k>0$ and $\beta+k\geqq-1$, the latter of which follows from the former.

In this corollary, the case $\gamma\geqq\beta+1$ is trivial when $\beta\geqq-1$. The restricted case $0<\gamma\leqq\beta<\gamma+1$ is due to Hyslop [**7**], and the general case due to Izumi [**8**].

## 6. Preliminary lemmas.

The following Lemmas 4, 5 are independent of Fourier series.

LEMMA 4. *Let* $\varphi(t)\in L$ *in* $(0,t_0)$ *and* $\Phi_\alpha^k(t)$ $(\alpha\geqq0,\ k\geqq0)$ *be defined as in* (0.2), *and let* $r>0$ *and* $q$ *be an arbitrary constant. Then, a necessary and sufficient condition for*

$$\Phi_r(t)=o(t^q) \qquad as \quad t\to0$$

*is that*

$$\Phi_r^k(t)=o(t^{k+q}) \qquad as \quad t\to0,$$

*for any positive number* $k$.

This is Theorem 1 in the paper [**15**].

LEMMA 5 (Riesz's theorem). *Let* $\varphi(t)\in L$ *in* $(0,t_0)$, *and* $\Phi_\alpha(t)$ $(\alpha>0)$ *be defined as in* (0.1), *and let* $0<\beta\leqq\gamma-c$. (I) *If as* $t\to0$

(6.1) $$\Phi_\beta(t)=o(t^\gamma)$$

(6.2) $$\varphi(t)=O(t^c),$$

*then we have*

(6.3) $$\Phi_r(t)=o(t^{c+r(\gamma-c)/\beta}), \qquad (t\to0),$$

*for* $0<r<\beta$.

(II) *If* (6.1) *holds, and* $\varphi(t)=O_L(t^c)$, *then we have* (6.3) *for* $r$ *with* $1\leqq r<\beta$, *provided that* $1<\beta$.

In the part (I), the case $c\geqq0$ is a modified result from Riesz [**1**, Theorem II], and the case $\gamma>-1$ is found in e. g. Bosanquet [**5**] and Sunouchi [**9**].

The part (II) is a result from Theorem 7 which will be proved in the last article.

(N. B. 6) In the above lemma, concerning the numbers $\beta$, $\gamma$ and $c$ one needs no restriction other than $0<\beta\leqq\gamma-c$, as it is easily verified by Lemma 4. If $c>-1$ then the condition (6.1) can be derived neither from $\Phi_1(t)=o(1)$ nor from $\Phi_1(t)=o(1)$ and (6.2). But, if $c<-1$ then one will have often some better conclusion than (6.3) on account of $\Phi_1(t)=o(1)$. Similarly, if $c=-1$ then the single condition (6.2) implies $\Phi_r(t)=o(t^{r-1})$, $r>0$.

LEMMA 4.° *Let* $-1\leqq\beta$, $0\leqq c$ *and* $0<\gamma+1-c\leqq\beta+1-b$. *Then, the two conditions in* (II) *of Theorem* 3, *i. e.*

$$\sum_{\nu=0}^{n}|s_\nu^\beta|=o(n^{\gamma+1}) \tag{6.4}$$

$$\Phi_c(t)=O_L(t^b) \tag{6.5}$$

*imply*

$$\Phi_c^k(t)=O_L(t^{k+b}), \tag{6.6}$$

*where* $k>2$ *is an arbitrary integer.*

PROOF. The case $c=0$ is trivial, and so we may suppose that $c>0$. By a theorem of Izumi, i. e. Corollary 4.2, the condition $s_n^{\beta+1}=o(n^{\gamma+1})$ which is a result from (6.4) implies

$$\Phi_{\gamma+k}(t)=o(t^{\beta+k}), \qquad k>2. \tag{6.7}$$

Observing that $\gamma+k>c+1$ and applying the part (II) of Lemma 5 to (6.5) and (6.7), we have

$$\Phi_{c+j}(t)=o(t^{b+j}) \qquad \text{for} \quad j=1,2,\cdots. \tag{6.8}$$

Hence, from the identity

$$\Phi_c^k(t)=\frac{1}{\Gamma(c)}\int_0^t(t-u)^{c-1}[t-(t-u)]^k\varphi(u)du$$
$$=t^k\Phi_c(t)+\frac{1}{\Gamma(c)}\sum_{j=1}^{k}\binom{k}{j}(-1)^j\Gamma(c+j)t^{k-j}\Phi_{c+j}(t),$$

we see that (6.5) and (6.8) imply (6.6), which proves the lemma.

LEMMA 6. *Let* $r>0$, $a$, $b$, $A$ *and* $B$ *be arbitrary constants independent of* $n$ *and* $t$, *and let* $a+b\geqq[r-0]$. *Then*

$$\int_0^t(t-u)^{r-1}u^a\left(2\sin\frac{1}{2}u\right)^b\cos((n+A)u+B)du=O\left(\frac{t^{a+b}}{n^r}\right)$$

*holds uniformly in* $n\geqq1$ *and* $0<t\leqq\pi$, *and this is the same matter as*

$$\int_0^1(1-w)^{r-1}w^a\left(2\sin\frac{1}{2}tw\right)^b\cos((n+A)tw+B)dw=O\left(\frac{t^b}{(nt)^r}\right).$$

The case where $0<r\leqq1$ and $a=b=0$ are both satisfied is well known. See Hobson [17. pp. 564-565].

This lemma is easily derived from the following lemma by induction, the proof being omitted.

LEMMA 6.1. *If* $0 < r \leqq 1$, *and* $g(u)$ *is a real function with* $g(0)=0$, *of bounded variation in every interval* $0 \leqq u \leqq t$, *and if* $G(t)=\int_0^t |dg(u)|$, *then*

$$\int_0^t (t-u)^{r-1} g(u) e^{inu} du = O\left(\frac{G(t)}{n^r}\right), \qquad i=\sqrt{-1},$$

*holds uniformly in* $n \geqq 1$ *and* $t>0$.

LEMMA 7. *If* $u_\nu \geqq 0$ $(\nu \geqq 1)$, *and* $c>a>b$, $a>0$, *then*

$$\sum_{\nu=1}^{n} u_\nu = o(n^a) \qquad as \quad n \to \infty$$

*implies*

$$\sum_{\nu=1}^{n} \frac{u_\nu}{\nu^b} = o(n^{a-b}) \quad and \quad \sum_{\nu=n}^{\infty} \frac{u_\nu}{\nu^c} = o\left(\frac{1}{n^{c-a}}\right),$$

*as* $n \to \infty$, *and conversely. All o's may be replaced by O's.*

The proof is easy.

## 7. Fundamental lemma (2).

LEMMA 8 (Fundamental lemma). *Let* $-1 \leqq \beta$, $0 \leqq r$ *and* $0 < \gamma+1-r \leqq \beta+1-q$. *If*

$$\sum_{\nu=0}^{n} |s_\nu^\beta| = o(n^{\gamma+1}) \qquad as \quad n \to \infty, \tag{7.1}$$

*and if for any assigned positive number* $\varepsilon$ *there exist an* $\varepsilon'=\varepsilon'(\varepsilon)>0$ *tending to zero with* $\varepsilon$ *and a* $t_\varepsilon$ *such that*

$$\Phi_r^l(t+u) - \Phi_r^l(t) > -\varepsilon' t^{l+q}, \qquad 0<u \leqq h, \tag{7.2}$$

$l$ *being an integer greater than* $\gamma+1$, *holds for* $h=\varepsilon t^{(\beta+1-q)/(\gamma+1-r)}$ *and* $0<t<t_\varepsilon$, *then one obtains*

$$\Phi_r(t) = o(t^q) \qquad as \quad t \to 0. \tag{7.3}$$

REMARK 4. Arguing similarly as in the proof of Lemma 8, we can easily obtain the following result.

If, in Lemma 8, $o$ in (7.1) is replaced by $O$, and the third assumption by "$\Phi_r^l(t+u)-\Phi_r^l(t) > -At^{l+q}$, $l>\gamma+1$ and $A>0$, holds for $0<u<t^{(\beta+1-q)/(\gamma+1-r)}$) and $0<t<1$", then we have $\Phi_r(t)=O(t^q)$ in place of (7.3).

To get this result, it is sufficient to take $h=(2k)^{-1}t^{(\beta+1-q)/(\gamma+1-r)}$, $k$ being an integer greater than $\gamma+1-r$.

In order to prove Lemma 8, we need some further lemmas.

LEMMA 8.1. *Let* $r \geqq 0$, *and*

$$(7.4)\qquad \Psi_r(t)=\begin{cases}\Psi_r^+(t)=h^{-k}\int_0^h du_1\int_0^h du_2\cdots\int_0^h \Phi_r^l(t+u_1+u_2+\cdots+u_k)du_k\\ \Psi_r^-(t)=h^{-k}\int_0^h du_1\int_0^h du_2\cdots\int_0^h \Phi_r^l(t-u_1-u_2-\cdots-u_k)du_k,\end{cases}$$

*where $l$ and $k$ are two fixed positive integers, $0<h=h(t,\varepsilon)<(2k)^{-1}t$, and let $h$ tend to zero with $t$ in as same order as or higher order than $t$. In these circumstances, if for a constant $q$*

$$(7.5)\qquad \Psi_r(t)=o(t^{l+q})\qquad as\quad t\to 0,$$

*and if* (7.2) *holds for $0<t<t_\varepsilon$, then we have* (7.3).

PROOF OF LEMMA 8.1. We first notice that, roughly speaking, (7.2) implies

$$\Phi_r^l(t+u)-\Phi_r^l(t)>-k\varepsilon' t^{l+q},\qquad 0<u\leqq kh,$$

for any fixed integer $k$. Substituting the relations (7.5) and (7.2) into the identities

$$\Phi_r^l(t)=\Psi_r^+(t)-h^{-k}\int_0^h du_1\int_0^h du_2\cdots\int_0^h[\Phi_r^l(t+u_1+u_2+\cdots+u_k)-\Phi_r^l(t)]du_k,$$

$$\Phi_r^l(t)=\Psi_r^-(t)+h^{-k}\int_0^h du_1\int_0^h du_2\cdots\int_0^h[\Phi_r^l(t)-\Phi_r^l(t-u_1-u_2-\cdots-u_k)]du_k,$$

we thus have

$$\limsup_{t\to 0}\frac{\Phi_r^l(t)}{t^{l+q}}\leqq 0,\qquad \liminf_{t\to 0}\frac{\Phi_r^l(t)}{t^{l+q}}\geqq 0$$

respectively. Hence, we get $\Phi_r^l(t)=o(t^{l+q})$, and then $\Phi_r(t)=o(t^q)$ by Lemma 4. This proves the lemma.

Now, we must investigate the property of $\Psi_r^\pm(t)$, and it may be restricted to consider $\Psi_r^+(t)$ only, since it is similar to $\Psi_r^-(t)$. In the case $r>0$, neglecting the numerical factor $1/\Gamma(r)$,

$$\Phi_r^l(u)=\int_0^u(u-v)^{r-1}v^l\varphi(v)dv,\qquad (v=uw),$$

$$=u^{r+l}\int_0^1(1-w)^{r-1}w^l\varphi(uw)dw,$$

and so substituting this expression with $u=t+u_1+u_2+\cdots+u_k$ for that of $\Psi_r(t)$ defined by (7.4), we have

$$(7.6)\qquad \begin{aligned}\Psi_r(t)=h^{-k}\int_0^h du_1\int_0^h du_2\cdots\int_0^h(t+u_1+\cdots+u_k)^{r+l}du_k\\ \times\int_0^1(1-w)^{r-1}w^l\varphi((t+u_1+\cdots+u_k)w)dw.\end{aligned}$$

And, clearly when $r=0$,

$$\Psi_0(t)=h^{-k}\int_0^h du_1\int_0^h du_2 \cdots \int_0^h (t+u_1+\cdots+u_k)^l \varphi(t+u_1+\cdots+u_k)du_k .$$

We may consider the case $r>0$ only, since the case $r=0$ is easier.

Concerning the particular function

$$\varphi(v)=\left(2\sin\frac{1}{2}v\right)^a \cos((n+b)v+c)$$

we have the following lemma, $l$ being replaced by $[r]+k$.

LEMMA 8.2. *Let $r>0$ be arbitrary, $k$ be a positive integer, $a\geqq 0$, $b$ and $c$ be three arbitrary constants, and let*

$$B_n=h^{-k}\int_0^h du_1\int_0^h du_2 \cdots \int_0^h u^{r+[r]+k}du_k \tag{7.7}$$
$$\times\int_0^1 (1-w)^{r-1}w^{[r]+k}\left(2\sin\frac{1}{2}uw\right)^a \cos((n+b)uw+c)dw ,$$

*where $u=t+u_1+\cdots+u_k$, and $h$ has the same meaning as in Lemma* 8.1. *Then, we have*

(7.8) $\qquad B_{n-1}=O\left(\dfrac{t^{[r]+k+a}}{n^r}\right)$ $\qquad$ (7.9) $\qquad B_n=O\left(\dfrac{t^{[r]+k+a}}{n^{r+k}h^k}\right)$

*uniformly in $n\geqq 1$ and $0<t\leqq\pi$.*

In order to study the general case where $r>0$ is arbitrary, we suppose that

$$[r]=[r-0] ,$$

i. e. that when $r$ is integral the notation $[r]$ means $r-1$ in place of $r$, so far as it is concerned with the present article.

PROOF OF LEMMA 8.2. Applying Lemma 6 to the last integral $\int_0^1 dw$ in (7.7), we see that

$$\int_0^1 (1-w)^{r-1}w^{[r]+k}\left(2\sin\frac{1}{2}uw\right)^a \cos((n+b)uw+c)dw=O\left(\frac{u^a}{(nu)^r}\right) , \tag{7.10}$$

which yields (7.8) immediately by (7.7) and $u\sim t$.

Next, applying integration by parts $[r]$-times to the left hand integral in (7.10), the integral becomes

$$\int_0^1 f(u,w)\frac{\cos((n+b)uw+c')}{(n+b)^{[r]}u^{[r]}}dw ,$$

where

$$f(u,w)=\left(\frac{\partial}{\partial w}\right)^{[r]}\left((1-w)^{r-1}w^{[r]+k}\left(2\sin\frac{1}{2}uw\right)^a\right) ,$$

and $c'=c+[r]\pi/2$. By an elaboration, we see that $f(u,w)$ may be written as an algebraic sum of functions of the type

$$(1-w)^{r-[r]-1}w^{k+p_1}(uw)^p\Big(2\sin\frac{1}{2}uw\Big)^{a-p+p_2}\Big(2\sin\frac{1}{4}uw\Big)^{2p_3},$$

where $0<r-[r]\leqq 1$ and $p_1, p, p_2, p_3$ are integers such that

$$0\leqq p_1\leqq [r],\quad 0\leqq p\leqq [r],\quad 0\leqq p_2\leqq p \quad\text{and}\quad p_3=0,1.$$

We may take as $f(u,w)$ a term corresponding to $p_1=p_2=p_3=0$

$$f_1(u,w)=(1-w)^{r-[r]-1}w^k g(uw),$$

where

$$g(uw)=(uw)^p\Big(2\sin\frac{1}{2}uw\Big)^{a-p},$$

since the estimation is similar for the other terms. $B_n$ in (7.7) then becomes

$$h^{-k}\int_0^h du_1\int_0^h du_2\cdots\int_0^h u^{r+[r]+k}du_k\int^1 f_1(u,w)\frac{\cos((n+b)uw+c')}{(n+b)^{[r]}u^{[r]}}dw,$$

and, exchanging the order of integration we have

$$B_n=\frac{1}{h^k(n+b)^{[r]}}\int_0^1(1-w)^{r-[r]-1}w^k dw\int_0^h du_1\int_0^h du_2\cdots$$

$$\times\int_0^h u^{r+k}g(uw)\cos((n+b)uw+c')du_k,$$

where $u=t+u_1+\cdots+u_k$. Now, observing that the function $u^{r+k}g(uw)$, $w>0$ being considered to be a parameter, is monotonously increasing with respect to each small $u_j>0$, and applying the second mean value theorem to the repeated integral, we obtain

$$B_n=\frac{(t+kh)^{r+k}}{h^k(n+b)^{[r]+k}}\int_0^1(1-w)^{r-[r]-1}g((t+kh)w)$$

$$\times\Big(\prod_{j=1}^k 2\sin\frac{1}{2}(n+b)(h-h_j)w\Big)\cos\Big[(n+b)\Big(t+\frac{1}{2}(kh+h_1+\cdots+h_k)\Big)w+c'\Big]dw,$$

where $0<h_j<h$. Expanding the product $(\prod_{j=1}^k)\cos$ into a linear sum of cosines, and applying Lemma 6 with $a=p$ and $b=a-p$ to each of the integrals we have

$$\int_0^1 dw=(t+kh)^p\cdot O\Big(\frac{t^{a-p}}{(nt)^{r-[r]}}\Big),$$

and then

$$B_n=\frac{(t+kh)^{r+k}}{h^k(n+b)^{[r]+k}}\cdot O\Big(\frac{t^a}{(nt)^{r-[r]}}\Big)=O\Big(\frac{t^{[r]+k+a}}{n^{r+k}h^k}\Big),$$

which is (7.9), and the lemma is proved.

PROOF OF LEMMA 8. By Lemma 8.1, it is sufficient to show that the condition (7.1) implies (7.5), while (7.5) is, in the case $r>0$, by (7.6) with $l=[r]+k$,

$$\Psi_r(t)\equiv h^{-k}\int_0^h du_1\int_0^h du_2\cdots\int_0^h u^{r+[r]+k}du_k \times\int_0^1(1-w)^{r-1}w^{[r]+k}\varphi(uw)dw=o(t^{[r]+k+q}), \tag{7.11}$$

where $u=t+u_1+\cdots+u_k$. Replacing $\varphi(uw)$, as we may, by its Fourier series the last integral in (7.11) is

$$\int_0^1(1-w)^{r-1}w^{[r]+k}\varphi(uw)dw=\sum_{\nu=0}^{\infty}{}' a_\nu\int_0^1(1-w)^{r-1}w^{[r]+k}\cos\nu uwdw,$$

where $\Sigma'$ means that $a_0$ is replaced by $a_0/2$.

Applying Abel's transformation $[\beta+1]$-times we have

$$\frac{1}{2}a_0+\sum_{\nu=1}^{n}a_\nu\cos\nu uw=-P_n+Q_n + \Big(2\sin\frac{1}{2}uw\Big)^{\beta+1}\sum_{\nu=0}^{n}s_\nu^\beta\cos\Big[\Big(\nu+\frac{1}{2}(\beta+1)\Big)uw-\frac{1}{2}(\beta+1)\pi\Big], \tag{7.12}$$

where

$$P_n=\Big(2\sin\frac{1}{2}uw\Big)^{[\beta+1]}\sum_{\nu=0}^{n}s_\nu^\beta\sum_{\mu=n+1}^{\infty}A_{\mu-\nu}^{[\beta]-\beta-1} \times\cos\Big(\Big(\mu+\frac{1}{2}[\beta+1]\Big)uw-\frac{1}{2}[\beta+1]\pi\Big),$$

$$Q_n=\sum_{j=0}^{[\beta]}\Big(2\sin\frac{1}{2}uw\Big)^j s_n^j\cos\Big(\Big(n+1+\frac{1}{2}j\Big)uw-\frac{1}{2}j\pi\Big),$$

in particular, $P_n$ vanishes when $\beta$ is integral, and $Q_n$ does when $-1\leqq\beta<0$. Cf. the cited paper [**12**]. Now, in view of $u=t+u_1+\cdots+u_k$, and Lemma 8.2 with

$$a=[\beta+1],\quad b=\frac{1}{2}[\beta+1]\quad\text{and}\quad c=-\frac{1}{2}[\beta+1]\pi,$$

we have

$$(P_n)=h^{-k}\int_0^h du_1\int_0^h du_2\cdots\int_0^h u^{r+[r]+k}du_k\int_0^1(1-w)^{r-1}w^{[r]+k}P_n dw$$
$$=\sum_{\nu=0}^{n}s_\nu^\beta\sum_{\mu=n+1}^{\infty}A_{\mu-\nu}^{[\beta]-\beta-1}B_\mu,$$

and by (7.9) $B_\mu=O(\mu^{-r-k})$ holds for fixed $t>0$ and $h>0$. Thus, taking $k>\gamma+1-r$, (7.1) yields

$$(P_n)=\sum_{\nu=0}^{n}s_\nu^\beta\sum_{\mu=n+1}^{\infty}A_{\mu-\nu}^{[\beta]-\beta-1}\cdot O(\mu^{-r-k})$$
$$=O(n^{-r-k}\sum_{\nu=0}^{n}|s_\nu^\beta|)=O(n^{-r-k}n^{\gamma+1})=o(1).$$

Similarly, when $\beta\geqq 0$,

$$(Q_n)=h^{-k}\int_0^h du_1\int_0^h du_2\cdots\int_0^h u^{r+[r]+k}du_k\int_0(1-w)^{r-1}w^{[r]+k}Q_n dw$$

$$=\sum_{j=0}^{[\beta]} s_n^j\cdot O(n^{-r-k})=\sum_{j=0}^{[\beta]} o(n^{\gamma+1})\cdot O(n^{-r-k})=o(1),$$

since $s_n^j=O(|s_n^{\beta+1}|)=o(n^{\gamma+1})$ for $j=0,1,\cdots,[\beta]$. This is also true when $-1\leqq\beta<0$, for $Q_n$ then vanishes.

Hence, following (7.12) we may replace the Fourier series of $\varphi(uw)$ by

$$\left(2\sin\frac{1}{2}uw\right)^{\beta+1}\sum_{\nu=0}^{\infty}s_\nu^\beta\cos\left[\left(\nu+\frac{1}{2}(\beta+1)\right)uw-\frac{1}{2}(\beta+1)\pi\right],$$

concerning the estimation of $\Psi_r(t)$. Replacing $\varphi(uw)$ in the integral of (7.11) by the last series, and using Lemma 8.2 with

$$a=\beta+1,\quad b=\frac{1}{2}(\beta+1)\quad\text{and}\quad c=-\frac{1}{2}(\beta+1)\pi,$$

we have

$$\Psi_r(t)=\sum_{\nu=0}^{\infty}s_\nu^\beta B_\nu=\sum_{\nu=0}^{n}+\sum_{\nu=n+1}^{\infty}=S_n+R_n, \tag{7.13}$$

where, uniformly in $\nu\geqq 1$ and $0<t\leqq\pi$,

$$B_{\nu-1}=O\left(\frac{t^{[r]+k+\beta+1}}{\nu^r}\right),\qquad B_\nu=O\left(\frac{t^{[r]+k+\beta+1}}{\nu^{r+k}h^k}\right).$$

We can easily verify that the succeeding argument remains unchanged also when $[r]=r=0$. Noting now that

$$h=\varepsilon t^\rho,\qquad \rho=(\beta+1-q)/(\gamma+1-r),$$

$$-1\leqq\beta,\quad 0\leqq r\quad\text{and}\quad 0<\gamma+1-r\leqq\beta+1-q,$$

taking

$$n=[h^{-1}],\qquad \text{i.e.}\quad nh\sim 1,$$

we then have by (7.1), Lemma 7 and $k>\gamma+1-r$, in turn

$$S_n=\sum_{\nu=0}^{n}s_\nu^\beta B_\nu=O(t^{[r]+k+\beta+1}\sum_{\nu=0}^{n}|s_\nu^\beta|(\nu+1)^{-r})$$

$$=o(t^{[r]+k+\beta+1}n^{\gamma+1-r})=o\left(\frac{t^{[r]+k+q}}{\varepsilon^{\gamma+1-r}}\right),$$

$$R_n=\sum_{\nu=n+1}^{\infty}s_\nu^\beta B_\nu=O(t^{[r]+k+\beta+1}h^{-k}\sum_{n+1}^{\infty}|s_\nu^\beta|\nu^{-r-k})$$

$$=o(t^{[r]+k+\beta+1}h^{-k}n^{\gamma+1-r-k})=o\left(\frac{t^{[r]+k+q}}{\varepsilon^{\gamma+1-r}}\right).$$

These relations together with (7.13) give (7.11), and the lemma is completely proved.

## 8. Proofs of Theorem 3 and Corollary 3.1.

PROOF OF THEOREM 3. Letting

$$\rho=(\beta+1-b)/(\gamma+1-r),$$

we see that

$$\rho\geqq 1 \quad \text{and} \quad q=b+(r-c)\rho.$$

For such $r$ and $q$, $(\beta+1-q)/(\gamma+1-r)=\rho$ holds, and the number $h$ used in Lemma 8 is written as

$$h=\varepsilon t^{\rho}. \tag{8.1}$$

We now put

$$r=c+\delta<\gamma+1, \quad q=b+\delta\rho \quad \text{and} \quad 0<\delta<1, \tag{8.2}$$

and use the letter $k$ in place of $l$ in Lemma 8.

Under these circumstances, we first prove that the condition

$$\Phi_c^k(t)=O(t^{k+b}), \qquad c\geqq 0, \tag{8.3}$$

which is by Lemma 4 equivalent to (5.2), implies

$$\Phi_{c+\delta}^k(t+h)-\Phi_{c+\delta}^k(t)=O(\varepsilon^{\delta}t^{k+b+\delta\rho}), \tag{8.4}$$

where $k$ is an integer greater than $\gamma+1$ and $-b$. Using the identity

$$\Gamma(\delta)\Phi_{c+\delta}^k(t)=\int_0^t(t-u)^{\delta-1}\Phi_c^k(u)du,$$

and neglecting the numerical factor $1/\Gamma(\delta)$, we have

$$\Phi_{c+\delta}^k(t+h)-\Phi_{c+\delta}^k(t)=\int_0^{t-h}[(t+h-u)^{\delta-1}-(t-u)^{\delta-1}]\Phi_c^k(u)du$$

$$-\int_{t-h}^t(t-u)^{\delta-1}\Phi_c^k(u)du+\int_{t-h}^{t+h}(t+h-u)^{\delta-1}\Phi_c^k(u)du=I_1-I_2+I_3,$$

say. Then, by (8.3),

$$I_1=O\left(t^{k+b}\int_0^{t-h}|(t+h-u)^{\delta-1}-(t-u)^{\delta-1}|du\right)$$

$$=O\left(t^{k+b}\int_0^{t-h}h(t-u)^{\delta-2}du\right), \qquad 0<\delta<1,$$

$$=O(t^{k+b}h^{\delta})=O(\varepsilon^{\delta}t^{k+b+\delta\rho}).$$

Similarly,

$$I_2=O\left(t^{k+b}\int_{t-h}^t(t-u)^{\delta-1}du\right)=O(\varepsilon^{\delta}t^{k+b+\delta\rho}),$$

and $I_3=O(\varepsilon^{\delta}t^{k+b+\delta\rho})$. These prove (8.4).

Clearly, the same argument as above gives for all $t$, $0<t<t_0$,

$$\Phi_{c+\delta}^k(t+u)-\Phi_{c+\delta}^k(t)=O(\varepsilon^{\delta}t^{k+b+\delta\rho}), \qquad 0<u\leqq h. \tag{8.5}$$

Here, it will be easily verified that $O$ depends only on $\delta$ and $O$ in (5.2). So, applying Lemma 8 with $\varepsilon'=\varepsilon^{\delta}$ to (8.5) and (5.1), i. e. $\Sigma_{\nu=0}^{n}|s_{\nu}^{\beta}|=o(n^{\gamma+1})$, we get

$$\Phi_{c+\delta}(t)=o(t^{b+\delta\rho}),$$

for $c+\delta<\gamma+1$ and $0<\delta<1$, and then for every value of $\delta$ such as $c<c+\delta<\gamma+1$. This proves the part (I) of the theorem.

Next, observing that the one-sided condition $\Phi_c(t)=O_L(t^b)$ and (5.1) imply $\Phi_c^k(t)=O_L(t^{k+b})$ by Lemma 4° in § 6, and letting $\delta=1$ in (8.2), as we may, we have

$$\Phi_{c+1}^k(t+h)-\Phi_{c+1}^k(t)=\int_t^{t+h}\Phi_c^k(u)du=O_L(ht^{k+b})=O_L(\varepsilon t^{k+b+\rho}),$$

by (8.1). Hence, Lemma 8 yields $\Phi_{c+1}(t)=o(t^{b+\rho})$, which proves the part (II) of the theorem by the part (I).

PROOF OF COROLLARY 3.1. The proof runs analogously as Corollary 1.1. Letting $c=1$ and $b=0$, the conditions $-1\leqq\beta$, $0\leqq c$ and $0<\gamma+1-c\leqq\beta+1-b$ in Theorem 3 are reduced to a single one $0<\gamma\leqq\beta+1$, the case $\gamma=\beta+1$ being trivial. And, the conditions

$$\sum_{\nu=0}^{n}|s_{\nu}^{\beta}|=O(n^{\gamma+1}) \quad \text{and} \quad \Phi_1(t)=O(1)$$

imply for $1\leqq r<\gamma+1$

(8.6) $$\Phi_r(t)=O(t^q), \qquad q=(r-1)(\beta+1)/\gamma,$$

by Remark 4 following Lemma 8 and the part (I) of Theorem 3. Applying Riesz's theorem in § 6 to (8.6) and $\Phi_1(t)=o(1)$, we get the desired result.

## 9. Proof of Theorem 4.

We need a lemma.

LEMMA 9. *Let $c\geqq0$, $b$ be arbitrary and $\rho\geqq1$ (except the case $c=\rho-1=0$). If*

(9.1) $$s_n^{b-1}=O(n^{c-1}) \quad or \quad \sum_{\nu=n}^{2n}|s_{\nu}^{b-1}|=O(n^c) \qquad as \quad n\to\infty,$$

*and if $0<\delta<1$ and $k\geqq\sup(0,c-b+1)$, then*

(9.2) $$|\Phi_{c+\delta}^k(t+u)+\Phi_{c+\delta}^k(t-u)-2\Phi_{c+\delta}^k(t)|\leqq A\varepsilon^{\delta}t^{k+b+\delta\rho}, \qquad 0<u\leqq h,$$

*holds for $h=\varepsilon t^{\rho}$, $\varepsilon>0$, and $0<t<1$, $A$ being a constant depending only on $\delta$ and $O$ in* (9.1).

In the exceptional case $c=\rho-1=0$, this lemma is still true if, in addition, $s_n^{b-1+\eta}=O(n^{c-1+\eta})$ holds for some positive $\eta$.

We sketch the proof of this lemma. Write

(9.3) $$C_n(t)=\int_0^t(t-u)^{c+\delta-1}u^k\left(2\sin\frac{1}{2}u\right)^b\cos(nu+b(u-\pi)/2)du.$$

Then, a similar argument as in the proof of Lemma 8.2 shows that, for $k \geqq \sup(0, c-b+1)$,

$$C_n(t) = O\Big(\frac{t^{k+b}}{n^{c+\delta}}\Big) \qquad (c \geqq 0), \tag{9.4}$$

and

$$C_{n-1}(t+h)+C_{n-1}(t-h)-2C_{n-1}(t) = O\Big(\frac{h^2 t^{k+b+c-[c]+\delta-2}}{n^{[c]}}\Big)+O\Big(\frac{ht^{k+b}}{n^{c+\delta-1}}\Big) \qquad (c \geqq 0)$$

$$= O\Big(\frac{h^2 t^{k+b+\delta-1}}{n^{c-1}}\Big)+O\Big(\frac{ht^{k+b}}{n^{c+\delta-1}}\Big) \qquad (c=1, 2, \cdots)$$

hold uniformly in $n \geqq 1$ and $0 \leqq t \leqq \pi$. If we replace $\varphi(u)$ by its Fourier series and take into account (9.4), then we obtain, for $k \geqq \sup(0, c-b+1)$

$$\Gamma(c+\delta)\Phi^k_{c+\delta}(t) = \int_0^t (t-u)^{c+\delta-1} u^k \varphi(u)du = \sum_{n=0}^{\infty} s_n^{b-1} C_n(t),$$

where $C_n(t)$ is defined by (9.3).

Arguing similarly as in the proof of Lemma 8, these relations give for $h = \varepsilon t^\rho$, $\varepsilon > 0$, and for given $\delta$ $(0 < \delta < 1)$

$$\Phi^k_{c+\delta}(t+h)+\Phi^k_{c+\delta}(t-h)-2\Phi^k_{c+\delta}(t) = O(\varepsilon^\delta t^{k+b+\delta\rho})$$

uniformly in $0 < t < 1$. From this follows the lemma.

We now prove Theorem 4. Putting $\rho = (\beta+1-b)/(\gamma+1-c)$, as we may, Lemma 9 and the remark after it conclude that the assumptions imply (9.2) for every positive $\delta < 1$. And, (9.2) is (8.5), the left side being slightly modified. So, the theorem follows from the part (I) of Theorem 3.

## Part III.

### 10. Theorems (3).

THEOREM 5. *Let* $0 \leqq b$ *and* $0 < \beta - b \leqq \gamma - c$. *If*

$$\Phi_\beta(t) = o(t^\gamma) \qquad as \quad t \to 0, \tag{10.1}$$

$$s_n^{c-1} = O(n^{b-1}) \qquad as \quad n \to \infty, \tag{10.2}$$

*then we have*

$$\Phi_r(t) = o(t^q), \qquad q = c+(r-b)\frac{\gamma-c}{\beta-b}, \tag{10.3}$$

*as* $t \to 0$, *for* $b < r < \beta$.

THEOREM 6. *Let* $-1 \leqq b$, $0 \leqq c$, $0 < \gamma - c \leqq \beta - b$ *and* $b - c < 1$. *If*

$$s_n^\beta = o(n^\gamma) \qquad as \quad n \to \infty, \tag{10.4}$$

$$\Phi_c(t) = O(t^b) \qquad as \quad t \to 0, \tag{10.5}$$

*then we have*

(10.6) $$s_n^r = o(n^q), \qquad q = c + (r-b)\frac{\gamma - c}{\beta - b},$$

*as* $n \to \infty$, *for* $b < r < \beta$.

(N. B. 7) Observing $a_n = o(1)$, one sees that Theorem 6 has a meaning in the sense noticed in Remark 3 in § 5, only when $c(\beta+1) < (b+1)\gamma$ is assumed in addition to the assumptions. Theorem 5 does when $c(\beta-1) > (b-1)\gamma$.

Discussing as in § 1, we have the following summability theorems from the above ones.

COROLLARY 5. *Let* $0 < \delta < 1$ *and* $\delta < \beta < \gamma$. *If*

$$\Phi_\beta(t) = o(t^\gamma) \quad and \quad a_n = O(n^{-(1-\delta)}),$$

*then we have*

$$\Phi_\alpha(t) = o(t^\alpha), \qquad \alpha = \gamma\delta/(\gamma - \beta + \delta).$$

COROLLARY 6. *Let* $0 < \gamma < \beta$ *and* $0 < \delta < 1$. *If*

$$s_n^\beta = o(n^\gamma) \quad and \quad \varphi(t) = O(t^{-\delta}),$$

*then we have*

$$s_n^\alpha = o(n^\alpha), \qquad \alpha = \gamma\delta/(\beta - \gamma + \delta).$$

## 11. Proof of Theorem 5.

We need two lemmas.

LEMMA 10. *Let* $\varphi(t) \in L$ *in* $(0, x)$ *and* $\Phi_\alpha(t)$, $\alpha > 0$, *be defined as in* (0.1), *and let* $0 < y < x$, $0 < \delta \leqq 1$. *Then, we have*

$$\left| \frac{1}{\Gamma(\delta)} \int_0^y (x-t)^{\delta-1} \varphi(t) dt \right| \leqq \max_{0 \leqq u \leqq x} |\Phi_\delta(u)|.$$

This is due to Riesz [1].

LEMMA 9'. *Let* $b \geqq 0$, $c$ *be arbitrary and* $\rho \geqq 1$ (*except the case* $b = \rho - 1 = 0$). *If*

(11.1) $$s_n^{c-1} = O(n^{b-1}) \qquad as \quad n \to \infty,$$

*and if* $0 < \delta < 1$ *and* $k \geqq \sup(0, b-c+1)$, *then*

(11.2) $$|\Phi_{b+\delta}^k(t+u) + \Phi_{b+\delta}^k(t-u) - 2\Phi_{b+\delta}^k(t)| \leqq A\varepsilon^\delta t^{k+c+\delta\rho}, \qquad 0 < u \leqq h,$$

*holds for* $h = \varepsilon t^\rho$, $\varepsilon > 0$, *and* $0 < t < 1$, *A being a constant depending only on* $\delta$ *and O in* (11.1).

This is an alternative form of Lemma 9 in § 9.

PROOF OF THEOREM 5. It is sufficient to show that, for every small $\delta > 0$

(11.3) $$\Phi_{b+\delta}(t) = o(t^{c+\delta\rho}),$$

where $b + \delta < \beta$ and $\rho = (\gamma - c)/(\beta - b) \geqq 1$, since the assumptions then imply (10.3) for $b + \delta \leqq r < \beta$, by Riesz's theorem in § 6.

We now put

(11.4) $$\beta-(b+\delta)=p+\Delta, \qquad 0<\Delta\leqq 1,$$

for non-negative integer $p$, and

(11.5) $$h=\varepsilon t^{\rho}, \qquad \varepsilon>0,$$

and proceed in a similar way as in Bosanquet [5] as follows. We then have the following identity

(11.6) $$\Phi_{b+\delta}^{k}(t)=\Psi(t)-\frac{\Delta}{h^{p+\Delta}}\int_0^h(h-u_0)^{\Delta-1}du_0\int_0^h du_1\cdots \times\int_0^h[\Phi_{b+\delta}^{k}(t+u_0+u_1+\cdots+u_p)-\Phi_{b+\delta}^{k}(t)]du_p,$$

where

$$\Psi(t)=\frac{\Delta}{h^{p+\Delta}}\int_0^h(h-u_0)^{\Delta-1}du_0\int_0^h du_1\cdots\int_0^h\Phi_{b+\delta}^{k}(t+u_0+u_1+\cdots+u_p)du_p.$$

By Lemma 9′, taking $k\geqq\sup(0, b-c+1)$ and excepting the case $b=\rho-1=0$, the assumptions imply (11.2). Hence, (11.6) is written as

(11.7) $$\Phi_{b+\delta}^{k}(t)=\Psi(t)+O(\varepsilon^{\delta}t^{k+c+\delta\rho}),$$

for every positive $\delta<1$.

On the other hand, calculating the repeated integral we have

(11.8) $$\Psi(t)=\frac{\Delta}{h^{p+\Delta}}\sum_{j=0}^{p}(-1)^{p-j}\binom{p}{j}\int_0^h(h-u_0)^{\Delta-1}\Phi_{p+b+\delta}^{k}(t+u_0+jh)du_0.$$

Observing that $p+b+\delta=\beta-\Delta$, the last integral with $j=0$ is, on account of $k+\gamma>0$,

$$\int_0^h(h-u_0)^{\Delta-1}\Phi_{p+b+\delta}^{k}(t+u_0)du_0, \qquad (t+u_0=u),$$

$$=\int_t^{t+h}(t+h-u)^{\Delta-1}\Phi_{\beta-\Delta}^{k}(u)du$$

$$=\Gamma(\Delta)\Phi_{\beta}^{k}(t+h)-\int_0^t(t+h-u)^{\Delta-1}\Phi_{\beta-\Delta}^{k}(u)du$$

$$=o(t^{k+\gamma})+o(t^{k+\gamma})=o(t^{k+\gamma}),$$

by Lemma 10, and the fact $\Phi_{\beta}^{k}(t)=o(t^{k+\gamma})$ which is equivalent to (10.1) owing to Lemma 4 in § 6. Same estimation holds for the other integrals in (11.8). Hence, from (11.4) and (11.5) we have

$$\Psi(t)=o\left(\frac{t^{k+\gamma}}{h^{p+\Delta}}\right)=o\left(\frac{t^{k+\gamma}}{h^{\beta-b-\delta}}\right)=o\left(\frac{t^{k+c+\delta\rho}}{\varepsilon^{\beta-b-\delta}}\right).$$

Thus, (11.7) yields for every small $\delta>0$

$$\Phi_{b+\delta}^{k}(t)=o(t^{k+c+\delta\rho}),$$

which is equivalent to (11.3) by Lemma 4.

In the exceptional case $b=\rho-1=0$, we see by an elaboration that (10.1) and (10.2) imply for every $\eta>0$

$$s_n^{c-1+\eta}=O(n^{b-1+\eta}) \qquad \text{if} \quad c-b\neq an \text{ odd integer},$$

and that the single condition (10.2) implies for every $\eta>0$

$$\Phi_{b+\eta}(t)=O(t^{c+\eta}) \qquad \text{if} \quad c-b=an \text{ odd integer}.$$

The result thus follows from these relations and (10.1), and the proof is completed.

## 12. Proof of Theorem 6.

The proof runs quite analogously as in Theorem 5, integration of a function being replaced by summation of a sequence. But, for the sake of completeness we reproduce the argument. We need two lemmas, the former of which is independent of Fourier series.

LEMMA 11. *If* $0<m<n$ *and* $0<\delta\leqq 1$, *then*

$$\Big|\sum_{\nu=0}^{m} A_{n-\nu}^{\delta-1} s_\nu\Big| \leqq \max_{0\leqq\mu\leqq n} |s_\mu^\delta|,$$

*where* $\{s_n\}$ *is an arbitrary sequence, and* $s_n^\alpha$ *denotes its n-th* $(\mathrm{C},\alpha)$ *sum.*

This is due to Bosanquet [4].

LEMMA 3′. *Let* $-1\leqq b$, $0\leqq c$ *and* $b-c<1$. *If*

$$\Phi_c(t)=O(t^b) \qquad as \quad t\to 0, \tag{12.1}$$

*and if* $0<\delta<1$ *and* $0<\rho\leqq 1$, *then*

$$|s_{n+\nu}^{b+\delta}-s_n^{b+\delta}|\leqq A\varepsilon^\delta n^{c+\delta\rho}, \qquad \nu=1,2,\cdots,m, \tag{12.2}$$

*holds for* $m=[\varepsilon n^\rho]$, $\varepsilon>0$, *and* $n>1$, *A being a constant depending only on* $\delta$ *and O in* (12.1).

This is an alternative form of Lemma 3 in § 4.

PROOF OF THEOREM 6. It is sufficient to show, in view of Dixson-Ferrar's theorem in § 3, that for every small $\delta>0$,

$$s_n^{b+\delta}=o(n^{c+\delta\rho}), \tag{12.3}$$

where $b+\delta<\beta$ and $\rho=(\gamma-c)/(\beta-b)\leqq 1$. We now define two numbers $p$, $\Delta$ as in (11.4), and put

$$m=[\varepsilon n^\rho], \qquad \varepsilon>0. \tag{12.4}$$

We then have the identity

$$s_n^{b+\delta}=T-U, \tag{12.5}$$

where

$$T=\frac{1}{m^p A_{m-1}^{\Delta}}\sum_{\nu_0=1}^{m} A_{m-\nu_0}^{\Delta-1}\sum_{\nu_1=1}^{m}\cdots\sum_{\nu_p=1}^{m} s_{n+\nu_0+\nu_1+\cdots+\nu_p}^{b+\delta},$$

$$U=\frac{1}{m^p A_{m-1}^{\Delta}}\sum_{\nu_0=1}^{m} A_{m-\nu_0}^{\Delta-1}\sum_{\nu_1=1}^{m}\cdots\sum_{\nu_p=1}^{m} (s_{n+\nu_0+\nu_1+\cdots+\nu_p}^{b+\delta}-s_n^{b+\delta}).$$

By (10.4) and Lemma 11 we have on account of $\gamma>0$

$$T=\frac{1}{m^p A_{m-1}^{\Delta}}\sum_{j=0}^{p}(-1)^{p-j}\sum_{\nu_0=1}^{m} A_{m-\nu_0}^{\Delta-1} s_{n+\nu_0+jm}^{b+\delta+p} \qquad \binom{b+\delta+p=\beta-\Delta}{n+\nu_0+jm=\nu}$$

$$=\frac{1}{m^p A_{m-1}^{\Delta}}\sum_{j=0}^{p}(-1)^{p-j}\sum_{\nu=n+jm+1}^{n+(j+1)m} A_{n+(j+1)m-\nu}^{\Delta-1} s_\nu^{\beta-\Delta}$$

$$=\frac{1}{m^p A_{m-1}^{\Delta}}\sum_{j=0}^{p}(-1)^{p-j}\cdot o(n^\gamma)=o\left(\frac{n^\gamma}{m^{p+\Delta}}\right)=o\left(\frac{n^{c+\delta\rho}}{\varepsilon^{\beta-b-\delta}}\right).$$

On the other hand, the assumptions imply (12.2) in Lemma 3′, and so it holds

$$U=O(\varepsilon^{\delta} n^{c+\delta\rho}),$$

for $0<\delta<1$.

Hence, from (12.5) we obtain (12.3) for every small $\delta>0$, and the theorem is proved.

### 13. Supplementary theorems.

Concerning Riesz's theorem mentioned in § 6, we shall prove the following theorems. In these theorems and their proofs, we suppose that $\varphi(t)\in L$ in $(0, t_0)$ and $\Phi_\alpha^k(t)$ $(\alpha\geqq 0,\ k\geqq 0)$ is defined as in (0.2).

THEOREM 7. *Let* $0<\beta$ *and* $\beta+1\leqq\gamma-c$. *If*

(13.1) $$\Phi_\beta(t)=o(t^\gamma) \quad \text{as} \quad t\to 0,$$

*and if, for* $0<u<t^{(\gamma-c)/(\beta+1)}$ *and* $0<t<t_0$,

(13.2) $$\varphi(t+u)-\varphi(t)>-At^c u, \qquad A>0,$$

*then we have, as* $t\to 0$,

(13.3) $$\Phi_r(t)=o(t^{c+(r+1)(\gamma-c)/(\beta+1)}) \quad \text{for} \quad 0\leqq r<\beta.$$

As in (N. B. 7) in § 6, concerning the numbers $\beta$, $\gamma$ and $c$ one needs no restriction other than $0<\beta\leqq\gamma-c-1$.

Theorem 7, $\varphi(t)$ being replaced by $\Phi_1(t)$, gives the part (II) of Riesz's theorem.

THEOREM 8. *If* $0<\beta\leqq\gamma-c$ *and* (13.1) *holds, and if for* $0<u<t^{(\gamma-c)/\beta}$ *and* $0<t<t_0$,

$$\varphi(t+u)-\varphi(t)>-At^c, \qquad A>0,$$

*then we have, as* $t\to 0$, $\varphi(t)=O(t^c)$, *and*

$$\Phi_r(t)=o(t^{c+r(\gamma-c)/\beta}) \quad \text{for} \quad 0<r<\beta.$$

This is a slight modification of the part (I) of Riesz's theorem. Concerning the "Cesàro sums"-analogue, see the paper [**12**, Lemma 2.1].

PROOF OF THEOREM 7. We sketch the proof. It is sufficient to show, in view of Riesz's theorem, that the assumptions imply

$$\varphi(t)=o(t^{c+\rho}), \qquad \rho=(\gamma-c)/(\beta+1). \tag{13.4}$$

(I) The case $\gamma\geqq 0$. Arguing similarly as in the proof of Theorem 5, (13.4) follows from the identity (11.6), $\Phi^k_{b+\delta}(t)$ being replaced by $\varphi(t)$ and from its analogue. So, we omit the proof of the present case. Indeed, this case is an illustration of loc. cit. [**5**, Theorem 1], in (14) of which it is supposed that $t_0^{\delta-1}$ should be taken in place of $(h-t_0)^{\delta-1}$.

(II) The case $\gamma<0$. Clearly, (13.1) implies

$$\Phi_{\beta+k}(t)=o(t^{\gamma+k}) \tag{13.5}$$

for every positive integer $k$. Taking $k>-\gamma$, and applying the case (I) to (13.5) and (13.2) we get $\varphi(t)=o(t^{c+1})$. From this and (13.2) one sees that

$$\begin{aligned}\Phi^k_0(t+u)-\Phi^k_0(t)&=[(t+u)^k-t^k]\varphi(t+u)+t^k[\varphi(t+u)-\varphi(t)]\\ &>[(t+u)^k-t^k]\cdot o(t^{c+1})-t^k\cdot At^c u\\ &>-(A+1)t^{k+c}u\end{aligned}$$

holds for $0<u<t^\rho$ and $0<t<t_0$. Applying again the case (I) to the last inequality and $\Phi^k_\beta(t)=o(t^{k+\gamma})$ which is equivalent to (13.1) owing to Lemma 4 in § 6, we have

$$\Phi^k_0(t)=o(t^{k+c+\rho}).$$

And, this is also equivalent to (13.4), which proves the theorem.

The proof of Theorem 8 runs quite analogously as above. The assumptions here imply $\varphi(t)=O(t^c)$ in place of $\varphi(t)=o(t^c)$.

Mathematical Institute
Nara Women's University

## References

[1] M. Riesz, Sur un théorème de la moyenne et ses application, Acta Szeged, **1** (1922), 114–126.

[2] A. L. Dixson and W. L. Ferrar, On Cesàro sums, J. London Math. Soc., **7** (1932), 87–93.

[3] M. N. Obrechkoff, Sur la sommation des séries trigonométriques de Fourier par les moyennes arithmétiques, Bull. Soc. Math. France, **62** (1934), 84–109.

[4] L. S. Bosanquet, A mean value theorem, J. London Math. Soc., **16** (1941), 146–148.

[5] L. S. Bosanquet, Note on convexity theorems, J. London Math. Soc., **18** (1943), 239–248.

[6] F. T. Wang, A note on Cesàro summability of Fourier series, Ann. of Math.

(2), **44** (1943), 379-400.

[7] J. M. Hyslop, Note on group of theorems in the theory of Fourier series, J. London Math. Soc., **24** (1949), 91-100.

[8] S. Izumi, Notes on Fourier analysis (XXVII) : A theorem on Cesàro summation, Tôhoku Math. J. (2), **3** (1951), 212-215.

[9] G. Sunouchi, Notes on Fourier analysis (XLVII) : Convexity theorems and Cesàro summability of Fourier series, J. Math., Tôkyo Metropolitan Univ., **1** (1953), 104-109.

[10] K. Kanno, On the Cesàro summability of Fourier series (II), Tôhoku Math. J. (2), **7** (1955), 265-278.

[11] K. Yano, On a method of Cesàro summation for Fourier series, Kôdai Math. Semi. Rep., **9** (1957), 49-58.

[12] K. Yano, Notes on Tauberian theorems for Riemann summability, Tôhoku Math. J. (2), **10** (1958), 19-31.

[13] K. Yano, On convergence criteria for Fourier series II, Tôhoku Math. J. (2), **11** (1959), 54-62.

[14] K. Yano, On Fejér kernels, Proc. Japan Acad., **35** (1959), 59-64.

[15] K. Yano, Some notes on Cesàro summation, Proc. Japan Acad., **35** (1959), 578-583.

[16] A. Zygmund, Trigonometrical series, Warszawa-Lwow, 1935.

[17] E. W. Hobson, The theory of functions of a real variable II, 1936, Cambridge.