mathematics

Article

Convolution, Correlation and Uncertainty Principle in the One-Dimensional Quaternion Quadratic-Phase Fourier Transform Domain

Mohammad Younus Bhat [1,*], Aamir H. Dar [1], Mohra Zayed [2] and Altaf A. Bhat [3]

[1] Department of Mathematical Sciences, Islamic University of Science and Technology, Kashmir 192122, India; ahdkul740@gmail.com
[2] Mathematics Department, College of Science, King Khalid University, Abha 61413, Saudi Arabia; mzayed@kku.edu.sa
[3] University of Technology and Applied Sciences, Salalah 324, Oman; altaf.sal@cas.edu.om
* Correspondence: younis.bhat@islamicuniversity.edu.in

Abstract: In this paper, we present a novel integral transform known as the one-dimensional quaternion quadratic-phase Fourier transform (1D-QQPFT). We first define the one-dimensional quaternion quadratic-phase Fourier transform (1D-QQPFT) of integrable (and square integrable) functions on **R**. Later on, we show that 1D-QQPFT satisfies all the respective properties such as inversion formula, linearity, Moyal's formula, convolution theorem, correlation theorem and uncertainty principle. Moreover, we use the proposed transform to obtain an inversion formula for two-dimensional quaternion quadratic-phase Fourier transform. Finally, we highlight our paper with some possible applications.

Keywords: quadratic-phase Fourier transform; quaternion quadratic-phase Fourier transform; convolution; two-dimensional inversion formula

MSC: 42C40; 42C15; 47G10; 42A38

check for updates

Citation: Bhat, M.Y.; Dar, A.H.; Zayed, M.; Bhat, A.A. Convolution, Correlation and Uncertainty Principle in the One-Dimensional Quaternion Quadratic-Phase Fourier Transform Domain. *Mathematics* **2023**, *11*, 3002. https://doi.org/10.3390/math11133002

Academic Editor: Luigi Rodino

Received: 7 June 2023
Revised: 30 June 2023
Accepted: 2 July 2023
Published: 5 July 2023

Copyright: © 2023 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction

Recently, Castro et al. [1,2] introduced a superlative generalized version of the Fourier transform(FT) coined as quadratic-phase Fourier transform (QPFT) which has overthrown all the applicable signal processing tools as it provides a unified analysis of both transient and non-transient signals in an easy and insightful fashion. We refer to [3,4] for information about integral transforms.

For parameter $\mu = (A, B, C, D, E)$, the QPFT of any signal $f \in L^1(\mathbb{R})$ is defined as

$$\mathcal{Q}_\mu[f](w) = \frac{1}{\sqrt{2\pi}} \int f(t)\Lambda_\mu(t, w)dt, \tag{1}$$

where $\Lambda_\mu(t, w)$ is called quadratic-phase Fourier kernel and is given by

$$\Lambda_\mu(t, w) = \exp\{i(At^2 + Btw + Cw^2 + Dt + Ew)\} \tag{2}$$

with $A, B, C, D, E \in \mathbf{R}, B \neq 0$.

Later on, much research has been carried out on quadratic-phase Fourier transform (see [5–8]). In the prospect of signal processing, it is evident that any signal processing tool converts the time-domain signals into frequency-domain. Further, in signal processing, convolution of two functions [9–13] is a most useful tool in constructing a filter for denoising the given noisy signals (see [14]).

In past decades, hypercomplex algebra has become a leading area of research with its applications in color image processing, image filtering, watermarking, edge detection and pattern recognition (see [15–21]). The Cayley–Dickson algebra of order four is labeled as quaternions which has wide applications in optical and signal processing. The extension of the Fourier transform in quaternion algebra is known as quaternion Fourier transform (QFT) [22] which is believed to be the substitute for the commonly used two-dimensional complex Fourier transform (CFT). The QFT has a wide range of applications: see [23,24]. The quaternion linear canonical transform (QLCT) and the quaternion offset linear canonical transform (QOLCT) are the generalized versions of the QFT and QLCT, respectively, which are both effective signal processing tools. The modern era of information processing is in dire need of quaternionic valued signals and, therefore, is a very hot area of research (see [25–28]). Recently, Bhat and Dar [29] introduced quaternion quadratic-phase Fourier transform Q-QPFT as a generilization of QFT. Since then it has gained much popularity in signal processing community [30–33]. Many fundamental properties of Q-QPFT are known, but the theory about quaternion one-dimensional QPFT as far as we know is still in its infancy. Therefore, it is worthwhile to study the theory of quaternion one-dimensional QPFT which can be productive for signal processing theory and applications.

In this paper, our main objective is to introduce the novel integral transform called the one-dimensional quaternion quadratic-phase Fourier transform (1D-QQPFT) and study its properties, such as inversion formula, linearity, Moyal's formula, convolution theorem, correlation theorem, Heisenberg–Pauli–Weyl uncertainty inequality and inversion formula for two-dimensional quaternion QPFT.

This paper is organized as follows: In Section 2, some general definitions and basic properties of quaternions are summarized. The definition and the properties of the 1D-QQPFT are studied in Section 3. In Section 4 we obtain inversion formula for the two-dimensional quadratic-phase Fourier transform. In Section 5, some potential applications of the proposed transform are highlighted. In Section 6, conclusions are drawn.

2. Preliminaries

Here, we introduce the already known results which are required in subsequent sections.

2.1. Quadratic-Phase Fourier Transform

In this subsection we introduce the quadratic-phase Fourier transform, which is a neoteric addition to the classical integral transforms and we also give its inversion formula and some other classical results which are already present in literature.

Definition 1. *Given a parameter $\mu = (A, B, C, D, E)$, the QPFT of any signal $f \in L^1(\mathbb{R})$ is defined by*

$$\mathcal{Q}_\mu[f](w) = \frac{1}{\sqrt{2\pi}} \int f(t)\Lambda_\mu(t, w)dt, \tag{3}$$

where the quadratic-phase Fourier kernel $\Lambda_\mu(t, w)$ is given by

$$\Lambda_\mu(t, w) = \exp\{i(At^2 + Btw + Cw^2 + Dt + Ew)\} \tag{4}$$

with $A, B, C, D, E \in \mathbf{R}, B \neq 0$.

Theorem 1. *The inversion formula of the quadratic-phase Fourier transform is given by*

$$f(t) = \frac{|B|}{\sqrt{2\pi}} \int \mathbf{Q}_\mu[f](w)\overline{\Lambda_\mu(t, w)}dw. \tag{5}$$

Using the inversion theorem, we can obtain the Parseval's relation given by

$$\langle f, g\rangle = |B|\langle \mathbf{Q}_\mu[f], \mathbf{Q}_\mu[g]\rangle \tag{6}$$

Theorem 2 ([5,6]). *Let $f, g \in L^2(\mathbf{R})$ and $\alpha, \beta, \tau \in \mathbf{R}$ then*

1. $\mathbf{Q}_\mu[\alpha f + \beta g](w) = \alpha\mathbf{Q}_\mu[f](w) + \beta\mathbf{Q}_\mu[g](w)$.

2. $\mathbf{Q}_\mu[f(t-\tau)](w) = \exp\{-i(A\tau^2 + B\tau w + D\tau)\}\mathbf{Q}_\mu[\exp\{-2iA\tau t\}f(t)](w).$
3. $\mathbf{Q}_\mu[f(-t)](w) = \mathbf{Q}_{\mu'}[f(t)](-w),$ *where* $\mu' = (A, B, C, -D, -E).$
4. $\mathbf{Q}_\mu[\exp\{i\alpha t\}f(t)](w) = \exp\{i(\alpha^2 + 2\alpha Bw + \alpha EB)\frac{1}{B}\}\mathbf{Q}_\mu[f]\left(w + \frac{A}{B}\right).$
5. $\mathbf{Q}_\mu[\overline{f(t)}](w) = \overline{\mathbf{Q}_{-\mu}[f(t)](w)},$ *where* $-\mu = (-A, -B, -C, -D, -E).$

Theorem 3 (QPFT Convolution [5])**.** *If* $f, g \in L^2(\mathbf{R})$ *then*

$$\mathbf{Q}_\mu[f *_\mu g(t)](w) = \exp\{-i(Cw^2 + Ew)\}\mathbf{Q}_\mu[f](w)\mathbf{Q}_\mu[g](w)$$

where

$$(f *_\mu g)(t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} f(z)g(t-z)\exp\{2iAz(t-z)\}dz. \tag{7}$$

Theorem 4 (Heisenberg–Pauli–Weyl inequality [6])**.** *Let* $\mathbf{Q}_\mu[f]$ *be the quadratic-phase Fourier transform of any signal* f*, then the following inequality holds:*

$$\int_{\mathbf{R}} w^2|\mathbf{Q}_\mu[f]|^2 dw \int_{\mathbf{R}} t^2|f(t)|^2 dt \geq \left\{\frac{1}{2|B|}\int_{\mathbf{R}}|f(t)|^2 dt\right\}^2.$$

2.2. Quaternions

Let $\mathbf{R}$ and $\mathbf{C}$ be the usual set of real numbers and set of complex numbers, respectively. The division ring of quaternions in the honor of Hamilton, is denoted by $\mathbf{H}$ and is defined as

$$\begin{aligned}\mathbf{H} &= \{a + e_1 b + e_2 c + e_3 d : a, b, c, d \in \mathbf{R}\} \\ &= \{(a + e_1 b) + e_2(c - e_1 d) : a, b, c, d \in \mathbf{R}\} \\ &= \{z_1 + e_2 z_2 : z_1, z_2 \in \mathbf{C}\} \quad (CayleyDicksonform)\end{aligned}$$

where e_1, e_2, e_3 satisfy Hamilton's multiplication rule

$$e_1e_2 = -e_2e_1 = e_3, \quad e_2e_3 = -e_3e_2 = e_1,$$
$$e_3e_1 = -e_1e_3 = e_2 \quad \text{and} \quad e_1^2 = e_2^2 = e_3^2 = -1.$$

Every member of $\mathbf{H}$ is known as quaternion. In quaternion algebra addition, multiplication, conjugate and absolute value of quaternions are defined by

$$(a_1 + e_2a_2) + (b_1 + e_2b_2) = (a_1 + b_1) + e_2(a_2 + b_2),$$
$$(a_1 + e_2a_2)(b_1 + e_2b_2) = (a_1b_1 - \bar{a}_2b_2) + e_2(a_2b_1 + \bar{a}_1b_2),$$
$$(a_1 + e_2a_2)^c = \bar{a}_1 - e_2a_2,$$
$$|a_1 + e_2a_2| = \sqrt{|a_1|^2 + |a_2|^2},$$

here, $\bar{a}_k$ is the complex conjugate of a_k and $|a_k|$ is the modulus of the complex number $a_k, k = 1, 2$. For all $a = a_1 + e_2a_2, \quad b = b_1 + e_2b_2 \in \mathbf{H}$, the following properties of conjugate and modulus and multiplicative inverse are well known.

$$(a^c)^c = a, \quad (a+b)^c = a^c + b^c, \quad (ab)^c = b^c a^c,$$
$$|a|^2 = aa^c = |a_1|^2 + |a_2|^2, \quad |ab| = |a||b|,$$
$$a^{-1} = \frac{\bar{a}}{|a|^2}.$$

We denote $L^p(\mathbf{R}, \mathbf{H})$, the Banach space of all quaternion-valued functions f satisfying

$$\|f\|_p = \left(\int |f_1(t)|^p + |f_2(t)|^p dt \right)^{1/p} < \infty, \quad p = 1, 2.$$

Moreover, on $L^2(\mathbf{R}, \mathbf{H})$ the inner product

$$\langle f, g \rangle = \int f(t)[g(t)]^c dt,$$

where integral of a quaternion valued function is defined by

$$\int (f_1 + e_2 f_2)(t) dt = \int f_1(x) dt + e_2 \int f_2(x) dt,$$

whenever the integral exists.

3. Quaternion One-Dimensional Quadratic-Phase Fourier Transform

In this section, we will introduce the definition of quaternion one-dimensional quadratic-phase Fourier transform (1D-QQPFT) by using [34–37]. Prior to that we note e_1, e_2 and e_3 (or equivalently i, j, k) denote the three imaginary ,units in the quaternion algebra [38].

Definition 2. *The 1D-QQPFT of any signal $f \in L^1(\mathbf{R}, \mathbf{H})$ with respect a parameter $\mu = (A, B, C, D, E)$ is defined by*

$$\mathbf{Q}_\mu^{\mathbf{H}}[f(t)](w) = \frac{1}{\sqrt{2\pi}} \int f(t) \Lambda_\mu^{e_2}(t, w) dt \tag{8}$$

where

$$\Lambda_\mu^{e_2}(t, w) = \exp\{e_2(At^2 + Btw + Cw^2 + Dt + Ew)\}. \tag{9}$$

With $A, B, C, D, E \in \mathbf{R}, B \neq 0$. Now we can find that if $f(t)$ is a real-valued signal in (8), then we can interchange the kernel (2) and (9).

By appropriately choosing parameters in $\mu = (A, B, C, D, E)$ the 1D-QQPFT(8) gives birth to the following existing time-frequency transforms:

- For $\mu = (0, 1, 0, 0, 0)$, the 1D-QQPFT (2) boils down to the quaternion one-dimensional Fourier Transform [34]
- For $\mu = (A/2B, -1/B, C/2B, 0, 0)$ and multiplying the right side of (9) by $1/\sqrt{e_2 B}$ the 1D-QQPFT(2) reduces to the quaternion one-dimensional linear canonical transform [37].
- For $\mu = (\cot\theta/2, -\csc\theta, \cot\theta/2, 0, 0), \theta \neq 0$ and multiplying the right side of (9) by $\sqrt{1 - e_2 \cot\theta}$, the 1D-QQPFT (2) reduces to the quaternion one-dimensional fractional Fourier transform [35].

Example 1. *The 1D-QQPFT of the signal* $f(t) = \exp\{-kt^2\}$, $k \geq 0$ *is given by*

$$
\begin{aligned}
&\mathbf{Q}_\mu^{\mathbf{H}}[f(t)](w) \\
&= \frac{1}{\sqrt{2\pi}} \int f(t) \Lambda_\mu^{e_2}(t,w) dt \\
&= \frac{1}{\sqrt{2\pi}} \int \exp\{-kt^2\} \exp\{e_2(At^2 + Btw + Cw^2 + Dt + Ew)\} dt \\
&= \frac{1}{\sqrt{2\pi}} \exp\{e_2(Cw^2 + Ew)\} \int \exp\{-kt^2\} \exp\{e_2(At^2 + Btw + Dt)\} dt \\
&= \frac{1}{\sqrt{2\pi}} \exp\{e_2(Cw^2 + Ew)\} \\
&\qquad \times \int \exp\left\{-(k + e_2 A)\left[t + e_2 \frac{(Bw + D)^2}{2(k + e_2 A)}\right]\right\} dt \exp\left\{-\frac{(Bw + D)^2}{4(k + e_2 A)}\right\}. \quad (10)
\end{aligned}
$$

Now, with the help of Gaussian integral, (10) yields

$$
\begin{aligned}
&\mathbf{Q}_\mu^{\mathbf{H}}[f(t)](w) \\
&= \frac{1}{\sqrt{2\pi}} \exp\{e_2(Cw^2 + Ew)\} \sqrt{\frac{\pi}{k + e_2 A}} \exp\left\{-\frac{(Bw + D)^2}{4(k + e_2 A)}\right\} \\
&= \exp\{e_2(Cw^2 + Ew)\} \sqrt{\frac{1}{2(k + e_2 A)}} \exp\left\{-\frac{(Bw + D)^2}{4(k + e_2 A)}\right\}.
\end{aligned}
$$

Definition 3 (Inversion)**.** *The inverse can be expressed in the form*

$$
\begin{aligned}
f(t) &= \{\mathbf{Q}_\mu^{\mathbf{H}}\}^{-1}[\mathbf{Q}_\mu^{\mathbf{H}}[f]](t) \\
&= \frac{|B|}{\sqrt{2\pi}} \int \mathbf{Q}_\mu^{\mathbf{H}}[f](w) \overline{\Lambda_\mu^{e_2}(t,w)} dw. \quad (11)
\end{aligned}
$$

Definition 4. *Let* $f = f_1 + e_2 f_2$ *be a quaternion valued signal in* $L^1(\mathbf{R}, \mathbf{H})$, *then the quaternion quadratic-phase Fourier transform is defined as*

$$
\mathbf{Q}_\mu^{\mathbf{H}}[f(t)](w) = \mathbf{Q}_\mu^{\mathbf{H}}[f_1(t)](w) + e_2 \mathbf{Q}_\mu^{\mathbf{H}}[f_2(t)](w). \quad (12)
$$

By above definition, it is consistent with the quadratic-phase Fourier transform on $L^1(\mathbf{R}, \mathbf{C})$. *Now it is clear from the definition of quaternion quadratic-phase Fourier transform and the properties of quadratic-phase Fourier transform on* $L^1(\mathbf{R}, \mathbf{H})$, *that* $\mathbf{Q}_\mu^{\mathbf{H}}\left(\mathbf{Q}_\nu^{\mathbf{H}}[f]\right) = \mathbf{Q}_{\mu+\nu}^{\mathbf{H}}[f]$ *and* $\{\mathbf{Q}_\mu^{\mathbf{H}}[f]\}^{-1} = \mathbf{Q}_{-\mu}^{\mathbf{H}}[f]$ *for every signal* $f \in L^1(\mathbf{R}, \mathbf{H})$.

Theorem 5. *The quaternion quadratic-phase Fourier transform* $\mathbf{Q}_\mu^{\mathbf{H}}$ *is* $\mathbf{H}-$*linear on* $L^1(\mathbf{R}, \mathbf{H})$.

Proof. Let us consider two quaternion signals $f = f_1 + e_2 f_2$ and $g = g_1 + e_2 g_2$ in $L^1(\mathbf{R}, \mathbf{H})$; now, by the linearity of $\mathbf{Q}_\mu^{\mathbf{H}}$ on $L^1(\mathbf{R}, \mathbf{C})$, we obtain

$$\begin{aligned}
&\mathbf{Q}_\mu^{\mathbf{H}}[f+g](w) \\
&= \mathbf{Q}_\mu^{\mathbf{H}}[(f_1 + e_2 f_2) + (g_1 + e_2 g_2)](w) \\
&= \mathbf{Q}_\mu^{\mathbf{H}}[(f_1 + g_1) + e_2(f_2 + g_2)](w) \\
&= \mathbf{Q}_\mu^{\mathbf{H}}[f_1](w) + \mathbf{Q}_\mu^{\mathbf{H}}[g_1](w) \\
&\quad + e_2\Big(\mathbf{Q}_\mu^{\mathbf{H}}[f_2](w) + \mathbf{Q}_\mu^{\mathbf{H}}[g_2](w)\Big) \\
&= \Big(\mathbf{Q}_\mu^{\mathbf{H}}[f_1](w) + e_2\mathbf{Q}_\mu^{\mathbf{H}}[f_2](w)\Big) + \Big(\mathbf{Q}_\mu^{\mathbf{H}}[g_1](w) \\
&\quad + e_2\mathbf{Q}_\mu^{\mathbf{H}}[g_2](w)\Big) \\
&= \mathbf{Q}_\mu^{\mathbf{H}}[f](w) + \mathbf{Q}_\mu^{\mathbf{H}}[g](w).
\end{aligned}$$

Now, to prove $\mathbf{H}-$linearity, we let $q = q_1 + e_2 q_2 \in \mathbf{H}$ and $f = f_1 + e_2 f_2 \in L^1(\mathbf{R}, \mathbf{H})$ be arbitrary, then we have

$$\begin{aligned}
\mathbf{Q}_\mu^{\mathbf{H}}[e_2 f](w) &= \mathbf{Q}_\mu^{\mathbf{H}}[e_2(f_1 + e_2 f_2)](w) \\
&= \mathbf{Q}_\mu^{\mathbf{H}}[e_2 f_1 - f_2](w) \\
&= e_2\mathbf{Q}_\mu^{\mathbf{H}}[f_1](w) - \mathbf{Q}_\mu^{\mathbf{H}}[f_2](w) \\
&= e_2\Big(\mathbf{Q}_\mu^{\mathbf{H}}[f_1](w) + e_2\mathbf{Q}_\mu^{\mathbf{H}}[f_2](w)\Big) \\
&= e_2\mathbf{Q}_\mu^{\mathbf{H}}[f](w).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbf{Q}_\mu^{\mathbf{H}}[qf](w) &= \mathbf{Q}_\mu^{\mathbf{H}}[q_1 f](w) + \mathbf{Q}_\mu^{\mathbf{H}}[e_2 q_2 f](w) \\
&= q_1\mathbf{Q}_\mu^{\mathbf{H}}[f](w) + e_2 q_2\mathbf{Q}_\mu^{\mathbf{H}}[f](w) \\
&= (q_1 + e_2 q_2)\mathbf{Q}_\mu^{\mathbf{H}}[f](w) \\
&= q\mathbf{Q}_\mu^{\mathbf{H}}[f](w).
\end{aligned}$$

Which completes the proof. □

Theorem 6 (Moyal's formula)**.** *Let $f, g \in L^1(\mathbf{R}, \mathbf{H}) \cap L^2(\mathbf{R}, \mathbf{H})$ be two signals functions with $\mathbf{Q}_\mu^{\mathbf{H}}[f] \in L^1(\mathbf{R}, \mathbf{H})$, $\langle f, g\rangle = |B|\langle \mathbf{Q}_\mu^{\mathbf{H}}[f], \mathbf{Q}_\mu^{\mathbf{H}}[g]\rangle$.*

Proof. For $f, g \in L^1(\mathbf{R}, \mathbf{H}) \cap L^1(\mathbf{R}, \mathbf{H})$ with $\mathbf{Q}^{\mathbf{H}}_{\mu}[f] \in L^1(\mathbf{R}, \mathbf{H})$,

$$\begin{aligned}
&\langle f, g\rangle \\
&= \int f(t)[g(t)]^c dt \\
&= \frac{|B|}{\sqrt{2\pi}} \int\int \mathbf{Q}^{\mathbf{H}}_{\mu}[f](w)\overline{\Lambda^{e_2}_{\mu}(t,w)}dw[g(t)]^c dt \quad (by\ (11)) \\
&= \frac{|B|}{\sqrt{2\pi}} \int\int \mathbf{Q}^{\mathbf{H}}_{\mu}[f](w)\overline{\Lambda^{e_2}_{\mu}(t,w)}[g(t)]^c dt dw \\
&= \frac{|B|}{\sqrt{2\pi}} \int \mathbf{Q}^{\mathbf{H}}_{\mu}[f](w) \int \{g(t)\Lambda^{e_2}_{\mu}(t,w)\}^c dt dw \\
&= \frac{|B|}{\sqrt{2\pi}} \int \mathbf{Q}^{\mathbf{H}}_{\mu}[f](w)\left\{\int g(t)\Lambda^{e_2}_{\mu}(t,w)dt\right\}^c dw \\
&= |B| \int \mathbf{Q}^{\mathbf{H}}_{\mu}[f](w)\left\{\frac{1}{\sqrt{2\pi}}\int g(t)\Lambda^{e_2}_{\mu}(t,w)dt\right\}^c dw \\
&= |B| \int \mathbf{Q}^{\mathbf{H}}_{\mu}[f](w)\{\mathbf{Q}^{\mathbf{H}}_{\mu}[g](w)\}^c dw \\
&= |B|\langle \mathbf{Q}^{\mathbf{H}}_{\mu}[f], \mathbf{Q}^{\mathbf{H}}_{\mu}[g]\rangle,
\end{aligned}$$

which completes the proof. □

Lemma 1. *For $f \in L^p(\mathbf{R}, \mathbf{C})$, $p = 1, 2$, we have $\mathbf{Q}^{\mathbf{H}}_{\mu}[\overline{f}](w) = \overline{\mathbf{Q}^{\mathbf{H}}_{-\mu}[f](w)}$.*

Proof. It follows from Definition 2 that

$$\begin{aligned}
\mathbf{Q}^{\mathbf{H}}_{\mu}[\overline{f}](w) &= \tfrac{1}{\sqrt{2\pi}} \int \overline{f(t)}\Lambda^{e_2}_{\mu}(t,w)dt \\
&= \tfrac{1}{\sqrt{2\pi}} \int \overline{f(t)\overline{\Lambda^{e_2}_{\mu}(t,w)}}dt \\
&= \tfrac{1}{\sqrt{2\pi}} \int \overline{f(t)\Lambda^{e_2}_{-\mu}(t,w)}dt \\
&= \overline{\tfrac{1}{\sqrt{2\pi}} \int f(t)\Lambda^{e_2}_{-\mu}(t,w)dt} \\
&= \overline{\mathbf{Q}^{\mathbf{H}}_{-\mu}[f](w)},
\end{aligned}$$

which completes the proof. □

Remark 1. *The Lemma 1 can also be written as $\mathbf{Q}^{\mathbf{H}}_{-\mu}[\overline{f}](w) = \overline{\mathbf{Q}^{\mathbf{H}}_{\mu}[f](w)}$.*

Definition 5. *For $f \in L^2(\mathbf{R}, \mathbf{H})$ and $g \in L^1(\mathbf{R}, \mathbf{H})$, define*

$$(f \otimes_{\mu} g) = (f_1 *_{\mu} g_1 - \mathbf{Q}^{\mathbf{H}}_{-2\mu}[\overline{f}_2 *_{\mu} g_2]) + e_2(f_2 *_{\mu} g_1 + \mathbf{Q}^{\mathbf{H}}_{-2\mu}[\overline{f}_1 *_{\mu} g_1]) \tag{13}$$

where $\otimes_{\mu}$ is the proposed definition of convolution.

Remark 2. $\mathbf{Q}^{\mathbf{H}}_{-2\mu} = \mathbf{Q}^{\mathbf{H}}_{-\mu} \circ \mathbf{Q}^{\mathbf{H}}_{-\mu}$ *is the composition of $\mathbf{Q}^{\mathbf{H}}_{-\mu}$ with itself.*

Theorem 7 (Convolution theorem)**.** *Let f, g be two given signal functions such that $f \in L^2(\mathbf{R}, \mathbf{H})$ and $g \in L^1(\mathbf{R}, \mathbf{H})$, then for all $w \in \mathbf{R}$ we have*

$$\mathbf{Q}^{\mathbf{H}}_{\mu}[f \otimes_{\mu} g](w) = \mathbf{Q}^{\mathbf{H}}_{\mu}[f](w)\mathbf{Q}^{\mathbf{H}}_{\mu}[g](w)\exp\{-e_2(Cw^2 + Ew)\}. \tag{14}$$

Proof. By applying Definition 5 and Theorem 3, we have

$$
\begin{aligned}
&\mathbf{Q}_\mu^{\mathbf{H}}[f \otimes_\mu g](w) \\
&= \mathbf{Q}_\mu^{\mathbf{H}}\Big[(f_1 *_\mu g_1 - \mathbf{Q}_{-2\mu}^{\mathbf{H}}[\overline{f}_2 *_\mu g_2])\Big](w) \\
&\quad + e_2\mathbf{Q}_\mu^{\mathbf{H}}\Big[(f_2 *_\mu g_1 + \mathbf{Q}_{-2\mu}^{\mathbf{H}}[\overline{f}_1 *_\mu g_1])\Big](w) \\
&= \mathbf{Q}_\mu^{\mathbf{H}}[f_1](w)\mathbf{Q}_\mu^{\mathbf{H}}[g_1](w)\exp\{-e_2(Cw^2+Ew)\} \\
&\quad - \mathbf{Q}_\mu^{\mathbf{H}}\mathbf{Q}_{-2\mu}^{\mathbf{H}}[\overline{f}_2](w)\mathbf{Q}_\mu^{\mathbf{H}}[g_2](w)\exp\{-e_2(Cw^2+Ew)\} \\
&\quad + e_2\Big\{\mathbf{Q}_\mu^{\mathbf{H}}[f_2](w)\mathbf{Q}_\mu^{\mathbf{H}}[g_1](w)\exp\{e_2(Cw^2+Ew)\} \\
&\quad + \mathbf{Q}_\mu^{\mathbf{H}}\mathbf{Q}_{-2\mu}^{\mathbf{H}}[\overline{f}_1](w)\mathbf{Q}_\mu^{\mathbf{H}}[g_1](w)\exp\{-e_2(Cw^2+Ew)\}\Big\} \\
&= \Big\{[\mathbf{Q}_\mu^{\mathbf{H}}[f_1]\mathbf{Q}_\mu^{\mathbf{H}}[g_1] - \mathbf{Q}_{-\mu}^{\mathbf{H}}[\overline{f}_2]\mathbf{Q}_\mu^{\mathbf{H}}[g_2]](w) \\
&\quad \times e_2[\mathbf{Q}_\mu^{\mathbf{H}}[f_2]\mathbf{Q}_\mu^{\mathbf{H}}[g_1] - \mathbf{Q}_{-\mu}^{\mathbf{H}}[\overline{f}_1]\mathbf{Q}_\mu^{\mathbf{H}}[g_1]](w)\Big\} \\
&\quad \times \exp\{-e_2(Cw^2+Ew)\} \\
&= \Big\{[\mathbf{Q}_\mu^{\mathbf{H}}[f_1]\mathbf{Q}_\mu^{\mathbf{H}}[g_1] - \overline{\mathbf{Q}_\mu^{\mathbf{H}}[f_2]}\mathbf{Q}_\mu^{\mathbf{H}}[g_2]](w) \\
&\quad \times e_2[\mathbf{Q}_\mu^{\mathbf{H}}[f_2]\mathbf{Q}_\mu^{\mathbf{H}}[g_1] + \overline{\mathbf{Q}_\mu^{\mathbf{H}}[f_1]}\mathbf{Q}_\mu^{\mathbf{H}}[g_1]](w)\Big\} \\
&\quad \times \exp\{-e_2(Cw^2+Ew)\} \\
&= \mathbf{Q}_\mu^{\mathbf{H}}[f](w)\mathbf{Q}_\mu^{\mathbf{H}}[g](w)\exp\{-e_2(Cw^2+Ew)\},
\end{aligned}
$$

which completes the proof. □

Remark 3. *By appropriate choice of parameters in* $\mu = (A, B, C, D, E)$*, Theorem 7 yields the corresponding convolution theorem for all the integral transforms ranging from the 1D quaternion Fourier transform to the much recent 1D quaternion special affine Fourier transforms.*

Prior to establishing the correlation theorem for the proposed 1D-QQPFT, we have the following lemma:

Lemma 2 (QPFT Correlation)**.** *If* $f, g \in L^2(\mathbf{R})$ *then*

$$
\begin{aligned}
&\mathbf{Q}_\mu[f \circ_\mu g(t)](w) \\
&= \exp\{e_2(Cw^2+Ew)\}\overline{\mathbf{Q}_{\mu'}[f](w)}\mathbf{Q}_\mu[g](w)
\end{aligned}
$$

where $\mu' = (-A, B, C, D, E)$ *and*

$$
(f \circ_\mu g)(t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} \overline{f(z)}g(t+z)\exp\{2e_2Az(t+z)\}dz. \tag{15}
$$

Proof. From the Definition 1, we have

$$
\begin{aligned}
&\mathbf{Q}_\mu[f \circ_\mu g(t)](w) \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} (f \circ_\mu g)(t) \exp\{e_2(At^2 + Btw + Cw^2 + Dt + Ew)\} dt \\
&= \frac{1}{(\sqrt{2\pi})^2} \int_{\mathbf{R}} \int_{\mathbf{R}} \overline{f(z)} g(t+z) \exp\{2e_2 Az(t+z)\} \\
&\qquad\qquad \times \exp\{e_2(At^2 + Btw + Cw^2 + Dt + Ew)\} dz dt \\
&= \frac{1}{(\sqrt{2\pi})^2} \int_{\mathbf{R}} \int_{\mathbf{R}} \overline{f(z)} g(u) \exp\{2e_2 Azu\} \\
&\qquad \times \exp\{e_2[A(u-z)^2 + B(u-z)w + Cw^2 + D(u-z) + Ew]\} dz du \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} g(u) \exp\{e_2[Au^2 + Buw + Cw^2 + Du + Ew]\} du \\
&\times \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \overline{f(z)} \exp\{e_2[Az^2 - Bzw - Cw^2 - Dz - Ew]\} dz e^{e_2(Cw^2+Ew)} \\
&= \exp\{i(Cw^2 + Ew)\} \frac{1}{\sqrt{2\pi}} \\
&\times \int_{\mathbf{R}} \overline{f(z)} \exp\{-e_2[(-A)z^2 + Bzw + Cw^2 + Dz + Ew]\} dz \\
&\times \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} g(u) \exp\{e_2[Au^2 + Buw + Cw^2 + Du + Ew]\} du \\
&= \exp\{e_2(Cw^2 + Ew)\} \frac{1}{\sqrt{2\pi}} \\
&\times \int_{\mathbf{R}} \overline{f(z) \exp\{e_2[(-A)z^2 + Bzw + Cw^2 + Dz + Ew]\}} dz \\
&\times \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} g(u) \exp\{e_2[Au^2 + Buw + Cw^2 + Du + Ew]\} du \\
&= \exp\{e_2(Cw^2 + Ew)\} \overline{\mathbf{Q}_{\mu'}[f](w)} \mathbf{Q}_\mu[g](w),
\end{aligned}
$$

which completes the proof. □

Definition 6. *For $f \in L^2(\mathbf{R}, \mathbf{H})$ and $g \in L^1(\mathbf{R}, \mathbf{H})$, define*

$$
\begin{aligned}
&(f \odot_\mu g) \\
&= (f_1 \circ_\mu g_1 - \mathbf{Q}^{\mathbf{H}}_{-2\mu'}[\overline{f}_2 \circ_\mu g_2]) + e_2(f_2 \circ_\mu g_1 + \mathbf{Q}^{\mathbf{H}}_{-2\mu'}[\overline{f}_1 \circ_\mu g_2])
\end{aligned}
$$

where $\mu' = (-A, B, C, D, E)$ and $\odot_\mu$ is the proposed definition of correlation.

Theorem 8 (Correlation theorem). *Let f, g be two given signal functions such that $f \in L^2(\mathbf{R}, \mathbf{H})$ and $g \in L^1(\mathbf{R}, \mathbf{H})$, then for all $w \in \mathbf{R}$ we have*

$$\mathbf{Q}^{\mathbf{H}}_\mu[f \odot_\mu g](w) = \overline{\mathbf{Q}^{\mathbf{H}}_{\mu'}[f](w)} \mathbf{Q}^{\mathbf{H}}_\mu[g](w) \exp\{e_2(Cw^2 + Ew)\} \tag{16}$$

Proof. By applying Definition 6 and Lemma 2, we have

$$
\begin{aligned}
&\mathbf{Q}_{\mu}^{\mathbf{H}}[f \odot_{\mu} g](w)\\
&= \mathbf{Q}_{\mu}^{\mathbf{H}}\Big[(f_1 \circ_{\mu} g_1 - \mathbf{Q}_{-2\mu'}^{\mathbf{H}}[\overline{f}_2 \circ_{\mu} g_2])\Big](w)\\
&\quad + e_2\mathbf{Q}_{\mu}^{\mathbf{H}}\Big[(f_2 \circ_{\mu} g_1 + \mathbf{Q}_{-2\mu'}^{\mathbf{H}}[\overline{f}_1 \circ_{\mu} g_2])\Big](w)\\
&= \overline{\mathbf{Q}_{\mu'}^{\mathbf{H}}[f_1](w)}\mathbf{Q}_{\mu}^{\mathbf{H}}[g_1](w)\exp\{e_2(Cw^2+Ew)\}\\
&\quad -\overline{\mathbf{Q}_{\mu'}^{\mathbf{H}}\mathbf{Q}_{-2\mu'}^{\mathbf{H}}[\overline{f}_2](w)}\mathbf{Q}_{\mu}^{\mathbf{H}}[g_2](w)\exp\{e_2(Cw^2+Ew)\}\\
&\quad +e_2\Big\{\overline{\mathbf{Q}_{\mu'}^{\mathbf{H}}[f_2](w)}\mathbf{Q}_{\mu}^{\mathbf{H}}[g_1](w)\exp\{e_2(Cw^2+Ew)\}\\
&\quad +\overline{\mathbf{Q}_{\mu'}^{\mathbf{H}}\mathbf{Q}_{-2\mu'}^{\mathbf{H}}[\overline{f}_1](w)}\mathbf{Q}_{\mu}^{\mathbf{H}}[g_2](w)\exp\{e_2(Cw^2+Ew)\}\Big\}\\
&= \Big\{[\overline{\mathbf{Q}_{\mu'}^{\mathbf{H}}[f_1]}\mathbf{Q}_{\mu}^{\mathbf{H}}[g_1] - \overline{\mathbf{Q}_{-\mu'}^{\mathbf{H}}[\overline{f}_2]}\mathbf{Q}_{\mu}^{\mathbf{H}}[g_2]](w)\\
&\quad \times e_2[\overline{\mathbf{Q}_{\mu'}^{\mathbf{H}}[f_2]}\mathbf{Q}_{\mu}^{\mathbf{H}}[g_1] - \overline{\mathbf{Q}_{-\mu'}^{\mathbf{H}}[\overline{f}_1]}\mathbf{Q}_{\mu}^{\mathbf{H}}[g_1]](w)\Big\}\\
&\quad \times \exp\{e_2(Cw^2+Ew)\}\\
&= \Big\{\Big[\overline{\mathbf{Q}_{\mu'}^{\mathbf{H}}[f_1]}\mathbf{Q}_{\mu}^{\mathbf{H}}[g_1] - \overline{\Big[\overline{\mathbf{Q}_{\mu'}^{\mathbf{H}}[f_2]}\Big]}\mathbf{Q}_{\mu}^{\mathbf{H}}[g_2]\Big](w)\\
&\quad \times e_2\Big[\overline{\mathbf{Q}_{\mu'}^{\mathbf{H}}[f_2]}\mathbf{Q}_{\mu}^{\mathbf{H}}[g_1] + \overline{\Big[\overline{\mathbf{Q}_{\mu'}^{\mathbf{H}}[f_1]}\Big]}\mathbf{Q}_{\mu}^{\mathbf{H}}[g_2]\Big](w)\Big\}\\
&\quad \times \exp\{e_2(Cw^2+Ew)\}\\
&= \overline{\mathbf{Q}_{\mu'}^{\mathbf{H}}[f](w)}\mathbf{Q}_{\mu}^{\mathbf{H}}[g](w)\exp\{e_2(Cw^2+Ew)\},
\end{aligned}
$$

which completes the proof. □

Theorem 9 (1D-QQPFT Uncertainty inequality)**.** *Let* $\mathbf{Q}_{\mu}^{\mathbf{H}}[f] \in L^2(\mathbf{R},\mathbf{H})$ *and* $f, tf(t) \in L^2(\mathbf{R},\mathbf{H})$*. Then, we have*

$$\int_{\mathbf{R}} w^2|\mathbf{Q}_{\mu}[f]|^2 dw \int_{\mathbf{R}} t^2|f(t)|^2 dt \geq \frac{1}{8|B|^2}\left(\int_{\mathbf{R}} |f(t)|^2 dt\right)^2.$$

Proof. For $f = f_1 + e_2 f_2$ in $L^2(\mathbf{R},\mathbf{H})$, we have $\|f\|_2^2 = \|f_1\|_2^2 + \|f_2\|_2^2$. Thus

$$
\begin{aligned}
&\int_{\mathbf{R}} w^2|\mathbf{Q}_{\mu}[f]|^2 dw \int_{\mathbf{R}} t^2|f(t)|^2 dt\\
&= \int_{\mathbf{R}}\Big(w^2|\mathbf{Q}_{\mu}[f_1]|^2 + w^2|\mathbf{Q}_{\mu}[f_2]|^2\Big)dw\\
&\quad \times \int_{\mathbf{R}}\Big(t^2|f_1|^2 + t^2|f_2|^2\Big)dt\\
&\geq \int_{\mathbf{R}} w^2|\mathbf{Q}_{\mu}[f_1]|^2 dw + \int_{\mathbf{R}} t^2|f_1|^2\\
&\quad + \int_{\mathbf{R}} w^2|\mathbf{Q}_{\mu}[f_2]|^2 dw + \int_{\mathbf{R}} t^2|f_2|^2.
\end{aligned}
$$

Now, using Theorem 4, we obtain

$$\begin{aligned}
&\int_{\mathbf{R}} w^2 |\mathbf{Q}_\mu[f]|^2 dw \int_{\mathbf{R}} t^2 |f(t)|^2 dt \\
&\geq \left\{ \frac{1}{2|B|} \int_{\mathbf{R}} |f_1(t)|^2 dt \right\}^2 + \left\{ \frac{1}{2|B|} \int_{\mathbf{R}} |f_2(t)|^2 dt \right\}^2 \\
&= \frac{1}{4|B|^2} \left\{ \left(\int_{\mathbf{R}} |f_1(t)|^2 dt \right)^2 + \left(\int_{\mathbf{R}} |f_2(t)|^2 dt \right)^2 \right\} \\
&\geq \frac{1}{8|B|^2} \left(\int_{\mathbf{R}} |f_1(t)|^2 dt + \int_{\mathbf{R}} |f_2(t)|^2 dt \right)^2 \\
&= \frac{1}{8|B|^2} \left(\int_{\mathbf{R}} |f(t)|^2 dt \right)^2
\end{aligned}$$

which completes the proof. □

4. Application to the 2-Dimensional Quaternion Quadratic-Phase Fourier Transform

In Section 3, we used 1D-QQPFT to define various topics but the main motive to define the proposed transform is to obtain an inversion formula for the two-dimensional quaternion quadratic-phase Fourier transform (2D-QQPFT) which will lay the foundation for applications of two-dimensional quaternion quadratic-phase Fourier transform in signal processing. Prior to that we shall define two-dimensional quaternion quadratic-phase Fourier transform(2D-QQPFT). Let us begin:

Definition 7. *The 2D-QQPFT of a quaternion-valued signal function $f \in L^1(\mathbf{R}^2, \mathbf{H})$, with respect to parameter $\mu_s = (A_s, B_s, C_s, D_s, E_s)$ for $s = 1, 2$ is defined as*

$$\mathbf{Q}^{\mathbf{H}}_{\mu_1,\mu_2}[f](\mathbf{w}) = \frac{1}{2\pi} \int_{\mathbf{R}^2} f(\mathbf{t}) \Lambda^{\mathbf{e}_1}_{\mu_1}(t_1, w_1) \Lambda^{\mathbf{e}_2}_{\mu_2}(t_2, w_2) d\mathbf{t}$$

where $\mathbf{w} = (w_1, w_2) \in \mathbf{R}^2$, $\mathbf{t} = (t_1, t_2) \in \mathbf{R}^2$ and $\Lambda^{\mathbf{e}_1}_{\mu_1}(t_1, w_1)$ and $\Lambda^{\mathbf{e}_2}_{\mu_2}(t_2, w_2)$ are kernel signals given by

$$\begin{aligned}
&\Lambda^{\mathbf{e}_1}_{\mu_1}(t_1, w_1) \\
&= \exp\{e_1(A_1 t_1^2 + B_1 t_1 w_1 + C_1 w_1^2 + D_1 t_1 + E_1 w_1)\}
\end{aligned}$$

$$\begin{aligned}
&\Lambda^{\mathbf{e}_2}_{\mu_2}(t_2, w_2) \\
&= \exp\{e_2(A_2 t_2^2 + B_2 t_2 w_2 + C_2 w_2^2 + D_2 t_2 + E_2 w_2)\}
\end{aligned}$$

where $A_s, B_s, C_s, D_s, E_s \in \mathbf{R}$, $B_s \neq 0$ and $s = 1, 2$.

Theorem 10. *Let $f \in L^2(\mathbf{R}^2, \mathbf{H})$ and $\mathbf{Q}^{\mathbf{H}}_{\mu_s}[f] \in L^1(\mathbf{R}^2, \mathbf{H})$. Then, 2-D QQPFT is invertible with inverse*

$$\begin{aligned}
&f(\mathbf{t}) \\
&= \{\mathbf{Q}^{\mathbf{H}}_{\mu_1,\mu_2}\}^{-1} [\mathbf{Q}^{\mathbf{H}}_{\mu_1,\mu_2}\{f\}](\mathbf{t}) \\
&= \frac{|B_1 B_2|}{2\pi} \int_{\mathbf{R}^2} \mathbf{Q}^{\mathbf{H}}_{\mu_1,\mu_2}[f]\}(\mathbf{w}) \Lambda^{\mathbf{e}_2}_{-\mu_2}(t_1, w_1) \Lambda^{\mathbf{e}_1}_{-\mu_1}(t_2, w_2) d\mathbf{w}.
\end{aligned}$$

Proof. From the Definition 1D-QQPFT and the 2D-QQPFT, we have

$$\begin{aligned}
\mathbf{Q}^{\mathbf{H}}_{\mu_1,\mu_2}\{f\}(\mathbf{w}) &= \int \mathbf{Q}^{\mathbf{H}}_{\mu_1}\{f\}(w_1) \Lambda^{e_2}_{\mu_2}(t_2, w_2) dt_2 \\
&= \mathbf{Q}^{\mathbf{H}}_{\mu_2}\left[\mathbf{Q}^{\mathbf{H}}_{\mu_1}\{f\}\right](\mathbf{w}).
\end{aligned}$$

Now it is given that $f \in L^2(\mathbf{R}^2, \mathbf{H})$ and $\mathbf{Q}^{\mathbf{H}}_{\mu_s}[f] \in L^1(\mathbf{R}^2, \mathbf{H})$; therefore, we have

$$\begin{aligned}
&\int_{\mathbf{R}^2} \mathbf{Q}^{\mathbf{H}}_{\mu_1,\mu_2}\{f\}(\mathbf{w})\Lambda^{e_2}_{-\mu_2}(t_2, w_2)\Lambda^{e_1}_{-\mu_1}(t_1, w_1)d\mathbf{w} \\
&= \int\int \mathbf{Q}^{\mathbf{H}}_{\mu_2}[\mathbf{Q}^{\mathbf{H}}_{\mu_1}\{f\}](\mathbf{w})\Lambda^{e_2}_{-\mu_2}(t_2, w_2)dw_2 \\
&\times\Lambda^{e_1}_{-\mu_1}(t_1, w_1)dw_1.
\end{aligned}$$

Using (11), we obtain

$$\begin{aligned}
&\int_{\mathbf{R}^2} \mathbf{Q}^{\mathbf{H}}_{\mu_1,\mu_2}\{f\}(\mathbf{w})\Lambda^{e_2}_{-\mu_2}(t_2, w_2)\Lambda^{e_1}_{-\mu_1}(t_1, w_1)d\mathbf{w} \\
&= \frac{\sqrt{2\pi}}{|B_2|}\int \mathbf{Q}^{\mathbf{H}}_{\mu_1}\{f\}(w_1)\Lambda^{e_1}_{-\mu_1}(t_1, w_1)dw_1 \\
&= \frac{2\pi}{|B_1 B_2|} f(\mathbf{t}),
\end{aligned}$$

which completes the proof. □

5. Potential Applications

The QQPFT which is a generalized version of the QFTs has gained its ground intermittently and profoundly influenced several disciplines of science and engineering, including harmonic analysis, quantum mechanics, differential equations, optics, pattern recognition, and so on [29,30]. The 1D-QQPFT can be widely applied in computer graphics, computer vision, robotics and even in astrophysics [39,40]. The 1D-QQPFT can be used in the recovery of bandlimited quaternion-valued signals in 1D-QQPFT domain from noisy samples; it is based on the oversampling theorem and without adding too much complexity, a reconstruction algorithm for bandlimited signals in 1D-QQPFT domain from noisy observations is obtained [41]. The potential application can be found in the detection of linear frequency-modulated(LFM) and non-transient quaternion-valued signals [31]. Moreover, the 1D-QQPFT can have a potential application in image preprocessing, color analysis and neural computing techniques for speech recognition [42]. On the other hand, the convolution and correlation type operators are very important mathematical objects which are used in the modelling of a great diversity of applied problems in signals, images and optics, especially in the design and implementation of multiplicative filters in the 1D-QQPFT [43]. The potential application can be found in in the theory of linear time-invariant (LTI) system [44].

The uncertainty inequalities have potential applications in signal processing as they can estimate the lower bound of integral [19,29,31]. The estimation of band-widths is another possible application of the proposed transform [45]. Theorem 9 can be used in the estimation of effective band-width in the 1D-QQPFT domain which states that the band-width of a system that performs 1D-QQPFT cannot be narrower than $\frac{1}{T_\mu^2 8B^2}$, where T_μ is the spread of the signal in the time domain.

6. Conclusions

In this paper, we have proposed the definition of the novel integral transform known as the one-dimensional quaternion quadratic-phase Fourier transform (1D-QQPFT) which is the embodiment of several well-known signal processing tools. We then obtained Moyal's formula and convolution theorem for the proposed transform. We then used theory 1D-QQPFT to obtain an inversion formula for two-dimensional quaternion quadratic-phase Fourier transform. Our future work on the two-sided quaternion quadratic-phase Fourier transform and two-sided Gabor quaternion quadratic-phase Fourier transform is in progress.

Author Contributions: Conceptualization, M.Y.B., A.H.D., M.Z. and A.A.B.; methodology, M.Y.B., A.H.D., M.Z. and A.A.B.; validation, M.Y.B., A.H.D., M.Z. and A.A.B.; formal analysis, M.Y.B., A.H.D., M.Z. and A.A.B.; investigation, M.Y.B., A.H.D., M.Z. and A.A.B.; resources, M.Z.; writing—original draft preparation, M.Y.B., A.H.D., M.Z. and A.A.B.; writing—review and editing, M.Y.B., A.H.D., M.Z. and A.A.B.; supervision, M.Y.B.; project administration, M.Y.B., A.H.D., M.Z. and A.A.B.; funding acquisition, M.Z. All authors have read and agreed to the published version of the manuscript.

Funding: This work is funded by the Deanship of Scientific Research at King Khalid University through large group Research Project under grant number RGP2/237/44.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: M. Zayed extends her appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through large group Research Project under grant number RGP2/237/44.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Castro, L.P.; Haque, M.R.; Murshed, M.M.; Saitoh, S.; Tuan, N.M. Quadratic Fourier transforms. *Ann. Funct. Anal.* **2014**, *5*, 10–23. [CrossRef]
2. Castro, L.P.; Minh, L.T.; Tuan, N.M. New convolutions for quadratic-phase Fourier integral operators and their applications. *Mediterr. J. Math.* **2018**, *15*, 13. [CrossRef]
3. Rashid, H.H.; Srivastava, H.M.; Hama, M.; Mohammed, P.O.; Sarairah, E.A. New Numerical Results on Existence of Volterra–Fredholm Integral Equation of Nonlinear Boundary Integro-Differential Type. *Symmetry* **2023**, *15*, 1144.
4. Srivastava, H.M.; Tariq, M.; Mohammed, P.O.; Alrweili, H.; Sarairah, E.A. On Modified Integral Inequalities for a Generalized Class of Convexity and Applications. *Axioms* **2023**, *12*, 162. [CrossRef]
5. Bhat, M.Y.; Dar, A.H. Quadratic-phase scaled Wigner distribution: convolution and correlation. *Signal Image Video Process.* **2023**, *17*, 2779–2788. [CrossRef]
6. Bhat, M.Y.; Dar, A.H.; Urynbassarova, D.; Urynbassarova, A. Quadratic-phase wave packet transform. *Opt. Int. J. Light Electron Opt.* **2022**, *261*, 169120. [CrossRef]
7. Bhat, M.Y.; Dar, A.H. Quadratic phase S-Transform: Properties and uncertainty principles. *e-Prime-Adv. Electr. Eng. Electron. Energy* **2023**, *4*, 100162. [CrossRef]
8. Prasad, A.; Sharma, P.B. The quadratic-phase Fourier wavelet transform. *Math. Meth. Appl. Sci.* **2019**, *43*, 1953–1969. [CrossRef]
9. Urynbassarova, D.; Li, B.Z.; Tao, R. Convolution and correlation theorems for Wigner-ville distribution associated with offset linear canonical transform. *Optik* **2018**, *157*, 455–466. [CrossRef]
10. Wei, D.; Li, Y.M. Generalized wavelet transform based on the convolution operator in the linear canonical transform domain. *Optik* **2014**, *125*, 4491–4496. [CrossRef]
11. Bahri, M.; Ashino, R.; Vaillancourt, R. Convolution theorems for quaternion fourier transform: Properties and applications. *Abstr. Appl. Anal.* **2013**, *2013*, 162769. [CrossRef]
12. Sharma, V.D.; Deshmukh, P.B. Convolution structure of fractional quaternion fourier transform. *IJESRT* **2016**, *5*, 176–182.
13. Bhat, M.Y.; Dar, A.H. Convolution and Correlation Theorems for Wigner-Ville Distribution Associated with the Quaternion Offset Linear Canonical Transform. *Signal Image Video Process.* **2022**, *16*, 1235–1242. [CrossRef]
14. Zhang, Z. Linear canonical wigner distribution based noisy LFM signals detection through the output SNR improvement analysis. *IEEE Trans. Signal Process.* **2019**, *67*, 5527–5542. [CrossRef]
15. Snopek, K.M. The study of properties of n-d analytic signals and their spectra in complex and hypercomplex domains. *Radio Eng.* **2012**, *21*, 29–36.
16. Sangwine, S.J.; Ell, T.A. Colour image filters based on hypercomplex convolution. *IEEE Proc. Vis. Image Signal Process.* **2000**, *49*, 89–93. [CrossRef]
17. Pei, S.C.; Chang, J.H.; Ding, J.J. Color pattern recognition by quaternion corre- lation. In Proceedings of the IEEE International Conference Image Process, Thessaloniki, Greece, 7–10 October 2001.
18. Sangwine, S.J.; Evans, C.J.; Ell T.A. Colour-sensitive edge detection using hyper- complex filters. In Proceedings of the 10th European Signal Processing Conference EUSIPCO, Tampere, Finland, 4–8 September 2000; Volume 1, pp. 107–110.
19. Gao, C.; Zhou, J.; Lang, F.; Pu, Q.; Liu, C. Novel approach to edge detection of color image based on quaternion fractional directional differentiation. *Adv. Autom. Robot.* **2012**, *1*, 163–170.
20. Took, C.C.; Mandic, D.P. The quaternion LMS algorithm for adaptive filtering of hypercomplex processes. *IEEE Trans. Signal Process.* **2009**, *57*, 1316–1327 [CrossRef]

21. Witten, B.; Shragge, J. Quaternion-based signal processing, stanford exploration project. In Proceedings of the SEG International Exposition and 76th Annual Meeting (SEG New Orleans 2006), New Orleans, LA, USA, 1–6 October 2006; pp. 2862–2866
22. Bülow, T.; Sommer, G. The hypercomplex signal-a novel extensions of the an- alytic signal to the multidimensional case. *IEEE Trans. Signal Process.* **2001**, *49*, 2844–2852. [CrossRef]
23. Bayro-Corrochano, E.; Trujillo, N.; Naranjo, M. Quaternion Fourier descriptors for preprocessing and recognition of spoken words using images of spatiotemporal representations. *J. Math. Imaging Vis.* **2007**, *28*, 179–190. [CrossRef]
24. Bas, P.; LeBihan, N.; Chassery, J.M. Color image water marking using quaternion Fourier transform. In Proceedings of the IEEE International Conference on Acoustics Speech and Signal and Signal Processing, ICASSP, Hong Kong, China, 6–10 April 2003; pp. 521–524.
25. Kou, K.I.; Ou, J.; Morais, J. Uncertainty principles associated with quaternionic linear canonical transforms. *Math. Meth. Appl. Sci.* **2016**, *39*, 2722–2736. [CrossRef]
26. Kassimi, M.E.; Haoui, Y.E.; Fahlaoui, S. The Wigner-Ville distribution associ- ated with the quaternion offset linear canonical transform. *Anal. Math.* **2019**, *45*, 787–802. [CrossRef]
27. Bhat, M.Y.; Dar, A.H. The algebra of 2D Gabor quaternionic offset linear canonical transform and uncertainty principles. *J. Anal.* **2022**, *30*, 637–649.. [CrossRef]
28. Bhat, M.Y.; Dar, A.H. Octonion spectrum of 3D short-time LCT signals. *Opt. Int. J. Light Electron Opt.* **2022**, *261*, 169156. [CrossRef]
29. Bhat, M.Y.; Dar, A.H. Towards quaternion quadratic-phase Fourier transform. *Math. Meth. Appl. Sci.* **2023**. [CrossRef]
30. Dar, A.H.; Bhat, M.Y. Donoho-Stark's and Hardy's Uncertainty Principles for the Short-time Quaternion Offset Linear Canonical Transform. *Filomat* **2023**, *37*, 4467–4480.
31. Bhat, M.Y.; Dar, A.H. The 2-D Hyper-complex Gabor quadratic-phase Fourier transform and uncertainty principles. *J. Anal.* **2023**, *31*, 243–260. [CrossRef]
32. Bhat, M.Y.; Dar, A.H.; Nurhidaya, I.; Pinelas, S. An interplay of Wigner-Ville distribution and 2D Hyper-complex quadratic-phase Fourier transform. *Fractal Fract.* **2023**, *7*, 159. [CrossRef]
33. Bhat, M.Y.; Dar, A.H.; Nurhidaya, I.; Pinelas, S. Uncertainty principles for the two-sided quaternion windowed quadratic-phase Fourier transform. *Symmetry* **2022**, *14*, 2650. [CrossRef]
34. Bahri, M.; Toaha, S.; Rahim, A.; Azis, M.I. On one-dimensional quaternion fourier transform. *Phys. Conf. Ser.* **2019**, *1341*, 062004. [CrossRef]
35. Roopkumar, R. Quaternionic one-dimensional fractional fourier transform. *Optik* **2016**, *127*, 11657–11661. [CrossRef]
36. Bhat, M.Y.; Dar, A.H. Quaternion offset linear canonical transform in one-dimensional setting. *J. Anal.* **2023**, 1–10.. [CrossRef]
37. Siddiqui, S.; Bing, L. Quaternionic one-dimensional linear canonical transform. *Optik* **2021**, *244*, 166914. [CrossRef]
38. Gao, W.B.; Li, B.Z. Quaternion windowed linear canonical transform of two-dimensional quaternionic signals. *Adv. Appl. Clifford Algebr.* **2020**, *30*, 1–6. [CrossRef]
39. Lengyel, E. *Mathematics for 3D Programming and Computer Graphics*; Charles River Media, Inc.: Newton, MA, USA, 2001.
40. Andreis, D.; Canuto, E. *Orbit Dynamics and Kinematics with Full Quaternions*; American Control Conference: Boston, MA, USA, 2004.
41. Zhao, H.; Wang, R.Y.; Song, D.P. Recovery of bandlimited signals in linear canon- ical transform domain from noisy samples. *Circ. Syst. Signal Process.* **2014**, *33*, 1997–2008. [CrossRef]
42. Pei, S.C.; Ding, J.J.; Chang, J.H. Efficient implementation of quaternion fourier transform,convolution and correlation by 2-D complex FFT. *IEEE Trans. Signal Process.* **2001**, *49*, 2783–2797.
43. Dar, A.H.; Bhat, M.Y.; Rahman, M. Generalized wave packet transform based on convolution operator in the quaternion quadratic-phase Fourier domain. *Optik—Int. J. Light Electron Opt.* **2023**, *286*, 171029. [CrossRef]
44. Wei, D.; Ran, Q.; Li, Y.; Ma, J.; Tan, L. A convolution and product theorem for the linear canonical transform. *IEEE Signal Process. Lett.* **2009**, *16*, 853–856.
45. Stern, A. Uncertainty principles in linear canonical transform domains and some of their implications in optics. *J. Opt. Soc. Am. A-Opt. Image Sci. Vis.* **2008**, *25*, 647–652. [CrossRef]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.