PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 99, Number 1, January 1987

# CONVOLUTION ESTIMATES FOR SOME MEASURES ON CURVES

DANIEL M. OBERLIN

ABSTRACT. Suppose that $\lambda$ is a smooth measure on a curve in $R^3$. It is shown that $\lambda * L^p(R^3) \subset L^q(R^3)$ under certain conditions on $\lambda, p$, and $q$.

For $1 \leq p \leq \infty$, let $L^p$ be the usual Lebesgue space formed with respect to Lebesgue measure on $R^n$. It is well known that every complex Borel measure $\lambda$ on $R^n$ acts as a convolution operator on $L^p$: $\lambda * L^p \subset L^p$. If $\lambda$ is absolutely continuous with density in $L^r$ for some $r > 1$, Young's inequality shows that for $1 \leq p < \infty$ we have $\lambda * L^p \subset L^q$ for some $q = q(p) > p$. One might say that such a measure is "$L^p$-improving." If $\lambda$ is singular, it is still possible for $\lambda$ to be $L^p$-improving when $p > 1$. The Cantor-Lebesgue measures on $R$ are examples [**1**]. The purpose of this paper is to investigate similar behavior for a different class of singular measures, smooth measures supported on curves in $R^3$. Our motivation comes from the following two results.

THEOREM 1. *Suppose $1 \leq k < n$ and $K$ is a smooth $k$-dimensional surface in $R^n$. Suppose that $\lambda$ is a smooth measure on $K$. If $\lambda * L^p \subset L^q$, then $(1/p, 1/q)$ lies in the closed triangle in $R^2$ with vertices $(0,0)$, $(1,1)$, and $(n/(2n-k), (n-k)/(2n-k))$.*

PROOF. A theorem of Hörmander [**3**] implies that $p \leq q$. Estimating the norms of $f$ and $\lambda * f$ when $f$ is the characteristic function of a small ball shows that $(1/p, 1/q)$ lies on or above the line $L$ joining (1,1) and $(n/(2n-k), (n-k)/(2n-k))$. Since $\lambda * L^p \subset L^q$ implies $\lambda * L^{q'} \subset L^{p'}$, where $p'$ and $q'$ are the conjugate indices, $(1/q', 1/p')$ also lies on or above $L$. This completes the proof.

THEOREM 2. *Suppose $K$ is a smooth $(n-1)$-dimensional surface in $R^n$ on which all $n-1$ principal curvatures are bounded away from zero. Suppose that $\lambda$ is a smooth finite measure compactly supported away from the boundary. Then $\lambda * L^p \subset L^q$ if and only if $(1/p, 1/q)$ lies in the closed triangle in $R^2$ with vertices $(0,0), (1,1)$, and $(n/(n+1), 1/(n+1))$.*

PROOF. A theorem of Littman [**4**] implies that $\lambda * L^{(n+1)/n} \subset L^{n+1}$. Interpolation with trivial cases completes half the proof, and Theorem 1 gives the rest.

Theorems 1 and 2 lead naturally to the conjecture that if $\lambda$ is a nice measure supported on a nice $k$-dimensional surface in $R^n$, then $\lambda * L^p \subset L^q$ precisely when $(1/p, 1/q)$ is in the closed triangle of Theorem 1. This conjecture seems difficult even when $k = 1, n = 3$. Our purpose here is to prove a partial result for this case. We consider certain measures $\lambda$ on some curves in $R^3$ and prove that $\lambda * L^p \subset L^q$ whenever $(1/p, 1/q)$ lies in the triangle of Theorem 1 and on or above the line

Received by the editors October 23, 1985.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 43A22.
Partially supported by the National Science Foundation.

through $(1/2, 1/3)$ and $(2/3, 1/2)$. To show this it is sufficient, by duality and interpolation, to establish that $\lambda * L^{3/2} \subset L^2$.

THEOREM 3. *Suppose that $I$ is a closed interval in $R$ and that the real-valued functions $\phi_1$ and $\phi_2$ are both polynomial functions or both trigonometric functions on $I$. Put $\phi(t) = (\phi_1(t), \phi_2(t))$ and, if $j = 2$ or $3$, write $\phi^{(j)}$ for the $j$th derivative of the function $\phi$. Suppose that given any $t_1, t_2 \in I$, the vectors $\phi^{(2)}(t_1)$ and $\phi^{(3)}(t_2)$ span $R^2$. Let $\lambda$ be the measure on $R^3$ defined by*

$$\langle \lambda, g \rangle = \int_I g(t, \phi_1(t), \phi_2(t))\, dt.$$

*Then $\lambda * L^{3/2} \subset L^2$.*

Here are two examples where Theorem 3 applies.

EXAMPLE 1. $I = [a, b]$, $\phi_1(t) = t^2$, $\phi_2(t) = t^3$.

EXAMPLE 2. $I = [0, \pi/6]$, $\phi_1(t) = \cos(t)$, $\phi_2(t) = \sin(t)$.

Following are some comments on the notation to be used in the proof of Theorem 3. The symbol $f$ (resp. $g$) will denote an arbitrary continuous function of compact support on $R^2$ (resp. $R^3$). The symbol $\|\cdot\|_p$ will denote the norm of the indicated function in $L^p(R^2)$ or $L^p(R^3)$, whichever is appropriate. A sector $\Gamma$ in the plane is defined to be the set of all points in the plane which have a polar representation $re^{i\theta}$ with $r \geq 0$, $c \leq \theta \leq d$, where $c$ and $d$ are fixed real numbers. The symbol $C$ will denote a positive constant which may increase from line to line but which depends only on the data $I$ and $\phi$ in the hypotheses of Theorem 3. Similarly, the symbol $C(\Gamma_1, \Gamma_2, \delta)$ appearing in the following lemma represents a "variable" constant which can always be chosen to depend only upon $\Gamma_1, \Gamma_2$, and $\delta$. The proof of this lemma is based on complex interpolation and follows fairly standard lines.

LEMMA. *Suppose $\Gamma_1$ and $\Gamma_2$ are sectors such that $\Gamma_2 \cap (\Gamma_1 \cup -\Gamma_1) = \{0\}$. Suppose $\delta > 0$. There is a positive constant $C(\Gamma_1, \Gamma_2, \delta)$ such that the following holds: Suppose $J \subset R$ is a closed interval of length not exceeding $\delta^{-1}$ and suppose that $\Psi(t) = (\Psi_1(t), \Psi_2(t))$ is a twice differentiable function from $J$ into $R^2$ such that*

(i) *for every $t \in J$: $\Psi'(t) \in \Gamma_1$, $\Psi^{(2)}(t) \in \Gamma_2$, $|\Psi'(t)| \geq \delta$, $|\Psi^{(2)}(t)| \geq \delta$;*

(ii) *for any $x \in R^2$, $J$ splits into disjoint subintervals $J_1, \ldots, J_K$ with $K \leq \delta^{-1}$ such that the scalar product $x \cdot \Psi^{(2)}(t)$ is of constant sign on each $t$-interval $J_n$.*

*The measure $\mu$ on $R^2$ defined by*

$$\langle \mu, f \rangle = \int_J f(\Psi(t))\, dt$$

*satisfies*

$$\|\mu * f\|_3 \leq C(\Gamma_1, \Gamma_2, \delta) \|f\|_{3/2}.$$

PROOF. For $z \in C$ consider the distribution $d_z(s) = |s|^z$ on $R$. The map $z \to d_z$ defines a meromorphic distribution-valued function on $C$ with simple poles at $z = -1, -3, -5, \ldots$. Thus $d_z/(\Gamma(z+1)/2)$ is an entire distribution-valued function of $z$ (see [**2**] for details).

Now let $y_1$ be a unit vector orthogonal to no nonzero vector in $\Gamma_1$ and let $y$ be a unit vector orthogonal to $y_1$.

For $z \in C$ and test functions $h$ on $R^2$, define the distribution $T_z$ by

$$\langle T_z, h\rangle = \frac{1}{\Gamma((z+1)/2)} \int_J \int_{-\infty}^{\infty} h(\Psi(t) + sy)|s|^z \, ds \, dt.$$

We will establish that

$$\|T_{iy} * h\|_\infty \le \left( C(\Gamma_1, \Gamma_2, \delta)/\Gamma\left(\frac{1+iy}{2}\right)\right) \|h\|_1, \tag{1}$$

$$\|T_{iy-3/2} * h\|_2 \le \left( C(\Gamma_1, \Gamma_2, \delta)/\Gamma\left(\frac{3-2iy}{4}\right)\right) \|h\|_2, \tag{2}$$

$$T_{-1} * h = \mu * h. \tag{3}$$

Then the conclusion of the lemma will follow from the interpolation theorem in [**5**].

To show (1) is to show that $T_{iy}$ is an $L^\infty$ function of appropriate norm. The map $(s,t) \to \Psi(t) + sy$ from $R \times J$ into $R^2$ is one-to-one by the assumptions on $y$ and $y_1$ and the mean value theorem. It has Jacobian of absolute value $|\Psi'(t) \cdot y_1|$, a quantity which exceeds $1/C(\Gamma_1, \Gamma_2, \delta)$. Thus

$$\left| \int_J \int_{-\infty}^{\infty} h(\Psi(t) + sy)\, ds\, dt \right| \le C(\Gamma_1, \Gamma_2, \delta)\|h\|_1.$$

This gives (1).

To obtain (2) we must estimate the Fourier transform of $T_{iy-3/2}$. Calculations in [**2**] show that the Fourier transform of $d_z/\Gamma((z+1)/2)$ at $s$ is

$$2^{z+1}\pi^{1/2}|s|^{-z-1}/\Gamma(-z/2).$$

Thus the Fourier transform of $T_z$ is given by

$$\hat{T}_z(x) = \hat{\mu}(x) \cdot 2^{z+1}\pi^{1/2}|x \cdot y|^{-z-1}/\Gamma(-z/2)$$

(from which (3) follows). To establish (2) it is therefore sufficient to show that

$$|x|^{1/2}|\hat{\mu}(x)| \le C(\Gamma_1, \Gamma_2, \delta) \quad \text{for } x \in R^2. \tag{4}$$

Note that the hypotheses guarantee that

$$|x \cdot \Psi'(t)| + |x \cdot \Psi^{(2)}(t)| \ge |x|/C(\Gamma_1, \Gamma_2, \delta), \qquad x \in R^2,\ t \in J. \tag{5}$$

Now fix nonzero $x \in R^2$ and put $p(t) = x \cdot \Psi(t)$. By (ii), $J$ splits into disjoint subintervals $J_1, \ldots, J_K$ with $K \le \delta^{-1}$ such that $p^{(2)}(t)$ has constant sign on each $J_n$. Write $\eta = |x|/2C(\Gamma_1, \Gamma_2, \delta)$ and let $I_n^1 = J_n \cap \{|p'(t)| < \eta\}$. Since $p'(t)$ is monotone on $J_n$, $I_n^1$ is an interval and so $J_n \backslash I_n^1$ is the disjoint union of $I_n^2$ and $I_n^3$, where each of $I_n^2$ and $I_n^3$ is an interval or empty. By (5) we have either $p^{(2)}(t) \ge \eta$ or $p^{(2)}(t) \le -\eta$ on $I_n^1$. Therefore

$$\eta^{1/2} \left| \int_{I_n^1} e^{-ix\cdot\Psi(t)}\, dt \right| \le 4 \tag{6}$$

by van der Corput's lemma [**7**]. If $j = 2$ or $j = 3$ we have $p'(t)$ monotone and either $p'(t) > \eta$ or $p'(t) < -\eta$ on $I_n^j$. Thus, by van der Corput's lemma again,

$$\eta^{1/2} \left| \int_{I_n^j} e^{-ix\cdot\Psi(t)}\, dt \right| \le \min\{\eta^{1/2} \cdot \text{length}(I_\eta^j), \eta^{-1/2}\} \le \delta^{-1/2}, \tag{7}$$

where the last inequality is because length $(J) \leq \delta^{-1}$. Adding all inequalities (6) and (7) gives (4) since there are at most $3K \leq 3\delta^{-1}$ such terms. This completes the proof of the lemma.

PROOF OF THEOREM 3. Let $\tilde{\lambda}$ be defined by

$$\langle \tilde{\lambda}, g \rangle = \int_{R^3} g(-x)\, d\lambda(x).$$

Since $\langle \tilde{\lambda} * \lambda * g, g \rangle = \|\lambda * g\|_2^2$, it is enough to show that

$$\|\tilde{\lambda} * \lambda * g\|_3 \leq C\|g\|_{3/2} \tag{8}$$

for continuous compactly supported functions $g$ on $R^3$. Writing $I = [a, b]$ and $I_u = [\max\{a, a-u\}, \min\{b, b-u\}]$ for $|u| \leq b-a$ we have

$$\begin{aligned}\langle \tilde{\lambda} * \lambda, g \rangle &= \int_a^b \int_a^b g(t-s, \phi_1(t) - \phi_1(s), \phi_2(t) - \phi_2(s))\, ds\, dt \\ &= \int_{a-b}^{b-a} \int_{I_u} g(u, \phi_1(s+u) - \phi_1(s), \phi_2(s+u) - \phi_2(s))\, ds\, du.\end{aligned}$$

For $|u| \leq b-a$, define the measure $\lambda_u$ on $R^2$ by

$$\langle \lambda_u, f \rangle = \int_{I_u} f(\phi_1(s+u) - \phi_1(s), \phi_2(s+u) - \phi_2(s))\, ds.$$

We will prove that

$$\|\lambda_u * f\|_3 \leq C|u|^{-2/3}\|f\|_{3/2}. \tag{9}$$

Assume (9) for the moment and write $g(t, x)$ ($t \in R$, $x \in R^2$) for a continuous compactly supported function on $R^3$. Then

$$\begin{aligned}\|\tilde{\lambda} * \lambda * g\|_3 &= \left\| \left\| \int_{a-b}^{b-a} \lambda_u * g(t-u, \cdot)(x)\, du \right\|_{3,x} \right\|_{3,t} \\ &\leq \left\| \int_{a-b}^{b-a} \|\lambda_u * g(t-u, \cdot)(x)\|_{3,x}\, du \right\|_{3,t} \\ &\leq C \left\| \int_{a-b}^{b-a} |u|^{-2/3} \|g(t-u, x)\|_{3/2,x}\, du \right\|_{3,t}.\end{aligned}$$

By the boundedness of the Riesz potential of order $\frac{1}{3}$ as a mapping from $L^{3/2}(R)$ to $L^3(R)$ (see p. 119 of [**6**]), this last term is dominated by

$$C\| \, \|g(t, x)\|_{3/2,x} \|_{3/2,t} = C\|g\|_{3/2}.$$

Thus (8), and so Theorem 3, will be proved when (9) is established.

By the hypotheses of Theorem 3 there is an $\eta > 0$ such that

$$|\phi^{(2)}(t)|, |\phi^{(3)}(t)| \geq \eta \quad \text{if } t \in I. \tag{10}$$

The sets

$$K_1 = \{\phi^{(2)}(t)/|\phi^{(2)}(t)| : t \in I\}, \qquad K_2 = \{\phi^{(3)}(t)/|\phi^{(3)}(t)| : t \in I\}$$

are disjoint closed intervals on the unit circle in $R^2$. If $i = 1$ or $2$, define $\Gamma_i$ to be the sector $\{rv : r \geq 0,\ v \in K_i\}$. The hypotheses of Theorem 3 imply that

$\Gamma_2 \cap (\Gamma_1 \cup -\Gamma_1) = \{0\}$ and thus that the $\Gamma_i$ are convex. By (10), for $i = 1$ or 2 the set $\phi^{(i+1)}(I)$ lies in a proper closed subset $\Gamma_i'$ of $\Gamma_i \backslash \{0\}$, and $\Gamma_i'$ can be chosen to be convex. Fix $\delta > 0$ such that

($\alpha$) $\{x \in R^2 \colon |x| < \delta\} \cap \Gamma_i' = \varnothing$ for $i = 1, 2$;

($\beta$) $b - a \le \delta^{-1}$;

($\gamma$) For $u \in [a-b, b-a]$ and $x \in R^2$, $I_u$ splits into disjoint subintervals $J_1, \ldots, J_K$ with $K \le \delta^{-1}$ such that $x \cdot (\phi^{(2)}(t+u) - \phi^{(2)}(t))$ is of constant sign on each $t$-interval $J_n$. (This is where we use the hypothesis that $\phi_1$ and $\phi_2$ be both polynomial or trigonometric functions.)

Now fix $u \in [a-b, b-a]$. Define $\Psi(t)$ on $I_u$ by

$$\Psi(t) = \frac{1}{u}(\phi(t+u) - \phi(t)).$$

Letting $\Gamma_1, \Gamma_2$, and $\delta$ be as above, we claim that $\Psi(t)$ satisfies the hypotheses of the lemma. The hypothesis (i) for $\Psi'(t)$ follows from

$$\Psi'(t) = \frac{1}{u} \int_t^{t+u} \phi^{(2)}(s)\, ds$$

combined with the convexity of $\Gamma_1'$ and ($\alpha$). The hypothesis (i) for $\Psi^{(2)}(t)$ follows similarly. The hypothesis (ii) is a consequence of ($\gamma$), and the hypothesis that length$(J)$ not exceed $\delta^{-1}$ is covered by ($\beta$). Thus it follows, with the notation of the lemma, that

$$\|\mu * f\|_3 \le C \|f\|_{3/2}. \tag{11}$$

For functions $f$ on $R^2$ and $u \in R$, put $D_u f(x) = f(ux)$, $x \in R^2$. Then

$$\lambda_u * f(x) = \int_{I_u} f(x - u\Psi(t))\, dt = \int_{I_u} f\left(u\left(\frac{x}{u} - \Psi(t)\right)\right)\, dt = D_{1/u}(\mu * (D_u f))(x).$$

Now (9) follows from (11) and the fact that

$$\|D_u f\|_p = |u|^{-2/p} \|f\|_p.$$

## References


1. M. Christ, *Convolution estimates for Cantor-Lebesgue measures*, preprint.
2. I. M. Gelfand and G. E. Shilov, *Generalized functions*, Academic Press, New York, 1964.
3. L. Hörmander, *Estimates for translation-invariant operators in $L^p$ spaces*, Acta Math. **104** (1960), 93–140.
4. W. Littman, *$L^p - L^q$ estimates for singular integral operators*, Proc. Sympos. Pure Math., vol. 23, Amer. Math. Soc., Providence, R.I., 1973.
5. E. M. Stein, *Interpolation of linear operators*, Trans. Amer. Math. Soc. **87** (1958), 159–172.
6. ——, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, N.J., 1970.
7. A. Zygmund, *Trigonometric series*, Cambridge Univ. Press, Cambridge, 1959.


Department of Mathematics, Florida State University, Tallahassee, Florida 32306-3027