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CONVOLUTION DIRICHLET SERIES AND A KRONECKER LIMIT FORMULA FOR SECOND-ORDER EISENSTEIN SERIES

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Abstract. In this article we derive analytic and Fourier aspects of a Kronecker limit formula for second-order Eisenstein series. Let Γ be any Fuchsian group of the first kind which acts on the hyperbolic upper half-space $\mathbf{H}$ such that the quotient $\Gamma\backslash\mathbf{H}$ has finite volume yet is non-compact. Associated to each cusp of $\Gamma\backslash\mathbf{H}$, there is a classically studied *first-order* non-holomorphic Eisenstein series $E(s, z)$ which is defined by a generalized Dirichlet series that converges for $\mathrm{Re}(s) > 1$. The Eisenstein series $E(s, z)$ admits a meromorphic continuation with a simple pole at $s = 1$. Classically, Kronecker's limit formula is the study of the constant term $\mathcal{K}_1(z)$ in the Laurent expansion of $E(s, z)$ at $s = 1$. A number of authors recently have studied what is known as the *second-order* Eisenstein series $E^*(s, z)$, which is formed by twisting the Dirichlet series that defines the series $E(s, z)$ by periods of a given cusp form f. In the work we present here, we study an analogue of Kronecker's limit formula in the setting of the second-order Eisenstein series $E^*(s, z)$, meaning we determine the constant term $\mathcal{K}_2(z)$ in the Laurent expansion of $E^*(s, z)$ at its first pole, which is also at $s = 1$. To begin our investigation, we prove a bound for the Fourier coefficients associated to the first-order Kronecker limit function $\mathcal{K}_1$. We then define two families of convolution Dirichlet series, denoted by L^+_m and L^-_m with $m \in \mathbb{N}$, which are formed by using the Fourier coefficients of $\mathcal{K}_1$ and the weight two cusp form f. We prove that for all m, L^+_m and L^-_m admit a meromorphic continuation and are holomorphic at $s = 1$. Turning our attention to the second-order Kronecker limit function $\mathcal{K}_2$, we first express $\mathcal{K}_2$ as a solution to various differential equations. Then we obtain its complete Fourier expansion in terms of the cusp form f, the Fourier coefficients of the first-order Kronecker limit function $\mathcal{K}_1$, and special values $L^+_m(1)$ and $L^-_m(1)$ of the convolution Dirichlet series. Finally, we prove a bound for the special values $L^+_m(1)$ and $L^-_m(1)$ which then implies a bound for the Fourier coefficients of $\mathcal{K}_2$. Our analysis leads to certain natural questions concerning the holomorphic projection operator, and we conclude this paper by examining certain numerical examples and posing questions for future study.

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§1. Introduction and statement of results

Let Γ contained in $\mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian group of the first kind acting on the upper half plane $\mathbf{H}$ with non-compact quotient $\Gamma\backslash\mathbf{H}$. As usual, we write $x+iy = z \in \mathbf{H}$. Set V equal to the hyperbolic volume of $\Gamma\backslash\mathbf{H}$. Assuming there is a cusp at ∞, let $\Gamma_\infty = \{\gamma \in \Gamma \mid \gamma\infty = \infty\}$, and, for simplicity we may assume that Γ_∞ is generated by $z \mapsto z+1$. The *first-order* non-holomorphic Eisenstein series is defined by the series

$$E(z,s) = \sum_{\gamma \in \Gamma_\infty\backslash\Gamma} \mathrm{Im}(\gamma z)^s$$

which converges for $\mathrm{Re}(s) > 1$ and has a meromorphic continuation to all s in $\mathbb{C}$ (see, for example, Chapter 6 of [Iw1]). The function $E(z,s)$ is known to have a simple pole at $s=1$ with residue V^{-1}, so then, when denoting the constant part at $s=1$ by $\mathcal{K}_1$, we can write

$$E(z,s) = \frac{V^{-1}}{s-1} + \mathcal{K}_1(z) + O(s-1) \quad \text{as} \quad s \to 1.$$

The first result which is known as *Kronecker's first limit formula* is the following. If $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, then

$$\mathcal{K}_1(z) = \frac{-1}{4\pi}\log(y^{12}|\Delta(z)|^2) + \frac{3}{\pi}(\gamma - \log 4\pi) \tag{1.1}$$

where

$$\Delta(z) = e^{2\pi i z}\prod_{n=1}^{\infty}\left(1 - e^{2\pi i n z}\right)^{24}$$

is the discriminant function, a weight 12 holomorphic cusp form for $\mathrm{PSL}_2(\mathbb{Z})$, and γ is Euler's constant. Kronecker's second limit formula, which for brevity we do not state here, is a determination of the constant term at the first pole of the first-order non-holomorphic Eisenstein series obtained by twisting the series definition of $E(z,s)$ with a unitary character of Γ. We refer to [La], [Si], or [Za1] for proofs of these classical results.

Many generalizations of the Kronecker limit formulas exist, and the results have diverse applications. In [La], [Si], and [Za2], formulas for class numbers of algebraic number fields are obtained; in [C-P] and [P-W], the limit formulas are used to find values of $|\eta(z)|$ at quadratic irrationalities; in [B-C-Z] and [Ra], special values of the Rogers-Ramanujan continued fraction are evaluated; and in [R-S], the limit formulas are used to explicitly evaluate analytic torsion for flat line bundles on elliptic curves. The analogue of Kronecker's first limit formula to Hilbert modular varieties has been studied, beginning with [As] for totally real fields and [E-G-M] for imaginary quadratic fields, then [J-L] for general number fields. Returning to the setting of $\mathrm{PSL}_2(\mathbb{R})$, the limit function $\mathcal{K}_1(z)$ has been determined for other groups in [Gn]; specific results for the Hecke congruence subgroups $\Gamma_0(N)$ are given in Section 10 below.

Our focus in this paper is to find formulas for the constant part at $s = 1$ of second-order Eisenstein series, which are defined by twisting the classical non-holomorphic Eisenstein series by a modular symbol. In general, a non-holomorphic second-order Eisenstein series $E^*(z, s)$ is associated to the following data: A Fuchsian group Γ of the first kind; a parabolic subgroup of Γ; and a weight two holomorphic form which vanishes in each cusp of Γ. The precise definition is given below. The series $E^*(z, s)$ was first defined and studied in [Gd] in order to provide another approach to the ABC-conjecture, which itself is connected to a number of fundamental and motivating problems in number theory, such as: Mordell's conjecture (a theorem of Faltings); Szpiro's conjecture; the degree conjecture; Goldfeld's period conjecture; and various questions and assertions regarding the Shafarevich-Tate group. In particular, we refer the reader to [Gd2] where Goldfeld states what he calls the Modular Symbol Conjecture, together with a summary of the inter-relations between the aforementioned conjectures as well as the role played by the Modular Symbol Conjecture. In [M-M], Manin and Marcolli generalized the classical Gauss-Kuzmin theorem having to do with the distribution of continued fractions. Going further, the authors develop connections between weighted averages of modular symbols, such as $E^*(z, s)$, and the distribution of continued fractions. The distribution of modular symbols themselves is elaborated by Petridis and Risager in [P-R] with their work on $E^*(z, s)$ and its generalizations. In [K-Z], Kleban and Zagier studied crossing probabilities and free energies for conformally invariant critical 2-D systems, which they derive from conformal field theory and certain stochastic integrals. It is shown in [K-Z] that the crossing probabilities and partition functions they encountered may be expressed as values of what should now be viewed as holomorphic second-order modular forms. As discussed in the concluding remarks of [K-Z], second-order forms in general can, in certain cases, be viewed as components of vector-valued modular forms associated to certain representations of the Fuchsian group

Γ into SL_2. In this way, the non-holomorphic second-order Eisenstein series, and second-order forms in general, are manifest in all aspects of the spectral theory, holomorphic function theory, number theory, and algebraic geometry of certain vector-valued functions on Riemann surfaces. In summary, second-order forms, which include $E^*(z,s)$, have at the present an established place in number theory [Gd2], [M-M], [P-R] and in physics [K-Z]; furthermore, additional connections to converse theorems in number theory, to spectral theory and to algebraic geometry are pending. As a result, any and all results regarding second-order forms should be viewed as interesting for their own sake as well as having wide yet unforeseen consequences.

For the purposes of narrowing our attention, we will concentrate on two aspects of the Kronecker limit formula: Differential equations, and Fourier expansions, with the latter necessarily requiring the study of the growth of the Fourier coefficients. Before stating our results, let us establish necessary background material and notation.

Let $S_k(\Gamma)$ be the space of holomorphic weight k cusp forms for Γ, meaning the vector space of holomorphic functions g on $\mathbf{H}$ which satisfy the transformation property

$$g(\gamma z) = j(\gamma, z)^k g(z) \quad \text{with} \quad j\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), z\right) = cz + d \quad \text{for} \quad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma,$$

and decay rapidly in each cusp in the quotient space $\Gamma\backslash\mathbf{H}$. As usual, we equip the vector space $S_k(\Gamma)$ with the well-known Petersson inner product. Since the analytic transformation $z \mapsto z+1$ corresponds to an element of Γ, we have that any $f \in S_k(\Gamma)$ admits a Fourier expansion, for which we use the notation

$$f(z) = \sum_{n=1}^{\infty} a_n e(nz) \quad \text{where} \quad e(z) = e^{2\pi i z},$$

and from which we define

$$F(z) = \sum_{n=1}^{\infty} \frac{a_n}{n} e(nz) = 2\pi i \int_{i\infty}^{z} f(w)\, dw.$$

For the remainder of this paper we set f to have weight two: $f \in S_2(\Gamma)$. The *modular symbol* $\langle\,\cdot\,, f\rangle$ associated to f is the homomorphism from Γ to $\mathbb{C}$ given by

$$\langle \gamma, f\rangle = 2\pi i \int_{z}^{\gamma z} f(w)\, dw = F(\gamma z) - F(z).$$

The *second-order* non-holomorphic Eisenstein series associated to f is defined as

$$E^*(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \langle\gamma, f\rangle \operatorname{Im}(\gamma z)^s.$$

In [O'S1] it is shown that this series converges absolutely and uniformly for $\operatorname{Re}(s) > 2$. The result (7.7) in Section 7 implies that

$$\langle\gamma, f\rangle \ll |\log \operatorname{Im}(\gamma z)| + |\log \operatorname{Im}(z)| + 1.$$

With this bound it follows that the series $E^*(z,s)$ is convergent for $\operatorname{Re}(s) > 1$. In [P-R] and [Ri] they also show convergence for $\operatorname{Re}(s) > 1$ using a different method.

For any $\gamma, \tau \in \Gamma$, the non-holomorphic Eisenstein series satisfy the transformation properties

$$E(\gamma z, s) - E(z,s) = 0, \tag{1.2}$$

$$E^*(\gamma\tau z, s) - E^*(\gamma z, s) - E^*(\tau z, s) + E^*(z,s) = 0. \tag{1.3}$$

In general, any function that transforms like (1.2) (resp. (1.3)) is called a *first-order* automorphic form (resp. *second-order* automorphic form). Both Eisenstein series are eigenfunctions of the hyperbolic Laplacian

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = -4y^2 \frac{d}{dz}\frac{d}{d\overline{z}},$$

meaning

$$\Delta E(z,s) = s(1-s)E(z,s),$$
$$\Delta E^*(z,s) = s(1-s)E^*(z,s).$$

The second-order Eisenstein series $E^*(z,s)$ is known to have a meromorphic continuation to all $s \in \mathbb{C}$ (see [Gd], [O'S1], [Pe]). In [G-O'S] it is shown that $E^*(z,s)$ has a simple pole at $s = 1$ with residue $-F(z)V^{-1}$, meaning

$$\lim_{s\to 1}\left(E^*(z,s) + F(z)\frac{V^{-1}}{s-1}\right) \quad \text{exists.}$$

Recalling that the first-order Eisenstein series $E(z,s)$ has a simple pole at $s = 1$ with residue V^{-1}, we also can say that

$$\lim_{s\to 1}\big(E^*(z,s) + F(z)E(s,z)\big) \quad \text{exists.}$$

We take as the second-order analogue of the Kronecker limit formula the study of the function

$$\mathcal{K}_2(z) = \lim_{s\to 1}\big(E^*(z,s) + F(z)E(z,s)\big)$$

for the following reason. By the definition of F, we have that

$$E^*(z,s) + F(z)E(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} F(\gamma z)\operatorname{Im}(\gamma z)^s,$$

which can be observed to be automorphic with respect to Γ for all s, in particular when s approaches 1. Therefore, the function $\mathcal{K}_2$ is necessarily Γ-invariant. Thus, in this notation,

$$\lim_{s\to 1}\left(E^*(z,s) + F(z)\frac{V^{-1}}{s-1}\right) = \mathcal{K}_2(z) - F(z)\mathcal{K}_1(z).$$

Before describing our results concerning the second-order Kronecker limit function $\mathcal{K}_2(z)$, we need the following theorem concerning the first-order Kronecker limit function $\mathcal{K}_1(z)$.

THEOREM 1.1. *The first-order Kronecker limit function $\mathcal{K}_1$ admits the Fourier expansion*

$$\mathcal{K}_1(z) = \sum_{n<0} k(n)e(n\overline{z}) + y + K - V^{-1}\log y + \sum_{n>0} k(n)e(nz) \tag{1.4}$$

with constants K and $k(n)$. Furthermore, $k(-n) = \overline{k(n)}$ and $k(n) \ll |n|^{1+\epsilon}$, with an implied constant which depends solely on Γ and $\epsilon > 0$.

We now can state the main results we obtain in our study of the second-order Kronecker limit function $\mathcal{K}_2$. To begin, we have the following theorem regarding the convolution Dirichlet series referred to in the title of the article.

THEOREM 1.2. *Fix a positive integer m, and let $k(0) = K + (\gamma + \log 4\pi m)/V$ where K refers to a component of the constant term in the Fourier expansion* (1.4) *of $\mathcal{K}_1$ and V is the hyperbolic volume of $\Gamma\backslash\mathbf{H}$. Formally, for $s \in \mathbb{C}$, define the convolution Dirichlet series*

$$L_m^+(s) = \sum_{n=1}^{\infty} \frac{a_n k(m-n)}{n^s}$$

and

$$L_m^-(s) = \sum_{n=1}^{\infty} \frac{a_n}{n} \frac{k(-m-n)}{(m+n)^{s-1}},$$

which are formed from the Fourier coefficients of f and $\mathcal{K}_1$. Then the series L_m^+ and L_m^- converge for $\mathrm{Re}(s) > 3$, *admit a meromorphic continuation to all s in $\mathbb{C}$ with* $\mathrm{Re}(s) > 1/2$, *and are holomorphic at $s = 1$.*

The usefulness of Theorem 1.2 will be evident in the results below regarding the Fourier expansion of $\mathcal{K}_2$.

It is known, and indeed is a elementary exercise, that by combining the differential equation for $E(z, s)$ with its Laurent expansion at $s = 1$, one can prove the differential equation $\Delta\mathcal{K}_1(z) = -V^{-1}$. As we will see below, the second-order analogue of this formula is the equation

$$\Delta\mathcal{K}_2(z) = -8\pi i y^2 f(z) \frac{d}{d\bar{z}}\mathcal{K}_1(z).$$

A more basic result would be to compute the differential equation satisfied by $\frac{d}{dz}\mathcal{K}_2(z)$ or by $\frac{d}{d\bar{z}}\mathcal{K}_2(z)$. We carry out these derivations, ultimately proving the following two theorems.

THEOREM 1.3. *Let Π_{hol} denote the holomorphic projection operator for the space of smooth, weight two automorphic forms into $S_2(\Gamma)$. Then*

$$\frac{1}{2\pi i} \frac{d}{dz}\mathcal{K}_2(z) = f(z)\mathcal{K}_1(z) - \Pi_{hol}\big(f(z)\mathcal{K}_1(z)\big).$$

Furthermore, if we set

$$\mathcal{K}_1^+(z) = \sum_{n>0} k(n) e(nz), \tag{1.5}$$

then we have

$$\Pi_{hol}\big(f(z)\mathcal{K}_1(z)\big) = \sum_{m=1}^{\infty} m L_m^+(1) e(mz) - \frac{1}{2\pi i} F(z) \frac{d}{dz}\mathcal{K}_1^+(z) + \frac{1}{4\pi} F(z).$$

THEOREM 1.4. *Let $W_s(z)$ be the classical Whittacker function associated to* $\mathrm{PSL}_2(\mathbb{R})$ *and set*

$$W^*(z) = \frac{d}{ds} W_s(z)\bigg|_{s=1} = \Gamma(0, 4\pi y) e^{4\pi y} e(z),$$

where $\Gamma(s,a)$ *denotes the incomplete gamma function*

$$\Gamma(s,a) = \int_a^\infty e^{-t}t^{s-1}\,dt.$$

Then

$$\frac{d}{d\overline{z}}\mathcal{K}_2(z) = \frac{-2\pi i}{V}\sum_{n=1}^{\infty} a_n W^*(nz) + \frac{i}{2Vy}F(z) + F(z)\frac{d}{d\overline{z}}\mathcal{K}_1(z) + 2\pi i\sum_{m=1}^{\infty} mL_m^-(1)e(-m\overline{z}).$$

As one would hope, Theorem 1.3 and Theorem 1.4 express the derivatives of $\mathcal{K}_2$ in terms of the initial information, namely $\mathcal{K}_1$ and f. Observe that either Theorem 1.3 or Theorem 1.4 can be used to compute $\Delta\mathcal{K}_2$; however, neither result can be used to derive the other.

Theorem 1.3 is appealing because of its relatively concise statement. Theorems 1.3 and 1.4 indicate the necessity in studying the Dirichlet series which are defined in Theorem 1.2. At this point, it must be noted that, in order to make sense out of Theorems 1.3 and 1.4, we need to have some idea as to the growth of the special values $L_m^+(1)$ and $L_m^-(1)$. Before doing so, we state the following result, which gives the complete Fourier expansion of the second-order Kronecker limit function $\mathcal{K}_2$.

THEOREM 1.5. *With notation as described above, the second-order Kronecker limit function* $\mathcal{K}_2(z)$ *admits the Fourier expansion*

$$\mathcal{K}_2(z) = \frac{-1}{V}\sum_{n=1}^{\infty}\frac{a_n}{n}W^*(nz) - \sum_{m=1}^{\infty} L_m^+(1)e(mz) - \sum_{m=1}^{\infty} L_m^-(1)e(-m\overline{z}) + F(z)\mathcal{K}_1(z).$$

Theorem 1.5 gives a complete description of the second-order Kronecker limit function associated to $E^*(z,s)$ at $s=1$. The new ingredients that are not fully understood are the special values $L_m^+(1)$ and $L_m^-(1)$. Theorem 1.2 asserts that $L_m^+(1)$ and $L_m^-(1)$ are finite for all m, but to show that the Fourier expansion in Theorem 1.5 makes sense we bound the special values $L_m^+(1)$ and $L_m^-(1)$. These bounds will imply that the series expansions in Theorems 1.3, 1.4 and 1.5 converge for all $z \in \mathbf{H}$.

THEOREM 1.6. *With the notation as above, we have the bounds*

$$L_m^+(1),\, L_m^-(1) \ll m^{1+\epsilon}$$

with an implied constant that depends solely on Γ, f *and* $\epsilon > 0$. *In addition, if*: (i) *the Fourier coefficients* a_n *of* f *are in* $\mathbb{R}$ *for all* n, *and* (ii) *we have that* $\iota(\Gamma) = \Gamma$ *where*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{\iota} \begin{pmatrix} -a & b \\ c & -d \end{pmatrix},$$

then the special values $L_m^+(1)$ *and* $L_m^-(1)$ *are also in* $\mathbb{R}$ *for all* $m \geqslant 1$.

To summarize, Theorem 1.1 establishes the Fourier expansion of the first-order Kronecker limit function $\mathcal{K}_1$ and sets notation to be used later. Theorem 1.2 defines two families of convolution Dirichlet series and asserts their meromorphic continuation and holomorphicity at $s = 1$. Theorem 1.3 and Theorem 1.4 state two different first-order differential equations which are satisfied by the second-order Kronecker limit function $\mathcal{K}_2$, and Theorem 1.5 gives its Fourier expansion. Bounds for the Fourier coefficients of $\mathcal{K}_1$ are given in Theorem 1.1, and Theorem 1.6 gives analogous bounds for the Fourier coefficients of $\mathcal{K}_2$. We believe that these results provide a complete investigation into analytic aspects of the Fourier series development for $\mathcal{K}_2$.

The outline of the paper is as follows. In Section 2 we initiate the development of the Fourier expansion of $\mathcal{K}_2$ and quickly find that

$$\mathcal{K}_2(z) = A(z) + B(z) + F(z)\mathcal{K}_1(z)$$

where $F\mathcal{K}_1$ is understood and A is very similar to F/V but non-holomorphic (in fact $\Delta A = F/V$). The main work in this paper is in understanding the term

$$B(z) = \sum_{n=1}^{\infty} \bigl(b_n e(nz) + b_{-n} e(-n\bar{z})\bigr).$$

The barrier to explicitly finding the constants b_n is that they come from the Fourier coefficients $\phi_n^*(s)$ of the second-order Eisenstein series $E^*(z,s)$. These coefficients involve Kloosterman sums twisted by modular symbols and their values are not known inside the critical strip $0 \leqslant \mathrm{Re}(s) \leqslant 1$ even for the simplest congruence groups.

In Section 3 we state, but do not prove, three key results: two on the analytic aspects of Poincaré series, both holomorphic of weight 2 and non-holomorphic, and a third concerning the holomorphic projection operator. Taken together these tools are powerful enough to probe the elements b_n. Because the proofs are so involved, we postpone verifying the statements of these results until later in the paper.

In Section 4, we obtain information about the holomorphic part of B by considering the holomorphic projection of the smooth, weight 2 function $\frac{d}{dz}\mathcal{K}_2(z)$. In the next section we show that the coefficients b_m, for $m > 0$, are given by the values of the convolution Dirichlet series $L_m^+(s)$ at $s = 1$. This proves Theorem 1.3. A similar idea is used in Section 6 to find the anti-holomorphic part of B in terms of $L_m^-(1)$, proving Theorem 1.4. Combining these two theorems produces Theorem 1.5. There seems to be no symmetry between the holomorphic and anti-holomorphic parts of B. This is to be expected since the definition of E^* includes a holomorphic cusp form f, breaking the symmetry.

In Section 7 we complete the proof of Theorem 1.1 (bounding the Fourier coefficients of $\mathcal{K}_1$) and prove Theorem 1.6 concerning the bounds on b_n. All results in Section 7 come from careful considerations involving the asyptotics of $E(z, s)$ and $E^*(z, s)$ as z approaches cusps. The crude bounds coming from the meromorphic continuation of these series are improved by a type of bootstrapping procedure. These results are independent of those in previous sections. At this time, there are a few remaining pieces to complete: The proofs of the results in Section 3 as well as the meromorphic continuations and regularity at $s = 1$ of $L_m^+(s)$ and $L_m^-(s)$. In Section 8 we use the spectral theory of automorphic forms to prove Theorem 3.1 and Theorem 3.2, and in Section 9 we prove Proposition 3.3 as well as the remaining properties regarding L_m^+ and L_m^- by introducing a type of non-holomorphic Poincaré series, $Q_m(z, s; F)$, that includes F in its definition.

Finally, in Section 10 we conclude with two types of examples: The first example shows how to explicitly evaluate the first-order Kronecker limit function $\mathcal{K}_1$ for the congruence subgroups $\Gamma_0(N)$ for square-free N, and the second example poses, as well as numerically investigates, a problem related to Theorem 1.3 involving the holomorphic projection operator.

The detailed, technical results in this paper begin in Section 7, then carry through to Sections 8 and 9. These precise calculations are used to prove the statements in Section 3 and the meromorphic continuation of L_m^+ and L_m^-, the details of which comprise the most difficult parts of our work. The arrangement of sections in this paper is meant to provide the motivation for each new result as it is needed and is purposefully consistent with our order of discovery.

§2. The Fourier expansion of $\mathcal{K}_2$

Our starting point is the Fourier expansions for the functions $E^*(z, s)$, $E(z, s)$ and $F(z)$, from which we obtain a somewhat general Fourier expansion for $\mathcal{K}_2(z)$. From [O'S1], page 164, we have that the Fourier expansion for the second-order Eisenstein series $E^*(z, s)$ is

$$E^*(z,s) = \sum_{n\neq 0} \phi_n^*(s) W_s(nz) \tag{2.1}$$

where W_s is the Whittaker function

$$W_s(nz) = 2|n|^{1/2} y^{1/2} K_{s-1/2}(2\pi|n|y) e(nx)$$

and K_s is the K-Bessel function

$$K_s(z) = \frac{1}{2}\int_0^\infty e^{-z(u+1/u)/2} u^s \frac{du}{u} \quad \text{for} \quad \mathrm{Re}(s) > 0.$$

Note that we have also used Corollary 4.3 of [O'S1] which proves that, in this instance, the second-order Eisenstein series has no constant term in its Fourier expansion. Exact formulas in terms of number theoretic functions are known for the Fourier coefficients of the first-order Eisenstein series $E(z, s)$ in the case when Γ is a congruence subgroup. In general, no such formulas are known for the coefficients $\phi_n^*(s)$. Let us use the following Laurent series:

$$\begin{aligned}
E(z,s) &= \frac{V^{-1}}{s-1} + \mathcal{K}_1(z) + O(s-1),\\
\phi_n^*(s) &= \frac{b_n(-1)}{s-1} + b_n(0) + O(s-1),\\
W_s(z) &= e(z) + W^*(z)(s-1) + O((s-1)^2)
\end{aligned}$$

where $W^*(z) = \frac{d}{ds} W_s(z)\big|_{s=1}$. In Corollary 2.2 below, we will prove the formula for $W^*(z)$ asserted in Theorem 1.4. Note that, by definition,

$W_s(z) = W_s(\bar{z})$ for z in the lower half plane. Substituting these expansions into the definition

$$\mathcal{K}_2(z) = \lim_{s\to 1}\big(E^*(z,s) + F(z)E(z,s)\big)$$

yields the expression

$$\mathcal{K}_2(z) = \lim_{s\to 1}\Bigg[\sum_{n\neq 0}\frac{b_n(-1)}{s-1}e(nz) + F(z)\frac{V^{-1}}{s-1} + \sum_{n=1}^{\infty}\big(b_n(0)e(nz) + b_{-n}(0)e(-n\bar{z})\big) + \sum_{n\neq 0} b_n(-1)W^*(nz) + F(z)\mathcal{K}_1(z)\Bigg].$$

Since the limit which defines $\mathcal{K}_2(z)$ exists, it is evident that we must have

$$b_n(-1) = \begin{Bmatrix} \dfrac{-a_n}{n}V^{-1} & n\geqslant 1 \\ 0 & \text{otherwise}\end{Bmatrix}.$$

Set $b_n = b_n(0)$ and, at this time, we can write

$$\mathcal{K}_2(z) = A(z) + B(z) + F(z)\mathcal{K}_1(z) \tag{2.2}$$

where

$$A(z) = \frac{-1}{V}\sum_{n=1}^{\infty}\frac{a_n}{n}W^*(nz), \tag{2.3}$$

$$B(z) = \sum_{n=1}^{\infty}\big(b_n e(nz) + b_{-n}e(-n\bar{z})\big). \tag{2.4}$$

To go further, we compute the resulting formula obtained by applying the Laplacian to $\mathcal{K}_1$, $\mathcal{K}_2$, A, and B. Using that

$$\begin{aligned}\Delta\left[\lim_{s\to 1}\left(E(z,s) - \frac{V^{-1}}{s-1}\right)\right] &= \lim_{s\to 1}\left[\Delta\left(E(z,s) - \frac{V^{-1}}{s-1}\right)\right] \\ &= \lim_{s\to 1}\big[s(1-s)E(z,s)\big]\end{aligned}$$

one shows that

$$\Delta\mathcal{K}_1(z) = -V^{-1}. \tag{2.5}$$

Similarly, we now consider

$$\Delta \lim_{s\to 1}\big(E^*(z,s)+F(z)E(z,s)\big) = \lim_{s\to 1}\Delta\big(E^*(z,s)+F(z)E(z,s)\big),$$

which can be easily computed. Since

$$\begin{aligned}&\Delta\big(E^*(z,s)+F(z)E(z,s)\big)\\&\quad= s(1-s)\big(E^*(z,s)+F(z)E(z,s)\big) - 8\pi i y^2 f(z)\frac{d}{d\bar z}E(z,s),\end{aligned}$$

we then obtain, by taking $s\to 1$, the formula

$$\Delta\mathcal{K}_2(z) = -8\pi i y^2 f(z)\frac{d}{d\bar z}\mathcal{K}_1(z). \tag{2.6}$$

Also $\Delta B(z)=0$, so then we have by (2.2) that

$$\Delta\mathcal{K}_2(z) = \Delta A(z) + \Delta(F(z)\mathcal{K}_1(z)),$$

which, when combined with (2.5) and (2.6), yields the equality

$$\Delta A(z) = F(z)V^{-1}. \tag{2.7}$$

In order to examine $A(z)$ more explicitly, we shall study $W^*(nz)$ by means of its definition in terms of K-Bessel functions. For this, we use that the K-Bessel function can be written as

$$K_{s-1/2}(2\pi y) = \frac{\sqrt{\pi}}{\Gamma(s)}(\pi y)^{s-1/2}\int_1^\infty (t^2-1)^{s-1}e^{-2\pi t y}\,dt. \tag{2.8}$$

(see page 205, [Iw1]). The integral in (2.8) converges absolutely for $\mathrm{Re}(s) > 0$ and $y>0$. We want to find $W^*(z) = \frac{d}{ds}W_s(z)\big|_{s=1}$.

LEMMA 2.1. *For all $y>0$, we have*

$$\frac{d}{ds}K_{s-1/2}(2\pi y)\bigg|_{s=1} = \Gamma(0,4\pi y)\frac{e^{2\pi y}}{2\sqrt{y}}.$$

Proof. Trivially, we have

$$\begin{aligned}&\frac{d}{ds}\int_1^\infty (t^2-1)^{s-1}e^{-2\pi t y}\,dt\bigg|_{s=1}\\&\quad= \int_1^\infty e^{-2\pi t y}\log(t-1)\,dt + \int_1^\infty e^{-2\pi t y}\log(t+1)\,dt\\&\quad= e^{-2\pi y}\int_0^\infty e^{-2\pi u y}\log u\,du + e^{2\pi y}\int_2^\infty e^{-2\pi u y}\log u\,du.\end{aligned}$$

Now
$$\int_0^\infty e^{-u} \log u \, du = \Gamma'(1),$$
where $\Gamma(s)$ denotes the classical gamma function. Therefore,
$$\int_0^\infty e^{-2\pi u y} \log u \, du = \frac{1}{2\pi y}(\Gamma'(1) - \log 2\pi y).$$
Through elementary computations, using integration by parts, one can show that
$$\int_2^\infty e^{-2\pi u y} \log u \, du = \frac{1}{2\pi y}\big(\Gamma(0, 4\pi y) + e^{-4\pi y} \log 2\big).$$
Combining these formulas, we obtain the relation
$$\begin{aligned} &\frac{d}{ds} \int_1^\infty (t^2-1)^{s-1} e^{-2\pi t y} \, dt \bigg|_{s=1} \\ &\quad = \frac{1}{2\pi y}\big(\Gamma(0, 4\pi y) e^{2\pi y} + (\Gamma'(1) - \log \pi y) e^{-2\pi y}\big). \end{aligned}$$
To complete the proof, one simply computes the derivative of (2.8) with respect to s and sets $s = 1$. Using that
$$\int_1^\infty (t^2-1)^{s-1} e^{-2\pi t y} \, dt \bigg|_{s=1} = \frac{e^{2\pi y}}{2\pi y},$$
the result follows from the standard rules of calculus. ∎

Corollary 2.2. *For z in $\mathbf{H}$ and $n \geqslant 1$, we have the following formulas*:
$$\begin{aligned} W^*(nz) &= \Gamma(0, 4\pi n y) e^{4\pi n y} e(nz), \\ \frac{d}{dz} W^*(nz) &= \frac{i}{2y} e(nz), \\ \frac{d}{d\overline{z}} W^*(nz) &= \frac{-i}{2y} e(nz) + 2\pi i n W^*(nz), \\ \Delta W^*(nz) &= -e(nz). \end{aligned}$$

Proof. The first identity follows directly from the definition of the Whittacker function in terms of the K-Bessel function, together with Lemma 2.1 and elementary calculus. The remaining three formulas are direct computations from the first expression, using nothing more than the fundamental theorem of calculus and standard formulas for differentiation of functions of one complex variable. ∎

These computations allow us to give a precise description of $A(z)$. Indeed, by definition we have

$$A(z) = \frac{-1}{V} \sum_{n=1}^{\infty} \frac{a_n}{n} W^*(nz),$$

so then Corollary 2.2 allows one to compute various derivatives of $A(z)$.

§3. Poincaré series and holomorphic projection

In order to continue studying the computations given in the previous section, we will use the holomorphic projection operator, whose basic properties we recall in the present section.

For any two smooth functions φ_1, φ_2 which transform with weight k, and have exponential decay at the cusps, the Petersson inner product between φ_1 and φ_2 is defined by

$$\langle \varphi_1, \varphi_2 \rangle_k = \int_{\Gamma \backslash \mathbf{H}} y^k \varphi_1(z) \overline{\varphi_2(z)} \, d\mu(z),$$

where $d\mu(z) = dxdy/y^2$ is the usual hyperbolic volume form. It can be shown that the Petersson inner product is non-degenerate on the space of holomorphic weight k cuspforms $S_k(\Gamma)$. Consequently, for any φ_1 as above, there exists a form $\Pi_{hol}(\varphi_1)$ in $S_k(\Gamma)$ such that for every g in $S_k(\Gamma)$

$$\langle \varphi_1, g \rangle_k = \langle \Pi_{hol}(\varphi_1), g \rangle_k.$$

The image $\Pi_{hol}(\varphi_1)$ of φ_1 into $S_k(\Gamma)$ is called the *holomorphic projection* of φ_1.

In the appendix of [Za1], beginning on page 286, it is shown that the Fourier coefficients of $\Pi_{hol}(\varphi_1)$ can be computed by taking g in the inner product above to be the weight k holomorphic Poincaré series. In Section 8 we construct and study aspects of these series relevant for our work. In order to proceed with the computations from the previous section, we shall state here various results regarding these Poincaré series, leaving the proofs of the assertions until Section 8.

Thus far we have only concerned ourselves with a single cusp, which we assumed was uniformized to be at the point at ∞. Let us now consider the possibility that an arbitrary (finite) number of Γ-inequivalent cusps exists. If there are other inequivalent cusps, let us fix representatives, label them

$\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \ldots$ and use the scaling matrices $\sigma_{\mathfrak{a}}, \sigma_{\mathfrak{b}}, \sigma_{\mathfrak{c}}, \ldots$ to give local coordinates near these cusps (see Chapter 2 of [Iw1] as well as [O'S1]). The subgroup $\Gamma_{\mathfrak{a}}$ is the set of elements of Γ which fixes the cusps equivalent to $\mathfrak{a}$, and we have that

$$\sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \Gamma_{\infty} = \left\{ \pm \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \,\middle|\, m \in \mathbb{Z} \right\}.$$

Following Selberg [Se], for each $m \geqslant 1$, we define the non-holomorphic Poincaré series associated to the cusp $\mathfrak{a}$ as

$$U_{\mathfrak{a}m}(z,s) = \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s e(m\sigma_{\mathfrak{a}}^{-1}\gamma z).$$

We shall also need $U'_{\mathfrak{a}m} = \frac{d}{dz} U_{\mathfrak{a}m}(z,s)$, the termwise derivative of $U_{\mathfrak{a}m}$.

To examine the growth of $U_{\mathfrak{a}m}$ and other automorphic functions, we follow the convention set in (2.42) of [Iw1] and introduce the useful notation

$$y_{\Gamma}(z) = \max_{\mathfrak{a}} \max_{\gamma \in \Gamma} (\operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)).$$

Heuristically, the function $y_{\Gamma}(z)$ measures how close the point $z \in \Gamma\backslash\mathbf{H}$ is to a cusp. If ψ (or $|\psi|$) is a smooth weight zero form (i.e., Γ-invariant function), then it is more convenient to write

$$\psi(z) \ll y_{\Gamma}(z)^A,$$

than, for example, writing that $\psi(\sigma_{\mathfrak{a}}z) \ll y^A$ for each cusp $\mathfrak{a}$ as $y \to \infty$.

THEOREM 3.1. *For all $m \geqslant 1$ and $\operatorname{Re}(s) > 1$, the series $U_{\mathfrak{a}m}(z,s)$ and $\frac{d}{dz}U_{\mathfrak{a}m}(z,s)$ are pointwise absolutely convergent and uniformly convergent for s in compact sets. Furthermore, both series admit meromorphic continuations to all $s \in \mathbb{C}$ which are analytic at $s = 1$. For $\operatorname{Re}(s) > 1/2$ we have the growth conditions*

$$U_{\mathfrak{a}m}(z,s) \ll |m|^{-1/2}\sqrt{y_{\Gamma}(z)}$$

and

$$yU'_{\mathfrak{a}m}(z,s) \ll |m|^{-1/2}\sqrt{y_{\Gamma}(z)}$$

with an implied constant depending on s and Γ alone.

Going further, let us define

$$V_{\mathfrak{a}m}(z,s) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s e(m\sigma_{\mathfrak{a}}^{-1}\gamma z) j(\sigma_{\mathfrak{a}}^{-1}\gamma, z)^{-2}$$

which can be viewed as a weight two version of $U_{\mathfrak{a}m}$. Formally, we would like to define our weight two holomorphic Poincaré series, which we will denote by $P_{\mathfrak{a}m}(z)_2$, to be given by $V_{\mathfrak{a}m}(z,0)$. However, as will be evident from the analysis of Section 8, the series defining $V_{\mathfrak{a}m}(z,s)$ is absolutely convergent only for $\mathrm{Re}(s) > 0$. In order to address this difficulty, we proceed as follows.

By a direct computation, one can easily show that for any $z \in \mathbf{H}$ and $\gamma \in \mathrm{PSL}_2(\mathbb{R})$, we have the identity

$$2i\frac{d}{dz}\big[\mathrm{Im}(\gamma z)^s e(m\gamma z)\big] = s\frac{\mathrm{Im}(\gamma z)^{s-1}}{j(\gamma,z)^2}e(m\gamma z) - 4\pi m\frac{\mathrm{Im}(\gamma z)^s}{j(\gamma,z)^2}e(m\gamma z).$$

By summing over all coset representatives $\gamma \in \Gamma_{\mathfrak{a}}\backslash\Gamma$, this implies the formula

$$sV_{\mathfrak{a}m}(z,s-1) = 2i\frac{d}{dz}U_{\mathfrak{a}m}(z,s) + 4\pi m V_{\mathfrak{a}m}(z,s), \tag{3.1}$$

which necessarily holds only in the half-plane of absolute convergence for both series which define $U_{\mathfrak{a}m}$ and $V_{\mathfrak{a}m}$. Therefore, in the light of Theorem 3.1, it makes (formal) sense to define the Poincaré series $P_{\mathfrak{a}m}(z)_2$ through the formula

$$P_{\mathfrak{a}m}(z)_2 = 2i\frac{d}{dz}U_{\mathfrak{a}m}(z,1) + 4\pi m V_{\mathfrak{a}m}(z,1).$$

We verify in Theorem 3.2 below that this does indeed give us a weight two holomorphic cusp form. Let us now examine how one can evaluate various inner products involving $P_{\mathfrak{a}m}(z)_2$.

Given a suitable function φ, we propose to evaluate $\langle \varphi, P_{\mathfrak{a}m}(\,\cdot\,)_2\rangle_2$ by first studying the meromorphic function

$$\langle \varphi, V_{\mathfrak{a}m}(\,\cdot\,,\overline{s}-1)\rangle_2 = \int_0^\infty\int_0^1 \varphi(z) y^{s-1}\overline{e(mz)}\,dxdy. \tag{3.2}$$

Under certain restrictions on φ the unfolded inner product on the right of (3.2) will converge for $\mathrm{Re}(s)$ large and may be computed to yield a function with a natural meromorphic continuation (for example involving gamma functions) to $s=1$. In this way (3.2) at $s=1$ yields an evaluation of $\langle \varphi, P_{\mathfrak{a}m}(\,\cdot\,)_2\rangle_2$. Indeed, we will follow this method to prove the following theorem.

THEOREM 3.2. *The weight two Poincaré series $P_{\mathfrak{a}m}(z)_2$ is in $S_2(\Gamma)$, the vector space of holomorphic weight two cusp forms with respect to Γ. Furthermore, for any f in $S_2(\Gamma)$ with*

$$j(\sigma_{\mathfrak{a}}, z)^{-2} f(\sigma_{\mathfrak{a}} z) = \sum_{n=1}^{\infty} a_{\mathfrak{a}}(n) e(nz)$$

we have that

$$\langle f, P_{\mathfrak{a}m}(\,\cdot\,)_2 \rangle_2 = a_{\mathfrak{a}}(m)/(4\pi m).$$

Frequently, we will assume that the cusp in question has been uniformized to be at ∞, so then, for ease of notation, we will set $U_m = U_{\infty m}$, $V_m = V_{\infty m}$ and $P_m = P_{\infty m}$. From the above discussion, we have the following. If φ is a smooth, bounded, continuous function on $\mathbf{H}$ which transforms like a weight two form with respect to the action by Γ, we then have

$$\Pi_{hol}(\varphi) = \sum_{m=1}^{\infty} d_m e(mz) \quad \text{where} \quad d_m = 4\pi m \langle \varphi, P_m(\,\cdot\,)_2 \rangle_2. \tag{3.3}$$

As stated above, the proofs of Theorem 3.1 and Theorem 3.2 will be given in Section 8 below.

In the forthcoming work, we will make use of the following proposition.

PROPOSITION 3.3. *Let φ_1 be a smooth weight zero form (function) and φ_2 a smooth form of weight two. Then:*

(i) *The form $\frac{d}{dz}\varphi_1$ is a weight two form;*

(ii) *The form $y^2 \frac{d}{d\bar{z}}\varphi_2$ is a weight zero form;*

(iii) *Assuming appropriate growth conditions on the functions near the cusps, we have the inner product formula*

$$\left\langle \frac{d}{dz}\varphi_1, \varphi_2 \right\rangle_2 = -\left\langle \varphi_1, y^2 \frac{d}{d\bar{z}}\varphi_2 \right\rangle_0.$$

The growth conditions are satisfied, for example, if φ_1 and $\frac{d}{dz}\varphi_1$ have at most polynomial growth in y in the cusps and if φ_2 and $y^2\frac{d}{d\bar{z}}\varphi_2$ have exponential decay in the cusps.

Proposition 3.3 will be proved as a corollary to Proposition 9.3, which states a more general result involving the Maass weight raising and lowering operators. We state the specific result here in order to continue with the calculations given in Section 2. We note that the proof of Proposition 2.1.3 of [Bu], which involves Stokes's theorem, may be adapted to yield a proof of Proposition 3.3. Rather than following this approach, our proof of Proposition 9.3 involves integration by parts together with some aspects of the first-order Eisenstein series, which gives an argument that extends to consider others pairs of forms with complementary weights.

Directly from Proposition 3.3, we have the following.

COROLLARY 3.4. *Let* φ *be a smooth, weight* 0 *function on* $\mathbf{H}$ *which is* Γ *invariant. Assume that* φ *and* $\frac{d}{dz}\varphi$ *have at most polynomial growth in the cusps of* $\Gamma\backslash\mathbf{H}$. *Then*

$$\Pi_{hol}\left(\frac{d}{dz}\varphi\right) = 0.$$

Proof. From Theorem 3.2, we have that the weight two Poincaré series is holomorphic, i.e.

$$\frac{d}{d\bar{z}}P_m(z)_2 = 0.$$

Corollary 3.4 now follows by using the second part of Theorem 3.2 together with Proposition 3.3. ∎

To re-iterate, the proofs of Theorem 3.1 and Theorem 3.2 will be given in Section 8, and the proof of Proposition 3.3 will be given in Section 9.

§4. $\mathcal{K}_2$ and the holomorphic projection of $f\mathcal{K}_1$

Using the material stated in Section 3, we now continue with the calculations from Section 2. Specifically, we will complete the proof of Theorem 1.3 in this section and the next.

Recall from (2.2) that we have written

$$\mathcal{K}_2(z) = A(z) + B(z) + F(z)\mathcal{K}_1(z),$$

with $A(z)$ and $B(z)$ defined in (2.3) and (2.4), respectively. Using Corollary 2.2, we then get the formula

$$\begin{aligned}(4.1)\quad \frac{d}{dz}\mathcal{K}_2(z) &= \frac{d}{dz}A(z) + \frac{d}{dz}B(z) + \frac{d}{dz}(F(z)\mathcal{K}_1(z)) \\ &= \frac{-i}{2Vy}F(z) + 2\pi i\sum_{n=1}^{\infty} nb_n e(nz) + F(z)\frac{d}{dz}\mathcal{K}_1(z) + 2\pi i f(z)\mathcal{K}_1(z).\end{aligned}$$

The right-hand side of (4.1) is a sum of two weight two forms. Since the holomorphic projection operator is linear, we then have

$$\begin{aligned}\Pi_{hol}\left(\frac{d}{dz}\mathcal{K}_2(z)\right) = \Pi_{hol}&\left(\frac{-i}{2Vy}F(z) + 2\pi i\sum_{n=1}^{\infty} nb_n e(nz) + F(z)\frac{d}{dz}\mathcal{K}_1(z)\right) \\ &+ \Pi_{hol}(2\pi i f(z)\mathcal{K}_1(z)).\end{aligned}$$

By Corollary 3.4, if $\mathcal{K}_2(z)$ and $\frac{d}{dz}\mathcal{K}_2(z)$ have polynomial growth at the cusps then

$$\Pi_{hol}\left(\frac{d}{dz}\mathcal{K}_2(z)\right) = 0.$$

We prove this polynomial growth in Lemma 4.1 at the end of this section. Continuing with our argument,

$$\begin{aligned}\Pi_{hol}&\left(\frac{-i}{2Vy}F(z) + 2\pi i\sum_{n=1}^{\infty} nb_n e(nz) + F(z)\frac{d}{dz}\mathcal{K}_1(z)\right) \\ &+ \Pi_{hol}(2\pi i f(z)\mathcal{K}_1(z)) = 0.\end{aligned}$$

Let

$$(4.2)\qquad g(z) = \frac{-i}{2Vy}F(z) + 2\pi i\sum_{n=1}^{\infty} nb_n e(nz) + F(z)\frac{d}{dz}\mathcal{K}_1(z).$$

We now will show that (4.2) is actually a holomorphic cusp form, and hence equal to its own holomorphic projection. Therefore $g = -2\pi i\,\Pi_{hol}(f\mathcal{K}_1)$ and substituting this back into (4.1) will complete the proof of the first part of Theorem 1.3.

Using the differential equation (2.5) for $\mathcal{K}_1$, we get

$$\begin{aligned}
\frac{d}{d\bar{z}}g(z) &= \frac{d}{d\bar{z}}\left(\frac{-i}{2Vy}F(z) + F(z)\frac{d}{dz}\mathcal{K}_1(z)\right) \\
&= \frac{-i}{2V}F(z)\frac{d}{d\bar{z}}(y^{-1}) + F(z)\frac{d^2}{dzd\bar{z}}\mathcal{K}_1(z) \\
&= \frac{-i}{2V}F(z)\frac{1}{2i}y^{-2} + F(z)\left(-\frac{1}{4y^2}(-V^{-1})\right) = 0,
\end{aligned}$$

hence g is holomorphic. It thus remains to show that g has exponential decay in each cusp, which will follow by studying its Fourier expansion in each cusp. In this generality, there are a number of analytic quantities associated with the cusp $\mathfrak{a}$. Using an obvious extension of notation established thus far, we define:

$$\begin{aligned}
E_{\mathfrak{a}}(z,s) &= \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s, \\
E^*_{\mathfrak{a}}(z,s) &= \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \langle\gamma, f\rangle \operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s, \\
F_{\mathfrak{a}}(z) &= 2\pi i\int_{\mathfrak{a}}^{z} f(w)\,dw, \\
\mathcal{K}_{1\mathfrak{a}}(z) &= \lim_{s\to 1}\left(E_{\mathfrak{a}}(z,s) - \frac{V^{-1}}{s-1}\right), \\
\mathcal{K}_{2\mathfrak{a}}(z) &= \lim_{s\to 1}\big(E^*_{\mathfrak{a}}(z,s) + F_{\mathfrak{a}}(z)E_{\mathfrak{a}}(z,s)\big).
\end{aligned}$$

The relevant Fourier expansions at the cusp $\mathfrak{b}$ are:

$$E_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z,s) = \delta_{\mathfrak{a}\mathfrak{b}}y^s + \phi_{\mathfrak{a}\mathfrak{b}}(s)y^{1-s} + \sum_{n\neq 0}\phi_{\mathfrak{a}\mathfrak{b}}(n,s)W_s(nz), \tag{4.3}$$

$$E^*_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z,s) = \phi^*_{\mathfrak{a}\mathfrak{b}}(0,s)y^{1-s} + \sum_{n\neq 0}\phi^*_{\mathfrak{a}\mathfrak{b}}(n,s)W_s(nz), \tag{4.4}$$

$$j(\sigma_{\mathfrak{b}},z)^{-2}f(\sigma_{\mathfrak{b}}z) = \sum_{n=1}^{\infty}a_{\mathfrak{b}}(n)e(nz),$$

$$F_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z) = T_{\mathfrak{a}\mathfrak{b}} + \sum_{n=1}^{\infty}\frac{a_{\mathfrak{b}}(n)}{n}e(nz), \tag{4.5}$$

where we define the period $T_{\mathfrak{a}\mathfrak{b}} = 2\pi i\int_{\mathfrak{a}}^{\mathfrak{b}} f(w)\,dw$. We refer to equation (3.20) [Iw1] for a proof of (4.3), and to equation (1.1) of [O'S1] for a proof of (4.4). Note that by Corollary 4.3 of [O'S1] we have that $\phi^*_{\mathfrak{a}\mathfrak{a}}(0,s) = 0$ which agrees with (2.1). We write the Laurent expansion of $\phi^*_{\mathfrak{a}\mathfrak{b}}(n,s)$ at $s = 1$ as

$$\phi^*_{\mathfrak{a}\mathfrak{b}}(n,s) = \frac{b_{\mathfrak{a}\mathfrak{b}}(n,-1)}{s-1} + b_{\mathfrak{a}\mathfrak{b}}(n,0) + O(s-1).$$

The analogue of (2.2) for $\mathcal{K}_{2\mathfrak{a}}$ at the cusp $\mathfrak{b}$ is then

$$\begin{aligned}(4.6)\quad \mathcal{K}_{2\mathfrak{a}}(\sigma_\mathfrak{b} z) = \frac{T_{\mathfrak{a}\mathfrak{b}}}{V}\log y + b_{\mathfrak{a}\mathfrak{b}}(0,0) + \frac{-1}{V}\sum_{n=1}^{\infty}\frac{a_\mathfrak{b}(n)}{n}W^*(nz) \\ + \sum_{n=1}^{\infty}\big(b_{\mathfrak{a}\mathfrak{b}}(n,0)e(nz) + b_{\mathfrak{a}\mathfrak{b}}(-n,0)e(-n\overline{z})\big) + F_\mathfrak{a}(\sigma_\mathfrak{b} z)\mathcal{K}_{1\mathfrak{a}}(\sigma_\mathfrak{b} z).\end{aligned}$$

Assuming the Fourier expansion

$$\begin{aligned}(4.7)\qquad \mathcal{K}_{1\mathfrak{a}}(\sigma_\mathfrak{b} z) = \sum_{n<0} k_{\mathfrak{a}\mathfrak{b}}(n)e(n\overline{z}) + \delta_{\mathfrak{a}\mathfrak{b}}y + k_{\mathfrak{a}\mathfrak{b}}(0) - V^{-1}\log y \\ + \sum_{n>0} k_{\mathfrak{a}\mathfrak{b}}(n)e(nz),\end{aligned}$$

which we will establish in this section below, we see that

$$\begin{aligned}(4.8)\qquad \frac{d}{dz}\mathcal{K}_{2\mathfrak{a}}(\sigma_\mathfrak{b} z) &= 2\pi i\sum_{n=1}^{\infty} nb_{\mathfrak{a}\mathfrak{b}}(n,0)e(nz) \\ &\quad + F_\mathfrak{a}(\sigma_\mathfrak{b} z)\left(\frac{-i}{2}\delta_{\mathfrak{a}\mathfrak{b}} + \frac{d}{dz}\left(\sum_{n>0} k_{\mathfrak{a}\mathfrak{b}}(n)e(nz)\right)\right) \\ &\quad + 2\pi i\ j(\sigma_\mathfrak{b}, z)^{-2} f(\sigma_\mathfrak{b} z)\mathcal{K}_{1\mathfrak{a}}(\sigma_\mathfrak{b} z).\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}(4.9)\quad j(\sigma_\mathfrak{b}, z)^{-2}g(\sigma_\mathfrak{b} z) &= \frac{d}{dz}\mathcal{K}_2(\sigma_\mathfrak{b} z) - 2\pi i\cdot j(\sigma_\mathfrak{b}, z)^{-2}f(\sigma_\mathfrak{b} z)\mathcal{K}_1(\sigma_\mathfrak{b} z) \\ &= 2\pi i\sum_{n=1}^{\infty} nb_{\mathfrak{a}\mathfrak{b}}(n,0)e(nz) \\ &\quad + F_\mathfrak{a}(\sigma_\mathfrak{b} z)\left(\frac{-i}{2}\delta_{\mathfrak{a}\mathfrak{b}} + \frac{d}{dz}\left(\sum_{n>0} k_{\mathfrak{a}\mathfrak{b}}(n)e(nz)\right)\right).\end{aligned}$$

If $\mathfrak{a} \neq \mathfrak{b}$, then (4.9) has rapid decay, which is seen by combining (4.5) together with the fact that $\delta_{\mathfrak{a}\mathfrak{b}} = 0$. If $\mathfrak{a} = \mathfrak{b}$, then in (4.5) we have that $T_{\mathfrak{a}\mathfrak{b}} = 0$, so we again conclude that (4.9) has rapid decay.

It needs to be verified that the expansion (4.7) holds. Indeed, this expansion follows directly from the Fourier expansion for the first-order non-holomorphic Eisenstein series, as stated in (4.3), together with the special function calculations given in the proof of Corollary 2.2. The important point is that the coefficient of $\log y$ is V^{-1}, which is implied by the fact that the Eisenstein series (4.3) has a first order pole at $s = 1$ with residue equal to V^{-1}. All of these properties of the Eisenstein series are proved in, for example, [Iw1] and [Kub]. This argument, which follows the method of calculation given in Section 2, gives the first part of Theorem 1.1. The bounds on the Fourier coefficients of $\mathcal{K}_1$, claimed in the second part of Theorem 1.1, are achieved in Section 7.

LEMMA 4.1.

$$\mathcal{K}_{1\mathfrak{a}}(z),\ y\frac{d}{dz}\mathcal{K}_{1\mathfrak{a}}(z),\ y\frac{d}{d\overline{z}}\mathcal{K}_{1\mathfrak{a}}(z) \ll y_\Gamma(z)$$
$$\mathcal{K}_{2\mathfrak{a}}(z),\ y\frac{d}{dz}\mathcal{K}_{2\mathfrak{a}}(z),\ y\frac{d}{d\overline{z}}\mathcal{K}_{2\mathfrak{a}}(z) \ll y_\Gamma(z).$$

Proof. With the Fourier expansion (4.7) we have that $\mathcal{K}_{1\mathfrak{a}}(z) \ll y_\Gamma(z)$. We used here the bound $k_{\mathfrak{a}\mathfrak{b}}(n) \ll |n|^{1+\epsilon}$ from Theorem 1.1 and proved in Section 7. Also, with (4.7), $\frac{d}{dz}\mathcal{K}_{1\mathfrak{a}}(\sigma_\mathfrak{b} z) \ll 1$ as $y \to \infty$ so that $y\frac{d}{dz}\mathcal{K}_{1\mathfrak{a}}(z) \ll y_\Gamma(z)$ and similarly for $y\frac{d}{d\overline{z}}\mathcal{K}_{1\mathfrak{a}}(z)$.

To treat $\mathcal{K}_{2\mathfrak{a}}(z)$ we use (4.6) and the following results:

(4.10) $$a_\mathfrak{b}(n) \ll n,$$

(4.11) $$F_\mathfrak{a}(\sigma_\mathfrak{b} z) \ll 1 \quad \text{as} \quad y \to \infty,$$

(4.12) $$W^*(nz) \ll (ny)^{-1/2} e^{-2\pi n y},$$

(4.13) $$b_{\mathfrak{a}\mathfrak{b}}(n,0) \ll |n|^{1+\epsilon}.$$

The estimate (4.11) follows from (4.5) and (4.10) which is the standard bound for the Fourier coefficients of cusp forms (see (5.7) of [Iw2]). The incomplete gamma function satisfies

$$|\Gamma(0,a)| = \left|\int_a^\infty e^{-t}t^{-1}\,dt\right| \leqslant \sqrt{\int_a^\infty e^{-2t}\,dt}\sqrt{\int_a^\infty t^{-2}\,dt} = \frac{e^{-a}}{\sqrt{2a}}.$$

Combine this with the first part of Corollary 2.2 to get (4.12). Lastly (4.13) is proved in Corollary 7.5.

Now (4.6) and (4.10)–(4.13) imply that $\mathcal{K}_{2\mathfrak{a}}(z) \ll y_\Gamma(z)$. Similarly, with (4.8) and the equivalent result for $\frac{d}{dz}\mathcal{K}_{2\mathfrak{a}}(z)$ we obtain the remaining parts of the lemma. ∎

With all this, the proof of the first statement of Theorem 1.3 is complete, and, indeed, we have shown that for any cusp $\mathfrak{a}$,

$$\frac{1}{2\pi i}\frac{d}{dz}\mathcal{K}_{2\mathfrak{a}}(z) = f(z)\mathcal{K}_{1\mathfrak{a}}(z) - \Pi_{hol}\big(f(z)\mathcal{K}_{1\mathfrak{a}}(z)\big).$$

It remains to give the stated expression for $\Pi_{hol}\big(f(z)\mathcal{K}_{1\mathfrak{a}}(z)\big)$ in the second part of Theorem 1.3. This is carried out next.

§5. The Dirichlet series L_m^+ and the holomorphic projection of $f\mathcal{K}_1$

We continue by studying the Fourier coefficients of $\Pi_{hol}\big(f(z)\mathcal{K}_{1\mathfrak{a}}(z)\big)$. As in the previous section, we will use the results stated in Section 3, whose proofs we will give in Section 8. In the notation established in Section 1, let us write

$$L_m^{++}(s) = \sum_{n=m+1}^{\infty} \frac{a_n k(m-n)}{n^s}, \tag{5.1}$$

so then, referring to the notation from Theorem 1.2, we have

$$L_m^+(s) = \sum_{n=1}^{m-1} \frac{a_n k(m-n)}{n^s} + \frac{a_m}{m^s}\left(K + \frac{\gamma + \log 4\pi m}{V}\right) + L_m^{++}(s).$$

Proposition 5.1. *With notation as above and for* $\mathrm{Re}(s)$ *sufficiently large, we have the identity*

$$\begin{aligned}\langle f\mathcal{K}_1, V_m(\,\cdot\,,\overline{s}-1)\rangle_2 &= \frac{\Gamma(s)}{(4\pi)^s}L_m^{++}(s) + \frac{\Gamma(s)}{(4\pi m)^s}\sum_{l=1}^{m-1} a_l k(m-l)\\ &\quad + \frac{a_m}{(4\pi m)^s}\left(\frac{\Gamma(s+1)}{4\pi m} + K\Gamma(s) + \frac{-\Gamma'(s) + \Gamma(s)\log 4\pi m}{V}\right).\end{aligned}$$

Proof. This is carried out using the ideas of Section 3, in particular (3.2), and follows the line of standard computations. First, expand f and

$\mathcal{K}_1$ in their Fourier expansions, i.e.

$$f(z)\mathcal{K}_1(z) = \left(\sum_{n=1}^{\infty} a_n e(nz)\right) \\ \times\left(\sum_{n<0} k(n)e(n\overline{z}) + y + K - V^{-1}\log y + \sum_{n>0} k(n)e(nz)\right).$$

Next, unfold the integral in question, similar to (3.2), and carry out the integral, ultimately using standard formulas for the classical Γ function. ∎

Remark 5.2. As already mentioned, the trivial bound for coefficients of a weight two cusp form states that $a_n \ll n$. In Section 7 below we will prove, as asserted in Theorem 1.1, that $k(n) \ll |n|^{1+\epsilon}$. Therefore, it follows that $L_m^+(s)$ is absolutely and uniformly convergent for $\mathrm{Re}(s) > 3$, as claimed in Theorem 1.2. The meromorphic continuation of $L_m^+(s)$ will follow from the expression derived in Proposition 5.1 together with a study of the Poincaré series $V_m(\,\cdot\,, s)$.

We now work with the expression

$$\langle f\mathcal{K}_1, P_m(\,\cdot\,)_2\rangle_2 = \langle f\mathcal{K}_1, V_m(\,\cdot\,, \overline{s}-1)\rangle_2|_{s=1},$$

in order to compute the Fourier coefficients of $\Pi_{hol}(f(z)\mathcal{K}_1(z))$. Recall Theorem 3.2 and the discussion preceding it where the natural holomorphic, weight 2 Poincaré series, $P_m(z)_2$, is defined as $V_m(z,s)$ analytically continued to $s = 0$. To begin, let us make sure that we have, or will, establish enough results regarding $V_m(z,s)$ to proceed. Assuming Theorem 3.1, from which we obtained (3.1), then $V_m(z,s)$ has a meromorphic continuation to all $s \in \mathbb{C}$. Again, Theorem 3.1 will be proved in Section 8, at which time it also will be shown that $V_m(z,s)$ has at most polynomial growth in y at the cusps. Therefore, $\langle f\mathcal{K}_1, V_m(\,\cdot\,, \overline{s}-1)\rangle_2$ converges to a meromorphic function. Proposition 5.1 holds for $\mathrm{Re}(s) > 3$, as stated in Remark 5.2, so then we now have the meromorphic continuation of the Dirichlet $L_m^{++}(s)$, and hence $L_m^+(s)$, to all $s \in \mathbb{C}$. The continuation $L_m^+(s)$ will not have a pole at $s = 1$ once it has been shown that $V_m(z, s-1)$ does not have a pole at $s = 1$. Hence, the stated results in Section 3, together with the growth condition for $V_m(z,s)$ which also comes from Section 8, are sufficient to allow us to continue our calculations.

Working directly with the Fourier expansion, (1.4), of $\mathcal{K}_1$ and recalling the definition of $\mathcal{K}_1^+$ in (1.5) we show that

$$\frac{d}{dz}\mathcal{K}_1(z) = \frac{d}{dz}\mathcal{K}_1^+(z) + \frac{i}{2Vy} - \frac{i}{2}.$$

Combining this with equation (4.2), as well as subsequent discussion, we get

$$\begin{aligned}\Pi_{hol}(f(z)\mathcal{K}_1(z)) \\ &= \frac{1}{2\pi i}\left(\frac{i}{2Vy}F(z) - F(z)\frac{d}{dz}\mathcal{K}_1(z)\right) - \sum_{n=1}^{\infty} nb_n e(nz), \\ &= \frac{1}{2\pi i}\left(\frac{i}{2}F(z) - F(z)\frac{d}{dz}\mathcal{K}_1^+(z)\right) - \sum_{n=1}^{\infty} nb_n e(nz).\end{aligned} \tag{5.2}$$

Let us write the Fourier expansion of $\Pi_{hol}(f(z)\mathcal{K}_1(z))$ as

$$\Pi_{hol}(f(z)\mathcal{K}_1(z)) = \sum_{m=1}^{\infty} d_m e(mz).$$

If we now substitute the Fourier expansions of F and $\mathcal{K}_1$ into (5.2) we find the formula

$$d_m = -mb_m + \frac{a_m}{4\pi m} - \sum_{l=1}^{m-1} \frac{a_l}{l}(m-l)k(m-l).$$

However, from Theorem 3.2 and Proposition 5.1, for all $m \geqslant 1$, we also have that

$$d_m = mL_m^{++}(1) + \frac{a_m}{4\pi m} + \sum_{l=1}^{m-1} a_l k(m-l) + a_m\left(K + \frac{\gamma + \log 4\pi m}{V}\right).$$

Therefore, by the definition of the Dirichlet series L_m^+, as first stated in Theorem 1.2, we conclude that for all $m \geqslant 1$, we have

$$b_m = -L_m^+(1). \tag{5.3}$$

Substituting (5.3) into (5.2) yields the Fourier expansion claimed in Theorem 1.3, whose proof is now complete.

§6. The Dirichlet series L_m^- and the proofs of Theorems 1.4 and 1.5

Let us first prove Theorem 1.4. To do so, we start with (2.2) and, using Corollary 2.2, obtain the formula

$$\frac{d}{dz}\overline{\mathcal{K}_2}(z) = \frac{-i}{2Vy}\overline{F}(z) + \frac{2\pi i}{V}\sum_{n=1}^{\infty}\overline{a_n}\overline{W^*}(nz) + \frac{d}{dz}\overline{B(z)} + \overline{F}(z)\frac{d}{dz}\overline{\mathcal{K}_1}(z) \tag{6.1}$$

which, in particular, implies that

$$\Pi_{hol}\left(\frac{d}{dz}\overline{\mathcal{K}_2}(z)\right) = \Pi_{hol}\left(\frac{-i}{2Vy}\overline{F}(z) + \frac{2\pi i}{V}\sum_{n=1}^{\infty}\overline{a_n}\overline{W^*}(nz) + \frac{d}{dz}\overline{B(z)} + \overline{F}(z)\frac{d}{dz}\overline{\mathcal{K}_1}(z)\right). \tag{6.2}$$

By Corollary 3.4 and Lemma 4.1, the left-hand-side of (6.2) is zero. Using Theorem 3.2 and equations (3.2) and (3.3), we can compute the m-th Fourier coefficient of the right-hand-side, which, since the left-hand-side vanishes, is necessarily zero. That is, we have that

$$0 = 4\pi m\int_0^{\infty}\int_0^1\left(\frac{-i\overline{F}(z)}{2Vy} + \frac{2\pi i}{V}\sum_{n=1}^{\infty}\overline{a_n}\overline{W^*}(nz) + \frac{d}{dz}\overline{B(z)} + \overline{F}(z)\frac{d}{dz}\overline{\mathcal{K}_1}(z)\right)y^{s-1}\overline{e(mz)}\,dxdy\bigg|_{s=1}.$$

In order to evaluate this, substitute the Fourier expansions for F and $\mathcal{K}_1$, as well as the formula,

$$W^*(nz) = \Gamma(0, 4\pi ny)e^{4\pi ny}e(nz).$$

Upon integrating with respect to x, we produce the equality

$$\begin{aligned}(2\pi i)4\pi m\int_0^{\infty}&\left(m\overline{b_{-m}}e^{-4\pi my} + \sum_{l=1}^{\infty}\frac{\overline{a_l}}{l}(m+l)\overline{k(-m-l)}e^{-4\pi(m+l)y}\right)y^{s-1}\,dy \\ &= \frac{2\pi im\Gamma(s)}{(4\pi)^{s-1}}\left(\frac{\overline{b_{-m}}}{m^{s-1}} + \overline{L_m^-(s)}\right),\end{aligned}$$

where

$$L_m^-(s) = \sum_{n=1}^{\infty}\frac{a_n}{n}\frac{k(-m-n)}{(m+n)^{s-1}}.$$

Now, by taking $s = 1$, we get that

$$b_{-m} = -L_m^-(1) \tag{6.3}$$

for all $-m < 0$ or $m > 0$, provided, of course, that $L_m^-(s)$ has an analytic continuation to $s = 1$ which would then allow for the above computations. The verification that $L_m^-(s)$ admits a meromorphic continuation to, (and is analytic at), $s = 1$ will be completed in Section 9. In effect, we will argue as follows. Recall that we have already used the bounds $a_n \ll n$ and $k(n) \ll |n|^{1+\epsilon}$. Observe that these bounds prove $L_m^-(s)$ is absolutely and uniformly convergent to an analytic function for $\mathrm{Re}(s) > 3$. In Section 9 we will prove the functional equation

$$L_m^-(s) = mL_m^-(s+1) + \frac{2i(4\pi)^s}{\Gamma(s+1)}\Big\langle y^2 f(z)\frac{d}{d\overline{z}}\mathcal{K}_1(z), \overline{U_m(z,s)}\Big\rangle, \tag{6.4}$$

where the Poincaré series $U_m(z,s)$ was introduced in Section 3. From (6.4), it is immediate that L_m^- does not have a pole at $s = 1$, which then completes the proof of Theorem 1.4.

Furthermore, this work yields Theorem 1.5. Indeed, from (5.3) and (6.3) we have shown that

$$b_m = \left\{\begin{matrix} -L_m^+(1) & m \geqslant 1 \\ \\ -L_{-m}^-(1) & m \leqslant -1 \end{matrix}\right\}.$$

Substituting into (2.2), and using the equations (2.3) and (2.4), then completes the proof of Theorem 1.5.

Remark 6.1. As an aside, let us study the right-hand-side of (6.2) and show that it can be reduced further. Let $H(z) = \overline{F}(z)\frac{d}{dz}\overline{\mathcal{K}_1}(z)$, so then, when using the relation $F(\gamma z) = F(z) + \langle \gamma, f\rangle$, we have that

$$H(\gamma z) = j(\gamma, z)^2\left(H(z) + \overline{\langle\gamma, f\rangle}\frac{d}{dz}\overline{\mathcal{K}_1}(z)\right). \tag{6.5}$$

In other words, $H(z)$ is a weight two, second-order automorphic form. For any $g \in S_2(\Gamma)$, we claim that $\langle H, g\rangle$ is well-defined. To see this, first choose a fundamental domain $\mathfrak{F}$ for $\Gamma\backslash\mathbf{H}$ and, for now, let

$$\langle H, g\rangle_{\mathfrak{F}} = \int_{\mathfrak{F}} y^2 H(z)\overline{g}(z)\, d\mu(z).$$

For any $\gamma \in \Gamma$, it is easy to show, using the transformation property for g and (6.5), that

$$\begin{aligned}\langle H, g\rangle_{\gamma\mathfrak{F}} &= \int_{\gamma\mathfrak{F}} y^2 H(z)\overline{g}(z)\, d\mu(z) \\ &= \int_{\mathfrak{F}} y^2 H(z)\overline{g}(z)\, d\mu(z) + \overline{\langle \gamma, f\rangle} \int_{\mathfrak{F}} y^2 \frac{d}{dz}\overline{\mathcal{K}_1}(z)\overline{g}(z)\, d\mu(z).\end{aligned}$$

By Corollary 3.4,

$$\left\langle \frac{d}{dz}\overline{\mathcal{K}_1}, g\right\rangle = 0,$$

which shows that $\langle H, g\rangle_{\mathfrak{F}}$ is Γ invariant, hence $\langle H, g\rangle$ is well-defined as claimed. Consequently, $\Pi_{hol}(H)$ makes sense and hence exists. Similar reasoning applies to the remaining part on the right-hand-side of (6.2). As a result, since

$$\Pi_{hol}\left(\frac{d}{dz}\overline{\mathcal{K}_2}(z)\right) = 0,$$

by Corollary 3.4, (6.2) can be written as

$$\begin{aligned} 0 = \Pi_{hol}&\left(-\frac{i}{2Vy}\overline{F}(z) + \frac{2\pi i}{V}\sum_{n=1}^{\infty}\overline{a_n}\overline{W^*}(nz) + \frac{d}{dz}\overline{B^-(z)}\right) \\ &+ \Pi_{hol}\left(\overline{F}(z)\frac{d}{dz}\overline{\mathcal{K}_1}(z)\right).\end{aligned}$$

Possible implications of this identity have not been investigated here.

§7. Bounding the Fourier coefficients of $\mathcal{K}_1$ and $\mathcal{K}_2$

In this section we estimate the size of the Fourier coefficients of $E(z,s)$ and $E^*(z,s)$. The calculations are used to bound $k(n)$, $L_m^+(1)$, and $L_m^-(1)$.

To begin, we need the following general result.

LEMMA 7.1. *Suppose $D(z) = D(x+iy)$ is a smooth function on $\mathbf{H}$ which is Γ invariant. Assume there is a continuous function $B(y)$ such that for each cusp $\mathfrak{a}$ of Γ, we have that $|D(\sigma_{\mathfrak{a}}z)| \leqslant B(y)$ as $y \to \infty$. Then we also have*

$$\begin{aligned} &D(\sigma_{\mathfrak{a}}z) \ll 1 && \text{as } y \to 0, \text{ if } B \text{ is decreasing and} \\ &D(\sigma_{\mathfrak{a}}z) \ll B(C/y) && \text{as } y \to 0, \text{ if } B \text{ is increasing}\end{aligned}$$

where both implied constants and $C > 0$ depend only on D and Γ (and are independent of x).

Proof. By conjugation we may assume (as we have been doing all along) that ∞ is a cusp of $\Gamma\backslash\mathbf{H}$ and that Γ_∞ is generated by the translation $z \mapsto z+1$. Let $\mathcal{F}_\infty = \{z \in \mathbf{H} \mid |\operatorname{Re}(z)| \leqslant 1/2\}$, and let $\mathcal{F}$ be the (Ford) fundamental domain for $\Gamma\backslash\mathbf{H}$ defined by $\mathcal{F} = \{z \in \mathcal{F}_\infty \mid 1 < |j(\gamma, z)| \text{ for all } \gamma \in \Gamma - \Gamma_\infty\}$.

The first statement of the Lemma is easy to prove. If B is decreasing then, since D is smooth, $D(z)$ is bounded on $\mathcal{F}$ and hence on $\mathbf{H}$ since it is Γ invariant. The bound $D(\sigma_{\mathfrak{a}}z) \ll 1$ as $y \to 0$ follows trivially.

Let us assume now that $B(y)$ is increasing as $y \to \infty$. For the cusp at infinity, by assumption,

$$D(w) \ll B(\operatorname{Im}(w)) \quad \text{as} \quad \operatorname{Im}(w) \to \infty.$$

We next consider what happens as $w \in \mathcal{F}$ approaches a cusp $\mathfrak{a} \in \mathbb{R}$. Set $w = \sigma_{\mathfrak{a}} w'$ so that $w \to \mathfrak{a}$ as $\operatorname{Im}(w') \to \infty$. It is easy to check that

$$\frac{1}{\operatorname{Im}(w')} \ll \operatorname{Im}(\sigma_{\mathfrak{a}} w') \ll \frac{1}{\operatorname{Im}(w')} \quad \text{as} \quad \operatorname{Im}(w') \to \infty$$

if $\sigma_{\mathfrak{a}}$ is not upper triangular and that

$$\operatorname{Im}(w') \ll \operatorname{Im}(\sigma_{\mathfrak{a}} w') \ll \operatorname{Im}(w') \quad \text{as} \quad \operatorname{Im}(w') \to \infty$$

if $\sigma_{\mathfrak{a}}$ is upper triangular. Since $\operatorname{Im}(w) \to 0$ as $\operatorname{Im}(w') \to \infty$ it must be the case that $\sigma_{\mathfrak{a}}$ is not upper triangular and hence, for some $C > 0$,

$$\operatorname{Im}(w') \leqslant \frac{C}{\operatorname{Im}(w)}.$$

By assumption

$$D(\sigma_{\mathfrak{a}} w') \ll B(\operatorname{Im}(w')) \quad \text{as} \quad \operatorname{Im}(w') \to \infty.$$

Therefore

$$D(w) \ll B(C/\operatorname{Im}(w)) \quad \text{as} \quad w \to \mathfrak{a}$$

in $\mathcal{F}$ and it follows that, for a possibly larger C,

$$D(w) \ll B(\operatorname{Im}(w) + C/\operatorname{Im}(w))$$

for all $w \in \mathcal{F}$.

Now, for any $z \in \mathcal{F}_\infty - \mathcal{F}$, there exists $\gamma \in \Gamma - \Gamma_\infty$ such that $\gamma z = w \in \mathcal{F}$. It can be show that

$$y \leqslant \operatorname{Im}(w) \ll \frac{1}{y},$$

where the first inequality comes from the definition of $\mathcal{F}$ and the second from Lemma 1.25 of [Sh] (see also Proposition 2.5 of [G-O'S]). The implied constant in the upper bound depends only on Γ. It now follows that, for any z in $\mathbf{H}$, we have

$$D(z) \ll B(C/y) \quad \text{as} \quad y \to 0$$

with the implied constant and (larger) C depending only on D and Γ. Finally, to prove the same bound for $D(\sigma_{\mathfrak{a}} z)$ we may use the same proof applied to the conjugate group $\Gamma' = \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}$. Specifically, let $D'(z) = D(\sigma_{\mathfrak{a}} z)$ then D' is a smooth Γ' invariant function. Now if $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \dots$ are a set of inequivalent cusps for $\Gamma\backslash\mathbf{H}$ then $\mathfrak{a}' = \sigma_{\mathfrak{a}}^{-1}\mathfrak{a} = \infty, \mathfrak{b}' = \sigma_{\mathfrak{a}}^{-1}\mathfrak{b}, \mathfrak{c}' = \sigma_{\mathfrak{a}}^{-1}\mathfrak{c}, \dots$ are a set of inequivalent cusps for $\Gamma'\backslash\mathbf{H}$ with corresponding scaling matrices $\sigma_{\mathfrak{a}'} = \sigma_{\mathfrak{a}}^{-1}\sigma_{\mathfrak{a}}, \sigma_{\mathfrak{b}'} = \sigma_{\mathfrak{a}}^{-1}\sigma_{\mathfrak{b}}, \sigma_{\mathfrak{c}'} = \sigma_{\mathfrak{a}}^{-1}\sigma_{\mathfrak{c}}, \dots$. Therefore, for any cusp $\mathfrak{b}'$ of Γ' we have

$$|D'(\sigma_{\mathfrak{b}'} z)| = |D(\sigma_{\mathfrak{b}} z)| \leqslant B(y) \quad \text{as} \quad y \to \infty.$$

It now follows from our previous work that $D'(z) \ll B(C/y)$ as $y \to 0$, completing the proof. $\square$

To continue, we recall that equation (6.19) of [Iw1] states an explicit bound for the Fourier coefficients of the first-order Eisenstein series, namely

$$\phi_{\mathfrak{a}\mathfrak{b}}(n,s) \ll |n|^{\sigma} + |n|^{1-\sigma},$$

with an implied constant depending on s and Γ. We will prove our stated bounds for the Fourier coefficients of $\mathcal{K}_2$ by making this bound for $\phi_{\mathfrak{a}\mathfrak{b}}(n,s)$ more precise, as well as extend the result to the functions $\phi^*_{\mathfrak{a}\mathfrak{b}}(n,s)$. The main technical result of this section is the following.

PROPOSITION 7.2. *For each compact set S in $\mathbb{C}$ there exist smooth functions $\psi_1(s)$, $\psi_2(s)$ and holomorphic functions $\xi_1(s)$, $\xi_2(s)$ so that*

$$|\phi_{\mathfrak{a}\mathfrak{b}}(n,s)| \leqslant \frac{\psi_1(s)}{|\xi_1(s)|}\big(|n|^{\sigma} + |n|^{1-\sigma}\big), \tag{7.1}$$

$$|\phi^*_{\mathfrak{a}\mathfrak{b}}(n,s)| \leqslant \frac{\psi_2(s)}{|\xi_2(s)|}(\log|n| + 1)\big(|n|^{\sigma} + |n|^{1-\sigma}\big) \tag{7.2}$$

for all s in S and all $n \neq 0$. The functions ψ_1, ξ_1 depend on S and Γ, and the functions ψ_2, ξ_2 depend on S, Γ and f.

Proof. The bound (7.1) will follow from the proof of the meromorphic continuation of the first-order Eisenstein series $E_{\mathfrak{a}}(z,s)$ as given in Proposition 6.1 of [Iw1]. After proving (7.1), we then employ the same method of proof, this time using the meromorphic continuation of the second-order Eisenstein series $E^*_{\mathfrak{a}}(z,s)$ as given in Theorem 3.8 of [O'S1]. For ease of notation, ψ and ξ will always represent smooth and holomorphic functions respectively, though the functions themselves may change from line to line.

From Proposition 6.1 of [Iw1], we have the following (weaker) form of the stated result. Given a compact subset S of $\mathbb{C}$, there exist functions $A_{\mathfrak{a}}(s) \not\equiv 0$ on S and $A_{\mathfrak{a}}(z,s)$ on $\mathbf{H} \times S$ such that:

(1) $A_{\mathfrak{a}}(z,s) = A_{\mathfrak{a}}(s)E_{\mathfrak{a}}(z,s)$ on $\{s \mid \mathrm{Re}(s) > 1\} \cap S$,
(2) $A_{\mathfrak{a}}(s)$ and $A_{\mathfrak{a}}(z,s)$ are holomorphic in s,
(3) $A_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z,s) \ll e^{\varepsilon y}$ for each cusp $\mathfrak{b}$ and $y \geqslant 1$ say. The implied constant depends on $\varepsilon > 0$, s and Γ.

Furthermore, from the proof of Proposition 6.1 in [Iw1], specifically (6.1), we conclude there exists a smooth function $\psi(s)$ on S so that

$$A_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z,s) \ll_{\varepsilon,\Gamma} \psi(s)\, e^{\varepsilon y}.$$

The Fourier expansion of $E_{\mathfrak{a}}(z,s)$, namely

$$E_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z,s) = \delta_{\mathfrak{a}\mathfrak{b}}y^s + \phi_{\mathfrak{a}\mathfrak{b}}(s)y^{1-s} + \sum_{n\neq 0} \phi_{\mathfrak{a}\mathfrak{b}}(n,s)W_s(nz)$$

then gives

$$\phi_{\mathfrak{a}\mathfrak{b}}(n,s) \cdot 2\sqrt{|n|y}K_{s-1/2}(2\pi|n|y) = \int_0^1 E_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z,s)e^{-2\pi inx}\,dx.$$

Using (1) above and Lemma 7.1, we then obtain the bound

$$\phi_{\mathfrak{a}\mathfrak{b}}(n,s) \cdot 2\sqrt{|n|y}K_{s-1/2}(2\pi|n|y) \ll \frac{\psi(s)}{|A_{\mathfrak{a}}(s)|}e^{\varepsilon/y} \tag{7.3}$$

as $y \to 0$. The K-Bessel function can be bounded using the estimate

$$\sqrt{y}K_{s-1/2}(y) = \sqrt{\frac{\pi}{2}}e^{-y}\Big(1 + O\Big(\frac{1+|s|^2}{y}\Big)\Big) \tag{7.4}$$

for $y > 1 + |s|^2$, which we quote from [Iw1], formula B.36. With this, and upon setting $y = 1/\sqrt{|n|}$, we get the auxiliary estimate

$$\phi_{\mathfrak{ab}}(n,s) \ll \frac{\psi(s)}{|A_{\mathfrak{a}}(s)|} e^{3\pi\sqrt{|n|}}.$$

Consequently,

$$\begin{aligned} |E_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z,s)| &= \left|\delta_{\mathfrak{ab}}y^s + \phi_{\mathfrak{ab}}(s)y^{1-s} + \sum_{n\neq 0}\phi_{\mathfrak{ab}}(n,s)W_s(nz)\right| \\ &\ll y^\sigma + |\phi_{\mathfrak{ab}}(s)|y^{1-\sigma} + \frac{\psi(s)}{|\xi(s)|}\sum_{n\neq 0} e^{3\pi\sqrt{|n|}-2\pi|n|y} \\ &\ll \frac{\psi(s)}{|\xi(s)|}\Big(y^\sigma + y^{1-\sigma} + e^{2\pi(1/2-y)}\Big) \end{aligned}$$

for $y \geqslant 1$ say. Now, repeat the argument yielding (7.3) with this new bound to get

$$\phi_{\mathfrak{ab}}(n,s) \ll \frac{\psi(s)}{|\xi(s)|}\frac{y^{-\sigma} + y^{-(1-\sigma)}}{\sqrt{|n|y}K_{s-1/2}(2\pi|n|y)}. \tag{7.5}$$

Letting $y = 1/|n|$, the proof of (7.1) is complete.

An easy consequence of (7.1) that we shall need shortly is the next result.

Corollary 7.3. *For each compact set S in $\mathbb{C}$ there exist ψ smooth and ξ holomorphic such that*

$$E_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z,s) - \delta_{\mathfrak{ab}}y^s - \phi_{\mathfrak{ab}}(s)y^{1-s} \ll \frac{\psi(s)}{|\xi(s)|}e^{-2\pi y}$$

as $y \to \infty$ and

$$E_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z,s) \ll \frac{\psi(s)}{|\xi(s)|}\big(y^\sigma + y^{-\sigma} + y^{1-\sigma} + y^{\sigma-1}\big) \tag{7.6}$$

for all y in $(0,\infty)$ and all $s \in S$. The implied constants depending only on S and Γ.

The proof of (7.2) follows the same pattern, in this case using Theorem 3.8 of [O'S1] rather than Proposition 6.1 of [Iw1]. To begin, for any compact $S \subset \mathbb{C}$, there are functions $A^*_{\mathfrak{a}}(s) \not\equiv 0$ on S and $A^*_{\mathfrak{a}}(z,s)$ on $\mathbf{H} \times S$ such that:

(1) $A^*_{\mathfrak{a}}(z,s) = A^*_{\mathfrak{a}}(s)E^*_{\mathfrak{a}}(z,s)$ on $\{s \mid \mathrm{Re}(s) > 2\} \cap S$,

(2) $A^*_{\mathfrak{a}}(s)$ and $A^*_{\mathfrak{a}}(z,s)$ are holomorphic in s,

(3) $A^*_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z,s) \ll e^{\varepsilon y}$ for each cusp $\mathfrak{b}$, $y \geqslant 1$ and implied constant depending on $\varepsilon > 0$, s, f and Γ.

Following the method of proof of Proposition 6.1 in [Iw1], the analysis in [O'S1] yields the bound

$$A^*_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z,s) \underset{\varepsilon,f,\Gamma}{\ll} \psi(s)e^{\varepsilon y}.$$

Lemma 7.1 applies to a weight zero function (Γ invariant). For this, we study

$$G_{\mathfrak{a}}(z,s) = E^*_{\mathfrak{a}}(z,s) + F_{\mathfrak{a}}(z)E_{\mathfrak{a}}(z,s),$$

which, as stated in Section 1, is Γ invariant. Let us write

$$F_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z) = 2\pi i \int_{\mathfrak{a}}^{\sigma_{\mathfrak{b}}z} f(w)\,dw = 2\pi i \int_{\sigma_{\mathfrak{b}}^{-1}\mathfrak{a}}^{z} g(w)\,dw$$

with $g(z) = f(\sigma_{\mathfrak{b}}z)/j(\sigma_{\mathfrak{b}},z)^2 \in S_2(\sigma_{\mathfrak{b}}^{-1}\Gamma\sigma_{\mathfrak{b}})$. Therefore,

$$\int_z^{z+1} g(w)\,dw = 0 \quad \text{and} \quad g(z) \ll 1/y$$

(see (5.3), [Iw2]). Consequently, we have, for each pair of cusps $\mathfrak{a}$, $\mathfrak{b}$ and all y in $(0,\infty)$, the bound

$$F_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z) \underset{f,\Gamma}{\ll} |\log y| + 1. \tag{7.7}$$

(Note: This estimate improves Lemma 1.1 of [O'S1]; see also [Ri], [P-R] for a different approach to this and similar bounds.) Continuing, the bounds for the Eisenstein series $E^*(z,s)$ and $E(z,s)$, together with (7.7) imply that as $y \to \infty$, we have

$$G_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z,s) \ll \frac{\psi(s)}{|\xi(s)|}e^{\varepsilon y}.$$

Thus by Lemma 7.1,

$$G_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z,s) \ll \frac{\psi(s)}{|\xi(s)|}e^{\varepsilon/y} \quad \text{as} \quad y \to 0.$$

With (7.6) and (7.7), we then obtain

$$E^*_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, s) \ll \frac{\psi(s)}{|\xi(s)|} e^{\varepsilon/y} \quad \text{as} \quad y \to 0.$$

By repeating the argument used to prove (7.1), we get the auxiliary estimate

$$\phi^*_{\mathfrak{ab}}(n, s) \ll \frac{\psi(s)}{|\xi(s)|} e^{3\pi\sqrt{|n|}}$$

so then

$$E^*_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, s) \ll \frac{\psi(s)}{|\xi(s)|} y^{1-\sigma} \quad \text{as} \quad y \to \infty.$$

Therefore

$$\begin{aligned} & G_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, s) \ll \frac{\psi(s)}{|\xi(s)|}(|\log y| + 1)(y^{\sigma} + y^{1-\sigma}) \quad \text{as} \quad y \to \infty, \\ \text{implies} \quad & G_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, s) \ll \frac{\psi(s)}{|\xi(s)|}(|\log y| + 1)(y^{-\sigma} + y^{\sigma-1}) \quad \text{as} \quad y \to 0, \\ \text{implies} \quad & E^*_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, s) \ll \frac{\psi(s)}{|\xi(s)|}(|\log y| + 1)(y^{-\sigma} + y^{\sigma-1}) \quad \text{as} \quad y \to 0. \end{aligned}$$

With this improved bound the equality (7.2) follows in the same manner that (7.1) was proved. This completes the proof of Proposition 7.2. ∎

The analogue of Corollary 7.3 follows from Proposition 7.2.

COROLLARY 7.4. *For s contained in a compact set S in $\mathbb{C}$ we have ψ smooth and ξ holomorphic with*

$$E^*_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, s) - \phi^*_{\mathfrak{ab}}(0, s) y^{1-s} \ll \frac{\psi(s)}{|\xi(s)|} e^{-2\pi y}$$

as $y \to \infty$ and the implied constant depending only on S, f and Γ.

Another consequence of Proposition 7.2 gives our desired bounds for the sequence $\{b_n\}$.

COROLLARY 7.5. *For every n, write*

$$\phi^*_{\mathfrak{ab}}(n, s) = \sum_{m=-1}^{\infty} b_{\mathfrak{ab}}(n, m)(s-1)^m.$$

Then for every $m \geqslant -1$ and every $\epsilon > 0$ we have

$$b_{\mathfrak{a}\mathfrak{b}}(n,m) \underset{m,\epsilon,f,\Gamma}{\ll} |n|^{1+\epsilon}.$$

In particular $b_n = b(n,0) \ll |n|^{1+\epsilon}$.

Proof. Let C_ϵ be a circular loop around 1 with small radius ϵ. We know that

$$b_{\mathfrak{a}\mathfrak{b}}(n,m) = \frac{1}{2\pi i}\int_{C_\epsilon} \frac{\phi^*_{\mathfrak{a}\mathfrak{b}}(n,s)}{(s-1)^{m+1}}\,ds$$

and by (7.2) the desired conclusion follows. ∎

Bounding the coefficients $k(n)$ of $\mathcal{K}_1$: The proof of Corollary 7.5 also applies directly to the definition of $\mathcal{K}_1$ (on replacing (7.2) with (7.1)) to give the bounds

$$k(n),\ k(-n) \ll n^{1+\epsilon}$$

for any $\epsilon > 0$, as asserted in Theorem 1.1. The identity $k(n) = \overline{k(-n)}$ follows from the symmetry $\overline{E(z,\overline{s})} = E(z,s)$ because $\overline{W_{\overline{s}}(z)} = W_s(-\overline{z})$ and therefore $\phi_{-n}(s) = \overline{\phi_n}(\overline{s})$. With this, the proof of Theorem 1.1 is complete. ∎

Bounding the Fourier coefficients of $\mathcal{K}_2$: The Fourier coefficients of $\mathcal{K}_2$ are expressed in terms of the Fourier coefficients of F, $\mathcal{K}_1$, and the sequence $\{b_n\}$. Known results bound the Fourier coefficients of F, Theorem 1.1 (whose proof is now complete) bounds the Fourier coefficients of $\mathcal{K}_1$, and Corollary 7.5 bounds the elements of the sequence $\{b_n\}$. Though it remains to prove that $b_m = L^+_m(1)$ for $m \geqslant 1$ and $b_m = L^-_{-m}(1)$ for $m \leqslant -1$, the Fourier coefficient bounds are complete nonetheless. To continue, let us further analyze the Fourier coefficients $\{b_m\}$.

If f has Fourier coefficients $\{a_n\}$ in $\mathbb{R}$ for all $n > 0$ then we want to show that the Fourier coefficients of $\mathcal{K}_2$ $\{b_m\}$ are also in $\mathbb{R}$, provided we have $\iota(\Gamma) = \Gamma$ for $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \xrightarrow{\iota} \left(\begin{smallmatrix} -a & b \\ c & -d\end{smallmatrix}\right)$. The map ι is an automorphism of $\mathrm{PSL}_2(\mathbb{R})$, and it is easily verified that $\gamma(-\overline{z}) = -\overline{(\iota(\gamma)z)}$ for any $\gamma \in \mathrm{PSL}_2(\mathbb{R})$. From this it follows that $E(-\overline{z},s) = E(z,s)$ for any subgroup Γ of $\mathrm{PSL}_2(\mathbb{R})$ with $\iota(\Gamma) = \Gamma$, and hence $\phi_m(s) = \phi_{-m}(s)$. Since f has real Fourier coefficients we see that

$$\langle \iota(\gamma), f\rangle = \overline{\langle \gamma, f\rangle},$$

and then

$$\begin{aligned} E^*(z,s) &= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \langle \gamma, f\rangle \operatorname{Im}(\gamma z)^s = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \langle \iota(\gamma), f\rangle \operatorname{Im}(\iota(\gamma) z)^s \\ &= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \overline{\langle \gamma, f\rangle} \operatorname{Im}(\gamma(-\overline{z}))^s = \overline{E^*(-\overline{z},\overline{s})}. \end{aligned}$$

Therefore, $\phi_m^*(s) = \overline{\phi_m^*(\overline{s})}$ which implies that $b_m = \overline{b_m}$.

§8. Poincaré series: Proofs of Theorems 3.1 and 3.2

We now prove Theorem 3.1 and Theorem 3.2. In essence, the material in this section is based on [Se], Chapter 17 of [Iw3] and [Ne]. The weight k Poincaré series is defined by the series

$$P_{\mathfrak{a}m}(z)_k = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \frac{e(m\sigma_{\mathfrak{a}}^{-1}\gamma z)}{j(\sigma_{\mathfrak{a}}^{-1}\gamma, z)^k}. \tag{8.1}$$

The series (8.1) converges absolutely and uniformly if $k > 2$ but not when $k = 2$. Hecke addressed this problem by introducing a complex parameter s and taking a limit. We will follow this approach employing the non-holomorphic Poincaré series $U_{\mathfrak{a}m}(z,s)$ from Section 3. If $m = 0$ we have that $U_{\mathfrak{a}0}(z,s) = E_{\mathfrak{a}}(z,s)$. Since the non-holomorphic Eisenstein series is absolutely convergent for $\operatorname{Re}(s) > 1$, we have that the function $E_{\mathfrak{a}}(z, \operatorname{Re}(s))$ is a majorant of $U_{\mathfrak{a}m}(z,s)$ for $m \geqslant 0$.

LEMMA 8.1. *For $m \geqslant 1$ and $\operatorname{Re}(s) > 1$ the Poincaré series $U_{\mathfrak{a}m}(z,s)$ is square integrable, i.e. $U_{\mathfrak{a}m}(z,s)$ is in $L^2(\Gamma\backslash\mathbf{H})$.*

Proof. We first examine the size of $U_{\mathfrak{a}m}$ in the neighborhood of each cusp. Setting $s = \sigma + it$, we have

$$\begin{aligned} |U_{\mathfrak{a}m}(\sigma_{\mathfrak{a}} z, s)| &\ll y^\sigma e^{-2\pi m y} + \sum_{\substack{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma \\ \gamma\neq\text{identity}}} \operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{a}} z)^\sigma \\ &\ll y^\sigma e^{-2\pi m y} + |E_{\mathfrak{a}}(\sigma_{\mathfrak{a}} z, \sigma) - y^\sigma| \ll 1. \end{aligned}$$

At any other cusp $\mathfrak{b} \neq \mathfrak{a}$

$$U_{\mathfrak{a}m}(\sigma_{\mathfrak{b}} z, s) \ll E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, \sigma) \ll \phi_{\mathfrak{a}\mathfrak{b}}(s) y^{1-\sigma} \ll 1.$$

In other words, $U_{\mathfrak{a}m}$ is bounded on $\Gamma\backslash\mathbf{H}$ and hence in $L^2(\Gamma\backslash\mathbf{H})$ since $\Gamma\backslash\mathbf{H}$ has finite volume. ∎

We will study the Poincaré series $U_{\mathfrak{a}m}(\sigma_{\mathfrak{a}}z, s)$ by means of its spectral expansion, which we now recall (see, for example, [Iw1] and references therein for further background information and complete proofs). The hyperbolic Laplacian Δ operates on the space $L^2(\Gamma\backslash\mathbf{H})$, and any element ξ of $L^2(\Gamma\backslash\mathbf{H})$ may be decomposed into constituent parts from the discrete and continuous spectrum of Δ. This decomposition, often referred to as the Roelcke-Selberg expansion, amounts to the identity

$$(8.2)\qquad \xi(z) = \sum_{j=0}^{\infty}\langle \xi, \eta_j\rangle \eta_j(z) + \frac{1}{4\pi}\sum_{\mathfrak{b}}\int_{-\infty}^{\infty}\langle \xi, E_{\mathfrak{b}}(\,\cdot\,, 1/2+ir)\rangle E_{\mathfrak{b}}(z, 1/2+ir)\,dr,$$

where $\{\eta_j\}$ denotes a complete orthonormal basis of Maass forms, with corresponding eigenvalues $\lambda_j = s_j(1-s_j)$, which forms the discrete spectrum. For notational convenience, we wrote $\langle\,\cdot\,,\,\cdot\,\rangle = \langle\,\cdot\,,\,\cdot\,\rangle_0$ for the inner product on $\Gamma\backslash\mathbf{H}$ of weight zero forms (i.e. Γ-invariant functions). As always, we will write $s_j = \sigma_j + it_j$, chosen so that $\sigma_j \geqslant 1/2$ and $t_j \geqslant 0$, and we enumerate the eigenvalues, counted with multiplicity, by $0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$. For each j, the Fourier expansion of η_j is

$$(8.3)\qquad \eta_j(\sigma_{\mathfrak{a}}z) = \rho_{\mathfrak{a}j}(0)y^{1-s_j} + \sum_{m\neq 0}\rho_{\mathfrak{a}j}(m)W_{s_j}(mz).$$

For all but finitely many of the j (corresponding to $\lambda_j < 1/4$) we have $\sigma_j = 1/2$ and $\rho_{\mathfrak{a}j}(0) = 0$. The expansion (8.2) is absolutely convergent for each fixed z and uniform on compact subsets of $\mathbf{H}$, provided ξ and $\Delta\xi$ are smooth and bounded (see, for example, Theorem 4.7 and Theorem 7.3 of [Iw1]). By taking $\xi = U_{\mathfrak{a}m}$, we then obtain the spectral expansion for the Poincaré series, which yields the identity

$$(8.4)\qquad \begin{aligned} &U_{\mathfrak{a}m}(z,s)\pi^{-1/2}(4\pi m)^{s-1/2}\Gamma(s)\\ &\quad = \sum_{j=1}^{\infty}\Gamma(s-s_j)\Gamma(s-1+s_j)\overline{\rho_{\mathfrak{a}j}}(m)\eta_j(z)\\ &\qquad + \frac{1}{4\pi}\sum_{\mathfrak{b}}\int_{-\infty}^{\infty}\Gamma(s-1/2-ir)\Gamma(s-1/2+ir)\\ &\qquad\qquad \times\overline{\phi_{\mathfrak{a}\mathfrak{b}}}(m, 1/2+ir)E_{\mathfrak{b}}(z, 1/2+ir)\,dr.\end{aligned}$$

The expansion (8.4) includes the identity

$$\langle U_{\mathfrak{a}m}(\,\cdot\,,s),\eta_j\rangle = \frac{\pi^{1/2}\Gamma(s-s_j)\Gamma(s-1+s_j)}{(4sm)^{s-1/2}\Gamma(s)}\overline{\rho_{\mathfrak{a}j}}(m),$$

with a similar formula which evaluates the inner product of the Poincaré series $U_{\mathfrak{a}m}(z,s)$ with the Eisenstein series $E_{\mathfrak{a}}(z,s)$. The proofs of these formulas come from unfolding the integrals under study and unfolding the series which defines the Poincaré series. These calculations we leave for the interested reader. When looking toward Theorem 3.2, the appearance of the coefficients $\rho_{\mathfrak{a}j}(m)$ and $\phi_{\mathfrak{a}\mathfrak{b}}(m,1/2+ir)$ is natural since $U_{\mathfrak{a}m}$ isolates m-th Fourier coefficients (see Theorem 3.2, and, more specifically, see [Ne] or Chapter 17 of [Iw3]).

Initially, (8.4) is valid for $\mathrm{Re}(s) > 1$. The remainder of this section shows that (8.4) converges absolutely and uniformly in s in compact subsets not containing a number of the form $s_j - n$ or $1 - s_j - n$ for $n \in \mathbb{N}$. These points are poles caused by the factors $\Gamma(s-s_j)\Gamma(s-1+s_j)$. Going further, we will prove bounds regarding the growth in z of $U_{\mathfrak{a}m}(z,s)$ and $\frac{d}{dz}U_{\mathfrak{a}m}(z,s)$. These computations will yield the proofs of Theorem 3.1 and Theorem 3.2.

To control the size of $\rho_{\mathfrak{a}j}(m)$ and $\phi_{\mathfrak{a}\mathfrak{b}}(m,1/2+ir)$ we appeal to the following formula of Bruggeman and Kuznetsov, as stated in (9.13) of [Iw1]. With notation as above, let

$$\begin{aligned} N_{\mathfrak{a}}(T) = &\sum_{|t_j|<T} |\Gamma(s_j)\Gamma(1-s_j)||\rho_{\mathfrak{a}j}(m)|^2 \\ &+ \frac{1}{4\pi}\sum_{\mathfrak{b}}\int_{-T}^{T} |\Gamma(1/2+ir)\Gamma(1/2-ir)||\phi_{\mathfrak{a}\mathfrak{b}}(m,1/2+ir)|^2\,dr. \end{aligned}$$

Then

$$N_{\mathfrak{a}}(T) = \frac{T^2}{2\pi|m|} + O(T) \quad \text{as} \quad T\to\infty, \tag{8.5}$$

with an implied constant which depends solely on the discrete group Γ. Recall that Stirling's formula states that the classical gamma function satisfies the bound

$$|\Gamma(\sigma+it)| \sim \sqrt{2\pi}|t|^{\sigma-1/2}e^{-\pi|t|/2} \quad \text{as} \quad |t|\to\infty.$$

For simplicity, we may assume that $T > 0$. By combining Stirling's formula with (8.5), we get the bounds

$$|\rho_{\mathfrak{a}j}(m)|^2 \ll \frac{|t_j|^2}{|m|} e^{\pi|t_j|}, \tag{8.6}$$

and

$$\int_T^{T+1} |\phi_{\mathfrak{a}\mathfrak{b}}(m, 1/2 + ir)|^2 \, dr \ll \frac{T^2}{|m|} e^{\pi T}. \tag{8.7}$$

We next need bounds concerning K-Bessel functions and Whittacker functions.

Lemma 8.2. *For any integer $k \geqslant 0$ we have, for $\sigma > 1/2 - k$, the bounds*

$$|K_{s-1/2}(y)| \ll \frac{|s|^{2k} + 1}{y^{2k-1/2+\sigma}} |\Gamma(s)| \tag{8.8}$$

and

$$\left| \frac{d}{dy} K_{s-1/2}(y) \right| \ll \frac{|s|^{2k+1} + 1}{y^{2k+1/2+\sigma}} |\Gamma(s)|, \tag{8.9}$$

where the implied constant depends solely on σ and k.

Proof. First consider the case $k = 0$. From page 205 of [Iw1], we have the expression

$$K_{s-1/2}(y) = \frac{1}{\sqrt{\pi}} \Gamma(s) \left(\frac{y}{2} \right)^{1/2-s} \int_0^\infty (u^2 + 1)^{-s} \cos(uy) \, du \tag{8.10}$$

which is absolutely convergent for $\sigma > 1/2$. Trivially, this gives (8.8) with $k = 0$. Next, we recall the recursive formula

$$K_{s-1/2}(y) = \frac{2s + 1}{y} K_{s+1/2}(y) - K_{s+3/2}(y)$$

which comes from integrating (8.10) through integration by parts. The recursive relation provides the inductive step by which (8.8) follows from (8.10) for all $k \geqslant 0$. Similarly, (8.9) follows from (8.10) with $k = 0$, and the general case is then derived using the identity

$$\frac{d}{dy} K_{s-1/2}(y) = \frac{s - 1/2}{y} K_{s-1/2}(y) - K_{s+1/2}(y).$$

$\square$

Recall that

$$W_s(z) = 2y^{1/2}K_{s-1/2}(2\pi y)e(x).$$

Therefore, from Lemma 8.2, we see that for any $k \geqslant 0$ and $\sigma > 1/2 - k$, we have the bounds

$$W_s(nz) \ll \frac{|s|^{2k}+1}{(|n|y)^{2k-1+\sigma}}|\Gamma(s)|, \tag{8.11}$$

and

$$\frac{d}{dz}W_s(nz) \ll \left(1 + \frac{|s|+1}{y}\right)\frac{|s|^{2k}+1}{(|n|y)^{2k-1+\sigma}}|\Gamma(s)|, \tag{8.12}$$

where the implied constants depend solely on σ and k.

Recall the definition of $y_\Gamma(z)$ before Theorem 3.1. The estimates (8.6), (8.11), and (8.12) now can be combined with the Fourier expansion (8.3) to show that if $\sigma_j = 1/2$, then

$$\eta_j(z) \ll y_\Gamma(z)^{1/2} + |t_j|^{7/2}y_\Gamma(z)^{-3/2}, \tag{8.13}$$

and

$$y\frac{d}{dz}\eta_j(z) \ll y_\Gamma(z)^{1/2} + |t_j|^{9/2}y_\Gamma(z)^{-3/2} \tag{8.14}$$

(compare, for example, with (8.3') and (8.4) of [Iw1]). Our argument at this point shows that

$$\sum_{j=1}^{\infty}\Gamma(s-s_j)\Gamma(s-1+s_j)\overline{\rho_{\mathfrak{a}j}}(m)\eta_j(z) \ll |m|^{-1/2}y_\Gamma(z)^{1/2},$$

where the implied constant depends on s and Γ. Clearly, the dependence of this bound on s is uniform on compact sets not containing $s_j - n$, $1 - s_j - n$ for $n \in \mathbb{N}$. In other words, the term in (8.4) associated to the discrete spectrum admits a meromorphic continuation to all $s \in \mathbb{C}$ and, as claimed in Theorem 3.1, we have the desired growth in the cusps. It remains to consider the integral term in (8.4). For this, we begin with the following proposition, which can be compared to (7.10) of [Iw1].

PROPOSITION 8.3. *For any cusp $\mathfrak{a}$ and $z \in \Gamma\backslash\mathbf{H}$, we have the bounds*

$$\int_T^{T+1} |E_{\mathfrak{a}}(z, 1/2+ir)|^2\, dr \ll y_\Gamma(z)T^{10}, \tag{8.15}$$

and

$$\int_T^{T+1} \left| y \frac{d}{dz} E_{\mathfrak{a}}(z, 1/2 + ir) \right|^2 dr \ll y_\Gamma(z) T^{12}, \tag{8.16}$$

where the implied constant depends solely on Γ.

Proof. The proof will follow by studying the Fourier expansion (4.3). From the functional equation for the scattering matrix (Theorem 6.6 of [Iw1]), we obtain the estimate

$$\phi_{\mathfrak{ab}}(1/2 + ir) \ll 1.$$

Therefore, with (8.11),

$$\begin{aligned} &E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, 1/2 + ir) \\ &\quad \ll \sqrt{y} + \sum_{m \neq 0} |\phi_{\mathfrak{ab}}(m, 1/2 + ir)|(|r|^{2k} + 1)|\Gamma(1/2 + ir)|(|m|y)^{2k-1/2}. \end{aligned}$$

Consequently

$$\begin{aligned} &\int_T^{T+1} |E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, 1/2 + ir)|^2 \, dr \\ &\quad \ll y + T^{2k} e^{-\pi T/2} y^{1-2k} \int_T^{T+1} \sum_{m \neq 0} |\phi_{\mathfrak{ab}}(m, 1/2 + ir)||m|^{1/2-2k} \, dr \\ &\qquad + T^{4k} e^{-\pi T} y^{1-4k} \int_T^{T+1} \sum_{m_1 \neq 0} \sum_{m_2 \neq 0} |\phi_{\mathfrak{ab}}(m_1, 1/2 + ir) \phi_{\mathfrak{ab}}(m_2, 1/2 + ir)| \\ &\qquad\qquad \times |m_1 m_2|^{1/2-2k} \, dr. \end{aligned}$$

To complete the proof of Proposition 8.3 we need to interchange integration and summation in the above expression. The following lemma allows one to employ the Lebesgue dominated convergence theorem, for example, to do this.

LEMMA 8.4. *For* $r \in [T, T+1]$ *we have*

$$\phi_{\mathfrak{ab}}(m, 1/2 + ir) \ll |m|^2$$

for an implied constant depending on T, Γ *only.*

Proof. As in Proposition 7.2, we write $E_{\mathfrak{a}}(z,s)$ as a quotient of holomorphic functions $A_{\mathfrak{a}}(z,s)/A_{\mathfrak{a}}(s)$, which is valid for s in S where, in this instance, S is the line segment between $1/2+iT$ and $1/2+i(T+1)$. Theorem 6.11 of [Iw1] states that $E_{\mathfrak{a}}(z,s)$ has no poles on S, in particular, so we may assume, after multiplying the numerator and denominator of $A_{\mathfrak{a}}(z,s)/A_{\mathfrak{a}}(s)$ by a polynomial if necessary, that $A_{\mathfrak{a}}(s)$ has no zeros on S. As in the proof of (7.5), and noting that $|\phi_{\mathfrak{a}\mathfrak{b}}(1/2+ir)| \leqslant 1$, we arrive at the bound

$$\phi_{\mathfrak{a}\mathfrak{b}}(m,1/2+ir) \ll \frac{\psi(1/2+ir)}{|A_{\mathfrak{a}}(1/2+ir)|}\frac{y^{-1/2}}{\sqrt{|m|y}K_{ir}(2\pi|m|y)}$$

where ψ is a smooth function, and the consideration is valid for $r\in[T,T+1]$ and $y<1$, say. If we set $y=(\log|m|)/(2\pi|m|)$, we get

$$\phi_{\mathfrak{a}\mathfrak{b}}(m,1/2+ir) \ll \frac{\sqrt{|m|}}{\log|m|K_{ir}(\log|m|)},$$

with an implied constant depending on T and Γ. By using the asymptotic (7.4), the proof of Lemma 8.4 is complete. ∎

Let us now continue with the proof of Proposition 8.3. We apply (8.7) to see that

$$\int_T^{T+1} |E_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z,1/2+ir)|^2\,dr \ll y + y^{1-2k}T^{2k+1} + y^{1-4k}T^{4k+2}$$

for any $k \geqslant 2$, with an implied constant depending on k and Γ. Estimate (8.15) of the proposition now follows when taking $k=2$. Estimate (8.16) is proved similarly using (8.12) instead of (8.11). With this, the proof of Proposition 8.3 is complete. ∎

We now analyze the integral in (8.4). Using (8.7), (8.15), and the Cauchy-Schwartz inequality we find

$$\begin{aligned}
&\int_T^{T+1} |\Gamma(s-1/2-ir)\Gamma(s-1/2+ir)\overline{\phi_{\mathfrak{a}\mathfrak{b}}}(m,1/2+ir)E_{\mathfrak{b}}(z,1/2+ir)|\,dr\\
&\quad\ll |\Gamma(s-1/2-iT)\Gamma(s-1/2+iT)|\\
&\qquad\times\left(\int_T^{T+1}|\overline{\phi_{\mathfrak{a}\mathfrak{b}}}(m,1/2+ir)|^2\,dr\int_T^{T+1}|E_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z,1/2+ir)|^2\,dr\right)^{1/2}\\
&\quad\ll |(T+t)(T-t)|^{\sigma-1}Te^{-\pi|T-t|/2-\pi|T+t|/2+\pi T/2}|m|^{-1/2}\sqrt{y_\Gamma(z)T^{10}}.
\end{aligned}$$

Thus, for s in a compact set S, the continuous spectrum contribution to the spectral expansion of $U_{\mathfrak{a}m}(z,s)$ is absolutely and uniformly convergent, and is bounded by $|m|^{-1/2}\sqrt{y_\Gamma(z)}$. The meromorphic continuation of $U_{\mathfrak{a}m}(z,s)$ is therefore given by (8.4) to the right of the line of integration at $\mathrm{Re}(s) = 1/2$. Were we to consider $s \in \mathbb{C}$ to the left of $\mathrm{Re}(s) = 1/2$, then we would express $U_{\mathfrak{a}m}(z,s)$ by (8.4) together with Eisenstein series that arise when the line of integration is crossed (see Satz 6.6 of [Ne] or Section 6 of [C-O'S]). However, we are only concerned with s near 1. It may now be seen from (8.4) that $U_{\mathfrak{a}m}(z,s)$ is holomorphic in s at $s = 1$.

We note that Selberg was the first to prove the meromorphic continuation of $U_{\mathfrak{a}m}(z,s)$, see [Se]. Our proof above shows that

$$U_{\mathfrak{a}m}(z,s) \ll |m|^{-1/2}\sqrt{y_\Gamma(z)} \tag{8.17}$$

for $\mathrm{Re}(s) > 1/2$ with an implied constant depending on s.

Let $U'_{\mathfrak{a}m}(z,s) = \frac{d}{dz}U_{\mathfrak{a}m}(z,s)$. By the same arguments, using (8.14) and (8.16), we see that $U'_{\mathfrak{a}m}(z,s)$ also has a meromorphic continuation to all s in $\mathbb{C}$ and satisfies

$$yU'_{\mathfrak{a}m}(z,s) \ll |m|^{-1/2}\sqrt{y_\Gamma(z)} \tag{8.18}$$

for $\mathrm{Re}(s) > 1/2$. It is also true that $U'_{\mathfrak{a}m}(z,s)$ is holomorphic in s at $s = 1$. With all this, the proof of Theorem 3.1 is complete.

For the reasons given in Section 3, we define the holomorphic, weight two, Poincaré series by

$$P_{\mathfrak{a}m}(z)_2 = 2iU'_{\mathfrak{a}m}(z,1) + 4\pi m V_{\mathfrak{a}m}(z,1). \tag{8.19}$$

It is elementary to show that the right hand side of (8.19) has weight two. Using the series definition for $V_{\mathfrak{a}m}$ and the differential equation

$$\big(\Delta - s(1-s)\big)U_{\mathfrak{a}m}(z,s) = 4\pi m s U_{\mathfrak{a}m}(z,s+1),$$

it is easy to show that $\frac{d}{d\bar{z}}P_{\mathfrak{a}m}(z)_2 = 0$, i.e. the form $P_{\mathfrak{a}m}(z)_2$ is holomorphic. Therefore, we have the Fourier expansion

$$j(\sigma_\mathfrak{b}, z)^{-2}P_{\mathfrak{a}m}(\sigma_\mathfrak{b} z)_2 = \sum_{n\in\mathbb{Z}} p_\mathfrak{b}(n)e(nz).$$

By adapting the proof of Lemma 8.1, one shows that

$$j(\sigma_\mathfrak{b}, z)^{-2}V_{\mathfrak{a}m}(\sigma_\mathfrak{b} z, 1) \ll y^{-1} \quad \text{as} \quad y \to \infty. \tag{8.20}$$

Using (8.18), (8.19), and (8.20), we conclude that we must have $p_{\mathfrak{b}}(n) = 0$ for $n \leqslant 0$. Consequently $P_{\mathfrak{a}m}(z)_2$ is in $S_2(\Gamma)$ as we wanted to show. This proves the first part of Theorem 3.2. The remaining aspect of Theorem 3.2 follows from a direct computation using (3.2) that we leave to the reader.

§9. Proofs of Proposition 3.3 and the meromorphic continuation of L_m^+ and L_m^-

In this section we tie up the remaining 'loose ends' by completing the proof of Proposition 3.3 and the meromorphic continuation of L_m^+ and L_m^-, as claimed in Theorem 1.2.

For $\mathrm{Re}(s)$ sufficiently large, $f \in S_2(\Gamma)$ and $F = 2\pi i \int f$, define the automorphic series

$$Q_m(z, s; f) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} f(\gamma z) \, \mathrm{Im}(\gamma z)^s e(m\gamma z),$$

and

$$Q_m(z, s; F) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} F(\gamma z) \, \mathrm{Im}(\gamma z)^s e(m\gamma z).$$

Proceeding formally, if we unfold the inner product of $\mathcal{K}_1$ and $Q_m(\,\cdot\,, s; F)$, we get

$$\langle \mathcal{K}_1, \overline{Q_m(\,\cdot\,, s; F)} \rangle = \int_0^\infty \int_0^1 \mathcal{K}_1(z) F(z) e(mz) y^{s-2} \, dxdy,$$

which in turn can be explicitly evaluated using the Fourier expansions of F and $\mathcal{K}_1$, yielding

$$\int_0^\infty \int_0^1 \mathcal{K}_1(z) F(z) e(mz) y^{s-2} \, dxdy = \frac{\Gamma(s-1)}{(4\pi)^{s-1}} L_m^-(s).$$

As we will see, we can manipulate this inner product to obtain (6.4), which will provide a meromorphic continuation of L_m^-.

From the bound (7.7), we get that

$$F(\gamma z) \ll 1 + |\log \mathrm{Im}(\gamma z)|.$$

By mimicking the proof of Lemma 8.1, we immediately arrive at the following estimate.

Lemma 9.1. *For any* $f \in S_2(\Gamma)$ *and integer* $m > 0$*, the series* $Q_m(z, s; F)$ *is absolutely convergent for* $\mathrm{Re}(s) > 1$*. Furthermore, if* $s = \sigma + it$ *with* $\sigma > 1$*, we have*

$$Q_m(z, s; F) \ll y_\Gamma(z)^{1-\sigma}$$

with the implied constant depending on s*,* f *and* Γ *alone.*

For the remainder of this section, we let $C^\infty(\Gamma\backslash\mathbf{H}, k)$ denote the space of smooth functions ψ on $\mathbf{H}$ that transform as

$$\psi(\gamma z) = \varepsilon(\gamma, z)^k \psi(z)$$

for γ in Γ and $\varepsilon(\gamma, z) = j(\gamma, z)/|j(\gamma, z)|$. For example, one element of this space is given by the series

$$U_{\mathfrak{a}m}(z, s, k) = \sum_{\gamma \in \Gamma_{\mathfrak{a}}\backslash\Gamma} \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s e(m\sigma_{\mathfrak{a}}^{-1}\gamma z)\varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma, z)^{-k}, \tag{9.1}$$

which is the weight k non-holomorphic Poincaré series, or, in particular, the Eisenstein series

$$E_{k\mathfrak{a}}(z, s) := U_{\mathfrak{a}0}(z, s, k)$$

in the special case when $m = 0$. (Warning: It should be clear from the context whether we mean this new notion of weight or the previous definition of weight.) Trivially, if $\psi \in C^\infty(\Gamma\backslash\mathbf{H}, k)$ then $|\psi|$ has weight zero (in either definition), and $\langle\cdot, \cdot\rangle = \langle\cdot, \cdot\rangle_0$ is an inner product for $C^\infty(\Gamma\backslash\mathbf{H}, k)$. We define the *Maass raising and lowering operators* by

$$R_k = 2iy\frac{d}{dz} + \frac{k}{2}, \quad L_k = -2iy\frac{d}{d\bar{z}} - \frac{k}{2}.$$

It is an elementary exercise to show that

$$R_k : C^\infty(\Gamma\backslash\mathbf{H}, k) \longrightarrow C^\infty(\Gamma\backslash\mathbf{H}, k + 2),$$
$$L_k : C^\infty(\Gamma\backslash\mathbf{H}, k) \longrightarrow C^\infty(\Gamma\backslash\mathbf{H}, k - 2),$$

and, furthermore, the hyperbolic Laplacian Δ can be realized as

$$\Delta = -L_2 R_0 = -R_{-2} L_0. \tag{9.2}$$

By direct verification we have the next lemma (see also Lemma 4.1 of [C-O'S]).

LEMMA 9.2. *For any $\gamma \in \mathrm{PSL}_2(\mathbb{R})$ and any smooth function F, let*

$$\mu(s,k,F) = F(\gamma z)\,\mathrm{Im}(\gamma z)^s e(m\gamma z)\varepsilon(\gamma,z)^{-k}.$$

Then

$$\begin{aligned} R_k\mu(s,k,F) &= 2i\mu\left(s+1,k+2,\frac{d}{dz}F\right) + (s+k/2)\mu(s,k+2,F) \\ &\qquad - 4\pi m\mu(s+1,k+2,F), \\ L_k\mu(s,k,F) &= -2i\mu\left(s+1,k-2,\frac{d}{d\overline{z}}F\right) + (s-k/2)\mu(s,k-2,F). \end{aligned}$$

Lemma 9.2 applies in the special case $F \equiv 1$ to yield the weight k non-holomorphic Poincaré series identities

$$R_k U_{\mathfrak{a}m}(z,s,k) = (s+k/2)U_{\mathfrak{a}m}(z,s,k+2) - 4\pi m U_{\mathfrak{a}m}(z,s+1,k+2)$$

and

$$L_k U_{\mathfrak{a}m}(z,s,k) = (s-k/2)U_{\mathfrak{a}m}(z,s,k-2).$$

Using this last identity, together with our established notational conventions, we see that

$$\begin{aligned} Q_m(z,s;f) &= f(z)\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j(\gamma,z)^2\,\mathrm{Im}(\gamma z)^s e(m\gamma z) \\ &= yf(z)\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \mathrm{Im}(\gamma z)^{s-1} e(m\gamma z)\varepsilon(\gamma,z)^2 \\ &= yf(z)U_m(z,s-1,-2) = yf(z)\frac{L_0U_m(z,s-1)}{s-1}. \end{aligned} \tag{9.3}$$

Next, combine Lemma 9.2 (this time with $F = 2\pi i\int f$ as usual) and the identity (9.2) to get

$$(\Delta - s(1-s))Q_m(z,s;F) = 4\pi s Q_m(z,s+1;f) + 4\pi m s Q_m(z,s+1;F).$$

Finally, by taking the inner product with $\mathcal{K}_1$, we get

$$\begin{aligned} &4\pi s\langle\mathcal{K}_1,\overline{Q_m(\,\cdot\,,s+1;f)}\rangle + 4\pi m s\langle\mathcal{K}_1,\overline{Q_m(\,\cdot\,,s+1;F)}\rangle \\ &\quad= \langle\mathcal{K}_1,(\Delta-\overline{s(1-s)})\overline{Q_m(\,\cdot\,,s;F)}\rangle \\ &\quad= \langle(\Delta-s(1-s))\mathcal{K}_1,\overline{Q_m(\,\cdot\,,s;F)}\rangle \\ &\quad= \langle\Delta\mathcal{K}_1,\overline{Q_m(\,\cdot\,,s;F)}\rangle - s(1-s)\langle\mathcal{K}_1,\overline{Q_m(\,\cdot\,,s;F)}\rangle. \end{aligned} \tag{9.4}$$

All calculations yielding (9.4) are correct providing the inner products make sense and we can justify moving Δ from one side to the other. For example, if all functions were bounded on $\Gamma\backslash\mathbf{H}$, then the manipulations are correct (see Lemma 4.1 of [Iw1]). Unfortunately, the functions in (9.4) are not bounded, so further analysis is required. The following proposition proves the bounds required to validate (9.4).

PROPOSITION 9.3. *Suppose* $\phi_1 \in C^\infty(\Gamma\backslash\mathbf{H}, k)$ *and* $\phi_2 \in C^\infty(\Gamma\backslash\mathbf{H}, k+2)$. *Let* $A, B \in \mathbb{R}$ *with* $A + B < 0$. *If*

$$\phi_1(z),\, R_k\phi_1(z) \ll y_\Gamma(z)^A \quad \text{and} \quad L_{k+2}\phi_2(z),\, \phi_2(z) \ll y_\Gamma(z)^B,$$

then

$$\langle R_k\phi_1,\, \phi_2\rangle + \langle \phi_1, L_{k+2}\phi_2\rangle = 0.$$

Proof. Let $\epsilon > 0$ be such that $A + B < -\epsilon$, and choose $s \in (1, 1+\epsilon)$. Since $E_{\mathfrak{a}}(z,s) \ll y_\Gamma(z)^s$ the inner products in the sum

$$\langle R_k\phi_1, \phi_2 E(\,\cdot\,, \overline{s})\rangle + \langle \phi_1, (L_{k+2}\phi_2)E(\,\cdot\,, \overline{s})\rangle$$

are absolutely convergent, so then we may unfold the integrals to get

$$\begin{aligned} &\langle R_k\phi_1, \phi_2 E(\,\cdot\,, \overline{s})\rangle + \langle \phi_1, (L_{k+2}\phi_2)E(\,\cdot\,, \overline{s})\rangle \\ &\quad = \int_0^\infty \int_0^1 (R_k\phi_1(z))\overline{\phi_2}(z) y^{s-2}\, dxdy \\ &\quad\quad + \int_0^\infty \int_0^1 \phi_1(z)\overline{(L_{k+2}\phi_2(z))} y^{s-2}\, dxdy. \end{aligned} \tag{9.5}$$

It is clearer to now replace $\int_0^\infty$ with $\int_{1/D}^D$ and then later let $D \to \infty$. With the definitions of R_k and L_{k+2}, (9.5) becomes

$$\int_{1/D}^D \int_0^1 \left[\left(\left(iy\frac{d}{dx}+y\frac{d}{dy}\right)\phi_1(z)\right)\overline{\phi_2}(z)+\phi_1(z)\left(iy\frac{d}{dx}+y\frac{d}{dy}\right)\overline{\phi_2}(z)\right] y^{s-2}\, dxdy$$
$$- \int_{1/D}^D \int_0^1 \phi_1(z)\overline{\phi_2}(z) y^{s-2}\, dxdy.$$

Now use integration by parts with respect to both x and y. Observing that most terms cancel, we are left with

$$\int_0^1 \left[\phi_1(x+iD)\overline{\phi_2}(x+iD)D^{s-1} - \phi_1(x+i/D)\overline{\phi_2}(x+i/D)D^{1-s}\right] dx$$
$$- s\int_{1/D}^D \int_0^1 \phi_1(z)\overline{\phi_2}(z) y^{s-2}\, dxdy.$$

By assumption, $\phi_1(z)\overline{\phi_2}(z) \ll y_\Gamma(z)^{A+B}$, hence we obtain the bounds

$$\phi_1(z)\overline{\phi_2}(z) \ll y^{A+B} \quad \text{as} \quad y \to \infty$$

and, by Lemma 7.1,

$$\phi_1(z)\overline{\phi_2}(z) \ll 1 \quad \text{as} \quad y \to 0,$$

and, indeed, the asymptotics are independent of x. These bounds are just enough to show that the first integral above vanishes as $D \to \infty$. Therefore

$$\begin{aligned} &\langle R_k\phi_1, \phi_2 E(\,\cdot\,, \overline{s})\rangle + \langle \phi_1, L_{k+2}\phi_2 E(\,\cdot\,, \overline{s})\rangle \\ &= -s\int_0^\infty \int_0^1 \phi_1(z)\overline{\phi_2}(z) y^{s-2}\, dxdy \\ &= -s\langle \phi_1, \phi_2 E_{-2}(\,\cdot\,, \overline{s})\rangle \end{aligned}$$

for the weight -2 Eisenstein series defined by (9.1) for $m = 0$, $k = -2$. This is valid for s in $(1, 1+\epsilon)$. By analytic continuation this is true for all s with $1/2 < \mathrm{Re}(s) < 1 + \epsilon$ say. Finally, equating residues at $s = 1$ yields the theorem because $E_{-2}(z, s)$ is holomorphic at $s = 1$. □

COROLLARY 9.4. *Assume $\phi_1(z)$ and $\phi_2(z)$ are smooth of weight zero with $A + B < 0$, and suppose*

$$\phi_1,\ R_0\phi_1,\ \Delta\phi_1 \ll y_\Gamma(z)^A,$$

and

$$\phi_2,\ R_0\phi_2,\ \Delta\phi_2 \ll y_\Gamma(z)^B.$$

Then

$$\langle \Delta\phi_1, \phi_2\rangle = \langle \phi_1, \Delta\phi_2\rangle.$$

Proof. One applies Proposition 9.3 twice and uses the identity which expresses the Laplacian in terms of the raising and lowering operators. □

Proof of Proposition 3.3. The proof is an immediate consequence of Proposition 9.3 when taking $k = 0$ together with the definitions of the functions under consideration. More specifically, given the functions in Proposition 3.3, one applies Proposition 9.3 with $\phi_1(z) = \varphi_1(z)$ and $\phi_2(z) = \mathrm{Im}(z) \cdot \varphi_2(z)$, after which one then easily computes the derivatives in question. □

Meromorphic continuation of L_m^-: Corollary 9.4 implies that (9.4) holds for $\mathrm{Re}(s)$ sufficiently large. Using (2.5), we then have that

$$\langle \Delta\mathcal{K}_1, \overline{Q_m(\,\cdot\,, s; F)}\rangle = \langle -V^{-1}, \overline{Q_m(\,\cdot\,, s; F)}\rangle = 0 \tag{9.6}$$

where the last equality comes from unfolding the integral in question and using that f is a holomorphic cusp form. If we now combine (9.3), (9.4) and (9.6), we then get

$$\frac{\Gamma(s+1)}{(4\pi)^{s-1}} L_m^-(s) = m\frac{\Gamma(s+1)}{(4\pi)^{s-1}} L_m^-(s+1) + 4\pi\langle \mathcal{K}_1, \overline{yf(z)L_0U_m(z,s)}\rangle. \tag{9.7}$$

However, the structure of the operators L and R are such that we have the relation

$$\begin{aligned}\langle \mathcal{K}_1, \overline{yf(z)L_0U_m(z,s)}\rangle &= \langle yf(z)\mathcal{K}_1, R_0\overline{U_m(z,s)}\rangle\\ &= -\langle yf(z)L_0\mathcal{K}_1, \overline{U_m(z,s)}\rangle.\end{aligned}$$

Substituting this into (9.7) completes the proof of the identity

$$L_m^-(s) = mL_m^-(s+1) + \frac{2i(4\pi)^s}{\Gamma(s+1)}\Big\langle y^2 f(z)\frac{d}{d\overline{z}}\mathcal{K}_1(z), \overline{U_m(z,s)}\Big\rangle. \tag{9.8}$$

We see that $Q_m(z,s;F)$ does not appear in (9.8) and a second proof of (9.8) is to simply unfold the inner product on the right side. The bounds on the Fourier coefficients $\{a_n\}$ and $\{k(n)\}$ are such that the Dirichlet series which defines $L_m^-(s)$ converges for $\mathrm{Re}(s) > 3$. The bound (8.17) and identity (9.8) provide the meromorphic continuation to $\mathrm{Re}(s) > 1/2$, as claimed in Theorem 1.2.

Meromorphic continuation of L_m^+: The argument to prove the meromorphic continuation of L_m^+ is similar, in spirit, to that of L_m^-. Recall equation (3.1), which shows that

$$sV_{\mathfrak{a}m}(z, s-1) = 2i\frac{d}{dz}U_{\mathfrak{a}m}(z,s) + 4\pi m V_{\mathfrak{a}m}(z,s) \tag{9.9}$$

for $\mathrm{Re}(s)$ sufficiently large. By comparing the series $V_{\mathfrak{a}m}(z,s)$ with $E_{\mathfrak{a}}(z,s+1)$ we see that it converges absolutely and uniformly to a holomorphic function of s for $\mathrm{Re}(s) > 0$. The techniques of Lemma 8.1 apply to $V_{\mathfrak{a}m}$ to give, for $\mathrm{Re}(s) > 0$,

$$yV_{\mathfrak{a}m}(z,s) \ll 1.$$

Combining this with (8.18) and (9.9) easily shows that the analytic continuation of $yV_{\mathfrak{a}m}(z,s-1)$ down to $\mathrm{Re}(s) > 1/2$ is bounded by a polynomial in $y_\Gamma(z)$. Therefore the inner product $\langle f\mathcal{K}_1, V_m(\,\cdot\,,\overline{s}-1)\rangle_2$ admits a meromorphic continuation for $\mathrm{Re}(s) > 1/2$, which is holomorphic at $s = 1$. By Proposition 5.1, this implies the meromorphic continuation of L_m^{++} to $\mathrm{Re}(s) > 1/2$. Since L_m^{++} and L_m^{+} differ by a Dirichlet polynomial, this part of Theorem 1.2 is now complete.

§10. Examples

To conclude this work, we will remind the reader of certain known computations as well as pose a question that can lead to future investigations.

Let us consider the discrete subgroup $\mathrm{PSL}_2(\mathbb{Z})$. In this case, the Fourier expansion of the first-order Kronecker limit function is well-known, namely

$$\mathcal{K}_1(z) = \sum_{n<0} k(n)e(n\overline{z}) + y + K - \frac{3}{\pi}\log y + \sum_{n>0} k(n)e(nz) \tag{10.1}$$

where $K = \frac{3}{\pi}(\gamma - \log 4\pi)$, $\sigma(n) = \sum_{d|n} d$ and

$$k(n) = \frac{6}{\pi}\frac{\sigma(|n|)}{|n|}.$$

Also, let us set the notation that for $l \geqslant 0$, we define the function $\sigma_l(n) = \sum_{d|n} d^l$. Now consider the congruence subgroup $\Gamma_0(N)$, and, for simplicity, assume that N is square-free. As stated in [C-I], one can express the first-order non-holomorphic Eisenstein series on $\Gamma_0(N)$ through the formula

$$E(z,s)_{\Gamma_0(N)} = \zeta_N(2s)\sum_{d|N} \mu(d)(dN)^{-s}E(Nz/d,s),$$

where $\zeta_N(s)$ is the incomplete zeta-function

$$\zeta_N(s) = \prod_{p|N}(1-p^{-s})^{-1}$$

where the product is over all primes p dividing N, μ is the Möbius function and $E(z,s)$ denotes the $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ Eisenstein series. In effect, this formula is a consequence of the Artin formalism associated to the spectral theory on the quotient space $\Gamma_0(N)\backslash\mathbf{H}$ viewed as a finite degree cover of

$\mathrm{PSL}_2(\mathbb{Z})$. In the special case when N is equal to a prime, which we denote by p, then we have that

$$E(z,s)_{\Gamma_0(p)} = \frac{1}{1-p^{-2s}}\big(p^{-s}E(pz,s) - p^{-2s}E(z,s)\big).$$

Recall that the volume of $\Gamma_0(p)\backslash\mathbf{H}$ is $p+1$ times the volume of $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbf{H}$. Therefore, one can compute the first-order Kronecker limit function on $\Gamma_0(N)$ to be

$$\mathcal{K}_1(z)_{\Gamma_0(p)} = \frac{1}{p^2-1}\big(p\mathcal{K}_1(pz) - \mathcal{K}_1(z)\big)$$

for p prime. Therefore, for any prime level p, we have, in effect, computed the Fourier coefficients of the second-order Kronecker limit function in terms of the divisor sums and the Fourier coefficients of the chosen degree two form $f \in S_2(\Gamma_0(N))$. Of course, the computations required to extract the special values $L_m^+(1)$ and $L_m^-(1)$, which require analytic continuation, could be formidable.

For general Fuchsian groups, the first-order Kronecker limit function $\mathcal{K}_1$ is studied in [Gn]. The analogue of the Dedekind η function and Dedekind sums are also studied there. We refer the interested reader to [Gn] for additional information.

Finally, we now highlight a question that arises from Theorem 1.3. Given a Fuchsian group Γ of the first kind and a parabolic subgroup, one then has a first-order Kronecker limit function $\mathcal{K}_1$. With this, consider the map from $S_k(\Gamma)$ to itself given by

$$f \longmapsto \Pi_{hol}(f\mathcal{K}_1). \tag{10.2}$$

Are there any interesting characteristics of this map which can then lead to further simplifications in Theorem 1.3 ? Consider the special case when $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ and $k = 24$. In this setting, we will examine two different holomorphic forms. The Dedekind delta function

$$\Delta(z) = e(z)\prod_{n=1}^{\infty}(1-e(nz))^{24}$$

is a weight twelve holomorphic form, as is the Eisenstein series

$$G_{12}(z) = -B_{12}/24 + \sum_{n=1}^{\infty}\sigma_{11}(n)e(nz),$$

with $-B_{12}/24 = 691/65520$. The vector space $S_{24}(\mathrm{PSL}_2(\mathbb{Z}))$ is two-dimensional with basis Δ^2, ΔG_{12}, [Za1]. The analogue of the inner product formula (3.3) for weight 24 forms is the identity

$$\Pi_{hol}(\varphi) = \sum_{m=1}^{\infty} d_m e(mz) \quad \text{for} \quad d_m = \frac{(4\pi m)^{23}}{22!} \langle \varphi, P_m(\,\cdot\,)_{24} \rangle_{24}$$

(see (8.1)). With this high weight there will be no problem with the convergence of the Poincaré series. In general, let $f(z) = \sum_{n>0} a_n e(nz) \in S_{24}(\Gamma)$ and let $\Pi_{hol}(f\mathcal{K}_1) = \sum_{m>0} d_m e(mz)$. Then when using (10.1), we can compute, as in the beginning of Section 6, the formula

$$\begin{aligned} \pi d_m = 6 \sum_{l=1}^{m} \frac{a_l \sigma(m-l)}{m-l} &+ 6m^{23} \sum_{l=m+1}^{\infty} \frac{a_l \sigma(l-m)}{l^{23}(l-m)} \\ &+ \frac{23 a_m}{4m} + 3a_m(2\gamma + \log m - H_{22}); \end{aligned}$$

H_n denotes the harmonic number $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, and we have used the formula

$$\int_0^{\infty} y^n \cdot \log y \cdot e^{-y}\, dy = n!(H_n - \gamma)$$

which holds for $n \geqslant 0$. This general formula allows for precise numerical computations. Specifically, we have computed that

$$\Pi_{hol}(\Delta^2 \mathcal{K}_1) \approx -0.852857\Delta^2 + 0.0000214526\Delta G_{12} \tag{10.3}$$

and

$$\Pi_{hol}(\Delta G_{12} \mathcal{K}_1) \approx 0.220305\Delta^2 + -0.591762\Delta G_{12}, \tag{10.4}$$

and these computations are correct to the number of decimal places shown. In conclusion, these computations suggest that the linear map $S_{24}(\mathrm{PSL}_2(\mathbb{Z})) \to S_{24}(\mathrm{PSL}_2(\mathbb{Z}))$ given by $f \mapsto \Pi_{hol}(f\mathcal{K}_1)$ to be neither zero nor diagonal.

At this time, a host of natural questions arise. For example, given a Fuchsian group Γ and a parabolic subgroup, is the map (10.2) diagonalizable? If so, then is there a natural basis of $S_k(\Gamma)$ such that the map (10.2) is diagonal? Is there any numerical significance to the coefficients in (10.3) and (10.4)? These issues certainly warrant future investigation.

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