[3] J. Górniak, *Dilations of positive definite operator valued functions in linear spaces* (in Polish), Komunikaty Inst. Mat. Polit. Wrocł. Nr 73 (1977). [See also: *Locally convex spaces with the factorization property*, Colloq. Math. (to appear).]
[4] — *Remarks on positive definite operator valued functions in linear spaces*, Lecture Notes in Math., vol. 656, Springer Verlag, 1978, 37–44.
[5] — *An analogue of Sz.-Nagy's dilation theorem II*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astrononom. Phys. 27 (1979), 287–289.
[6] J. Górniak and A. Weron, *An analogue of Sz.-Nagy's dilation theorem*, ibid. 24 (10) (1976), 867–872.
[7] A. N. Kolmogorov, *Stationary sequences in Hilbert space* (in Russian), Bull. Math. Univ. Moscow 2 (1941), 1–40.
[8] D. Kölzow, *A survey of Abstract Wiener Spaces* (preprint, 1974).
[9] P. Masani, *An explicit treatment of dilation theory* (preprint, Autumn 1975). [See also an enlarged version: *Dilations as propagators of Hilbertian varieties*, SIAM J. Math. Analysis 9 (1978), 414–456.]
[10] — *Propagators and dilations*, Lecture Notes in Math. vol. 656, Springer Verlag, 1978, 95–117.
[11] W. Mlak, *Dilations of Hilbert space operators (General theory)*, Dissertationes Math. 158 (1978), 1–61.
[12] W. Mlak and A. Weron, *Dilations of Banach spaces operator valued functions*, Ann. Polon. Math. 38 (1980), 295–303.
[13] N. N. Vakhania, *Probability distributions on linear spaces* (in Russian), Mecniereba, Tbilisi 1971. [English edition: North-Holland, 1981.]
[14] N. N. Vakhania and V. I. Tarieladze, *Covariance operators of probability measures in locally convex spaces* (in Russian), Teor. Verojatnost. i Primenen. 23 (1) (1978), 3–26.
[15] A. Weron, *Prediction theory in Banach spaces*, Lecture Notes in Math., vol. 472, Springer Verlag, 1975, 207–228.
[16] — *Second order stochastic processes and dilation theory in Banach spaces*, Ann. Inst. H. Poincaré 16 (1980), 29–38.
[17] — *Remarks on positive-definite operator valued functions in Banach spaces*, Bull. Acad. Polon. Sci. 24 (10) (1976), 873–876.

INSTITUTE OF MATHEMATICS
WROCŁAW TECHNICAL UNIVERSITY
50-370, WROCŁAW, POLAND

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Convexity, type and the three space problem

by

N. J. KALTON (Columbia, Mo.)*

Abstract. A twisted sum of two quasi-Banach spaces X and Y is a quasi-Banach space Z with a closed subspace $X_0 \cong X$ such that $Z/X_0 \cong Y$.

We show that if X is p-convex and Y is q-convex where $p \neq q$, then Z is $\min(p, q)$ convex. Similarly, if X is a type p Banach space and Y is a type q Banach space where $p \neq q$ then Z is type $\min(p, q)$.

If X and Y are Banach spaces, we show that Z is *log convex*, i.e., for some $C < \infty$

$$\|z_1 + \ldots + z_n\| \leqslant C\left(\sum_{k=1}^{n} \|z_k\|\left(1 + \log \frac{1}{\|z_k\|}\right)\right)$$

where $\|z_1\| + \ldots + \|z_n\| = 1$. Conversely, every log convex space is the quotient of a subspace of a twisted sum of two Banach spaces.

If X and Y are type p Banach spaces $(1 < p \leqslant 2)$ and one is the quotient of a subspace of some L_p-space, then Z is *log type* p, i.e.,

$$\left\{\int_0^1 \|\varepsilon_1(t) z_1 + \ldots + \varepsilon_n(t) z_n\|^p \, dt\right\}^{1/p} \leqslant c\left\{\sum \|z_k\|^p \left(1 + \left(\log \frac{1}{\|z_k\|}\right)^p\right)\right\}^{1/p}$$

where $\|z_1\|^p + \ldots + \|z_n\|^p = 1$. This result is best possible in a certain sense.

We also show that if $p < 1$ type p implies p-convexity, but if $p = 1$ a type 1 space need not be convex.

We investigate which Orlicz sequence spaces and Köthe sequence spaces are $\mathscr{K}$-spaces, i.e., such that every twisted sum with $\boldsymbol{R}$ is a direct sum.

1. Introduction. A quasi-Banach space Z is a twisted sum of X and Y if it has a subspace $X_0 \cong X$ such that $Z/X_0 \cong Y$. The so-called *three space problem* is to study the properties of Z in terms of those of X and Y.

In [1], Enflo, Lindenstrauss and Pisier showed that a Banach space which is a twisted sum of two Hilbert spaces need not be a Hilbert space. Independently, the author [6], Ribe [15] and Roberts [16] showed that a twisted sum of a line and a Banach space need not be locally convex. In [9] the author and Peck showed that these results are related by describing a general construction which shows that for every p, $0 < p < \infty$, there is a twisted sum of l_p with l_p which is not a direct sum. In particular, for $0 < p < 1$, there is a non p-convex space which is a twisted sum of two p-convex spaces.

* University of Missouri. Research done while the author held visiting positions at Michigan State University and the University of Illinois.

In contrast to these negative results there are a number of theorems which say that a twisted sum cannot be too bad. In [1] the twisted sum of two type 2 Banach spaces is shown to be type p for all $p<2$. In [6] it is shown that if X is a Banach space and Y is a type p Banach space for some $p>1$, then a twisted sum of X and Y (in *that* order) is convex (i.e., a Banach space). Also if X is p-convex $(0<p\leqslant 1)$ and Y is q-convex $(0<q<p)$, then any twisted sum of X and Y (in *that* order) is again q-convex.

These suggest a general principle, if we regard p-convexity $(0<p\leqslant 1)$ or type p $(0<p\leqslant 2)$ as an index of "roundness". The twisted sum of two spaces of differing degrees of roundness will retain the properties of the less round space; the twisted sum of two spaces of equal roundness may however be less round than either. The main aim of this paper is to establish such a pattern, and to examine more precisely the case of equal roundness.

First in Section 3, we introduce a new class of quasi-Banach spaces which we name *logconvex*. A space X is logconvex if either of the following two equivalent conditions holds for some C, $C^*<\infty$

$$\|x_1+\ldots+x_n\|\leqslant C\sum_{i=1}^{n}\|x_i\|\Big(1+\log\frac{1}{\|x_i\|}\Big) \tag{1.0.1}$$

wherever $\|x_1\|+\ldots+\|x_n\|=1$, $x_1,\ldots,x_n\in X$ or

$$\|x_1+\ldots+x_n\|\leqslant C^*\sum_{j=1}^{n}\|x_j\|\,(1+\log j). \tag{1.0.2}$$

Logconvex spaces play an important role in this paper; they are, in a sense, the next best thing to being Banach spaces. An example is the space $L(1,\infty)$ (i.e., weak L_1).

In Section 4 we show that if $p<1$, type p is equivalent to p-convexity, so that we reduce the study of type to the case $1\leqslant p\leqslant 2$.

Section 5 contains some initial technical results in twisted sums which contain very little that is new. Lemma 5.2 essentially reproduces a result of [1] in rather more generality.

In Section 6 we show that if X and Y are p-convex and q-convex, respectively, where $p<q$, then any twisted sum of X and Y or of Y and X is p-convex. (One half of this result is in [6], see above). In a similar vein, if X and Y are type p and type q, respectively, where $1\leqslant p<q\leqslant 2$, then any twisted sum Z of X and Y or Y and X is type p. Let us remark here that the methods of [1] (cf. also [13]) show that in either case if $z_1,\ldots,z_n\in Z$

$$\Big\{\int_0^1\Big\|\sum\varepsilon_i z_i\Big\|^2dt\Big\}^{1/2}\leqslant Cn^{1/p-1/2}\Big(\sum_{i=1}^{n}\|z_i\|^2\Big)^{1/2}. \tag{1.0.3}$$

The main step in the argument here is to pass from this inequality to establishing type p (but only for twisted sums where the other space is type $q>p$). As shown in [13] (1.0.3) implies type r for $r<p$. We note that a type 1 space need not be convex.

In Section 7 we study the case $p=q$. We show that any twisted sum of two Banach spaces is logconvex, and this result is best possible. In fact, a space is logconvex if and only if it is a quotient of some subspace of a twisted sum of two Banach spaces. The corresponding results for type p are right if we assume that one of the spaces X and Y is a quotient of a subspace of a space $L_p(\mu)$. In that case any twisted sum Z is *log type* p, i.e.,

$$\Big\{\int_0^1\Big\|\sum\varepsilon_i z_i\Big\|^p dt\Big\}^{1/p}\leqslant C\Big(1+\sum\|z_i\|^p\Big(\log\frac{1}{\|z_i\|}\Big)^p\Big)^{1/p} \tag{1.0.4}$$

whenever $\|z_1\|^p+\ldots+\|z_n\|^p=1$, or equivalently

$$\Big\{\int_0^1\Big\|\sum_{i=1}^{n}\varepsilon_i z_i\Big\|^p dt\Big\}^{1/p}\leqslant C^*\Big(\sum_{k=1}^{n}\|z_k\|^p(1+\log k)^p\Big)^{1/p} \tag{1.0.5}$$

for $z_1,\ldots,z_n\in Z$. Furthermore this is best possible for the twisted sum of l_p and l_p $(1\leqslant p\leqslant 2)$ constructed in [9] contains a copy of the Orlicz space l_φ where

$$\varphi(t)=t^p\Big[1+\Big(\log\frac{1}{t}\Big)^p\Big]$$

near zero, and for this space (1.0.4) cannot be improved.

In Section 8 we examine twisted sums of $\boldsymbol{R}$ and l_1 more closely, showing in particular that the examples in [6] and [15] are non-equivalent but both are best possible in a certain sense.

In Section 9 we classify those non-locally convex Orlicz spaces $l_f\subset l_1$ which are $\mathscr{K}$-spaces, i.e., for which every twisted sum of $\boldsymbol{R}$ and l_f is a direct sum. In particular, this applied to examining "galb" conditions of the type $\big(\sum f(\|x_i\|)<\infty\Rightarrow\sum x_i$ converges$\big)$ which are preserved under twisted sums with $\boldsymbol{R}$. It is shown that if f is submultiplicative, this condition will be preserved if and only if $f(x)\geqslant cx^p$ for some $p<1$.

In Section 10, we examine those locally convex F-spaces X which are not locally bounded, but are $\mathscr{K}$-spaces, so that they have the property that if Y is locally convex any twisted sum of Y and X is locally convex. It is shown that every nuclear space is a $\mathscr{K}$-space, and Köthe spaces which are $\mathscr{K}$-spaces are characterized exactly.

2. Quasi-Banach spaces. Throughout this paper all vector spaces will be real, although most arguments may be modified without difficulty to the complex case.

A *quasi-norm* on a real vector space X is a map $x \to \|x\|$ $(X \to \mathbf{R})$ such that for some $K < \infty$,

(2.0.1) $$\|x\| > 0, \quad x \neq 0, \quad x \in X,$$

(2.0.2) $$\|tx\| = |t|\,\|x\|, \quad t \in \mathbf{R},\ x \in X,$$

(2.0.3) $$\|x+y\| \leqslant K(\|x\|+\|y\|), \quad x, y \in X.$$

A quasi-norm induces a locally bounded topology on X and conversely any locally bounded topology is given by a quasi-norm. A complete quasi-normed space is called a *quasi-Banach space.* If in addition we have for some $0 < p \leqslant 1$

(2.0.4) $$\|x+y\|^p \leqslant \|x\|^p + \|y\|^p, \quad x, y \in X,$$

then X is called a *p-Banach space* (or if $p = 1$ a Banach space).

A quasi-Banach space X is said to be *p-convex* for some $0 < p \leqslant 1$ if there is a constant A such that

(2.0.5) $$\|x_1 + x_2 + \ldots + x_n\| \leqslant A(\|x_1\|^p + \ldots + \|x_n\|^p)^{1/p}$$

for $x_1, \ldots, x_n \in X$. If X is p-convex, it may be equivalently quasi-normed to be a p-Banach space. A theorem of Aoki and Rolewicz (see [17]) states that every quasi-Banach space is p-convex for some $p > 0$. We shall repeatedly exploit this by assuming the quasi-norm on a given space satisfies (2.0.4) for some $p > 0$.

We denote by $(\varepsilon_n\colon n \in \mathbf{N})$ a sequence of independent random variables (or measurable functions) on $[0, 1]$ such that $\lambda(\varepsilon_n = +1) = \lambda(\varepsilon_n = -1) = \frac{1}{2}$ where λ is Lebesgue measure. We then say that a quasi-Banach space X is *type p* $(0 < p \leqslant 2)$ ([12], [13]) if for some constant $K < \infty$ we have

(2.0.6) $$\Big(\int_0^1 \Big\|\sum_{i=1}^n \varepsilon_i(t)x_i\Big\|^p dt\Big)^{1/p} \leqslant K\Big(\sum_{i=1}^n \|x_i\|^p\Big)^{1/p}.$$

If X is p-convex, then X is certainly type p.

We remark here that Kahane [4] shows that for a Banach space X and $0 < p < q < \infty$ there is a constant $K = K(p, q)$ such that

(2.0.7) $$\Big\{\int_0^1 \Big\|\sum \varepsilon_i x_i\Big\|^p dt\Big\}^{1/p} \leqslant \Big\{\int_0^1 \Big\|\sum \varepsilon_i x_i\Big\|^q dt\Big\}^{1/q} \leqslant K\Big\{\int_0^1 \Big\|\sum \varepsilon_i x_i\Big\|^p dt\Big\}^{1/p}.$$

This means that we can change the exponent on the left of (2.0.6) without altering the definition.

In fact, (2.0.7) holds for quasi-Banach spaces; the modifications is Kahane's argument are minor but we include a proof for completeness.

THEOREM 2.1. *Let X be a quasi-Banach space. Then* (2.0.7) *holds.*

Proof. Let $\tilde{L}_0(X)$ be the space of X-valued simple functions on $[0, 1]$ equipped with the topology of convergence in measure. Let $\mathrm{Rad}(X)$ be the subspace of functions of the form $\varepsilon_1 x_1 + \ldots + \varepsilon_n x_n$ for $x_1, \ldots, x_n \in X$ and $n \in \mathbf{N}$. We show that on $\mathrm{Rad}(X)$, the L_0-topology coincides with the stronger topology induced by any quasi-norm

$$f \to \Big\{\int_0^1 \|f(t)\|^p dt\Big\}^{1/p}.$$

We see this, we need only show that the set of $f \in \mathrm{Rad}(X)$ with $\lambda(\|f\| \geqslant 1) < \frac{1}{8}$ is bounded in each L_p-norm.

Suppose $f = \varepsilon_1 x_1 + \ldots + \varepsilon_n x_n$ and

$$\lambda(\|f\| > r) = a.$$

Let

$$M(t) = \max_{1\leqslant k\leqslant n} \Big\|\sum_{i=1}^k \varepsilon_i(t)x_i\Big\|, \quad 0 \leqslant t \leqslant 1,$$

$$N(t) = \max_{1\leqslant k\leqslant n} \Big\|\sum_{i=k}^n \varepsilon_i(t)x_i\Big\|, \quad 0 \leqslant t \leqslant 1.$$

Let A_k $(1 \leqslant k \leqslant n)$ be the set of t such that

$$\Big\|\sum_{i=1}^l \varepsilon_i(t)x_i\Big\| < Kr, \quad 1 \leqslant l \leqslant k-1,$$

$$\Big\|\sum_{i=1}^k \varepsilon_i(t)x_i\Big\| \geqslant Kr$$

(where K is the modulus of concavity of the quasi-norm given by (2.0.3)).

Since f has the same distribution as

$$f^* = \varepsilon_1 x_1 + \ldots + \varepsilon_k x_k - \varepsilon_{k+1}x_{k+1} - \ldots - \varepsilon_n x_n$$

and

$$\lambda\big(A_k \cap (\|f+f^*\| \geqslant 2Kr)\big) \leqslant \lambda\big(A_k \cap (\|f\| \geqslant r)\big) + \lambda\big(A_k \cap (\|f^*\| \geqslant r)\big) = 2\lambda\big(A_k \cap (\|f\| \geqslant r)\big)$$

and hence

$$\lambda(A_k) \leqslant 2\lambda\big(A_k \cap (\|f\| \geqslant r)\big),$$

so that, summing over k,

$$\lambda(M > Kr) \leqslant 2a.$$

Similarly, $\lambda(N > Kr) \leqslant 2a$.

Now if $t \in A_k$ and $\|f\| \geqslant 2K^2r$,

$$\Big\|\sum_{i=1}^{k} \varepsilon_i x_i\Big\| \geqslant Kr$$

and

$$\Big\|\sum_{i=k}^{n} \varepsilon_i x_i\Big\| \geqslant Kr \quad \Big(\text{since} \Big\|\sum_{i=1}^{k-1} \varepsilon_i x_i\Big\| < Kr\Big).$$

Hence

$$\lambda\big(A_k \cap (\|f\| \geqslant 2K^2r)\big) \leqslant \lambda\Big(A_k \cap \Big(\Big\|\sum_{i=k}^{n} \varepsilon_i x_i\Big\| \geqslant Kr\Big)\Big) \leqslant 2a\lambda(A_k)$$

since these sets are independent. Summing over k,

$$\lambda(\|f\| \geqslant 2K^2r) \leqslant 4a^2.$$

Thus if

$$\lambda(\|f\| \geqslant 1) < \tfrac{1}{8},$$
$$\lambda\big(\|f\| \geqslant (2K^2)^n\big) < 4^{2^n-1}(\tfrac{1}{8})^{2^n} < (\tfrac{1}{2})^{2^n}$$

and

$$\int_0^1 \|f\|^p dt < 1 + \sum_{n=0}^{\infty} (2K^2)^{np}(\tfrac{1}{2})^{2^n} = S_p < \infty.$$

The *galb* $G(X)$ of a quasi-Banach space is the space of all sequences $\{a_n\}$ such that if $\|x_n\| \leqslant 1$, then $\{\sum_{k=1}^{n} a_k x_k\}$ is bounded. $G(X)$ is a quasi-Banach space when quasi-normed by

$$\|(a_1, a_2, \ldots)\| = \sup_{\|x_k\| \leqslant 1} \sup_n \Big\|\sum_{k=1}^{n} a_1 x_k\Big\|.$$

X is said to be *galbed* by a space of sequences E if, given $\|x_n\| \leqslant 1$, and $\{a_n\} \in E$, then $\sum_{k=1}^{n} a_k x_k$ is bounded (see Turpin [17] for a more detailed study of these notions).

An *F-space* is a complete metric topological vector space. A twisted sum of two F-spaces X and Y is a space Z which has a closed subspace $X_0 \cong X$ such that $Z/X_0 \cong Y$. Thus there is a short exact sequence $0 \to X \to Z \to Y \to 0$. If every twisted sum of X and Y is a direct sum (i.e. $Z \cong X \oplus Y$ in the natural way) then we say that (X, Y) *splits* (the order is important here). If $(\boldsymbol{R}, X)$ splits, then X is a $\mathscr{K}$-space ([8]).

If X is a locally convex $\mathscr{K}$-space, then every twisted sum of Y and X with Y locally convex is also locally convex ([6], Theorem 4.10) and this property characterizes locally convex $\mathscr{K}$-spaces.

3. Logconvex spaces and related classes. Let φ denote the Orlicz function

$$\varphi(t) = \begin{cases} t\Big(1+\log\dfrac{1}{t}\Big), & 0 \leqslant t \leqslant 1, \\ t, & t \geqslant 1 \end{cases}$$

(where $0 \log \infty = 0 \log 0 = 0$ by convention). Then l_φ is a locally bounded but non-locally convex Orlicz sequence space. The quasi-norm inducing the topology on l_φ may be given by

$$\|x\|_\varphi = \sup\Big\{\xi\colon \sum_{i=1}^{\infty} \varphi(\xi^{-1}|x_i|) \leqslant 1\Big\}.$$

Our first result gives an equivalent quasi-norm.

THEOREM 3.1. *An equivalent quasi-norm on l_φ is given by*

$$\|x\|_\varphi^* = \|x\|_1 + \sum_{i=1}^{\infty} |x_i| \log \frac{\|x\|_1}{|x_i|}$$

where $\|x\|_1 = \sum_{i=1}^{\infty} |x_i|$.

Proof. Since $\|\cdot\|_\varphi^*$ is easily seen to be homogeneous, it suffices to show that

$$0 < \inf(\|x\|_\varphi^*\colon \|x\|_\varphi = 1) \leqslant \sup(\|x\|_\varphi^*\colon \|x\|_\varphi = 1) < \infty.$$

If $\|x\|_\varphi = 1$, then $\sum \varphi(|x_i|) = 1$ and hence

$$\sum |x_i| \Big(1 + \log \frac{1}{|x_i|}\Big) = 1.$$

Hence $\|x\|_1 \leqslant 1$ and

$$\|x\|_\varphi^* \leqslant \|x\|_1 + \sum_{i=1}^{\infty} |x_i| \log \frac{1}{|x_i|} = \|x\|_\varphi = 1.$$

Conversely,

$$\|x\|_\varphi^* = \|x\|_\varphi - \|x\|_1 \log \|x\|_1 = 1 - \|x\|_1 \log \|x\|_1 \geqslant 1 - \frac{1}{e}$$

since $0 \leqslant \|x\|_1 \leqslant 1$.

DEFINITION 3.2. A quasi-Banach space X is *logconvex* if it is galbed by l_φ, i.e., whenever $x_n \in X$ and

$$\sum \varphi(\|x_n\|) < \infty$$

then $\sum x_n$ converges.

EXAMPLE. The space l_φ itself is logconvex; this follows easily from the fact that φ is submultiplicative at 0 (cf. Turpin [19], p. 79).

THEOREM 3.3. *A quasi-Banach space X is logconvex if and only if for some constant C and any $x_1, \ldots, x_n \in X$*

$$(3.3.1)\qquad \|x_1+x_2+\ldots+x_n\| \leqslant C\left[\sum_{i=1}^{n} \|x_i\|\left(1+\log\frac{S}{\|x_i\|}\right)\right]$$

where $S = \sum_{i=1}^{n} \|x_i\|$.

Remark. (3.3.1) is equivalent to

$$(3.3.2)\qquad \|x_1+\ldots+x_n\| \leqslant C\left(1+\sum_{i=1}^{n}\|x_i\|\log\frac{1}{\|x_i\|}\right)$$

whenever $\|x_1\|+\ldots+\|x_n\| \leqslant 1$.

Proof. Let I be an infinite set with $|I| = |X|$ and let $(x_i\colon i\in I)$ be the unit ball of X. If X is logconvex the map $T\colon l_\varphi(I)\to X$ (where $l_\varphi(I)$ is the generalized sequence space of all $(\xi_i\colon i\in I)$ such that $\sum\varphi(|\xi_i|)<\infty$ defined by

$$T(\xi) = \sum_{i\in I}^{\infty} \xi_i x_i$$

is well-defined and continuous. Hence for some $C<\infty$

$$\|T(\xi)\| \leqslant C\left(\|\xi\|_1+\sum_{i\in I}^{\infty}|\xi_i|\log\frac{\|\xi\|_1}{|\xi_i|}\right)$$

and (3.3.1) follows easily.

Conversely, if (3.3.1) holds and

$$\sum_{n=1}^{\infty}\varphi(\|x_n\|)\leqslant 1,$$

then

$$\sum_{n=1}^{\infty}\|x_n\|\left(1+\log\frac{1}{\|x_n\|}\right)\leqslant 1,$$

and hence

$$\left\|\sum_{n=k+1}^{l} x_n\right\| \leqslant C\sum_{n=k+1}^{l}\|x_n\|\left(1+\frac{1}{\log\|x_n\|}\right)\to 0 \quad\text{as}\quad k,l\to\infty$$

so that $\sum x_n$ converges.

$L(1,\infty)$ denotes the space of measurable functions on $[0,1]$ such that

$$\|f\| = \sup x\lambda(|f|>x)<\infty.$$

THEOREM 3.4. *The space $L(1,\infty)$ is logconvex.* [Added in proof: see [20].]

Proof. Suppose $x_1,\ldots,x_n\in L(1,\infty)$ and let

$$f = x_1+\ldots+x_n.$$

Suppose also $\|x_1\|+\ldots+\|x_n\| = 1$.

Fix τ, $0<\tau<1$ and let $A\subset(0,1)$ be a set of measure τ. For each $i=1,2,\ldots,n$ let

$$E_i = (|x_i|>2\tau^{-1}).$$

The $\lambda(E_i)\leqslant\frac{1}{2}\tau\|x_i\|$; hence if $E = E_1\cup\ldots\cup E_{n'}$, then $\lambda(E)\leqslant\frac{1}{2}\tau$. Now

$$\begin{aligned}
\inf_{t\in A}|f(t)| &\leqslant \inf_{t\in A\setminus E}|f(t)|\leqslant\frac{2}{\tau}\int_{A\setminus E}|f(t)|\,dt\\
&\leqslant\frac{2}{\tau}\sum_{i=1}^{n}\int_{A\setminus E}|x_i(t)|\,dt\leqslant\frac{2}{\tau}\sum_{i=1}^{n}\int_{A\setminus E_i}|x_i(t)|\,dt\\
&\leqslant\frac{2}{\tau}\sum_{i=1}^{n}\int_{A}\min(|x_i(t)|,2\tau^{-1})\,dt\leqslant\frac{2}{\tau}\sum_{i=0}^{n}\int_{0}^{\tau}\min\left(\frac{\|x_i\|}{u},2\tau^{-1}\right)du\\
&\leqslant\frac{2}{\tau}\sum_{i=1}^{n}\left(\|x_i\|+\int_{1/2\tau\|x_i\|}^{\tau}\frac{\|x_i\|}{u}\,du\right)=\frac{2}{\tau}\sum_{i=1}^{n}\left(\|x_i\|+\|x_i\|\log\frac{2}{\|x_i\|}\right)\\
&=\frac{1}{\tau}\left(2\log 2+2+2\sum_{i=1}^{n}\|x_i\|\log\frac{1}{\|x_i\|}\right).
\end{aligned}$$

Hence

$$\|f\|\leqslant(2\log 2+2)+2\sum_{i=1}^{n}\|x_i\|\log\frac{1}{\|x_i\|}$$

and so $L(1,\infty)$ is logconvex.

EXAMPLE. Let $(g_n\colon n=1,2,\ldots)$ be a sequence of independent random variables each with the Cauchy distribution (i.e., with probability density function

$$f(x) = \frac{1}{\pi}\cdot\frac{1}{1+x^2},\qquad -\infty<x<\infty.$$

Then $(|g_n|: n \in N)$ is bounded in $L(1, \infty)$ and so if $a_n \geqslant 0$, $\sum a_n|g_n|$ converges in $L(1, \infty)$ if

$$\sum_{n=1}^{\infty} a_n\left(1+\log\frac{1}{a_n}\right) < \infty \tag{3.4.1}$$

and then

$$\sum_{n=1}^{\infty} a_n|g_n(t)| < \infty \text{ a.e.} \tag{3.4.2}$$

L. Schwartz [18] shows that (3.4.1) is equivalent to (3.4.2). See Kahane [4] p. 97 for a similar example.

We now give another characterization of logconvex spaces; for this we require the following lemma.

Lemma 3.5. *Suppose $\varepsilon > 0$ and*

$$C_\varepsilon = \log\left[\sum_{k=1}^{\infty}\left(\frac{1}{k}\right)^{1+\varepsilon}\right] \quad (= \log\zeta(1+\varepsilon)).$$

If $\xi_1 \geqslant \xi_2 \geqslant \ldots \geqslant \xi_n \geqslant 0$ and

$$\xi_1+\xi_2+\ldots+\xi_n = 1$$

then

$$\sum_{k=1}^{n}\xi_k\log k \leqslant \sum_{k=1}^{n}\xi_k\log\frac{1}{\xi_k} \leqslant (1+\varepsilon)\sum_{k=1}^{n}\xi_k\log k + C_\varepsilon. \tag{3.5.1}$$

Proof. Since $\xi_k \leqslant 1/k$ the first inequality is clear. To prove the second, fix n and let C_n be the maximum of

$$F(\xi_1, \xi_2, \ldots, \xi_n) = \sum_{k=1}^{n}\xi_k\log\frac{1}{\xi_k} - (1+\varepsilon)\sum_{k=1}^{n}\xi_k\log k$$

subject to $\xi_1 \geqslant \ldots \geqslant \xi_n \geqslant 0$ and $\xi_1+\ldots+\xi_n = 1$. Then for some $(u_1, \ldots, u_n)$, $F(u_1, \ldots, u_n) = C_n$.

We claim first that $u_1 > u_2 > u_3 > \ldots > u_n > 0$. For if $l \leqslant n$ in the first index such that $u_l = 0$ then a small increase in u_l and decrease in u_{l-1} increases F; a similar argument shows that $u_i \neq u_j$ if $i \neq j$. It follows that $(u_1, \ldots, u_n)$ is a local maximum of F subject to the single condition $\xi_1+\ldots+\xi_n = 1$. Hence there is a Lagrange multiplier λ such that

$$\frac{\partial F}{\partial \xi_k}(u_1, \ldots, u_n) = \lambda, \qquad k = 1, 2, \ldots, n,$$

i.e.,

$$\log\frac{1}{u_k} - (1+\varepsilon)\log k = \lambda+1.$$

Here $u_k = e^{-1-\lambda}\left(\frac{1}{k}\right)^{1+\varepsilon}$ and so

$$\left(\sum_{k=1}^{n}\left(\frac{1}{k}\right)^{1+\varepsilon}\right) = e^{\lambda+1}, \qquad F(u_1, \ldots, u_n) = (\lambda-1) = C_n,$$

and hence $C_n \leqslant C_\varepsilon$ and the result is proved.

Theorem 3.6. *A quasi-Banach space X is logconvex if and only if for some $C < \infty$, whenever $x_1, \ldots, x_n \in X$*

$$\|x_1+\ldots+x_n\| \leqslant C\sum_{k=1}^{n}\|x_k\|(1+\log k). \tag{3.6.1}$$

Proof. This follows immediately from the preceding lemma and Theorem 3.3.

Remark. This theorem essentially means that the Orlicz space l_φ is identical to the Lorentz space of all sequences (a_n) such that $\sum a_n^*(1+\log n) < \infty$ where (a_n^*) is decreasing re-arrangement of $(|a_n|)$. See [11] for similar results for *convex* Orlicz spaces and Lorentz spaces.

4. Type in quasi-Banach spaces.

Theorem 4.1. *Suppose $1 < p \leqslant 2$; then a quasi-Banach space of type p is convex.*

Proof. Clearly if

$$b_n = \sup_{\|x_i\|\leqslant 1}\inf_{\sigma_i=\pm 1}\|\sigma_1x_1+\sigma_2x_2+\ldots+\sigma_nx_n\|,$$

then $b_n = o(n)$, and the result follows from Theorem 2.5 of [6].

Theorem 4.2. *Suppose $0 < p < 1$; then a quasi-Banach space X of type p is p-convex.*

Proof. We can and do suppose X is an r-Banach space where $0 < r < p$. For each $n \in N$, let d_n be the least constant such that

$$\|x_1+\ldots+x_n\| \leqslant d_n(\|x_1\|^p+\ldots+\|x_n\|^p)^{1/p}$$

for $x_1, \ldots, x_n \in X$. Suppose for any n

$$\left\{\int_0^1\left\|\sum_{i=1}^{n}\varepsilon_i(t)x_i\right\|^p dt\right\}^{1/p} \leqslant C\left(\sum_{i=1}^{n}\|x_i\|^p\right)^{1/p}.$$

Then for any $x_1, \ldots, x_n \in X$ there exists $\sigma_i = \pm 1$ $(1 \leqslant i \leqslant n)$ such that

$$\|\sigma_1x_1+\ldots+\sigma_nx_n\| \leqslant C\left(\sum_{i=1}^{n}\|x_i\|^p\right)^{1/p}.$$

We may suppose that if $F = \{i: \sigma_i = -1\}$ then $\sum_{i\in F} \|x_i\|^p \leqslant \frac{1}{2} \sum_{i=1}^{n} \|x_i\|^p$. Then

$$\Big\|\sum_{i\in F} x_i\Big\| \leqslant 2^{-1/p} d_n \Big(\sum_{i=1}^{n} \|x_i\|^p\Big)^{1/p},$$

and hence

$$\Big\|\sum_{i=1}^{n} x_i\Big\|^r \leqslant \Big\|\sum_{i=1}^{n} \sigma_i x_i\Big\|^r + 2^r \Big\|\sum_{i\in F} x_i\Big\|^r \leqslant (C^r + 2^{r(1-1/p)} d_n^r) \Big(\sum_{i=1}^{n} \|x_i\|^p\Big)^{r/p}.$$

Thus

$$d_n^r \leqslant C^r + 2^{r(1-1/p)} d_n^r$$

so that

$$d_n \leqslant \frac{C}{[1-(\frac{1}{2})^{r(1/p-1)}]^{1/r}}.$$

As $\{d_n\}$ is bounded, X is p-convex.

Remark. As is easily seen the hypothesis actually used in the proof is that X satisfies

$$\min_{\sigma_i=\pm 1} \|\sigma_1 x_1 + \ldots + \sigma_n x_n\| \leqslant C(\|x_1\|^p + \ldots + \|x_n\|^p)^{1/p}.$$

The same argument shows that if $b_n(X) = O(n^{1/p})$ $(p<1)$ then $a_n(X) = O(n^{1/p})$ where

$$b_n(X) = \sup_{\|x_i\|\leqslant 1} \min_{\sigma_i=\pm 1} \|\sigma_1 x_1 + \ldots + \sigma_n x_n\|,$$

$$a_n(X) = \sup_{\|x_i\|\leqslant 1} \|x_1 + \ldots + x_n\|.$$

We do not know, however, if $a_n = O(n^{1/p})$ implies X is p-convex when $p<1$.

When $p=1$ the above proof breaks down and we shall see later that a type 1 space need not be convex. It is tempting to conjecture that a type 1 space must at least be logconvex in view of the following theorem (the converse is clearly false-consider l_φ).

THEOREM 4.3. *Let X be a type 1 quasi-Banach space isomorphic to a subspace of L_0. Then X is log convex.*

Proof. By Nikisin's theorem ([7], [14]), X embeds in $L(1,\infty)$. Now apply Theorem 3.5.

We have not been able to substantiate this conjecture and have only the following, whose proof we omit. It depends on rather more delicate handling of the argument in Theorem 4.2.

THEOREM 4.4. *Suppose X is a type 1 quasi-Banach space; then for some $C<\infty$ and any $x_1,\ldots,x_n \in X$*

$$\|x_1 + \ldots + x_n\| \leqslant C(1+\log n)(\|x_1\| + \ldots + \|x_n\|).$$

In view of the above results we shall only consider type when $1 \leqslant p \leqslant 2$.

5. Twisted sums. Suppose X and Y are quasi-Banach spaces and Z is a twisted sum of Y and X, so that Z has a subspace isomorphic to Y such that $Z/Y \cong X$. Then (cf. [6], [8]) there is a map $F: X\to Y$ satisfying

(5.0.1) $$F(tx) = tF(x), \quad t\in\mathbf{R},\ x\in X,$$

(5.0.2) $$\|F(x_1+x_2) - F(x_1) - F(x_2)\| \leqslant K(\|x_1\| + \|x_2\|), \quad x_1, x_2 \in X,$$

where K is independent of x_1, x_2, such that Z is isomorphic to the space $Y\oplus_F X$, i.e., the Cartesian sum $Y\oplus X$ quasinormed by

$$\|(y,x)\| = \|y - F(x)\| + \|x\|.$$

Conversely, given any such *quasilinear map* F satisfying (5.0.1) and (5.0.2), then $Y\oplus_F X$ is a twisted sum of Y and X.

Suppose then $F: X\to Y$ is a fixed quasilinear map. We define for a finite subset $\{x_1,\ldots,x_n\}$ of X

$$\Delta(x_1,\ldots,x_n) = F(x_1+\ldots+x_n) - \sum_{i=1}^{n} F(x_i).$$

We now state the properties of Δ.

LEMMA 5.1. (1) *If $A_1, A_2, \ldots, A_m$ are disjoint subsets of $\{1,2,\ldots,n\}$ such that $A_1\cup\ldots\cup A_m = \{1,2,\ldots,n\}$ and*

$$u_i = \sum_{j\in A_i} x_j,$$

then

$$\Delta(x_1,\ldots,x_n) = \Delta(u_1,u_2,\ldots,u_m) + \sum_{i=1}^{m} \Delta(x_j: j\in A_i).$$

(2) $\Delta(x) = 0$.

(3) $\|\Delta(x_1,x_2)\| \leqslant K(\|x_1\| + \|x_2\|)$.

(4) *For some s, $0<s\leqslant 1$ and $M<\infty$*

$$\|\Delta(x_1,\ldots,x_n)\| \leqslant M(\|x_1\|^s + \ldots + \|x_n\|^s)^{1/s}$$

for $x_1,\ldots,x_n\in X$.

Proof. (1)–(3) are obvious and (4) is shown in [6].

Now suppose that W is any quasi-Banach space. We define $d_n = d_n(W)$ to be a least constant such that

$$\|w_1 + \ldots + w_n\| \leqslant d_n(\|w_1\| + \ldots + \|w_n\|)$$

whenever $w_1, \ldots, w_n \in W$. We also define $\delta_n = \delta_n(W)$ to be the least constant such that

$$\Big\{\int_0^1 \Big\|\sum_{i=1}^n \varepsilon_i w_i\Big\|^2 dt\Big\}^{1/2} \leqslant \delta_n\Big(\sum_{i=1}^n \|w_i\|^2\Big)^{1/2}.$$

The sequence $\{\delta_n\}$ has been studied for Banach space in [1], [2] and [13]. It is easy enough to see that both sequences $\{d_n\}$ and $\{\delta_n\}$ are submultiplicative ($d_{mn} \leqslant d_m d_n$, $\delta_{mn} \leqslant \delta_m \delta_n$).

For a quasilinear map $F\colon X \to Y$ we define $c_n = c_n(F)$ to be the least constant such that

$$\|\Delta(x_1, \ldots, x_n)\| \leqslant c_n(\|x_1\| + \ldots + \|x_n\|), \qquad x_1, \ldots, x_n \in X,$$

and $\gamma_n = \gamma_n(F)$ to be the least constant such that

$$\Big\{\int_0^1 \|\Delta(\varepsilon_1 x_1, \varepsilon_2 x_2, \ldots, \varepsilon_n x_n)\|^2 dt\Big\}^{1/2} \leqslant \gamma_n(\|x_1\|^2 + \ldots + \|x_n\|^2)^{1/2}, \quad x_1, \ldots, x_n \in X.$$

Our first result is simply a generalization of a result of Enflo, Lindenstrauss and Pisier.

THEOREM 5.2. *Suppose $0 < r \leqslant 1$ and Y is an r-Banach space. For $m, n \in N$*

(5.2.1) $$c^r_{mn} \leqslant c^r_m d^r_n(X) + c^r_n d^r_m(Y),$$

(5.2.2) $$\gamma^r_{mn} \leqslant \gamma^r_m \delta^r_n(X) + \gamma^r_n \delta^r_m(Y).$$

Proof. (5.2.1). Suppose $x_1, x_2, \ldots, x_{mn} \in X$. Let

$$u_i = \sum_{(i-1)n+1}^{in} x_j, \quad i = 1, 2, \ldots, m.$$

Then $\|\Delta(u_1, \ldots, u_m)\| \leqslant c_m \sum_{i=1}^m \|u_i\| \leqslant c_m d_n(X) \sum_{i=1}^{mn} \|x_i\|$ and

$$\Big\|\sum_{i=1}^m \Delta\big(x_j\colon (i-1)n < j \leqslant in\big)\Big\| \leqslant d_m(Y) \sum_{i=1}^m \big\|\Delta\big(x_j\colon (i-1)n < j \leqslant in\big)\big\|$$
$$\leqslant d_m(Y) c_n \sum_{i=1}^{mn} \|x_i\|$$

and (5.2.1) follows from Lemma 5.1.

(5.2.2). Let

$$u_i(t) = \sum_{(i-1)n+1}^{in} \varepsilon_j(t) x_j.$$

Then

$$\int_0^1 \|\Delta(u_1(t), u_2(t), \ldots, u_m(t))\|^2 dt = \int_0^1\int_0^1 \|\Delta(\varepsilon_1(s)u_1(t), \ldots, \varepsilon_m(s)u_m(t))\|^2 ds dt$$

(by symmetry)

$$\leqslant \int_0^1 \gamma_m^2 \sum_{i=1}^m \|u_i(t)\|^2 dt \leqslant \gamma_m^2 \delta_n^2(X) \sum_{i=1}^{mn} \|x_i\|^2.$$

Also, by a similar argument,

$$\int_0^1 \Big\|\sum_{i=1}^m \Delta\big(\varepsilon_j x_j\colon (i-1)n < j \leqslant in\big)\Big\|^2 dt \leqslant \delta_m^2(Y)\gamma_n^2 \sum_{i=1}^{mn} \|x_i\|^2$$

and (5.2.2) now follows from Lemma 5.1 and the convexity of the $L_{2/r}$-norm.

LEMMA 5.3. *If $p > 0$, there exists $\alpha = \alpha(p) \geqslant 0$ and $C = C(p)$ such that for $x_1, \ldots, x_n \in X$*

(5.3.1) $$\|\Delta(x_1, \ldots, x_n)\| \leqslant C\Big(\sum_{k=1}^n k^\alpha \|x_k\|^p\Big)^{1/p}$$

Proof. By Lemma 5.1 we can find $s > 0$ and $M < \infty$ such that

$$\|\Delta(x_1, \ldots, x_n)\| \leqslant M\Big(\sum_{k=1}^n \|x_k\|^s\Big)^{1/s}.$$

Thus for $0 < p \leqslant s$, $\alpha = 0$ will suffice. Now suppose $s < p < \infty$, and choose $\theta > 1$. Let $c = \big(\sum_{k=1}^\infty k^{-\theta}\big)^{-1}$. Then $\sum_{k=1}^\infty ck^{-\theta} = 1$ and hence

$$\Big(\sum_{k=1}^n \|x_k\|^s\Big)^{1/s} = \Big(\sum_{k=1}^n ck^{-\theta}\|x_k\|^s c^{-1}k^\theta\Big)^{1/s} \leqslant \Big(\sum_{k=1}^n ck^{-\theta}\|x_k\|^p c^{-p/s}k^{\theta p/s}\Big)^{1/p}$$

and hence $\alpha = \theta(p/s - 1)$ will suffice.

6. Twisted sums with unequal convexity.

LEMMA 6.1. *Suppose $\mu > \nu \geqslant 0$, and Y is an r-Banach space.*

(6.1.1) *If $d_n(Y) = O(n^\mu)$ and $d_n(X) = O(n^\nu)$, then $c_n = O(n^\mu)$.*

(6.1.2) *If $\delta_n(Y) = O(n^\mu)$ and $\delta_n(X) = O(n^\nu)$, then $\gamma_n = O(n^\mu)$.*

Remark. The roles of X and Y may be interchanged in this lemma.

Proof. Suppose $d_n(Y) \leqslant an^\mu$ and $d_n(X) \leqslant bn^\nu$. Select N so that $bN^{\nu-\mu} < 1$. Let $\theta_k = (c_{N^k}N^{-k\mu})^r$. Then

$$c^r_{N^k} \leqslant c^r_{N^{k-1}} d^r_N(X) + c^r_N d^r_{N^{k-1}}(Y)$$

so that

$$\theta_k \leqslant (bN^{\nu-\mu})^r \theta_{k-1} + a\theta_1$$

and hence $\{\theta_k\}$ is bounded. Hence $e_n = O(n^\mu)$.

(6.1.2) has a similar proof.

THEOREM 6.2. *Suppose that $0 < p, q \leqslant 1$ and $p \neq q$, and that X is a p-convex quasi-Banach space and Y is a q-convex quasi-Banach space. Then any twisted sum $Y \oplus_F X$ of Y and X is $\min(p, q)$-convex.*

Proof. The case $q > p$ is proved in [6]. We therefore assume that $q < p$ and that Y is a q-Banach space and X is a p-Banach space. Then

$$d_n(Y) \leqslant n^{1/q-1}, \qquad d_n(X) \leqslant n^{1/p-1},$$

and hence for any quasilinear map $F\colon X \to Y$

$$c_n(F) \leqslant Cn^{1/q-1}$$

for some C.

Now suppose $x_1, \ldots, x_n \in X$ is non-zero and

$$\|x_1\|^q + \ldots + \|x_n\|^q = 1.$$

Let $A_m = \{i\colon 2^{-m} < \|x_i\| \leqslant 2\cdot 2^{-m}\}$, $m = 1, 2, 3, \ldots$ Then for some N, $A_1, \ldots, A_N$ partitions $\{1, 2, \ldots, n\}$. Let

$$u_m = \sum_{i\in A_m} x_i.$$

Then, if we make the convention $\Delta(\emptyset) = 0$ and $\sum_{\emptyset} x_i = 0$,

$$\|\Delta(x_1, \ldots, x_n)\|^q \leqslant \|\Delta(u_1, \ldots, u_N)\|^q + \sum_{i=1}^{N} \|\Delta(x_j\colon j \in A_i)\|^q.$$

Now

$$\|\Delta(x_j\colon j \in A_i)\| \leqslant C|A_i|^{1/q-1} \sum_{j\in A_i} \|x_j\| \leqslant 2C|A_i|^{1/q} 2^{-i},$$

so that

$$\|\Delta(x_j\colon j \in A_i)\|^q \leqslant (2C)^q |A_i| 2^{-iq} \leqslant (2C)^q \sum_{j\in A_i} \|x_j\|^q.$$

Hence

$$\sum_{i=1}^{N} \|\Delta(x_j\colon j \in A_i)\|^q \leqslant (2C)^q \sum_{i=1}^{n} \|x_i\|^q = (2C)^q.$$

Now by Lemma 5.3 there is a constant M and $\alpha \geqslant 0$ so that

$$\|\Delta(w_1, \ldots, w_l)\|^p \leqslant M \sum_{k=1}^{l} k^\alpha \|w_k\|^p, \qquad w_1, \ldots, w_l \in X.$$

Hence

$$\|\Delta(u_1, \ldots, u_N)\|^p \leqslant M \sum_{k=1}^{N} k^\alpha \|u_k\|^p \leqslant M \sum_{k=1}^{N} k^\alpha \sum_{i\in A_k} \|x_i\|^p$$
$$\leqslant M^* \sum_{i=1}^{n} \|x_i\|^p \left(\log \frac{2}{\|x_i\|}\right)^\alpha$$

where $M^* = M/\log 2$.

Now $\sup_{0\leqslant\xi\leqslant 1} \xi^{p-q}(\log(2/\xi))^\alpha = \theta < \infty$ and hence if $M^{**} = \theta M^*$

$$\|\Delta(u_1, \ldots, u_N)\|^p \leqslant M^{**} \sum_{i=1}^{n} \|x_i\|^q = M^{**}.$$

Hence

$$\|\Delta(x_1, \ldots, x_n)\|^q \leqslant (M^{**})^q + (2C)^q$$

where both M^{**} and C are independent of $x_1, \ldots, x_n$. We conclude that for *any* $x_1, \ldots, x_n$

$$\|\Delta(x_1, \ldots, x_n)\|^q \leqslant D \sum_{i=1}^{n} \|x_i\|^q.$$

Now suppose $(y_i, x_i) \in Y \oplus_F X$. Then

$$\Big\|\Big(\sum y_i, \sum x_i\Big)\Big\|^q = \Big(\Big\|\sum y_i - F\Big(\sum x_i\Big)\Big\| + \Big\|\sum x_i\Big\|\Big)^q$$
$$\leqslant \Big(\Big\|\sum y_i - F\Big(\sum x_i\Big)\Big\|^q + \Big\|\sum x_i\Big\|^q\Big)$$
$$\leqslant \Big(\Big\|\sum (y_i - F(x_i))\Big\|^q + \Big\|\Delta(x_1, \ldots, x_n)\Big\|^q + \Big\|\sum x_i\Big\|^q\Big)$$
$$\leqslant 2D\Big(\sum \|x_i\|^q + \sum \|y_i - F(x_i)\|^q\Big)$$
$$\leqslant 2^{1/q-1} D \sum \|(x_i, y_i)\|^q$$

and so $Y \oplus_F X$ is q-convex.

THEOREM 6.4. *Suppose that X is a quasi-Banach space of type p ($1 \leqslant p \leqslant 2$) and Y is a quasi-Banach space of type q ($1 \leqslant q \leqslant 2$). Then if $q < p$, any twisted sum $Y \oplus_F X$ is of type q.*

Proof. This proof mimics Theorem 6.2. We assume that Y is an r-Banach space where $r \leqslant 1$; of course, if $q > 1$, we may take $r = 1$. We suppose that if $x_1, \ldots, x_n \in X$ and $y_1, \ldots, y_n \in Y$, then

$$\Big\{\int_0^1 \Big\|\sum_{i=1}^{n} \varepsilon_i(t) x_i\Big\|^p dt\Big\}^{1/p} \leqslant c\Big(\sum \|x_i\|^p\Big)^{1/p},$$

$$\Big\{\int_0^1 \Big\|\sum_{i=1}^{n} \varepsilon_i(t) y_i\Big\|^q dt\Big\}^{1/q} \leqslant c\Big(\sum \|x_i\|^q\Big)^{1/q},$$

and that

$$\|\Delta(x_1,\dots,x_n)\| \leqslant M\Big(\sum_{k=1}^{n} k^{\alpha}\|x_k\|^p\Big)^{1/p}$$

as in Lemma 5.3.

We have $\delta_n(Y) = O(n^{1/q-1/2})$ and $\delta_n(X) = O(n^{1/p-1/2})$ and hence

$$\gamma_n(F) \leqslant Cn^{1/q-1/2}, \qquad n \in N.$$

Suppose $x_1,\dots,x_n \in X$ are non-zero and

$$\|x_1\|^q + \dots + \|x_n\|^q = 1.$$

Let $A_m = \{i\colon 2^{-m} < \|x_i\| \leqslant 2\cdot 2^{-m}\}$ and suppose $A_1,\dots,A_N$ positions $\{1,2,\dots,N\}$. As before we make the convention $\Delta(\emptyset) = 0$ and $\sum_{i\in\emptyset} x_i = 0$. Then if

$$u_i(t) = \sum_{j\in A_i} \varepsilon_j(t)x_i, \qquad 0 \leqslant t \leqslant 1,$$

$$\Delta\big(\varepsilon_1(t)x_1,\dots,\varepsilon_n(t)x_n\big) = \Delta\big(u_1(t),\dots,u_N(t)\big) + \sum_{i=1}^{N} \Delta\big(\varepsilon_j(t)x_j\colon j \in A_i\big).$$

Now by symmetry

$$\begin{aligned}\int_0^1 \Big\|\sum_{i=1}^{N} \Delta\big(\varepsilon_j(t)x_j\colon j\in A_i\big)\Big\|^q dt &= \int_0^1\int_0^1 \Big\|\sum_{i=1}^{N} \varepsilon_i(s)\,\Delta\big(\varepsilon_j(t)x_j\colon\ j\in A_i\big)\Big\|^q ds\,dt\\ &\leqslant c^q \int_0^1 \sum_{i=1}^{N} \|\Delta\big(\varepsilon_j(t)x_j\colon j\in A_i\big)\|^q dt\\ &\leqslant c^q \sum_{i=1}^{N}\Big\{\int_0^1 \|\Delta\big(\varepsilon_j(t)x_j\colon j\in A_i\big)\|^2 dt\Big\}^{q/2}\\ &\leqslant c^q \sum_{i=1}^{N} C^q|A_i|^{1-q/2}\Big(\sum_{j\in A_i}\|x_j\|^2\Big)^{q/2}\\ &\leqslant 2^{q/2}c^qC^q\sum_{i=1}^{N}|A_i|2^{-iq} \leqslant 2^{q/2}c^qC^q\sum_{i=1}^{n}\|x_i\|^q \leqslant 2^{q/2}c^qC^q.\end{aligned}$$

Also

$$\begin{aligned}\int_0^1 \|\Delta\big(u_1(t),\dots,u_N(t)\big)\|^p dt &\leqslant M^p\int_0^1 \sum_{k=1}^{N} k^{\alpha}\|u_k(t)\|^p dt \leqslant M^pc^p\sum_{k=1}^{N} k^{\alpha}\sum_{j\in A_k}\|x_j\|^p\\ &\leqslant M^*\sum_{j=1}^{n}\|x_j\|^p\Big(\log\frac{2}{\|x_j\|}\Big)^{\alpha} \leqslant M^{**}\end{aligned}$$

as in Theorem 6.2, where M^{**} is independent of $x_1,\dots,x_n$. Hence

$$\begin{aligned}\Big\{\int_0^1 \|\Delta(\varepsilon_1x_1,\dots,\varepsilon_nx_n)\|^q dt\Big\}^{r/q} &\leqslant \Big\{\int_0^1 \|\Delta\big(u_1(t),\dots,u_N(t)\big)\|^q dt\Big\}^{r/q} +\\ &\quad + \Big\{\int_0^1 \Big\|\sum_{i=1}^{N}\Delta(\varepsilon_jx_j\colon j\in A_i)\Big\|^q dt\Big\}^{r/q}\\ &\leqslant (M^{**})^r + 2^{r/2}c^rC^r.\end{aligned}$$

It follows easily that for *any* $x_1,\dots,x_n$

$$\Big\{\int_0^1 \|\Delta(\varepsilon_1x_1,\dots,\varepsilon_nx_n)\|^q dt\Big\}^{1/q} \leqslant D\Big\{\sum\|x_i\|^q\Big\}^{1/q}.$$

Now suppose $(y_i, x_i) \in Y\oplus_F X$ $(1 \leqslant i \leqslant n)$. Then

$$\begin{aligned}&\Big\{\int_0^1 \Big\|\sum_{i=1}^{n}\varepsilon_i(t)(y_i, x_i)\Big\|^q dt\Big\}^{1/q}\\ &\qquad = \Big\{\int_0^1 \Big(\Big\|\sum\varepsilon_i(t)y_i - F\Big(\sum\varepsilon_i(t)x_i\Big)\Big\| + \Big\|\sum\varepsilon_i(t)x_i\Big\|\Big)^q dt\Big\}^{1/q}\\ &\qquad \leqslant \Big\{\int_0^1 \Big\|\sum\varepsilon_i(t)y_i - F\Big(\sum\varepsilon_i(t)x_i\Big)\Big\|^q dt\Big\}^{1/q} + c\Big(\sum\|x_i\|^q\Big)^{1/q},\end{aligned}$$

$$\Big\|\sum\varepsilon_i(t)y_i - F\Big(\sum\varepsilon_i(t)x_i\Big)\Big\| \leqslant 2^{1/r-1}\Big(\Big\|\sum\varepsilon_i(t)\big(y_i - F(x_i)\big)\Big\| + \|\Delta(x_1,\dots,x_n)\|\Big)$$

so

$$\begin{aligned}&\Big\{\int_0^1 \Big\|\sum\varepsilon_i(t)(y_i, x_i)\Big\|^q dt\Big\}^{1/q}\\ &\qquad \leqslant 2^{1/r-1}c\Big(\sum\Big\|y - F(x_i)\Big\|^q\Big)^{1/q} + (2^{1/r-1}D + c)\Big(\sum\|x_i\|^q\Big)^{1/q}\\ &\qquad \leqslant \big(2^{1/r-1}(c+D)+c\big)\Big(\sum\big(\|x_i\| + \|y - F(x_i)\|\big)^q\Big)^{1/q}\\ &\qquad \leqslant K\Big(\sum\|(y_i, x_i)\|^q\Big)^{1/q},\end{aligned}$$

i.e., $Y\oplus_F X$ is type q.

We now show that a type 1 space need not be convex.

In [6], [15] and [16] it is shown that one can construct a non-convex twisted sum of $\boldsymbol{R}$ and a Banach space. This is type 1 by the next theorem.

THEOREM 6.5. *Suppose X is a type p quasi-Banach space and Y is a type q Banach space where $q > p$. Then any twisted sum $Y\oplus_F X$ is type p.*

Proof. Here our techniques are rather different. We use a result of Kahane [4] that there is a constant $K = K(p, q)$ such that for any elements $u_1,\dots,u_n$ of X

$$\Big\{\int_0^1 \Big\|\sum \varepsilon_i u_i\Big\|^q dt\Big\}^{1/q} \leqslant K\Big\{\int_0^1 \Big\|\sum \varepsilon_i u_i\Big\|^p dt\Big\}^{1/p}.$$

[For the case $p=1$, we apply Theorem 2.1.]

Suppose c has the same meaning as in Theorem 6.4. Let θ_n be the least constant such that

$$\Big\{\int \|\Delta(\varepsilon_1 x_1,\dots,\varepsilon_n x_n)\|^q dt\Big\}^{1/q} \leqslant \theta_n\Big\{\sum_{i=1}^n \|x_i\|^p\Big\}^{1/p}.$$

Suppose $\|x_1\|^p+\dots+\|x_n\|^p=1$ and $\|x_n\|^p\geqslant 1/N$. Then

$$\Delta(\varepsilon_1x_1,\dots,\varepsilon_nx_n) = \Delta\big(u(t),\ \varepsilon_n(t)x_n\big)+\Delta(\varepsilon_1x_1,\dots,\varepsilon_{n-1}x_{n-1})$$

where $u(t)=\sum_{i=1}^{n-1}\varepsilon_i(t)x_i$. Now

$$\Big\{\int_0^1\big\|\Delta\big(u(t),\ \varepsilon_n(t)x_n\big)\big\|^q dt\Big\}^{p/q}\leqslant \theta_2^p\Big\{\int_0^1\big(\|u(t)\|^p+\|x_n\|^p\big)^{q/p}dt\Big\}^{p/q}$$

by the usual symmetrization argument.

Hence

$$\Big\{\int_0^1\big\|\Delta\big(u(t),\ \varepsilon_n(t)x_n\big)\big\|^q dt\Big\}^{p/q}\leqslant\theta_2^p\Big[\Big(\int_0^1\|u(t)\|^q dt\Big)^{p/q}+\|x_n\|^p\Big]$$

$$\leqslant \theta_2^p\Big[K^pc^p\sum_{i=1}^{n-1}\|x_i\|^p+\|x_n\|^p\Big]\leqslant K^p\theta_2^pc^p.$$

Hence if $\|x_1\|^p+\dots+\|x_n\|^p=1$ and $\max\|x_i\|^p\geqslant N^{-1}$,

$$\Big\{\int_0^1\|\Delta(\varepsilon_1x_1,\dots,\varepsilon_nx_n)\|^qdt\Big\}^{1/q}\leqslant\theta_{n-1}(1-1/N)^{1/p}+K\theta_2c.$$

Now suppose $\|x_1\|^p+\dots+\|x_n\|^p=1$ and $\max\|x_i\|^p<N^{-1}$. Then it is possible to subdivide $\{1,2,\dots,n\}$ into N sets $A_1,\dots,A_N$ such that

$$\sum_{i\in A_j}\|x_i\|^p\leqslant 2/N,\qquad j=1,2,\dots,N.$$

Then let

$$u_j(t)=\sum_{i\in A_j}\varepsilon_i(t)x_i,$$

$$\Delta(\varepsilon_1x_1,\dots,\varepsilon_nx_n)=\Delta\big(u_1(t),\dots,u_N(t)\big)+\sum_{i=1}^N\Delta(\varepsilon_ix_i:\ i\in A_j),$$

and by symmetrization

$$\Big\{\int_0^1\|\Delta(u_1(t),\dots,u_N(t))\|^qdt\Big\}^{p/q}\leqslant\theta_N^p\Big\{\int_0^1\Big(\sum_{i=1}^N\|u_i(t)\|^p\Big)^{q/p}dt\Big\}^{p/q}$$

$$\leqslant\theta_N^p\sum_{i=1}^N\Big\{\int_0^1\|u_i(t)\|^qdt\Big\}^{p/q}$$

$$\leqslant\theta_N^p\sum_{i=1}^N K^pc^p\sum_{i\in A_j}\|x_i\|^p\leqslant\theta_N^pK^pc^p,$$

$$\Big\{\int_0^1\Big\|\sum_{i=1}^N\Delta(\varepsilon_ix_i:\ i\in A_j)\Big\|^qdt\Big\}^{1/q}\leqslant c\Big\{\sum_{i=1}^N\int_0^1\|\Delta(\varepsilon_ix_i:\ i\in A_j)\|^qdt\Big\}^{1/q}$$

(by symmetrization)

$$\leqslant c\theta_n\Big\{\sum_{i=1}^N\Big(\sum_{i\in A_j}\|x_i\|^q\Big)^{p/q}\Big\}^{1/q}$$

$$\leqslant c\theta_n\Big\{\sum_{i=1}^N\Big(\frac{2}{N}\Big)^{q/p-1}\sum_{i\in A_j}\|x_i\|^p\Big\}^{1/q}$$

$$\leqslant\Big(\frac{2}{N}\Big)^{1/p-1/q}c\theta_n.$$

Hence

$$\Big\{\int_0^1\|\Delta(\varepsilon_1x_1,\dots,\varepsilon_nx_n)\|^qdt\Big\}^{1/q}\leqslant cK\theta_N+\Big(\frac{2}{N}\Big)^{1/p-1/q}c\theta_n.$$

Thus

$$\theta_n\leqslant\max\Big\{\theta_n\Big(1-\frac{1}{N}\Big)^{1/r}+K\theta_2c;\ cK\theta_N+\Big(\frac{2}{N}\Big)^{1/p-1/q}c\theta_n\Big\}.$$

If we choose N so that

$$c\Big(\frac{2}{N}\Big)^{1/p-1/q}<1,$$

this implies a bound on θ_n.

The fact that θ_n is bounded implies that $Y\oplus_F X$ is type p in the usual way, as in Theorem 6.4.

7. Twisted sums with equal convexity. Since the twisted sum of two Banach spaces may not be convex we may ask what class it does belong to. It turns out that we can give a complete answer to this. We require first the following lemma. We use the notation of Section 5.

LEMMA 7.1. *Suppose $\mu \geqslant 0$ and Y is a Banach space.*

(7.1.1) *If $d_n(X) = 1$ and $d_n(Y) = 1$, then $c_n = O(\log n)$.*

(7.1.2) *If $\delta_n(X) \leqslant n^\mu$ and $\delta_n(Y) = O(n^\mu)$ (or $\delta_n(X) = O(n^\mu)$ and $\delta_n(Y) \leqslant n^\mu$), then $\gamma_n = O(n^\mu \log n)$.*

Proof. We prove only (7.1.2) (as the same argument then proves (7.1.1)). If $\delta_n(X) \leqslant n^\mu$ and $\delta_n(Y) \leqslant Cn^\mu$, then by Theorem 5.2

$$\frac{\gamma_{mn}}{(mn)^\mu} \leqslant \frac{\gamma_m}{m^\mu} + C\frac{\gamma_n}{n^\mu}$$

and hence $\gamma_{n^k} n^{-k\mu} \leqslant Ck\gamma_n n^{-\mu}$ and the result follows easily.

THEOREM 7.1. *Suppose X and Y are Banach spaces. Then any twisted sum $Y \oplus_F X$ is logconvex.*

Proof. Here we have $d_n(X) = d_n(Y) = 1$ for all n and hence $c_n \leqslant C(\log n + 1)$. Induction on n as in Lemma 3.2 of [6] shows that

$$\|\Delta(x_1, \ldots, x_n)\| \leqslant M \sum_{k=1}^{n} k\|x_k\|, \qquad x_1, \ldots, x_n \in X,$$

in this case, for some M independent of $x_1, \ldots, x_n$.

Suppose $\|x_1\| + \|x_2\| + \ldots + \|x_n\| = 1$ and suppose $\|x_1\| \geqslant \|x_2\| \geqslant \ldots \ldots \geqslant \|x_n\| > 0$. Let N_k be the greatest suffix such that $\|x_{N_k}\| > 2^{-k}$ ($k = 1, 2, \ldots$) and let

$$u_k = \sum_{i=N_{k-1}+1}^{N_k} x_i, \qquad k = 1, 2, \ldots$$

(where $N_0 = 0$). Suppose $N_l = n$. Then

$$\Delta(x_1, \ldots, x_n) = \Delta(u_1, \ldots, u_l) + \sum_{k=1}^{l} \Delta(x_{N_{k-1}+1}, \ldots, x_{N_k}),$$

$$\Big\|\sum_{k=1}^{l} \Delta(x_{N_{k-1}+1}, \ldots, x_{N_k})\Big\| \leqslant C \sum_{k=1}^{l} \big(1 + \log(N_k - N_{k-1})\big) \sum_{N_{k-1}+1}^{N_k} \|x_i\|$$
$$\leqslant C + C\sum_{k=1}^{l} \log N_k \sum_{N_{k-1}+1}^{N_k} \|x_i\|.$$

Clearly, $N_k 2^{-k} \leqslant 1$ so that $N_k \leqslant 2/\|x_i\|$ for $N_{k-1}+1 < i \leqslant N_k$. Hence

$$\Big\|\sum_{k=1}^{l} \Delta(x_{N_{k-1}+1}, \ldots, x_{N_k})\Big\| \leqslant C + C\log 2 + C\sum_{i=1}^{n} \|x_i\| \log\frac{1}{\|x_i\|}.$$

Also

$$\|\Delta(u_1, \ldots, u_l)\| \leqslant M\sum_{k=1}^{l} k\|u_k\| \leqslant \frac{M}{\log 2}\sum_{i=1}^{n} \|x_i\| \log\frac{2}{\|x_i\|}$$
$$\leqslant M + \frac{M}{\log 2}\sum_{i=1}^{n} \|x_i\| \log\frac{1}{\|x_i\|}.$$

Thus

$$\|\Delta(x_1, \ldots, x_n)\| \leqslant M + C + C\log 2 + \Big(C + \frac{M}{\log 2}\Big)\sum_{i=1}^{n} \|x_i\| \log\frac{1}{\|x_i\|}$$

whenever $\sum \|x_i\| = 1$. Hence for general $x_1, \ldots, x_n$

$$\|\Delta(x_1, \ldots, x_n)\| \leqslant B_1\sum_{i=1}^{n} \|x_i\| + B_2\sum_{i=1}^{n} \|x_i\| \log\frac{S}{\|x_i\|}$$

where $S = \sum_{i=1}^{n} \|x_i\|$.

Now it easily follows that $Y \oplus_F X$ is logconvex for

$$\Big\|\sum_{i=1}^{n} (y_i, x_i)\Big\| = \Big\|\sum y_i - F\Big(\sum x_i\Big)\Big\| + \Big\|\sum x_i\Big\|$$
$$= \Big\|\sum \big(y_i - F(x_i)\big)\Big\| + \|\Delta(x_1, \ldots, x_n)\| + \Big\|\sum x_i\Big\|$$
$$\leqslant \sum_{i=1}^{n} \|(y_i, x_i)\| + B_1\sum \|x_i\| + B_2\sum \|x_i\| \log\frac{S}{\|x_i\|},$$

and the result follows from the fact that the function

$$\Phi(\xi_1, \ldots, \xi_n) = \sum_{i=1}^{n} \xi_i + \sum_{i=1}^{n} \xi_i \log\frac{\sum \xi_j}{\xi_i}, \qquad \xi_1, \ldots, \xi_n \geqslant 0$$

is monotone in each ξ_i, and $\|x_i\| \leqslant \|(y_i, x_i)\|$.

Remark. If we take $X = Y = l_1$ and $F\colon l_1 \to l_1$ is given on the finitely non-zero sequences by

$$F(x) = \Big(x_n \log\frac{\|x\|}{|x_n|}\Big),$$

then $l_1 \oplus_F l_1$ contains l_φ (where as usual $\varphi(x) = x(1 + \log(1/x))$ near zero). (See [8].) This shows that the result of Theorem 7.1 is best possible.

THEOREM 7.2. *A quasi-Banach space is logconvex if and only if it is the quotient of a subspace of a twisted sum of two Banach spaces.*

Proof. By Theorem 7.1 such a space must be logconvex. The above example generalized to $l_1(I)$ for arbitrary index sets I enables one to obtain $l_\varphi(I)$ as a subspace of a twisted sum of Banach spaces and hence every logconvex space as a quotient (cf. [19] or the method of Theorem 3.3).

Definition 7.3. We say a quasi-Banach space X is of *logtype* p $(1 \leqslant p \leqslant 2)$ if for some constant $C < \infty$ we have

$$(7.3.1)\qquad \Big\{\int_0^1 \Big\|\sum \varepsilon_i(t)x_i\Big\|^p dt\Big\}^{1/p} \leqslant C\Big(1+\sum \|x_i\|^p\Big(\log\frac{1}{\|x_i\|}\Big)^p\Big)^{1/p}$$

whenever $\|x_1\|^p+\ldots+\|x_n\|^p = 1$.

Remark. In order that X is of logtype p it is sufficient that

$$(7.3.2)\qquad \Big\{\int_0^1 \Big\|\sum \varepsilon_i(t)x_i\Big\|^p dt\Big\}^{1/p} \leqslant C'\Big\{\sum \|x_k\|^p(1+\log k)^p\Big\}^{1/p}.$$

To see this arrange x_k so that $\|x_k\|$ decreases. Then if $\sum \|x_k\|^p = 1$, $\|x_k\|^p \leqslant k^{-1}$ and hence

$$\log\frac{1}{\|x_k\|} \geqslant \frac{1}{p}\log k.$$

We will see later that (7.3.1) and (7.3.2) are equivalent; of course, for $p=1$ this is immediate from Section 3, and for $1 < p \leqslant 2$ could be established directly in a similar manner. However our indirect methods also establish this result without difficulty.

Definition 7.4. A Banach space X is of *exact type* p if

$$\|x_1-x_2\|^p+\|x_1+x_2\|^p \leqslant 2(\|x_1\|^p+\|x_2\|^p), \qquad x_1, x_2 \in X.$$

Remarks. If $p=1$, this is automatic. If $p=2$, it implies that X is a Hilbert space, for in this case

$$\|2x_1\|^2+\|2x_2\|^2 \leqslant 2(\|x_1-x_2\|^2+\|x_1+x_2\|^2)$$

and hence the parallelogram law holds; then apply a result of Jordan and von Neumann [3]. For $1 < p < 2$ it is sufficient that X is a quotient of a subspace of an L_p-space.

Theorem 7.5. *Suppose X and Y are Banach spaces of type p where $1 \leqslant p \leqslant 2$. Suppose that either X or Y is of exact type p. Then any twisted sum $Z = Y \oplus_F X$ satisfies*

$$\Big\{\int_0^1 \Big\|\sum_{i=1}^n \varepsilon_i(t)z_i\Big\|^p dt\Big\}^{1/p} \leqslant C\Big\{\sum_{k=1}^n \|z_k\|^p + \sum \|z_k\|^p(\log k)^p\Big\}^{1/p}$$

and hence is of logtype p.

Proof. If X is of exact type p, then

$$\Big\{\int \Big\|\sum \varepsilon_i(t)x_i\Big\|^p dt\Big\}^{1/p} \leqslant \Big(\sum \|x_i\|^p\Big)^{1/p}$$

for $x_1, \ldots, x_n$ so that $\delta_n(x) \leqslant n^{1/p-1/2}$. Hence Lemma 7.1 implies that

$$\gamma_n \leqslant Bn^{1/p-1/2}(\log n+1).$$

By Lemma 5.3 there exists $a > 0$ and $M < \infty$ such that

$$\|\Delta(x_1, \ldots, x_n)\| \leqslant M\Big(\sum_{k=1}^n k^a\|x_k\|^p\Big)^{1/p}$$

for $x_1, \ldots, x_n \in X$.

Now suppose $\|x_1\|^p+\ldots+\|x_n\|^p = 1$ and $\|x_1\| \geqslant \ldots \geqslant \|x_n\| > 0$. Let N_k be the greatest suffix such that $\|x_{N_k}\| > 2^{-k}$ and let

$$u_k(t) = \sum_{N_{k-1}+1}^{N_k} \varepsilon_i(t)x_i, \qquad k = 1, 2, \ldots$$

Suppose $N_l = n$. Then

$$\Delta(\varepsilon_1x_1, \ldots, \varepsilon_nx_n) = \Delta(u_1(t), \ldots, u_l(t)) + \sum_{k=1}^{N_k} \Delta(\varepsilon_{N_{k-1}+1}x_{N_{k-1}+1}, \ldots, \varepsilon_{N_k}x_{N_k}).$$

Now let

$$a = \Big\{\int_0^1 \Big\|\sum_{k=1}^l \Delta(\varepsilon_{N_{k-1}+1}x_{N_{k-1}+1}, \ldots, \varepsilon_{N_k}x_{N_k})\Big\|^p dt\Big\}^{1/p}$$
$$\leqslant B_2\Big\{\int_0^1 \sum_{k=1}^l \|\Delta(\varepsilon_{N_{k-1}+1}x_{N_{k-1}+1}, \ldots, \varepsilon_{N_k}x_{N_k})\|^p dt\Big\}^{1/p}$$

by the symmetrization argument, where B_2 is the type p constant of Y. Hence

$$a \leqslant BB_2\Big\{\sum_{k=1}^l (N_k-N_{k-1})^{1-p/2}[\log(N_k-N_{k-1})+1]^p\Big(\sum_{i=N_{k-1}+1}^{N_k} \|x_i\|^2\Big)^{p/2}\Big\}^{1/p}$$
$$\leqslant BB_2\Big\{\sum_{k=1}^l [\log(N_k-N_{k-1})+1]^p \sum_{i=N_{k-1}+1}^{N_k} \|x_i\|^p\Big\}^{1/p}.$$

Now observe

$$\sum_{m+1}^{n}(1+\log k)^p \geqslant \int_m^n (1+\log x)^p\,dx$$
$$= (n-m)(1+\log n)^p - p\int_m^n \frac{x-m}{x}(1+\log x)^{p-1}dx$$
$$\geqslant (n-m)(1+\log n)^p - p(n-m)(1+\log n)^{p-1}$$
$$\geqslant \tfrac{1}{2}(n-m)(1+\log n)^p$$

provided $n \geqslant n_0$ where n_0 depends on p.

Hence there is a constant c such that $c > 0$ and

$$\sum_{k=m+1}^{n}(1+\log k)^p \geqslant c(n-m)(1+\log n)^r$$

for all n, m. Thus we have

$$\sum_{N_{k-1}+1}^{N_k}(1+\log j)^p\|x_j\|^p \geqslant 2^{-kp}\sum_{N_{k-1}+1}^{N_k}(1+\log j)^p$$
$$\geqslant c2^{-kp}(N_k - N_{k-1})(1+\log N_k)^p$$
$$\geqslant c2^{-p}(1+\log N_k)^p \sum_{N_{k-1}+1}^{N_k}\|x_j\|^p.$$

Thus

$$a \leqslant 2c^{-1/p}BB_2\Big\{\sum_{k=1}^{n}\|x_k\|^p(1+\log k)^p\Big\}^{1/p}.$$

Now we shall show that if

$$b = \Big\{\int_0^1 \|\Delta(u_1(t), \ldots, u_l(t))\|^p dt\Big\}^{1/p},$$

then

$$b \leqslant D\Big\{\sum_{k=1}^{n}\|x_k\|^p(1+\log k)^p\Big\}^{1/p}$$

for some D independent of $x_1, \ldots, x_n$. We have

(7.5.1) $$b \leqslant M\Big\{\int_0^1 \sum_{k=1}^{l} k^a\|u_k(t)\|^p dt\Big\}^{1/p} \leqslant MB_3\Big\{\sum_{k=1}^{l} k^a \sum_{N_{k-1}+1}^{N_k}\|x_i\|^p\Big\}^{1/p}$$

where B_3 is the type p constant of X. Hence

(7.5.2) $$b \leqslant M_1\Big\{\sum_{i=1}^{n}\|x_i\|^p\Big(\log\frac{2}{\|x_i\|}\Big)^a\Big\}^{1/p}$$

(for some constant M_1)

$$\leqslant M_2\Big(\sum_{i=1}^{n}\|x_i\|^q\Big)^{1/p}$$

where $\tfrac{1}{2}p < q < p$. Let $\theta = q/(p-q) > 1$. Then

$$b \leqslant M_2\Big(\sum_{k=1}^{n}k^{q/p}\|x_k\|^q k^{-q/p}\Big)^{1/p}$$
$$\leqslant M_2\Big(\sum_{k=1}^{n}k\|x_k\|^p\Big)^{q/p^2}\Big(\sum_{k=1}^{n}k^{-\theta}\Big)^{1/p-q/p^2} \leqslant M_3\Big(\sum_{k=1}^{n}k\|x_k\|^p\Big)^{1/p}.$$

Combining

$$a+b \leqslant M_4\Big(\sum_{k=1}^{n}k\|x_k\|^p\Big)^{1/p}$$

so that

$$\Big\{\int\|\Delta(\varepsilon_1x_1, \ldots, \varepsilon_nx_n)\|^p dt\Big\}^{1/p} \leqslant M_4\Big(\sum_{k=1}^{n}k\|x_k\|^p\Big)^{1/p}$$

and this must hold for *any* $x_1, \ldots, x_n \in X$. Returning to (7.5.1) it is clear that (after a symmetrization argument) we may now take $a = 1$, and in (7.5.2)

$$b \leqslant M_5\Big\{\sum_{i=1}^{n}\|x_i\|^p\log\frac{2}{\|x_i\|}\Big\}^{1/p}.$$

By Lemma 3.5

$$b \leqslant M_6\Big\{\sum_{k=1}^{n}\|x_k\|^p(1+\log k)\Big\}^{1/p}$$

and combining we now have the estimate

$$\Big\{\int_0^1\|\Delta(\varepsilon_1x_1, \ldots, \varepsilon_nx_n)\|^p dt\Big\}^{1/p} \leqslant C\Big\{\sum_{k=1}^{n}\|x_k\|^p(1+\log k)^p\Big\}^{1/p}.$$

We omit the verification this implies the desired property of $Z = Y \oplus_F X$.

Remark. We observe that Theorem 7.5 is best possible, in the sense that each p, $1 \leqslant p \leqslant 2$, there is a twisted sum Z_p of l_p and itself and a constant $c > 0$ such that if $\xi_1^p + \ldots + \xi_n^p = 1$ there are $z_i \in Z_p$, $i = 1, 2, \ldots, n$ with $\|z_i\| = \xi_i$ and

$$\Big\{\int_0^1\Big\|\sum_{i=1}^{n}\varepsilon_iz_i\Big\|^p dt\Big\}^{1/p} \geqslant c\Big(1+\sum\|z_i\|^p\Big(\log\frac{1}{\|z_i\|}\Big)^p\Big)^{1/p}.$$

Indeed, consider the spaces Z_p of [9]. Then $Z_p = l_p \oplus_F l_p$ where $F\colon l_p^0 \to l_p^0$ is defined by

$$F(x) = \left(x_n \log \frac{\|x\|}{|x_n|}\right).$$

If e_n is the nth basis vector of l_p, then

$$\begin{aligned}\left\|\left(0, \sum \pm \xi_n e_n\right)\right\| &= \left\|F\left(\sum \pm \xi_n e_n\right)\right\| + \left\|\sum \pm \xi_n e_n\right\| \\ &= \left(\sum \xi_n^p \left(\log \frac{1}{\xi_n}\right)^p\right)^{1/p} + \left(\sum \xi_n^p\right)^{1/p} \\ &\geqslant \left(1 + \sum \xi_n^p \left(\log\left(\frac{1}{\xi_n}\right)\right)^p\right)^{1/p}.\end{aligned}$$

Similarly this implies

$$\left(1 + \sum \xi_n^p \left(\log\left(\frac{1}{\xi_n}\right)\right)^p\right)^{1/p} \leqslant C' \left(\sum \xi_n^p (1+\log n)^p\right)^{1/p}$$

so that (7.3.1) and (7.3.2) are equivalent.

8. Twisted sums of l_1 and R. We now recall two ways of forming a twisted sum $\boldsymbol{R} \oplus_F l_1$. One method due to the author is by defining $F_1\colon l_1^0 \to \boldsymbol{R}$ by

$$F_1(x) = \sum_{n=1}^{\infty} \tilde{x}_n \log n, \qquad x \geqslant 0,$$

where $(\tilde{x}_n)$ is the decreasing rearrangement of x, and

$$F_1(x) = F_1(x^+) - F_1(x^-)$$

where $x = x^+ - x^-$ where $x^+ \geqslant 0$, $x^- \geqslant 0$ and $|x^+| \wedge |x^-| = 0$; see [6]. The other functional due to Ribe [15] is $F_2\colon l_1^0 \to \boldsymbol{R}$ given by

$$F_2(x) = \sum_{n=1}^{\infty} x_n \log \frac{\|x\|}{|x_n|}.$$

(Actually Ribe uses the equivalent functional

$$\tilde{F}_2(x) = \sum_{n=1}^{\infty} x_n \log \frac{1}{x_n} + \left(\sum_{n=1}^{\infty} x_n\right) \log\left(\sum_{n=1}^{\infty} x_n\right).)$$

$\| \ \|_1$ and $\| \ \|_2$ denote the induced norms on $\boldsymbol{R} \oplus l_1$. Then if e_n is the nth basis vector of l_1 and $t_n \geqslant 0$ $(n = 1, 2, \ldots, N)$

$$\left\|\left(0, \sum_{i=1}^{N} t_n e_n\right)\right\|_1 = \sum_{n=1}^{N} |t_n| + \sum_{n=1}^{N} \tilde{t}_n \log n$$

and

$$\left\|\left(0, \sum_{n=1}^{N} t_n e_n\right)\right\|_2 = \sum_{n=1}^{N} (t_n) + \sum_{n=1}^{\infty} t_n \log \frac{(\sum t_k)}{t_n}.$$

Thus we see for both F_1 and F_2 we have examples to show

THEOREM 8.1. *There is a twisted sum of $\boldsymbol{R}$ and l_1 where galb is l_φ.*

In view of this we remark that these two twisted sums are not projectively equivalent ([9]) in the sense that there is no isomorphism $S\colon \boldsymbol{R} \oplus_{F_1} l_1 \to \boldsymbol{R} \oplus_{F_2} l_1$ such that the diagram

$$\begin{array}{ccc} \boldsymbol{R} \oplus_{F_1} l_1 & \xrightarrow{S} & \boldsymbol{R} \oplus_{F_2} l_1 \\ \downarrow & & \downarrow \\ l_1 & \xrightarrow{\alpha} & l_1 \end{array}$$

commutes where $\alpha \neq 0$. For (see [9]), projective equivalence implies the existence of $a \neq 0$, $N < \infty$ and a linear map $t\colon l_1^0 \to \boldsymbol{R}$ so that

$$|F_1(ax) - F_2(x) - t(x)| \leqslant N\|x\|, \qquad x \in l_1^0.$$

Since $F_1(e_n) = F_2(e_n) = 0$ for all n, this would imply $t(e_n)$ bounded so that t is continuous and

$$|F_1(ax) - F_2(x)| \leqslant (N + \|t\|)\,\|x\|, \qquad x \in l_1^0.$$

Now by Lemma 3.5 we see the only possible value of a is $a = 1$. Thus

$$|F_1(x) - F_2(x)| \leqslant (N + \|t\|)\,\|x\|.$$

Now let $x_N = \sum_{n=1}^{N} \frac{1}{n} e_n$.

$$F_1(x_N) = \sum_{n=1}^{N} \frac{\log n}{n},$$

$$F_2(x_N) = \sum_{n=1}^{N} \frac{1}{n} (\log n + \log S_N)$$

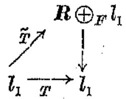
where $S_N = \sum_{n=1}^{N} \frac{1}{n}$. Hence

$$F_2(x_N) - F_1(x_N) = S_N \log S_N$$

while $\|x_N\| = S_N \to \infty$ and so we have a contradiction.

To conclude this short section we consider the following:

DEFINITION 8.2. An operator $\tilde{T}$: $l_1 \to l_1$ is *liftable* if for every twisted sum $\boldsymbol{R} \oplus_F l_1$ there is a map $\tilde{T}$: $l_1 \to \boldsymbol{R} \oplus_F l_1$ such that the diagram



commutes.

THEOREM 8.3. *Let* T: $l_1 \to l_1$ *be given by* $Te_n = d_n e_n$ *where* $\|T\| = \sup |d_b| < \infty$. *Then the following are equivalent:*

(a) T *is liftable.*

(b) *For some* $\tau < \infty$, $\sum_{n=1}^{\infty} \exp(-\tau/|d_n|) < \infty$.

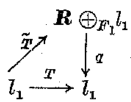
(c) $[d_n] \in c_0$ *and if* d_n^* *is the decreasing re-arrangement of* (d_n), *then* $\{d_n^* \log n\}$ *is bounded.*

(d) $T(l_1) \subset l_\varphi$.

(e) *For any logconvex space* X *and quotient map* q: $X \to l_1$ *there is an operator* S: $l_1 \to X$ *such that* $qS = T$.

Proof. It clearly suffices to consider the case $d_n \geqslant 0$. Note first that if $d_n \nrightarrow 0$, then there is a projection P onto a subspace isomorphic to l_1 such that $P = ST$ for some bounded S. Then T is liftable so is P and this clearly contradicts the fact that $(\boldsymbol{R}, l_1)$ does not split. Here we may assume $d_n \to 0$ and then we may suppose $\{d_n\}$ decreasing.

(a) ⇒ (c). Consider



Suppose $\tilde{T}e_n = (c_n, d_n e_n)$. Then

$$\|\tilde{T}e_n\|_1 = |c_n| + |d_n| \leqslant \|\tilde{T}\|$$

and

$$\|\tilde{T}(e_1 + \ldots + e_n)\|_1 = \sum_{k=1}^{n} d_k + \Big|\sum_{k=1}^{n} c_k - \sum_{i=1}^{n} d_k \log k\Big|$$
$$\geqslant \sum_{k=1}^{n} d_k + \sum_{k=1}^{n} d_k \log k - \sum_{k=1}^{n} c_k$$
$$\geqslant n d_n \log n - n\|\tilde{T}\|.$$

Hence

$$d_n \log n \leqslant 2\|\tilde{T}\|.$$

(c) ⇒ (b). If $d_n \leqslant b(\log n)^{-1}$, $n \geqslant 2$, then if $\tau > b$

$$\exp\Big(-\frac{\tau}{d_n}\Big) \leqslant n^{-(\tau/b)}.$$

(b) ⇒ (d). Suppose $|t_1| + \ldots + |t_n| \leqslant 1$. Then

$$\|T(t_1e_1 + \ldots + t_ne_n)\|_\varphi^* = \sum_{i=1}^{n} d_i|t_i| + \sum_{i=1}^{n} d_i|t_i| \log \frac{\sum_{j=1}^{n} d_i|t_j|}{d_i|t_i|}$$
$$= S + S\log S - \sum_{i=1}^{n} d_i|t_i| \log d_i -$$
$$- \sum_{i=1}^{n} d_i|t_i| \log |t_i|$$

where $S = \sum_{i=1}^{n} d_i|t_i| \leqslant \|T\|$. Also $-d_i \log d_i \leqslant e^{-1}$. Hence

$$\|T(t_1e_1 + \ldots + t_ne_n)\|_\varphi^* \leqslant \|T\| + \|T\| \log \|T\| + e^{-1} + \sum_{i=1}^{n} d_i|t_i| \log \frac{1}{|t_i|}.$$

Now suppose $\xi_1, \ldots, \xi_n \geqslant 0$ are chosen to maximize $\psi(\xi_1, \ldots, \xi_n) = \sum_{i=1}^{n} d_i \xi_i \log \frac{1}{\xi_i}$ subject to $\xi_1 + \ldots + \xi_n = 1$.

Then there is a Lagrange multiplier λ such that if $\xi_i \neq 0$,

$$d_i \log \frac{1}{\xi_i} - d_i = \lambda,$$

i.e.,

$$\log \frac{1}{\xi_i} = 1 + \frac{\lambda}{d_i}$$

so that

$$\xi_i = e^{-(1+\lambda/d_i)}.$$

Let $A = \{i:\ \xi_i > 0\}$. Then

$$\sum_{i \in A} e^{-(1+\lambda/d_i)} = 1,$$

$$\psi(\xi_1, \ldots, \xi_n) = \sum_{i \in A} d_i e^{-(1+\lambda/d_i)}(1 + \lambda/d_i) \leqslant \|T\| + \lambda.$$

Now

$$\sum_{i \in A} e^{-\lambda/d_i} = e.$$

Hence

$$\sum_{i=1}^{\infty} e^{-\lambda/d_i} \geqslant e.$$

Since for some $\tau < \infty$

$$\sum e^{-\lambda/d_i} < \infty,$$

there exists λ_0 such that $\lambda > \lambda_0$ implies

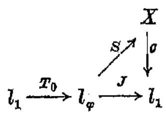
$$\sum_{i=1}^{\infty} e^{-\lambda/d_i} < e.$$

Thus $\gamma \leqslant \lambda_0$ and so

$$\|T(t_1e_1 + \ldots + t_ne_n)\|_\varphi^* \leqslant \|T\|(2+\log\|T\|) + e^{-1} + \lambda_0.$$

Hence T maps l_1 into l_φ.

(d) $\Rightarrow$ (e).



T factors $T = JT_0$ where $J: l_\varphi \to l_1$ is the inclusion map. The existence of a lift S follows from the fact that X is logconvex.

(e) $\Rightarrow$ (a). Theorem 7.1.

9. Orlicz sequence spaces. We recall that an F-space X is a $\mathscr{K}$-space if $(\boldsymbol{R}, X)$ splits ([6], [8]). In this section we classify completely those locally banded Orlicz sequence spaces $l_f \subset l_1$ which are $\mathscr{K}$-spaces. It is known ([6]) that l_p is a $\mathscr{K}$-space if $0 < p < 1$ and fails to be a $\mathscr{K}$-space if $p = 1$.

We shall suppose throughout that f is a twice-differentiable strictly increasing Orlicz function with $f(1) = 1$ such that $xf(x)$ is convex (cf. [5]); these assumptions may be made without loss of generality. We also suppose that f satisfies the Δ_2-condition, i.e. for some K

$$f(2x) \leqslant Kf(x), \quad 0 \leqslant x < \infty.$$

We define

$$\alpha_f = \sup\{p: \exists M,\ f(ax) \leqslant Ma^pf(x),\ 0 < a, x < 1\},$$

$$\beta_f = \inf\{p: \exists M,\ f(ax) \geqslant Ma^pf(x),\ 0 < a, x < 1\}.$$

Since l_f is locally bounded, $\alpha_f > 0$, and the Δ_2-condition implies $\beta_f < \infty$.

Since $l_f \subset l_1$, we shall suppose

$$f(x) \leqslant Mx, \quad 0 \leqslant x \leqslant 1,$$

for some M.

Now let $h: \boldsymbol{R} \to \boldsymbol{R}$ be defined by

$$h(x) = x\int_x^1 \frac{f(t)}{t^2}\,dt, \quad x > 0,$$

$$h(0) = 0,$$

$$h(x) = -h(-x), \quad x < 0.$$

LEMMA 9.1. *h has the following properties:*

(i) *h is continuous, and twice differentiable for $x \neq 0$.*

(ii) $h''(u) \leqslant 0$, $u \neq 0$.

(iii) *If $u+v+w = 0$,*

(9.1.1) $$h(u)+h(v)+h(w) \leqslant 2\big(f(|u|)+f(|v|)+f(|w|)\big).$$

(iv) *If $0 \leqslant a \leqslant 1$ and $x \in \boldsymbol{R}$,*

(9.1.2) $$|h(ax) - ah(x)| \leqslant f(x).$$

Proof. (i) For continuity at 0, observe if $0 < x < 1$

$$h(x) \leqslant x\int_x^1 \frac{M}{t}\,dt \leqslant Mx\log\frac{1}{x}.$$

The other assertion is clear.

(ii) $$h'(u) = \int_a^1 \frac{f(t)}{t^2}\,dt - \frac{f(u)}{u} \quad u > 0,$$

$$h''(u) = -\frac{f'(u)}{u} \leqslant 0.$$

(iii) Suppose without loss of generality $u > 0$, $v > 0$ and $w = -(u+v)$. Since $h'' \leqslant 0$,

$$h(u+v) \leqslant h(u)+h(v) \leqslant 2h\big(\tfrac{1}{2}(u+v)\big),$$

so that

$$0 \leqslant h(u)+h(v)+h(w) \leqslant 2h\big(\tfrac{1}{2}(u+v)\big) - h(u+v),$$

$$2h\big(\tfrac{1}{2}(u+v)\big) - h(u+v) = (u+v)\int_{\frac{1}{2}(u+v)}^{u+v} \frac{f(x)}{x^2}\,dx \leqslant 2f(u+v).$$

Hence

$$h(u)+h(v)+h(w) \leqslant 2f(|w|) \leqslant 2\big(f(|u|)+f(|v|)+f(|w|)\big).$$

(iv) For $x > 0$.

$$h(ax) - ah(x) = ax \int_{ax}^{x} \frac{f(t)}{t^2}\,dt \leqslant axf(x) \int_{ax}^{x} \frac{1}{t^2}\,dt = f(x)(1-a) \leqslant f(x).$$

LEMMA 9.2. *Suppose for some $B < \infty$ we have for $0 \leqslant x \leqslant 1$*

$$h(x) \leqslant Bf(x).$$

Then $\beta_f < 1$.

Proof. Let $C_f \subset C[0,1]$ be defined by

$$C_f = \overline{\text{co}}\{f_t\colon 0 < t \leqslant 1\}$$

where

$$f_t(x) = \frac{f(tx)}{f(t)}$$

(cf. [5], [10]). Since $l_f \subset l_1$; $a_f \leqslant 1$. If $\beta_f \geqslant 1$, then $x \in C_f$ ([10]).

Now

$$\int_x^1 \frac{f(t)}{t^2}\,dt \leqslant \frac{f(x)}{x}.$$

and if $0 < s \leqslant 1$

$$\int_x^1 \frac{f_s(t)}{t^2}\,dt = \int_x^1 \frac{f(st)}{t^2 f(s)}\,dt = \int_{sx}^{f} \frac{sf(u)}{u^2 f(s)}\,du \leqslant \frac{s}{f(s)} \int_{sx}^{1} \frac{f(u)}{u^2}\,du$$

$$\leqslant B\,\frac{s}{f(sx)}\,\frac{f(sx)}{sx} = \frac{Bf_s(x)}{x}, \qquad 0 < x \leqslant 1.$$

Hence, if $g \in C_f$

$$\int_x^1 \frac{g(t)}{t^2}\,dt \leqslant B\,\frac{g(x)}{x}, \quad 0 < x \leqslant 1.$$

In particular if $\beta_f \geqslant 1$, we may let $g(t) = t$

$$\int_x^1 \frac{1}{t}\,dt \leqslant B, \qquad 0 < x \leqslant 1$$

and this contradiction shows $\beta_f < 1$.

THEOREM 9.3. *Suppose f is a Orlicz function satisfying the Δ_2-condition and that l_f is locally bounded and contained in l_1. Then l_f is a $\mathscr{K}$-space if and only if $\beta_f < 1$.*

Proof. If $\beta_f < 1$, l_f is a $\mathscr{K}$-space [6]. Conversely, suppose l_f is a $\mathscr{K}$-space. We define

$$H\colon l_f^0 \to \boldsymbol{R}$$

(where l_f^0 is the finitely non-zero sequences in l_f) by

$$H(x) = \sum_{i=1}^{\infty} h(x_i) \quad \text{if} \quad \sum_{i=1}^{\infty} f(|x_i|) = 1$$

and extend so that

$$H(ax) = aH(x), \qquad a \in \boldsymbol{R}.$$

We first assert that $H\colon l_f^0 \to \boldsymbol{R}$ is quasilinear. To see this we show that if $u, v, w \in l_f^0$

$$u+v+w = 0$$

and

$$\|u\|+\|v\|+\|w\| \leqslant 1,$$

then

$$|H(u)+H(v)+H(w)| \leqslant 9.$$

Indeed,

$$\Big|H(u) - \sum_{i=1}^{\infty} h(u_i)\Big| \leqslant 1$$

and similarly for v, w while

$$\Big|\sum_{i=1}^{\infty} h(u_i)+h(v_i)+h(w_i)\Big| \leqslant 6.$$

Now l_f is a $\mathscr{K}$-space so that there a linear map $\psi\colon l_f^0 \to \boldsymbol{R}$ such that

$$\sup_{\|x\|\leqslant 1} |H(x) - \psi(x)| < \infty.$$

Thus, $\{\psi(e_n)\colon n = 1, 2, \ldots\}$ is bounded since $H(e_n) = 0$ and so ψ is continuous. Hence,

$$|H(x)| \leqslant L\|x\|, \qquad x \in l_f^0,$$

for some $L < \infty$.

Suppose $0 < \xi \leqslant 1$; choose n so that $n \in \boldsymbol{N}$ and $\frac{1}{2} < n\xi \leqslant 1$. Choose η so that $f(\eta) = 1 - n\xi$ and let

$$x = \xi(e_1 + \ldots + e_n) + \eta e_{n+1}.$$

Then $\|x\| = 1$ and

$$H(x) \geqslant n\xi \int_{\xi}^{1} \frac{f(t)}{t^2}\,dt \geqslant \frac{1}{2}\,\frac{\xi}{f(\xi)} \int_{\xi}^{1} \frac{f(t)}{t^2}\,dt.$$

Hence,

$$\int_{\xi}^{1} \frac{f(t)}{\xi}\,dt \leqslant 2L\,\frac{f(\xi)}{\xi}, \qquad 0<\xi\leqslant 1,$$

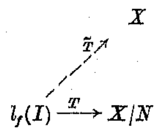
and so by Lemma 9.2, $\beta_f<1$.

Suppose now f is submultiplicative at 0. Then we say X is *f-convex* if $\sum f(\|x_i\|)<\infty$ implies $\sum x_i$ converges, i.e., X is galbed by l_f.

COROLLARY 9.4. *Suppose f is submultiplicative at 0; then every twisted sum of $\boldsymbol{R}$ and a f-convex space is also f-convex if and only if*

$$\beta_f=\lim_{x\to 0}\frac{\log f(x)}{\log x}<1.$$

Proof. Observe that $l_f(I)$ (for an index set I) is projective among f-convex, i.e.



Hence if every twisted sum of $\boldsymbol{R}$ and l_f is f-convex, then l_f is a $\mathscr{K}$-space. Conversely, suppose l_f is a $\mathscr{K}$-space and X is any f-convex space and $Y=\boldsymbol{R}\oplus_F X$ is a twisted sum of $\boldsymbol{R}$ and X. Then there is a quotient map $T\colon l_f(I)\to X$ for some index set I. Now $l_f(I)$ is also a $\mathscr{K}$-space (this is easy to show) and so there is a lift $\tilde{T}\colon l_f(I)\to Y$. If $\tilde{T}$ fails to be surjective Y splits, while if $\tilde{T}$ is surjective, Y is f-convex.

Remark. In this case $\beta_f<1$ implies $f(x)\geqslant cx^p$ for some $p<1$, and $c>0$ for all $0\leqslant x\leqslant 1$.

10. Locally convex $\mathscr{K}$-spaces. Let X be a metrizable locally convex space. Let $\|\cdot\|_n$ be a sequence of semi-norms on X which define the topology of X and such that

$$\|x\|_n\leqslant\|x\|_{n+1}, \qquad n\in\boldsymbol{N}.$$

Define a map $F\colon X\to\boldsymbol{R}$ be quasilinear if for some $n\in\boldsymbol{N}$, $K<\infty$

$$F(tx)=tF(x), \qquad t\in\boldsymbol{R},\ x\in X,$$

$$|F(x+y)-F(x)-F(y)|\leqslant K(\|x\|_n+\|y\|_n)$$

if F is quasilinear we define $\boldsymbol{R}\oplus_F X$ to be the space $\boldsymbol{R}\oplus X$ equipped with the quasi-semi-norms

$$\|(t,x)\|_m^*=|t-F(x)|+\|x\|_m, \qquad m\geqslant n.$$

Then if $q\colon \boldsymbol{R}\oplus_F X\to X$ is given by

$$q(t,x)=x,$$

q is a quotient map and $q^{-1}(0)=\{(t,0),\ t\in\boldsymbol{R}\}$. Thus we have a short exact sequence $0\to\boldsymbol{R}\to\boldsymbol{R}\oplus_F X\to X\to 0$ and $\boldsymbol{R}\oplus_F X$ is a twisted sum of $\boldsymbol{R}$ and X. It is easy to show that if X is complete, then so is $\boldsymbol{R}\oplus_F X$ (since it is such a twisted sum).

The twisted sum $\boldsymbol{R}\oplus_F X$ will split if and only if there is a linear map $\psi\colon X\to\boldsymbol{R}$ such that

$$|F(x)-\psi(x)|\leqslant M\|x\|_m, \qquad x\in X,$$

for some $m\in\boldsymbol{N}$ and $M<\infty$.

THEOREM 10.1. *Let X be a Fréchet space (complete metrizable locally convex space) and suppose an F-space Y is a twisted sum of $\boldsymbol{R}$ and X. Then there exists a quasilinear map $F\colon X\to\boldsymbol{R}$ such that Y is isomorphic to $\boldsymbol{R}\oplus_F X$ (as a twisted sum).*

Proof. It is convenient to write $Y=\boldsymbol{R}\oplus X$ algebraically so that the quotient map $q\colon Y\to X$ is given by $q(t,x)=x$.

Let $\{V_n\colon n\in\boldsymbol{N}\}$ be a base of balanced neighborhoods of 0 such that $V_{n+1}+V_{n+1}\subset V_n$ for all n and $V_1\cap\boldsymbol{R}e$ is bounded where $e=(1,0)$. Let $!\cdot!_n$ be the Minkowski functional of V_n. Then we have

(10.1.1) $$!(t,x)!_n\leqslant !(t,x)!_{n+1}, \qquad t\in\boldsymbol{R},\ x\in X,$$

(10.1.2) $$!(s+t,x+y)!_n\leqslant !(s,x)!_{n+1}+!(t,x)!_{n+1},$$

(10.1.3) $$!(1,0)!_n=\alpha_n \quad\text{where}\quad \alpha_1>0 \text{ and } \alpha_n\uparrow\infty.$$

Also since q is open, there exist increasing sequences $\{m(n)\}$ and $\{\beta_n\}$ such that

(10.1.4) For $x\in X$ there exists $t_n\in\boldsymbol{R}$

$$!(t_n,x)!_n\leqslant\beta_n\|x\|_{m(n)}.$$

In view of (10.1.4) there is a map $F_n\colon X\to\boldsymbol{R}$ such that $F(tx)=tF(x)$, $t\in\boldsymbol{R}$ and

$$!(F_n(x),x)!_n\leqslant\beta_n\|x\|_{m(n)}.$$

Now if $n>p>1$,

$$|F_n(x)-F_p(x)|=\alpha_{p-1}^{-1}!(F_n(x)-F_p(x),0)!_{p-1}$$
$$\leqslant\alpha_{p-1}^{-1}\big(!(F_n(x),x)!_p+!(F_p(x),x)!_p\big)$$

(10.1.5) $$\leqslant\alpha_{p-1}^{-1}(\beta_n+\beta_p)\|x\|_{m(n)}.$$

Also if $n \geqslant 3$ and $x, y \in X$

(10.1.6) $$!(F_n(x)+F_n(y),\ x+y)!_{n-1} \leqslant \beta_n(\|x\|_{m(n)}+\|y\|_{m(n)}).$$

(10.1.7) $$!(F_{n-1}(x+y),\ x+y)!_{n-1} \leqslant \beta_{n-1}(\|x\|_{m(n)}+\|y\|_{m(n)}).$$

Hence combining (10.1.6) and (10.1.7) with (10.1.2)

$$\alpha_{n-2}|F_n(x)+F_n(y)-F_n(x+y)| \leqslant (\beta_n+\beta_{n-1})(\|x\|_{m(n)}+\|y\|_{m(n)}).$$

Thus

$$|F_n(x)+F_n(y)-F_n(x+y)| \leqslant C_n(\|x\|_{m(n)}+\|y\|_{m(n)}).$$

In particular $F = F_3$ in quasilinear, and for $n \geqslant 3$

$$|F_n(x)-F(x)| \leqslant D_n\|x\|_{m(n)}.$$

Thus

$$\begin{aligned} !(u, x)!_n &\leqslant \alpha_{n+1}|u-F_n(x)|+\beta_{n+1}\|x\|_{m(n+1)} \\ &\leqslant \alpha_{n+1}|u-F(x)|+(\alpha_{n+1}D_n+\beta_{n+1})\|x\|_{m(n+1)} \\ &\leqslant A_n(|u-F(x)|+\|x\|_{m(n+1)}). \end{aligned}$$

Hence the identity i: $\boldsymbol{R}\oplus_F X \to Y$ is continuous. By the closed graph theorem i is an isomorphism.

THEOREM 10.2. *Any nuclear Fréchet space is a $\mathscr{K}$-space.*

Proof. If X is a nuclear Fréchet space and $F: X \to \boldsymbol{R}$ is quasilinear, then there is a Hilbertian semi-norm $\|\cdot\|_n$ on X such that

$$|F(x+y)-F(x)-F(y)| \leqslant \|x\|_n+\|y\|_n.$$

Since a Hilbert space is a $\mathscr{K}$-space ([6]), there is a linear map ψ: $X \to \boldsymbol{R}$ such that

$$|F(x)-\psi(x)| \leqslant M\|x\|_n.$$

Let (a_{mn}) be a matrix with non-negative entries such that $a_{m+1,n} \geqslant a_{m,n}$ for all n and for each n there exists n with $a_{mn} > 0$. Then the Köthe sequence space $l_1[a_{mn}]$ is the space of sequences (x_n) such that

$$\|x\|_m = \sum_{n=1}^{\infty}|a_{mn}|\,|x_n| < \infty, \qquad m = 1, 2, \ldots$$

THEOREM 10.3. *$l_1[a_{mn}]$ is a $\mathscr{K}$-space if and only if given $m \in \boldsymbol{N}$ there exists $\tau < \infty$ and $r > m$ with*

(10.3.1) $$\sum_{n=1}^{\infty}\exp\left(-\tau\frac{a_{rn}}{a_{mn}}\right) < \infty \qquad (0/0 = \infty).$$

Proof. If (10.3.1) fails, there exists m such that for all $r > m$ and $\tau < \infty$

(10.3.2) $$\sum_{n=1}^{\infty}\exp\left(-\tau\frac{a_{rn}}{a_{mn}}\right) = \infty.$$

Define F: $l_1^0[a_{mn}] \to \boldsymbol{R}$ by

$$F(x) = \sum_{n=1}^{\infty}a_{mn}x_n\log\frac{\|x\|_m}{a_{mn}|x_n|}$$

(for finitely non-zero sequences). If $\boldsymbol{R}\oplus_F l_1^0[a_{mn}]$ splits then there is a linear map ψ: $l_1^0[a_{mn}] \to \boldsymbol{R}$ such that

$$|F(x)-\psi(x)| \leqslant K\|x\|_r$$

for some $r > m$. This implies

$$\psi(e_n) \leqslant Ka_{rn}$$

and so

$$|\psi(x)| \leqslant K\|x\|_r$$

so that

$$|F(x)| \leqslant 2K\|x\|_r.$$

Hence

$$\sum_{n=1}^{\infty}a_{mn}|x_n|\log\frac{\|x\|_m}{a_{mn}|x_n|} \leqslant 2K\sum_{n=1}^{\infty}a_{rn}|x_n|.$$

This means the diagonal map $\{x_n\} \to \{d_n x_n\}$ maps l_1 into l_φ where

$$d_n = \frac{a_{mn}}{a_{rn}} \qquad (= 0 \text{ if } a_{rn} = 0).$$

Hence by Theorem 8.3 we have contradicted (10.3.2).

Conversely if (10.3.1) holds and F is quasilinear, then if

$$|F(x+y)-F(x)-F(y)| \leqslant K(\|x\|_m+\|y\|_m),$$

choose $r > m$ to satisfy (10.3.1). Let

$$d_n = \frac{a_{mn}}{a_{rn}}.$$

Then D: $\{x_n\} \to \{d_n x_n\}$ is liftable. If we define G: $l_1 \to \boldsymbol{R}$ by

$$G(x) = F(\{a_{mn}^{-1}x_n\}) \qquad (1/0 = 0!).$$

G is quasilinear on l_1 and hence there is a linear map $\tilde{D}$

$$\begin{array}{ccc} & & \boldsymbol{R}\oplus_a l_1 \\ & \overset{\tilde{D}}{\swarrow} & \downarrow \\ l_1 & \xrightarrow{D} & l_1 \end{array}$$

If $\tilde{D}x = (\psi(x),\ Dx)$, then

$$|\psi(x) - G(Dx)| \leqslant C\|x\|, \qquad x \in l_1.$$

Hence

$$|\psi\{a_{mn}x_n\} - F(x)| \leqslant C\|x\|_r, \qquad x \in l_1[a_{mn}]$$

so that $\boldsymbol{R}\oplus_F l_1[a_{mn}]$ splits.

Remark. Condition (10.3.1) is thus a topological invariant of $l_1[a_{mn}]$.

References

[1] P. Enflo, J. Lindenstrauss and G. Pisier, *On the three space problem*, Math. Scand. 36 (1975), 199–210.
[2] T. Figiel, J. Lindenstrauss and V. D. Milman, *The dimension of almost spherical sections of convex bodies*, Acta Math. 139 (1977), 53–94.
[3] P. Jordan and J. von Neumann, *On inner products in linear metric spaces*, Ann. Math. 36 (1935), 719–723.
[4] J. P. Kahane, *Some random series of functions*, Heath Mathematical Monographs, 1968.
[5] N. J. Kalton, *Orlicz sequences without local convexity*, Math. Proc. Cambridge Philos. Soc. 81 (1977), 253–277.
[6] — *The three space problem for locally bounded F-spaces*, Comp. Math. 37 (1978), 243–276.
[7] — *Operators on L_p, $0 < p < 1$*, Trans. Amer. Math. Soc. 259 (1980), 319–355.
[8] N. J. Kalton and N. T. Peck, *Quotients of $L_p(0,1)$, $0 \leqslant p < 1$*, Studia Math. 64 (1978).
[9] —, — *Twisted sums of sequence spaces and the three space problem*, Trans. Amer. Math. Soc. 255 (1979), 1–30.
[10] J. Lindenstrauss and L. Tzafriri, *On Orlicz sequence space III*, Israel J. Math. 14 (1973) 368–389.
[11] —, — *Classical Banach Spaces*, Vol. I *Sequence Spaces*, Springer, 1977.
[12] B. Maurey, *Théorèmes des factorization pour les opérateurs lineaires a valeurs dans les espaces L^p*, Asterisque 11 (1974), 1–163.
[13] B. Maurey and G. Pisier, *Series de variables aléatoires vectorielles independantes et propriétés geométriques des espaces de Banach*, Studia Math. 58 (1976), 45–90.
[14] E. M. Nikisin, *Resonance theorems and expansions in eigenfunctions of the Laplace operator*, Izv. Akad. Nauk. U.S.S.R. Math. 36 (1972), 795–813 (= Math. U.S.S.R. Izv. 6 (1972), 788–806).

[15] M. Ribe, *Examples for the nonlocally convex three-space problem*, Proc. Amer. Math. Soc. 79 (1979), 351–355.
[16] J. W. Roberts, *A non-locally convex F-space with the Hahn–Banach approximation property*, Proc. Kent State Conference, Springer Lecture Notes 604 (1977).
[17] S. Rolewicz, *Metric linear spaces*, Warsaw 1972.
[18] L. Schwartz, *Les applications 0-radonifiantes dans les espaces de suite*, Seminaire Laurent Schwartz Applications Radonifiantes 1964–1970, Exposé No. 26.
[19] P. Turpin, *Convexités dans les espaces vectoriels topologiques généraux*, Diss. Math. 131 (1976).
[20] E. M. Stein and N. J. Weiss, *On the convergence of Poisson integrals*, Trans. Amer. Math. Soc. 140 (1969), 35–54.

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