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# Convolution Filters for Triangles

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**Abstract**. The construction of a new triangle by erecting similar ears on the sides of a given triangle (as in Napoleon's theorem) can be considered as the convolution of the initial triangle with another triangle. We use the discrete Fourier transformation and a shape function to give a complete and explicit description of such convolution filters and their iterates. Our method leads to many old and new results in a very direct way.

## 1. Introduction

Johnson [17, pp. 284, 294], citing Emmerich [5, p. 129], formulates and proves the following property: "If points divide the sides of a given triangle in equal ratios, they are vertices of a triangle having the same Brocard angle as the given triangle. [ . . . ] The triangles whose vertices divide in equal ratios the sides of a given triangle constitute all the different forms of triangle having the same Brocard angle." (See also [29, 11] and our Theorem 2.) We analyze this kind of transformations of an initial triangle into a new triangle by considering convolutions of two triangles: with one exception, such a convolution simply erects three similar triangular ears on the sides of the initial triangle before transforming the triangle of the ears' apices by a direct similarity. A circulant linear transformation of a triangle in the complex plane, given by the coefficients $c_0$, $c_1$, $c_2$ of the circulant linear combination of the vertices, is simply the convolution of the initial triangle and the triangle with vertices $c_0$, $c_{\mathbf{2}}$, $c_{\mathbf{1}}$.

Here is a sketch of our method. We use the spectral decomposition in the Fourier base of $\mathbf{C}^3$ to represent any triangle of the complex plane as the sum of its centroid and of two positively and negatively oriented equilateral triangles: the convolution with this triangle is then a diagonal linear map, and we call *shape* of the triangle the quotient of the eigenvalues belonging to the negatively and positively oriented equilateral base vectors; this shape is also the quotient of the corresponding spectral values of the convolving triangle, *i.e.*, the ratio of the negatively and positively oriented equilateral quantities in this triangle. It is then immediate that the shape of a convolution product of two triangles is equal to the product of the triangles' shapes. Two directly similar triangles (with vertices in order) have the same shape; moreover, when restricted to normalized triangles with vertices 0, 1, and $z$, the

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shape function is a Möbius transformation as function of $z$: the equivalence classes of directly similar triangles (with respect to the given order of their vertices) are thus parametrized by their shape. Every triangle transformation given by a convolution can be described by analyzing this Möbius transformation. Emmerich's introducing result, for example, becomes almost immediate (see also [11]): the transformation is a convolution with a degenerate triangle; since degenerate triangles are characterized by shapes of modulus $1$, the convolution acts on the shape of the initial triangle as a rotation and does not change the shape's modulus; since triangles are equibrocardal and equally oriented exactly when their shapes have the same modulus, the Brocard angle is invariant under a convolution with a degenerate triangle; the converse follows from the fact that the shapes of the degenerate triangles needed for the Emmerich transformations cover the whole unit circle ($-1$ being half an exception). By iterating the convolution with a degenerate triangle, one simply rotates the shape by a constant angle at each step: the successive triangles are directly similar to triangles with a common base whose apices turn on the same Neuberg circle, and these apices are periodic or dense on the Neuberg circle according as the rotation angle for the shape is a rational or irrational multiple of $\pi$.

Many triangle transformations in the literature are in fact convolutions with a fixed triangle and could thus have been analyzed by the present method in a very efficient and standardized way. Moreover, if the convolving triangle is degenerate, *i.e.*, if the shape of the convolving triangle lies on the unit circle, the transformation behaves as the Emmerich transformation with the same shape (as far as only the triangle's form is concerned), and the work is done as soon as the rotation angle, *i.e.*, the argument of the shape of the convolving degenerate triangle, is determined! The $s$-Rooth triangles of [11] for example, which already appear in [22], are in fact given by the convolution with a degenerate triangle, like any circulant transformation of a triangle where the new vertices are a $real$ linear combination of the old ones, and such convolutions are included once for all in the Emmerich transformations (as far as only the triangle's form is concerned).

The Fourier decomposition of a triangle or polygon and circulant matrices have been used for a long time for studying triangle and polygon transformations with a circulant structure, beginning with Darboux in 1878 [3], [1, 4, 6, 7, 8, 14, 25, 27, 28, 30, 31, 32, 33, 36]; they are in general more efficient than purely geometric and trigonometric approaches [16, 34, 35]. Our presentation is free from matrix algebra since we only need convolutions. The shape function we use seems to appear for the first time in 2003 in a paper by Nakamura and Oguiso [22] (from a sixteen years older preprint), and later independently in [10]: we found only in [10] an indirect relation between the shape function and some eigenfunction as tentative explanation of the nature of this shape function. As far as we know, the fact that the shape of a convolution product is the product of the shapes is noticed and exploited here for the first time, and this is the key point for the success of our method. Note that Hajja *et al.* [10, 11, 12, 13], and Nakamura and Oguiso [22] establish *ad hoc* for each analyzed transformation that the shape of the transformed triangle is the shape of the initial triangle multiplied by some function independent of the initial triangle, without noticing that this function is in fact the shape of a triangle and that the transformation is a convolution. We extended this shape function to polygons in [24].

## 2. Fourier transform of a triangle

Consider a triangle $\Delta$ of the complex plane as a point $\Delta = (z_0, z_1, z_2) \in \mathbf{C}^3$ representing the closed polygonal line $z_0 \to z_1 \to z_2 \to z_0$ starting at $z_0$: there are up to six triangles with the same vertices. A triangle is called *degenerate* when its vertices are collinear, *trivial* when it is reduced to a triple point, and *proper* when it is nondegenerate; a degenerate triangle is both positively and negatively oriented.

Endow $\mathbf{C}^3$ with the inner product $\langle (z_0, z_1, z_2), (w_0, w_1, w_2) \rangle = \frac{1}{3}\sum_{k=0}^{2} z_k \overline{w_k}$ and set $\zeta = e^{i2\pi/3}$. The vectors

$$e_0 = (1, 1, 1), \quad e_1 = (1, \zeta, \zeta^2), \quad e_2 = (1, \zeta^2, \zeta^4) = (1, \zeta^2, \zeta) = \overline{e_1}$$

form the orthonormal *Fourier base* of $\mathbf{C}^3$: $e_0$ is a trivial triangle; $e_1$ and $e_2$ are a positively and a negatively oriented equilateral triangle centered at the origin, respectively. The *discrete Fourier transform* or *spectrum* of $\Delta$ is the triangle $\widehat{\Delta} = (\hat{z}_0, \hat{z}_1, \hat{z}_2)$ given by the *spectral representation* of $\Delta$ in the Fourier base:

$$\Delta = \sum_{k=0}^{2} \hat{z}_k e_k \quad \text{with} \quad \hat{z}_k = \langle \Delta, e_k \rangle\,,\ k = 0, 1, 2.$$

$\hat{z}_0 = \frac{1}{3}(z_0 + z_1 + z_2)$ is the centroid of $\Delta$, and

$$\hat{z}_1 = \tfrac{1}{3}(z_0 + \zeta^2 z_1 + \zeta z_2), \quad \hat{z}_2 = \tfrac{1}{3}(z_0 + \zeta z_1 + \zeta^2 z_2).$$

A triangle is trivial if $\hat{z}_1 = \hat{z}_2 = 0$; it is *pequilateral* if it is equilateral and positively oriented, *i.e.*, if $\hat{z}_1 \neq 0$, $\hat{z}_2 = 0$; it is *nequilateral* if it is equilateral and negatively oriented, *i.e.*, if $\hat{z}_1 = 0$, $\hat{z}_2 \neq 0$; it is *mixed* if $\hat{z}_1 \neq 0$, $\hat{z}_2 \neq 0$, *i.e.*, if it is neither trivial nor equilateral. A spectrum is *full* if all *Fourier coefficients* $\hat{z}_k$ are different from $0$.

## 3. Shape of a triangle

We define the shape of a nontrivial triangle $\Delta$ by

$$\sigma_\Delta = \sigma(z_0, z_1, z_2) = \frac{\hat{z}_2}{\hat{z}_1} = \frac{z_0 + \zeta z_1 + \zeta^2 z_2}{z_0 + \zeta^2 z_1 + \zeta z_2} \in \mathbf{C} \cup \{\infty\},$$

the ratio of the contributions to $\Delta$ of the nequilateral $e_2$ and of the pequilateral $e_1$ [22, 10]. Note that [22, 10] define this shape function without any reference to Fourier transforms or circulant matrices. The first properties of the shape function presented below can already be found in [10]: some errors are corrected in [13]. Since $(w_0, w_1, w_2) = a(z_0, z_1, z_2) + be_0$ for some $b$ is equivalent to $(\hat{w}_1, \hat{w}_2) = a(\hat{z}_1, \hat{z}_2)$, $\Delta_1$ has the shape of $\Delta$ if and only if $\Delta_1 = a\Delta + be_0$ with $a, b \in \mathbf{C}$, $a \neq 0$: $\Delta = (z_0, z_1, z_2)$ has thus the shape of a unique (directly similar) normalized triangle $\Delta' = \Delta'(z) = (0, 1, z)$, namely $\Delta'\big(\frac{z_2 - z_0}{z_1 - z_0}\big)$, where $\Delta'(\infty) = (0, 1, \infty)$

means $\Delta' = (0,0,1)$ for $z_0 = z_1$. The shape $f(z)$ of the normalized triangle $\Delta'(z)$ is the Möbius transformation

$$s = \sigma_{\Delta'(z)} = f(z) = \frac{\zeta z + 1}{z + \zeta} = \zeta \frac{z - e^{i\pi/3}}{z - e^{-i\pi/3}} = \frac{1}{\overline{f(\overline{z})}}$$

of the extended complex plane with inverse

$$z = f^{-1}(s) = \frac{\zeta s - 1}{\zeta - s} = -\zeta \frac{s - \zeta^2}{s - \zeta} = -f(-s) = \frac{1}{\overline{f^{-1}(\overline{s})}}.$$

Triangles corresponding to different normalized triangles have different shapes: the equivalence classes of triangles are parametrized by their shape. $\sigma_\Delta = 0$ or $\infty$ if and only if $\Delta$ is pequilateral or nequilateral, respectively. A cyclic left shift $(z_0, z_1, z_2) \mapsto (z_1, z_2, z_0)$, *i.e.*, a start from the next vertex of the triangle, causes a change $(\hat{z}_0, \hat{z}_1, \hat{z}_2) \mapsto (\hat{z}_0, \zeta\hat{z}_1, \zeta^2\hat{z}_2)$ in the spectrum; an orientation reversing, *i.e.*, a permutation of the last two vertices $\Delta = (z_0, z_1, z_2) \mapsto (z_0, z_2, z_1) = 3\hat{\hat{\Delta}}$, causes the same permutation of the Fourier coefficients:

$$\sigma(z_1, z_2, z_0) = \zeta\sigma(z_0, z_1, z_2), \quad \sigma(z_0, z_2, z_1) = \frac{1}{\sigma(z_0, z_1, z_2)}, \quad \sigma_{\overline{\Delta}} = \frac{1}{\overline{\sigma_\Delta}}.$$

A triangle $\Delta_1$ is thus directly similar to the proper $\Delta$ of shape $\sigma_\Delta = s$ with the same orientation if and only if $\sigma_{\Delta_1} = \zeta^k s$, $k \in \{0,1,2\}$; $\Delta_1$ is directly similar to the proper $\Delta$ with the inverse orientation if and only if $\sigma_{\Delta_1} = \zeta^k \frac{1}{s}$; $\Delta_1$ is inversely similar to the proper $\Delta$ with the same orientation if and only if $\sigma_{\Delta_1} = \zeta^k \overline{s}$; and $\Delta_1$ is inversely similar to the proper $\Delta$ with the inverse orientation if and only if $\sigma_{\Delta_1} = \zeta^k \frac{1}{\overline{s}}$.

Since $f(0) = \zeta^2$, $f(1) = 1$, $f(\infty) = \zeta$, and $f(e^{i\pi/3}) = 0$, $f$ maps the extended real axis (corresponding to the normalized degenerate triangles) to the unit circle and the extended upper half-plane to the unit disc: the modulus $|\sigma_\Delta|$ is thus $< 1$, $> 1$, or $= 1$ according as $\Delta$ is positively oriented and proper, negatively oriented and proper, or degenerate, respectively. Two nontrivial degenerate triangles $\Delta$ and $\Delta_1$ are similar if and only if $\sigma_{\Delta_1} = \zeta^k \sigma_\Delta$ or $\sigma_{\Delta_1} = \zeta^k \overline{\sigma_\Delta}$, $k \in \{0,1,2\}$. The degenerate normalized triangle $\Delta'(x)$, $x \in \mathbf{R} \cup \{\infty\}$, and its shape $e^{i\varphi}$ are linked by

$$\varphi = \arg \sigma_{\Delta'(x)} = \frac{2\pi}{3} + 2\arg\big(x - e^{i\pi/3}\big) = \frac{2\pi}{3} + 2\arctan \frac{\sqrt{3}}{1 - 2x} \tag{1}$$

and $x = \frac{\sin\left(\frac{\pi}{3} + \frac{\varphi}{2}\right)}{\sin\left(\frac{\pi}{3} - \frac{\varphi}{2}\right)}$: when $x$ runs rightwards over the whole extended real axis, $f(x) = \sigma_{\Delta'(x)}$ turns counterclockwise on the unit circle starting and ending at $\zeta$, which is the shape of $\Delta'(\infty) = (0,0,1)$.

For the normalized right-angled triangles (Figure 1), $f$ maps the extended imaginary axis to the circle $\mathcal{C}$ of radius $\sqrt{3}$ centered at $-2$, the circle of radius $\frac{1}{2}$ centered at $\frac{1}{2}$ to the circle $\zeta\mathcal{C}$ of radius $\sqrt{3}$ centered at $1 - i\sqrt{3}$, and the extended vertical line through 1 to the circle $\zeta^2\mathcal{C}$ of radius $\sqrt{3}$ centered at $1 + i\sqrt{3}$: the shape function $\sigma$ maps thus the right-angled triangles to the three circles $\mathcal{C}$, $\zeta\mathcal{C}$, and $\zeta^2\mathcal{C}$.

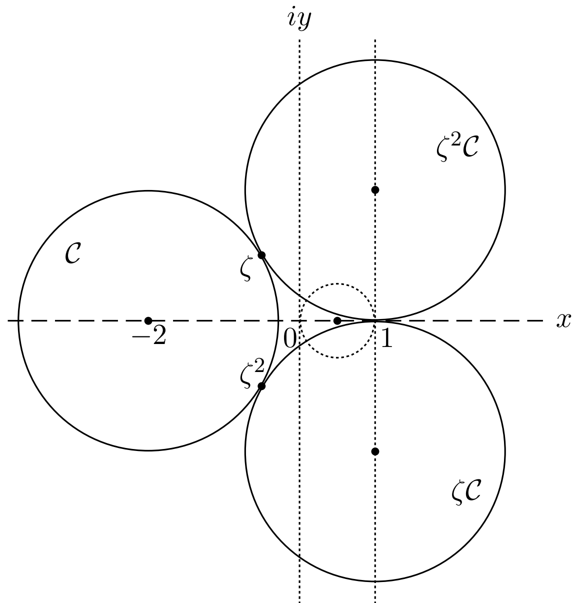


Figure 1. Normalized right-angled triangles $(0, 1, z)$: loci of their vertex $z$ (dotted curves) and of their shape (plain circles)

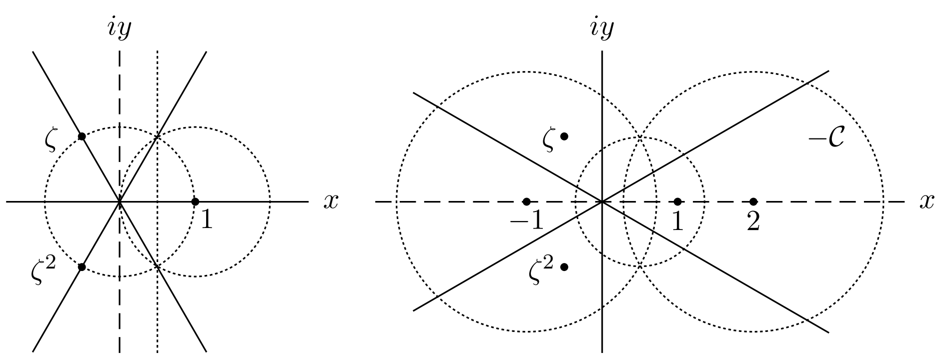


Figure 2. Normalized isosceles (on the left) and automedian triangles $(0, 1, z)$: loci of their vertex $z$ (dotted curves) and of their shape (plain straight lines)

For the normalized isosceles triangles (Figure 2), $f$ maps the unit circle to the extended real axis, the extended vertical line through $\frac{1}{2}$ to the extended line through $0$ and $\zeta$, and the circle of radius $1$ centered at $1$ to the extended line through $0$ and $\zeta^2$: $\sigma$ maps thus the nonequilateral isosceles triangles to the punctured lines $\lambda\zeta^k$, $\lambda \in \mathbf{R} \setminus \{0\}$, $k = 0, 1, 2$. $s$ is the shape of an isosceles triangle if and only if $\overline{s} \in \{s, \zeta s, \zeta^2 s\}$. The normalized isosceles $\Delta'(z)$ with apex $z = \frac{1+i\tan\theta}{2}$, $|\theta| < \frac{\pi}{2}$, and base angles $\theta$ ($< 0$ when $\operatorname{Im} z < 0$) has the shape $f(z) = \frac{\tan\theta-\sqrt{3}}{\tan\theta+\sqrt{3}}\zeta$; the shape

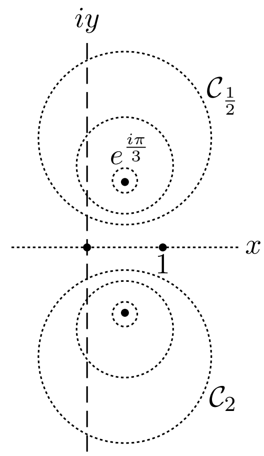


Figure 3. Neuberg circles $\mathcal{C}_R\colon |\sigma_{\Delta'}| = \left|\frac{z-e^{i\pi/3}}{z-e^{-i\pi/3}}\right| = R$ for the normalized triangles $\Delta' = (0,1,z)$, $R = 0, \frac{1}{10}, \frac{1}{3}, \frac{1}{2}, 1, 2, 3, 10, \infty$

of the isosceles $\Delta'\big(\frac{1}{\sqrt{3}}e^{i\pi/6}\big)$ with base angles $\frac{\pi}{6}$ is $-\frac{\zeta}{2}$. We set $\xi = \frac{1}{\sqrt{3}}e^{i\pi/6} = \frac{1}{2} + \frac{i}{2\sqrt{3}} = \frac{1}{1-\zeta} = \frac{-1}{\sqrt{3}}i\zeta$.

A nontrivial triangle is *automedian* when it is (inversely) similar to its median triangle: by the median theorem, this is the case if and only if the vertex opposite to the middle side $u$ lies on the circle of radius $\frac{\sqrt{3}}{2}u$ centered at the midpoint of $u$ (as does the apex of the equilateral triangle erected on $u$), and also if and only if $2u^2$ is the sum of the other squared sides. The sides of a right-angled automedian triangle are proportional to $1 : \sqrt{2} : \sqrt{3}$, and an isosceles triangle is automedian exactly when it is equilateral. For the normalized automedian triangles (Figure 2), $f$ maps the circle of radius $\frac{\sqrt{3}}{2}$ centered at $\frac{1}{2}$ to the extended line through $0$ and $i\zeta = -e^{i\pi/6}$, the circle of radius $\sqrt{3}$ centered at $-1$ to the extended line through $0$ and $i\zeta^2$, and the circle $-\mathcal{C}$ of radius $\sqrt{3}$ centered at 2 to the extended imaginary axis: $\sigma$ maps thus the nonequilateral automedian triangles to the punctured lines $\lambda i\zeta^k$, $\lambda \in \mathbf{R} \setminus \{0\}$, $k = 0, 1, 2$. $s$ is the shape of an automedian triangle if and only if $-\overline{s} \in \{s, \zeta s, \zeta^2 s\}$. The normalized automedian triangle with vertex $z = \frac{1}{2} + \frac{\sqrt{3}}{2}e^{i\varphi}$ has the shape $f(z) = \tan(\frac{\pi}{4} - \frac{\varphi}{2})e^{i\pi/6}$.

For $R \in [0,\infty]$, $f$ maps the circle $|z| = R$ to the Apollonius circle $\left|\frac{s-\zeta^2}{s-\zeta}\right| = R$, *i.e.*, to the circle $\left|s + \frac{1}{2} - i\frac{\sqrt{3}}{2}\frac{R^2+1}{R^2-1}\right| = \frac{\sqrt{3}R}{|R^2-1|}$ if $R \neq 1$ and to the extended real axis if $R = 1$. Conversely, $f^{-1}$ maps the circle $|s| = R$ to the Apollonius circle

$$\mathcal{C}_R\colon \left|\frac{z+\zeta^2}{z+\zeta}\right| = \left|\frac{z-e^{i\pi/3}}{z-e^{-i\pi/3}}\right| = R, \tag{2}$$

*i.e.*, to the circle $\left|z - \frac{1}{2} + i\frac{\sqrt{3}}{2}\frac{R^2+1}{R^2-1}\right| = \frac{\sqrt{3}R}{|R^2-1|}$ if $R \neq 1$ and to the extended real axis if $R = 1$ (Figure 3). If $R \neq 1$, $\mathcal{C}_R$ is a Neuberg circle [17, p. 287], [29], *i.e.*, the locus of the vertex $z$ of the triangles $\Delta'(z)$ with an appropriate constant Brocard angle $\omega$ in the upper or lower half-plane according as $R<1$ or $R>1$: since the base side 1 of $\Delta'(z)$ subtends the angle $2\omega$ from the center of $\mathcal{C}_R$, one has (see also [10])

$$\cot\omega=\sqrt{3}\left|\frac{R^2+1}{R^2-1}\right| \text{ for } R=\left|\sigma_{\Delta'(z)}\right|. \tag{3}$$

If $\left|\sigma_{\Delta'(z_0)}\right|=R\neq 0,1,\infty$, the sides of $\Delta'(z_0)$ issued from $z_0\in\mathcal{C}_R$ cut $\mathcal{C}_R$ in one or two other points: these are vertices of normalized triangles inversely similar to $\Delta'(z_0)$; the reflections of $z_0$ and of these vertices in the line $\operatorname{Re} z=\frac{1}{2}$ give the other normalized triangles similar to $\Delta'(z_0)$ with the same orientation [17, p. 289].

Figures 1, 2, and 3 show how many right-angled, isosceles, and automedian normalized triangles are equibrocardal with a shape of given modulus $R\neq 0,1,\infty$. There are two inversely similar automedian triangles with opposite shapes (and their companions with cyclically shifted vertices); there are two isosceles triangles with opposite shapes (and their companions), whose base angles are

$$\theta_1=\arctan\frac{\sqrt{3}(1-R)}{1+R},\quad \theta_2=\arctan\frac{\sqrt{3}(1+R)}{1-R}. \tag{4}$$

Note that $\tan\theta_1\cdot\tan\theta_2=3$ and that $\theta_1$, $\theta_2$ are $<0$ when the orientation is negative. For $R=2\pm\sqrt{3}$ or $R\in\left]2-\sqrt{3},2+\sqrt{3}\right[$ there are one (isosceles) right-angled triangle or two inversely similar right-angled triangles with conjugate shapes (and their companions), respectively. Note that some of the above triangles may be simultaneously right-angled and isosceles or automedian.

We will prove later that two nontrivial triangles have opposite shapes (possibly after multiplying one of the shapes by $\zeta^{\pm 1}$) if and only if they are directly similar to the median triangle of each other.

$f$ maps the circle $\mathcal{C}'$ through $e^{i\pi/3}$, $e^{-i\pi/3}$, and $z_0\in\mathbf{C}\setminus\left\{e^{i\pi/3},e^{-i\pi/3}\right\}$ to the line $\lambda f(z_0)$, $\lambda\in\mathbf{R}\cup\{\infty\}$, and $Z_0=\frac{z_0-2}{2z_0-1}$ lies on this circle since $f\left(\frac{z-2}{2z-1}\right)=-f(z)$. When $z_0$ is not real, $\Delta'(z_0)$ is automedian if and only if the line through $z_0$ and $Z_0$ is horizontal or contains 0 or 1, because two equibrocardal triangles with a common angle are similar and because $\Delta'(Z_0)$ is directly similar to the median triangle of $\Delta'(z_0)$. For each $L\geq 0$, $f$ admits the values $\pm Lf(z_0)$ at the points $z_\pm$ given by the intersections of $\mathcal{C}'$ with the Apollonius circle $\mathcal{C}_{L|f(z_0)|}$ (Figure 4).

$\widetilde{\sigma}(z_0,z_1,z_2)=\frac{z_2-z_0}{z_1-z_0}$ [36], which is invariant under triangle translation and under homothety and rotation, is a shape function that is more intuitive than $\sigma$. One has $\widetilde{\sigma}(0,1,z)=z$, $\sigma(0,1,z)=\zeta\widetilde{\sigma}(z,e^{-i\pi/3},e^{i\pi/3})$, and $\sigma(z_0,z_1,z_2)=\sigma(0,1,\widetilde{\sigma}(z_0,z_1,z_2))$.

## 4. Convolution filters

We consider a *filter* $T_\Gamma\colon \mathbf{C}^3\to\mathbf{C}^3$ given by the cyclic convolution $*$ with a fixed triangle $\Gamma=(c_0,c_1,c_2)$, *i.e.*, by a circulant matrix:

$$T_\Gamma\colon \Delta\mapsto\Delta*\Gamma=(z_0,z_1,z_2)*(c_0,c_1,c_2)=(z_0,z_1,z_2)\begin{pmatrix}c_0 & c_1 & c_2\\ \boldsymbol{c_2} & c_0 & c_1\\ \boldsymbol{c_1} & c_2 & c_0\end{pmatrix}.$$

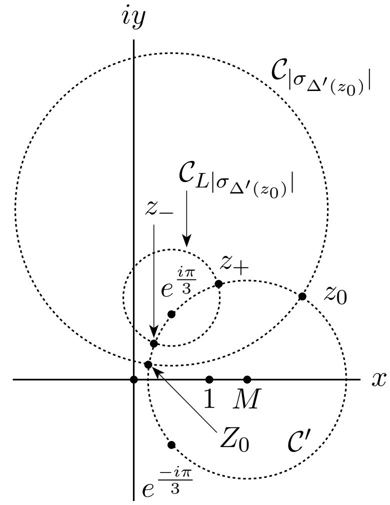


Figure 4. $\Delta'(z_0)$ and $\Delta'(Z_0)$ have opposite shapes; $\Delta'(z_\pm)$ have opposite shapes $\pm L\sigma_{\Delta'(z_0)}$ for $L = \frac{1}{2}$.

The $k$th entry of $\Delta * \Gamma = \Gamma * \Delta$ is $\sum_{\ell=0}^{2} z_\ell c_{k-\ell \pmod 3}$, $k = 0, 1, 2$.

Since the operator $*$ is bilinear and since $e_k * e_\ell = \begin{cases} 3e_k & (k = \ell) \\ (0,0,0) & (k \neq \ell) \end{cases}$, one has

$$T_\Gamma(\Delta) = \Delta * \Gamma = \left( \sum_{k=0}^{2} \hat{z}_k e_k \right) * \left( \sum_{\ell=0}^{2} \hat{c}_\ell e_\ell \right) = \sum_{k=0}^{2} 3\hat{c}_k \hat{z}_k e_k,$$

*i.e.*, $\widehat{\Delta * \Gamma} = 3\widehat{\Delta} \cdot \widehat{\Gamma}$, where $\cdot$ is the entrywise product: the Fourier base is a base of eigenvectors of the convolution $T_\Gamma$ with eigenvalues $3\hat{c}_k$ (and the ratio $\frac{3\hat{c}_2}{3\hat{c}_1}$ of the two "equilateral" eigenvalues is the shape of $\Gamma$). $T_\Gamma$ maps thus trivial, pequilateral and nequilateral triangles to triangles of the same category or to trivial triangles. $T_\Gamma(\Delta)$ and $\Delta$ always have the same centroid if and only if $c_0 + c_1 + c_2 = 1$, which means $\hat{c}_0 = \frac{1}{3}$; the centroid is always translated to the origin if and only if $\hat{c}_0 = 0$. The image by $T_\Gamma$ of the *Dirac triangle* $(1, 0, 0) = \frac{1}{3}(e_0 + e_1 + e_2)$ of shape 1 is $\Gamma$, the *impulse response* of the filter $T_\Gamma$, and the filter output is the convolution of the input with the impulse response, as for every linear time-invariant filter; the convolution with the Dirac triangle is the identity. It is immediate that

$$\sigma_{\Delta * \Gamma} = \sigma_\Delta \sigma_\Gamma$$

when $\Gamma$ and $\Delta$ are not trivial, except that $\Delta * \Gamma$ is trivial when $\Gamma$ and $\Delta$ are equilateral with opposite orientations (*i.e.*, $0 \cdot \infty =$ trivial). When $\Gamma$ and $\Delta$ are mixed, $\Delta * \Gamma$ is degenerate if and only if $|\sigma_\Delta \sigma_\Gamma| = 1$. When $\Delta$ is mixed, $T_\Gamma(\Delta)$ can have any prescribed shape $\sigma_{\Delta_1}$, and this is the case if and only if $\sigma_\Gamma = \frac{\sigma_{\Delta_1}}{\sigma_\Delta}$.

One has $T_{\Gamma_1} \circ T_{\Gamma_2} = T_{\Gamma_2} \circ T_{\Gamma_1} = T_{\Gamma_1 * \Gamma_2}$. The iterates of $T_\Gamma$ are the convolution filters $T_\Gamma^n \colon (z_0, z_1, z_2) \mapsto \sum_{k=0}^{2} (3\hat{c}_k)^n \hat{z}_k e_k$, $n \in \mathbf{N}$, with centroid $(3\hat{c}_0)^n \hat{z}_0$. The sum of the squared distances between the centroid and the vertices of $T_\Gamma^n(\Delta)$ is

$3\sum_{k=1}^{2}(3|\hat{c}_k|)^{2n}|\hat{z}_k|^2$: the diameter of $T_\Gamma^n(\Delta)$ tends to 0 for all $\Delta$ when $n \to \infty$ if and only if $|\hat{c}_1| < \frac{1}{3}$ and $|\hat{c}_2| < \frac{1}{3}$; this diameter remains bounded for every $\Delta$ exactly when $|\hat{c}_1| \le \frac{1}{3}$ and $|\hat{c}_2| \le \frac{1}{3}$, and it tends to $\infty$ for all nontrivial $\Delta$ if and only if $|\hat{c}_1| > \frac{1}{3}$ and $|\hat{c}_2| > \frac{1}{3}$. When $\Gamma$ and $\Delta$ are neither trivial nor both equilateral with opposite orientations, $T_\Gamma^n(\Delta)$ has the shape $\sigma_\Delta \sigma_\Gamma^n$ for $n \ge 1$. The behavior of the shape of a mixed triangle under iterated convolution with $\Gamma$ is thus a matter of domination between the eigenvalues $3\hat{c}_1$ and $3\hat{c}_2$, *i.e.*, this behavior depends on $|\sigma_\Gamma|$ (Theorem 2). We call the filter trivial, equilateral, degenerate, and so on when $\Gamma$ is trivial, equilateral, degenerate, and so on. A trivial filter maps any triangle to a trivial one.

We now show that a nontrivial convolution filter (with half an exception) simply adds three similar ears of fixed shape to every triangle $\Delta = (z_0, z_1, z_2)$ before submitting the triangle $\Delta_1$ of the ears' apices to a direct similarity $a\Delta_1 + b\hat{z}_0 e_0$ with fixed $a \ne 0$ and fixed $b$. A (generalized) *Kiepert triangle* consists of the apices of ears that are erected on the sides of the initial triangle (opposite to the vertices in order) and that all have the shape of the normalized $\Delta'(z) = (0, 1, z)$ with apex $z \in \mathbf{C}$: the ear's apex for the side $z_1 \to z_2$ is defined as $z_2 + z(z_1 - z_2)$; it is a right-hand ear if $\operatorname{Im} z > 0$. The corresponding Kiepert triangle is thus given by the centroid-preserving convolution with $K(z) = (0, 1 - z, z)$ of spectrum $\frac{1}{3}(1, \zeta^2 + \sqrt{3}iz, \zeta - \sqrt{3}iz)$ and of shape $\frac{\xi - z}{z - \overline{\xi}} = -\widetilde{\sigma}(z, \overline{\xi}, \xi)$, where $\xi = \frac{1}{\sqrt{3}}e^{i\pi/6}$. One has $\sigma_{K(1-z)} = 1/\sigma_{K(z)}$, since $K(1-z) = (0, z, 1-z)$, and $K(\overline{z}) = \overline{K(z)}$. Thus $\sigma_{K(1-x)} = \overline{\sigma_{K(x)}}$ if $x \in \mathbf{R}$.

$K(z)$ is orthogonal to $e_2$, hence pequilateral since nontrivial, exactly for $z = \xi = \frac{1}{1-\zeta}$; $K(z)$ is orthogonal to $e_1 = \overline{e_2}$, hence nequilateral, exactly for $z = \overline{\xi} = \frac{1}{1-\zeta^2}$. The filters $T_{K(\xi)}$ and $T_{K(\overline{\xi})}$ add right-hand and left-hand isosceles ears with base angles $\frac{\pi}{6}$ and shape $-\frac{\zeta}{2}$ and $-2\zeta$, respectively. Napoleon's theorem [9, 26] is now obvious: the convolution of the initial triangle with the pequilateral triangle $K(\xi) = (0, \overline{\xi}, \xi) = \frac{1}{3}(e_0 - e_1)$ and with the nequilateral triangle $K(\overline{\xi}) = \overline{K(\xi)} = \frac{1}{3}(e_0 - e_2)$, respectively, are equilateral (or trivial).

Since $(0, 1, z) = (z+1)\big(0, 1 - \frac{z}{z+1}, \frac{z}{z+1}\big)$ for $z \in \mathbf{C} \setminus \{-1\}$, every normalized triangle $\Delta'(z)$ different from $(0, 1, -1)$ and from $(0, 0, 1)$ has the shape of $K\big(\frac{z}{z+1}\big)$. One has further $(0, 0, 1) = K(1) = \lim_{z\to\infty} K\big(\frac{z}{z+1}\big)$, and $(0, 1, -1) = \frac{i}{\sqrt{3}}(e_2 - e_1)$ of shape $-1$ is equal to $\lim_{z\to -1}(z+1)K\big(\frac{z}{z+1}\big)$.

**Theorem 1.** *A nontrivial filter $T_\Gamma$ of shape $\sigma_\Gamma \ne -1$ is the convolution with* $\Gamma =$

$$aK\left(\frac{\overline{\xi}\sigma_\Gamma + \xi}{\sigma_\Gamma + 1}\right) + be_0 \text{ for some fixed complex } a \ne 0 \text{ and } b, \text{ where } \xi = \tfrac{1}{\sqrt{3}}e^{i\pi/6}.$$

*If $\sigma_\Gamma = -1$, $T_\Gamma$ is the convolution with some $\Gamma = a(0, 1, -1) + be_0$, $a \ne 0$.*

*A triangle $\Gamma = (c_0, c_1, c_2)$ can be written as $\Gamma = (c_1 - c_0)(0, 1, -1) + c_0 e_0$ if $c_1 + c_2 = 2c_0$ and as $\Gamma = (c_1 + c_2 - 2c_0)K\big(\frac{c_2 - c_0}{c_1 + c_2 - 2c_0}\big) + c_0 e_0$ otherwise.*

The convolutions with $K(1) = (0, 0, 1)$ of shape $\zeta$ and $K(0) = (0, 1, 0)$ of shape $\zeta^2$ are simply a left and a right cyclic shift of the vertices, respectively; the only shape-preserving Kiepert filter is the convolution with $K(\frac{1}{2}) = (0, \frac{1}{2}, \frac{1}{2}) = \frac{1}{6}(2e_0 - e_1 - e_2)$ of shape 1, which maps a triangle $\Delta$ to its medial triangle, *i.e.*, to $\frac{1}{2}$ times the half-turned $\Delta$ (shrunk and rotated about the centroid). More generally, $T_{K(x)}(\Delta)$, $x \in \mathbf{R}$, is the $(1-x)$-medial triangle of $\Delta$ [11]: this is the introducing Emmerich transformation! Since $K(x)$ has the shape of $\Delta'\big(\frac{x}{1-x}\big)$, $\sigma_{K(x)}$ turns anticlockwise on the unit circle, starting and ending at $-1 = \sigma(0, 1, -1)$ excluded, as $x$ grows on the real axis.

Note that $\sigma_{K(z)}$ is real if and only if $\operatorname{Re} z = \frac{1}{2}$, *i.e.*, if and only if the added ears are isosceles with equal angles at 0 and 1 (this corresponds to the *classical* Kiepert triangles). To get isosceles ears with base angles $|\theta| < \frac{\pi}{2}$, one has to convolve with the isosceles $K_{\text{iso}}(\theta) = K\big(\frac{1+i\tan\theta}{2}\big)$ with apex angle $2|\theta|$ and shape $\frac{1-\sqrt{3}\tan\theta}{1+\sqrt{3}\tan\theta}$: the ears are right-hand or left-hand according as $\theta \geq 0$ or $\theta \leq 0$. Since $\widehat{K_{\text{iso}}(\frac{\pi}{3})} = \frac{1}{3}(1, -2, 1)$, one can retrieve $\Delta$ from $\Delta * K_{\text{iso}}(\frac{\pi}{3})$ by constructing $\frac{1}{2}\big(T_{K_{\text{iso}}(\frac{\pi}{3})}(\Delta) + T^2_{K_{\text{iso}}(\frac{\pi}{3})}(\Delta)\big) = \Delta$: this is Lemoine's problem [21]. With the same idea, one finds $\Delta = \frac{1}{2}\big(T_{K_{\text{iso}}(-\frac{\pi}{3})}(\Delta) + T^2_{K_{\text{iso}}(-\frac{\pi}{3})}(\Delta)\big)$, $\Delta = 2T^2_{K(\frac{1}{2})}(\Delta) - T_{K(\frac{1}{2})}(\Delta)$, and $\Delta = T_{K(z)}(\Delta) + \frac{1}{3(z-\xi)(z-\overline{\xi})}\big(T^3_{K(z)}(\Delta) - T_{K(z)}(\Delta)\big)$ for $z \neq \xi, \overline{\xi}$. A triangle $\Delta$ and its classical Kiepert triangle $\Delta * K_{\text{iso}}(\theta)$ are always perspective [4, 20] and, if $\Delta$ is proper and nonisosceles, the perspectors form the equilateral Kiepert hyperbola of $\Delta$ (Figure 5) as $\theta$ runs from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, *i.e.*, the hyperbola through the vertices of $\Delta$, the centroid $G$, and the orthocenter $H$ (which is the perspector in the limit case $|\theta| = \frac{\pi}{2}$). We now look at this limit case more closely.

The vertices of $\Delta * K_{\text{iso}}(\theta)$ tend for $|\theta| \to \frac{\pi}{2}-$ to the infinite points of the altitudes of $\Delta$ on the line at infinity (when $\Delta$ has three different vertices). On the other hand, $\lim_{|\theta|\to\frac{\pi}{2}-} \sigma_{K_{\text{iso}}(\theta)} = -1 = \sigma(0, 1, -1)$: the limit shape of $\Delta * K_{\text{iso}}(\theta)$ when $|\theta| \to \frac{\pi}{2}-$ is thus the shape of $\Delta * (0, 1, -1)$ for any nontrivial triangle $\Delta$. This is clear geometrically: with $\Delta = (z_0, z_1, z_2)$, the angles of $\Delta * K_{\text{iso}}(\theta)$ tend for $|\theta| \to \frac{\pi}{2}-$ to the angles of the (quarter-turned) triangle

$$(z_2 - z_1, z_0 - z_2, z_1, z_1 - z_0) = \Delta * (0, 1, -1),$$

whose vertices are the tips of the vectors $z_1 \to z_2$, $z_2 \to z_0$, $z_0 \to z_1$ starting from 0 (Figure 6). Convolving $\Delta$ with the normalized $(0, 1, -1)$ gives a scaled down and quarter-turned "equally shaped" bounded copy of "$\Delta$ with infinite similar isosceles ears". Here is another description of the filter $T_{(0,1,-1)}$: it translates $\Delta = (z_0, z_1, z_2)$ to $\Delta_1 = \Delta - \hat{z}_0 e_0 = \big(w_0, w_1, w_2\big)$ with centroid at the origin and then blows up the $\frac{1}{3}$-medial triangle (with cyclically shifted vertices) $\Delta_2 = \Delta_1 * \big(\frac{1}{3}, \frac{2}{3}, 0\big)$ to $3\Delta_2 = (W_0, W_1, W_2) = \Delta * (0, 1, -1)$ (Figure 6). $T^2_{(0,1,-1)}(\Delta)$ is three times the half-turned $\Delta_1$ (enlarged and rotated about the origin). $\Delta * (0, 1, -1)$ is thus directly similar to $\Delta * (0, \frac{1}{3}, \frac{2}{3}) = \Delta * K(\frac{2}{3})$, the $\frac{1}{3}$-medial triangle of $\Delta$, and two triangles have opposite shapes (possibly after multiplying one of the shapes by $\zeta^{\pm 1}$) if and only if they are directly similar to the $\frac{1}{3}$-medial triangle of each other: since the $\frac{1}{3}$-medial triangle is directly similar to the $\frac{2}{3}$-medial triangle and to the median triangle (Figure 6), this proves that two triangles have

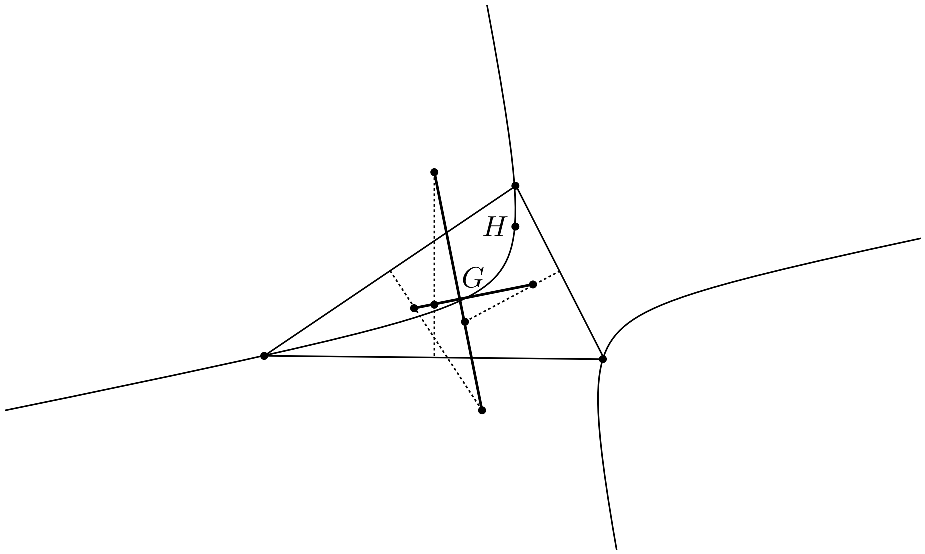


Figure 5. Kiepert hyperbola and both degenerate classical Kiepert triangles with dotted perpendicular bisectors

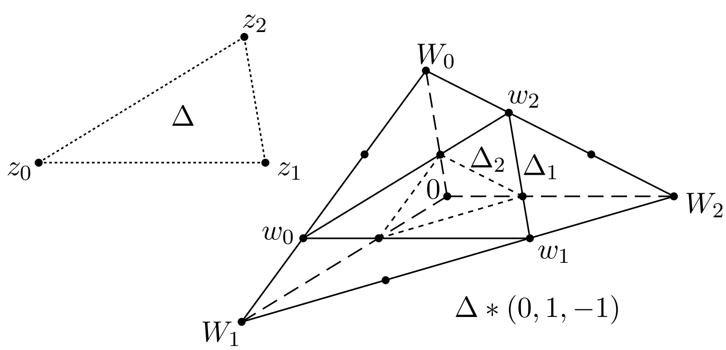


Figure 6

opposite shapes (possibly up to a factor $\zeta^{\pm 1}$) if and only if they are directly similar to the median triangle of each other. The well-known fact that the median triangle of the median triangle is directly similar to the start triangle is equivalent to the fact that the shape of $(0, 1, -1) * (0, 1, -1)$ is $1$. If $\Delta'(z)$ is a normalized triangle, remember that $\Delta'(z) * (0, 1, -1)$ and $\Delta'\left(\frac{z-2}{2z-1}\right)$ have the same shape $-\sigma_{\Delta'(z)}$ (Figure 4).

The following theorem is almost immediate.

**Theorem 2.** (1) *The convolution filter $T_\Gamma$ is bijective if and only $\Gamma$ has a full spectrum,* i.e., *if and only if $\Gamma = a(0, 1, -1) + be_0$ with $a, b \neq 0$, or $\Gamma = aK(z) + be_0$ with $z \notin \{\xi, \overline{\xi}\}$, $a \neq 0$, $a + 3b \neq 0$; if $\Gamma = (c_0, c_1, c_2)$, the inverse filter is then the convolution with $\sum_{k=0}^{2} \frac{1}{9\hat{c}_k} e_k$.*

(2) *A pequilateral filter* (*of shape* $0$)*,* i.e.*, a convolution with* $aK(\xi)+be_0$*,* $a\neq 0$*, maps nequilateral triangles to trivial triangles and all other nontrivial triangles to pequilateral triangles. A nequilateral filter* (*of shape* $\infty$)*,* i.e.*, a convolution with* $aK\left(\overline{\xi}\right)+be_0$*,* $a\neq 0$*, maps pequilateral triangles to trivial triangles and all other nontrivial triangles to nequilateral triangles.*
(3) *A proper nonequilateral filter* $T_\Gamma$*,* i.e.*, with* $|\sigma_\Gamma|\neq 0,1,\infty$*, is smoothing: its action on equilateral triangles is shape-preserving; and according as the filter is positively or negatively oriented,* i.e.*, according as* $0<|\sigma_\Gamma|<1$ *or* $1<|\sigma_\Gamma|<\infty$*, the iterates* $T_\Gamma^n(\Delta)$ *of every mixed* $\Delta$ *are eventually positively or negatively oriented and have a (never reached) pequilateral or nequilateral limit in shape, respectively.*
(4) *A nontrivial filter* $T_\Gamma$ *is degenerate* (*with* $|\sigma_\Gamma|=1$) *if and only if* $\Gamma=aK+be_0$ *for some* $a\neq 0$ *and some degenerate* $K=K(x)$*,* $x\in\mathbf{R}$*, or* $K=(0,1,-1)$*.*
(5) *If* $\Gamma=a(0,1,-1)+be_0$*,* $a\neq 0$*, and if* $\Delta=(z_0,z_1,z_2)$*, one has*

$$T_\Gamma^n(\Delta)=\left(-i\sqrt{3}\,a\right)^n\left(\hat{z}_1e_1+(-1)^n\hat{z}_2e_2\right)+(3b)^n\hat{z}_0e_0.$$

*If* $\Delta$ *is mixed, these iterates* $T_\Gamma^n(\Delta)$ *are* $2$*-periodic in shape with shape* $(-1)^n\sigma_\Delta$ (Figures 6 and 4).
(6) *If* $x\in\mathbf{R}$*, one has*

$$\widehat{K(x)}=(\hat{c}_0,\hat{c}_1,\overline{\hat{c}_1})=\tfrac{1}{6}\left(2,-1+i\sqrt{3}(2x-1),-1-i\sqrt{3}(2x-1)\right)\quad\textit{and}$$

$$\sigma_{K(x)}=-e^{i2\arg(\xi-x)}=e^{i2\kappa(x)}\ \textit{with}\ \kappa(x)=\arctan\left(\sqrt{3}(2x-1)\right).$$

*If* $\Gamma=aK(x)+be_0$ *for some* $a\neq 0$*, one has*

$$T_\Gamma^n(\Delta)=\left(-3|\hat{c}_1|e^{-i\kappa(x)}a\right)^n\left(\hat{z}_1e_1+e^{i2n\kappa(x)}\hat{z}_2e_2\right)+(a+3b)^n\hat{z}_0e_0\qquad(5)$$

*for every* $\Delta=(z_0,z_1,z_2)$*, where* $3|\hat{c}_1|=\sqrt{3x^2-3x+1}$*. When* $\Delta$ *is mixed, these iterates* $T_\Gamma^n(\Delta)$ *are periodic or nonperiodic in shape (with chaotic behavior) according as* $\kappa(x)/\pi$ *is rational or irrational, respectively (the period in similarity may be shorter than the period in shape). The period length is* $m=1$ *if and only if* $\kappa(x)=0$*,* i.e.*, if and only if* $K(x)=K(\frac{1}{2})$*;* $m\geq 2$ *is the minimal period length* (*the same for all mixed* $\Delta$) *if and only if* $\sigma_\Gamma=\sigma_{K(x)}=e^{i2\pi\ell/m}$ *for some integer* $\ell\in[1,m-1]$ *coprime to* $m$*,* i.e.*, if and only* $m\geq 3$ *and*

$$x=\frac{1}{2}+\frac{1}{2\sqrt{3}}\tan\left(\frac{\ell}{m}\pi\right)\qquad(6)$$

*for some integer* $\ell\in[1,m-1]$ *coprime to* $m$ (*note that the period* $2$ *corresponds to* $K=(0,1,-1)$ *and that* $x\in[0,1]$ *exactly when* $\frac{\ell}{m}\notin\left]\frac{1}{3},\frac{2}{3}\right[$)*;* $T_{K(x)}^m(\Delta)$ *is then given by the homothety of ratio* $(-1)^{m+\min(\ell,m-\ell)}(3|\hat{c}_1|)^m$ *about the centroid of* $\Delta$*.*
(7) *When* $\Delta$ *is a mixed triangle and when* $\Gamma$ *is degenerate but not trivial, the shapes* $(-1)^n\sigma_\Delta$ *or* $e^{i2n\kappa(x)}\sigma_\Delta$ *of the iterates* $T_\Gamma^n(\Delta)$ *lie on the circle* $|s|=|\sigma_\Delta|=R\in\left]0,\infty\right[$ *and correspond to (equibrocardal) normalized triangles* $(0,1,z)$ *with vertex* $z$ *on the Neuberg circle* $\mathcal{C}_R$ *given by* (2) *if* $R\neq 1$*, and to nontrivial degenerate triangles if* $R=1$ (Figures 3 and 4)*. If* $m$*-periodic,* $m\geq 1$*, these shapes are the vertices (in order) of a regular oriented* $\{m/\ell\}$*-gon with start at* $\sigma_\Delta$ *for some*

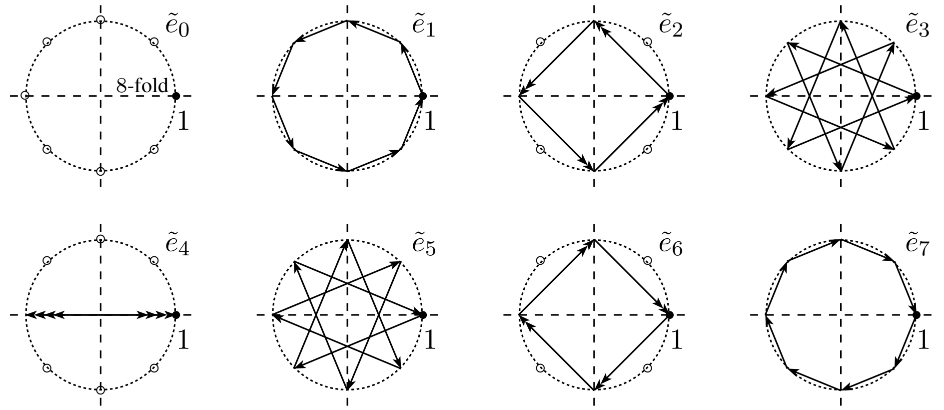


Figure 7. The Fourier base vectors $\tilde{e}_k$ of $\mathbf{C}^8$, $0 \le k \le 7$, are the regular $\{8/k\}$-gons $\left(\left(e^{i2\pi/8}\right)^{k\cdot 0}, \left(e^{i2\pi/8}\right)^{k\cdot 1}, \ldots, \left(e^{i2\pi/8}\right)^{k\cdot 7}\right)$.

$\ell \in [0, m-1]$ *coprime to* $m$ *(note that this is the Fourier base polygon* $\tilde{e}_\ell$ *of* $\mathbf{C}^m$ (Figure 7) *multiplied by* $\sigma_\Delta$ *and that the choice of another* $\ell$ *coprime to* $m$ *only changes the order of the shapes); if nonperiodic,* i.e.*, if* $\kappa(x)$ *is an irrational multiple of* $\pi$*, these shapes* $T_\Gamma^n(\Delta)$ *are dense on the circle* $|s| = |\sigma_\Delta| = R$*,* i.e.*, the accumulation triangles (in shape) of the sequence* $\left(T_\Gamma^n(\Delta)\right)_{n\ge 0}$ *are the (equibrocardal) normalized triangles* $(0, 1, z)$ *with vertex* $z$ *on the Neuberg circle* $\mathcal{C}_R$ *if* $R \neq 1$ *and the nontrivial degenerate triangles if* $R = 1$*.*

(8) *When* $\Delta$ *is a proper positively oriented mixed triangle with the shape of* $\Delta'(z_0)$ *and when* $x$ *grows on the whole real axis, the shape of* $\Delta * K(x)$ *travels counterclockwise over the whole circle* $|s| = |\sigma_\Delta|$ *starting and ending at* $-\sigma_\Delta$ *excluded, whereas the vertex* $z$ *of the normalized triangle* $\Delta'(z)$ *with the shape of* $\Delta * K(x)$ *turns counterclockwise over the whole Neuberg circle* $\mathcal{C}_{|\sigma_{\Delta'(z_0)}|}$ *of* Figure 4 *starting and ending at* $Z_0$ *excluded.* $\Delta * (0, 1, -1)$ *fills the holes* $-\sigma_\Delta$ *and* $Z_0$*. The rotation on the Neuberg circle is clockwise if* $\Delta$ *is negatively oriented. In the degenerate case* $|\sigma_\Delta| = 1$*,* $z$ *runs rightwards over the whole extended real axis starting and ending at* $Z_0 = \frac{z_0 - 2}{2z_0 - 1}$ *excluded.*

Note that the result (6) or an equivalent one can be found in [22, 36, 16, 11] and that the last two parts of Theorem 2 probably furnish the solution that the quite incomprehensible paper [19] aimed at. Note also that the iterates $T_\Gamma^n(\Delta)$ of a mixed $\Delta$ are 3-periodic in shape if and only if $\Gamma = aK(1) + b$ or $aK(0) + b$, $a \neq 0$, $K(1)$ and $K(0)$ causing the left and right shifts of $\Delta$'s vertices.

The proper nonequilateral triangle $\Delta$ is directly similar to its $(1-t)$-medial triangle $\Delta * K(t)$, $t \in \mathbf{R}$, if and only if $\sigma_{K(t)} = \zeta^k$, $k \in \{0, 1, -1\}$, *i.e.*, if and only if $t = \frac{1}{2}, 1, 0$: $\Delta * K(t)$ is then the medial triangle or a copy of $\Delta$ with cyclically shifted vertices. The proper nonequilateral triangle $\Delta$ is inversely similar to $\Delta * K(t)$ if and only if $\sigma_{K(t)} = \zeta^k e^{-i2\arg\sigma_\Delta}$, *i.e.*, if and only if

$$t = \tfrac{1}{2} + \tfrac{1}{2\sqrt{3}} \tan(k\tfrac{\pi}{3} - \arg\sigma_\Delta),\ k \in \{0, 1, -1\} : \tag{7}$$

the solutions $t$ depend only on $\arg\sigma_\Delta \pmod{\pi}$, *i.e.*, the set of solutions for the nonequilateral normalized triangle $\Delta'(z_0) = (0, 1, z_0)$ remains the same for all nonequilateral $\Delta'(z)$ with $z$ on the circle $\mathcal{C}'$ through $e^{i\pi/3}$, $e^{-i\pi/3}$, and $z_0$ (Figure 4). These solutions $t$ are again $0, \frac{1}{2}, 1$ if $\Delta$ is isosceles, they are $\frac{1}{3}, \frac{2}{3}$, and the infinite point of the $t$-axis if $\Delta$ is automedian (then $\sigma_{K(t)} = \zeta^k e^{i\pi/3}$, $k = -1, 0, 1$, and the infinite solution corresponds to $\Delta * (0, 1, -1)$), and the solutions are three different real numbers $t \neq 0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1$ in the other cases. It is clear that $\frac{1}{3}$ and $\frac{2}{3}$ appear only for automedian triangles, because the $\frac{2}{3}$- and $\frac{1}{3}$- medial triangles $\Delta * K(\frac{1}{3})$ and $\Delta * K(\frac{2}{3})$, of shape $e^{\mp i\pi/3}\sigma_\Delta$, are directly similar to the median triangle (Figure 6). If $\Delta$ is degenerate and nontrivial, $\Delta$ and $\Delta * K(t)$ are similar if and only if $t$ is 0, $\frac{1}{2}$, 1, or a solution of (7). Figure 8 shows the values of $t$ for wich $\Delta'(z_0) * K(t)$ is similar to $\Delta'(z_0)$ as functions of the midpoint $M$ of $\mathcal{C}'$, whose radius is $r = \sqrt{M^2 - M + 1}$: by considering a real point $M \pm r$ of $\mathcal{C}'$ and by plugging $\arg\sigma_{\Delta'(M\pm r)}$ given by (1) into (7), one obtains for $k = 0, 1, -1$ the solutions

$$t_0 = \frac{M-1}{M-2}, \quad t_1 = \frac{1}{M+1}, \quad t_{-1} = \frac{M}{2M-1} \tag{8}$$

given cyclically by $t_{k+1}(M) = t_k\big(1 - \frac{1}{M}\big)$ (plain, dashed, and dotted hyperbolas of Figure 8, respectively); the values of $M$ corresponding to isosceles or automedian triangles are the midpoints of the dotted circles of Figure 2. If $\Delta$ is neither trivial, nor isosceles, nor automedian, exactly two of the $(1-t)$-medial triangles $\Delta * K(t)$ inversely similar to $\Delta$ are inscribed in $\Delta$, and the sum of these two values of $t$, on each side of $\frac{1}{2}$, is never 1. The cosine law in $\Delta'(z_0)$ with $z_0 = x_0 + iy_0$ on $\mathcal{C}'$ and sides $a = |z_0 - 1|$, $b = |z_0|$, $c = 1$, and the equation $M^2 - M + 1 = r^2 = (x_0 - M)^2 + (b^2 - x_0^2)$ give $M = \frac{1-b^2}{a^2-b^2}$, and thus by (8)

$$1 - t_0 = \frac{a^2 - b^2}{2a^2 - b^2 - c^2}, \; 1 - t_1 = \frac{b^2 - c^2}{2b^2 - c^2 - a^2}, \; 1 - t_{-1} = \frac{c^2 - a^2}{2c^2 - a^2 - b^2}, \tag{9}$$

which are correct by similarity for any nonequilateral and nontrivial triangle $ABC$ with sides $a, b, c$ opposite to the vertices (then $M = \frac{c^2-b^2}{a^2-b^2}c$, $t_0 = \frac{M-c}{M-2c}$, $t_1 = \frac{c}{M+c}$, $t_{-1} = \frac{M}{2M-c}$ if $A = 0$, $B = c > 0$). Since the set $\{t_0, t_1, t_{-1}\}$ is invariant under a cyclic shift of the vertices, each set of solutions $t$ appears for three different $M$ in Figure 8: for $M = \frac{a^2-b^2}{a^2-c^2}$, $M = \frac{b^2-c^2}{b^2-a^2}$, and $M = \frac{c^2-a^2}{c^2-b^2}$ (whose product is $-1$ in the nonisosceles case) given cyclically by $M \mapsto 1 - \frac{1}{M}$ (once in $\big[\frac{1}{2}, 2\big[$, once in $\big[-1, \frac{1}{2}\big[$, and once in the rest of the extended real axis). Note that (9) already appeared in [18] with another proof.

Start for example from a proper nonequilateral *isosceles* $\Delta$ with real shape $\lambda$ (after a cyclic permutation of its vertices, if necessary); choose $x = \frac{1}{2} + \frac{1}{2\sqrt{3}}\tan(\frac{\ell}{m}\pi)$ for some *odd* $m \geq 3$ and some integer $\ell \in [1, m-1]$ coprime to $m$: the $m$-periodic shapes of the equibrocardal $T^n_{K(x)}(\Delta)$ are the vertices in order of $\lambda\tilde{e}_\ell$ in $\mathbf{C}^m$; the cycle contains no automedian triangle; $T^k_{K(x)}(\Delta)$ and $T^{m-k}_{K(x)}(\Delta)$ form a pair of inversely similar triangles for each integer $k \in [1, \frac{m-1}{2}]$ since their shapes are

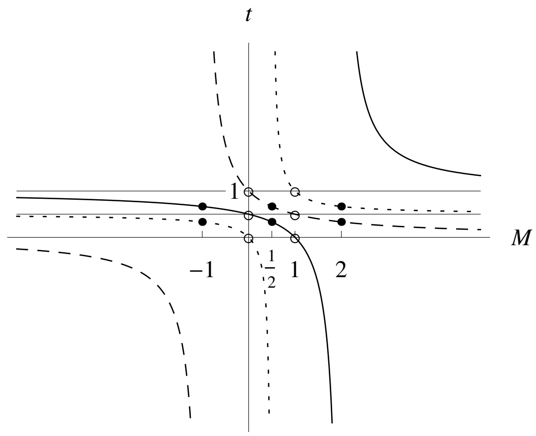


Figure 8. The three or six different values $t$ for which the nonequilateral $\Delta'(z_0)$ and $\Delta'(z_0)*K(t)$ are similar as function of the midpoint $M$ of the circle through $e^{i\pi/3}$, $e^{-i\pi/3}$, and $z_0$: only $t = 0, \frac{1}{2}, 1$ if $\Delta'(z_0)$ is isosceles ($M = 0, 1, \infty$); $t = \frac{1}{3}, \frac{2}{3}, \infty$ ($M = -1, \frac{1}{2}, 2$) besides $t = 0, \frac{1}{2}, 1$ if $\Delta'(z_0)$ is automedian.

complex conjugate; if $m$ is moreover coprime to 3, the cycle contains no two directly similar triangles (hence no other isosceles triangle than $\Delta$) and the minimal period length in similarity is also $m$; if $m$ is divisible by 3, the first third of the cycle in shape contains no two directly similar triangles, the other thirds are obtained by multiplying the shapes of the first third by $\zeta^{\pm 1}$, and the minimal period length in similarity is $\frac{m}{3}$. (Since similarity does not depend on the order of the vertices, the condition that the isosceles $\Delta$ has a real shape can in fact be dropped.)

Start now from a proper nonequilateral *automedian* $\Delta$ with, say, purely imaginary shape $i\lambda$ (after a cyclic permutation of its vertices, if necessary); choose $x$ and $\ell$ as above, but with an *even* $m \geq 2$: the $m$-periodic shapes of the equibrocardal $T^n_{K(x)}(\Delta)$ are the vertices in order of $i\lambda\tilde{e}_\ell$ in $\mathbf{C}^m$; each triangle in the second half of the cycle is directly similar to the $\frac{1}{3}$-medial triangle of the triangle with the same rank in the first half (and conversely), since they have opposite shapes; $T^{m/2}_{K(x)}(\Delta)$ is thus inversely similar to the automedian $\Delta$; each pair $T^k_{K(x)}(\Delta)$, $T^{m/2-k}_{K(x)}(\Delta)$ of the first half consists of inversely similar triangles (except $T^{m/4}_{K(x)}(\Delta)$ if $m$ is divisible by 4), and this property is inherited by the second half. If $m$ is moreover coprime to 3, the cycle contains no two directly similar triangles and no other automedian triangles than $\Delta$ and $T^{m/2}_{K(x)}(\Delta)$; if $m$ is divisible by 3 (hence by 6), there are no two directly similar triangles in the same third of the cycle in shape and the only automedian triangles are the $T^{km/6}_{K(x)}(\Delta)$, which are directly and inversely similar to $\Delta$ for $k = 0, 2, 4$ and $k = 1, 3, 5$, respectively. We consider for example the automedian triangle $\Delta_0$ with sides 1, $\sqrt{2}$, $\sqrt{3}$ and $x = \frac{3+\sqrt{6}-\sqrt{3}}{6} \approx 0.61957$ corresponding to $m = 8$, $\ell = 1$ to construct the iterated $(1-x)$-medial triangle

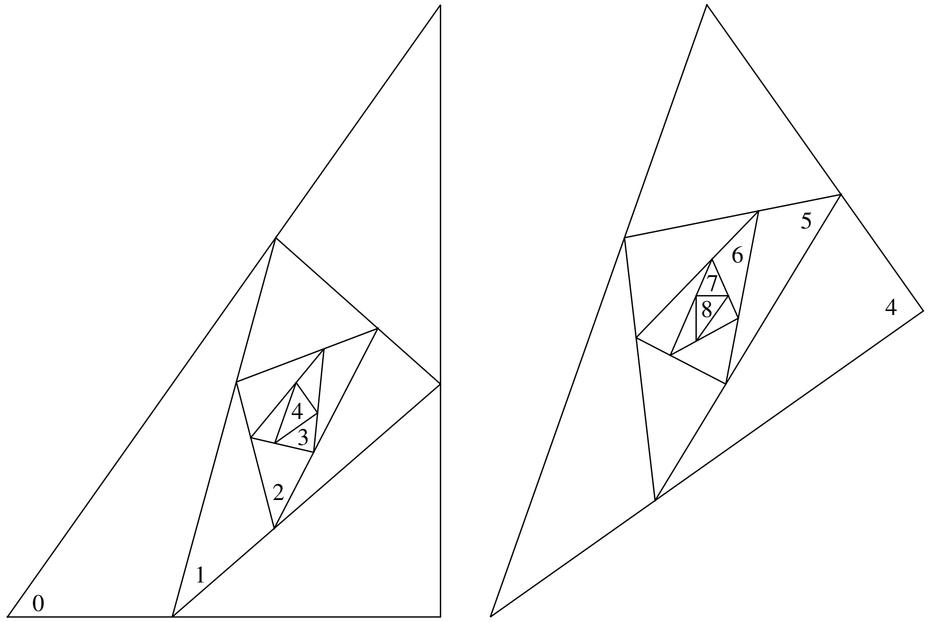


Figure 9

of $\Delta_0$, whose minimal period in shape has length 8: $\sigma_{\Delta_0}$ is $(\sqrt{3} - \sqrt{2}\,)i$ when $\Delta_0 = (0, 1, 1 + \sqrt{2}\, i)$, and $\sigma_{\Delta_{k+1}} = e^{i\pi/4}\sigma_{\Delta_k}$ (Figure 9). $\Delta_k$ and $\Delta_{k+4}$ are similar to the $\frac{1}{3}$-medial triangle of each other and are thus nonsimilar if not automedian; $\Delta_0$ and $\Delta_4$ are right-angled, automedian and inversely similar; $\Delta_1$ and $\Delta_3$ are inversely similar, as are $\Delta_5$ and $\Delta_7$, and these four triangles are neither isosceles, nor automedian, nor right-angled (Figures 1 and 2); $\Delta_2$ and $\Delta_6$ are isosceles; by (5), $\Delta_8$ is obtained from $\Delta_0$ by a half-turn about the centroid of $\Delta_0$ followed by a homothety of ratio $\frac{17}{4} - 3\sqrt{2} \approx 0.00736$ about this centroid. Figure 10 shows the corresponding normalized triangles with their vertices $f^{-1}\big(e^{ik\pi/4}\sigma_{\Delta_0}\big)$ on the Neuberg circle: note the position of the vertices of the inversely similar and of the isosceles triangles [17, p. 289].

*Remark.* In [22], the triangle $S_{p,q}(\Delta) = (w_0, w_1, w_2)$ is constructed cyclically from the proper $\Delta = (z_0, z_1, z_2)$ for real $p$ and $q$ with $pq \neq 1$: $w_0$ is the intersection of the cevian issued from $z_1$ dividing the side $z_2 \to z_0$ in the ratio $p : (1-p)$ and of the cevian issued from $z_0$ dividing the side $z_1 \to z_2$ in the ratio $(1-q) : q$. $S_{p,1-p}(\Delta)$ is the $p$-Rooth triangle of [11].
The centroid-preserving transformation $S_{p,q}$ amounts to the convolution of $\Delta$ with the degenerate triangle $\frac{1}{1-pq}\big(p(1-q), (1-p)(1-q), q(1-p)\big)$: by Theorem 1, $S_{p,q}(\Delta)$ is thus obtained for $p(2q-3) \neq -1$ by submitting $\Delta * K\big(\frac{q-p}{p(2q-3)+1}\big)$ to a homothety of ratio $\frac{p(2q-3)+1}{1-pq}$ with respect to the centroid of $\Delta$. Remember that $\Delta * K(x) = S_{0,x}(\Delta)$ is the $(1-x)$-medial triangle of $\Delta$ for $x \in \mathbf{R}$. If

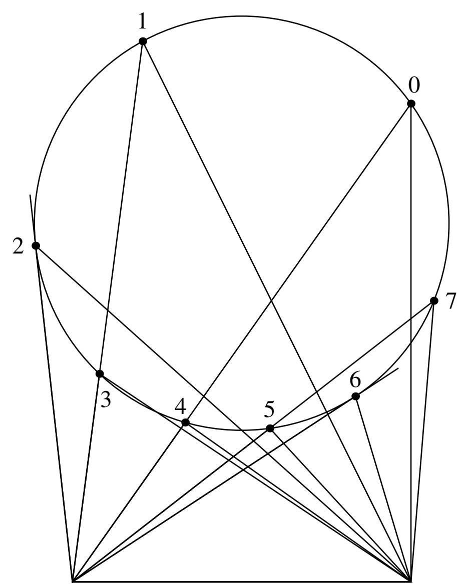


Figure 10

$p(2q-3) = -1$, $S_{p,q}$ is the convolution with $\frac{1-2q}{3}(0,1,-1) + \frac{1}{3}e_0$ and $S_{p,q}(\Delta)$ is the $\frac{1}{3}$-medial triangle of $\Delta$ transformed by a cyclic left shift of its vertices and by a homothety of ratio $1-2q$ with respect to the centroid of $\Delta$; $S_{\frac{1}{2},\frac{1}{2}}$ maps in particular every triangle to its centroid.

For every fixed real $h \neq 0, -1, \pm 2$, the different pairs $(p_1, q_1) = \left(\frac{h-1}{h-2}, \frac{h+1}{h+2}\right)$ and $(p_2, q_2) = \left(\frac{1}{h+2}, \frac{1-h}{2}\right)$ are such that $S_{p_1,q_1}(\Delta)$ and $S_{p_2,q_2}(\Delta)$ have the same vertices: $S_{p_1,q_1}(\Delta)$ is the $\frac{1}{3}$-medial triangle transformed by a homothety of ratio $h$ with respect to its centroid, and $S_{p_2,q_2}(\Delta)$ is the same triangle after a cyclic left shift of the vertices; $h=-3$ corresponds in particular to $S_{\frac{4}{5},2}$ and $S_{-1,2}$.

For $p \in \mathbf{R}\setminus\left\{\frac{1}{2}\right\}$, the triangle $\frac{1-p(1-p)}{1-2p} S_{p,1-p}(\Delta) = \Delta * \frac{1}{1-2p}\left(p^2, p(1-p), (1-p)^2\right)$ is a $p$-median triangle of $\Delta$ [11], *i.e.*, a triangle whose sides are parallel to and as long as the cevians connecting the corresponding vertices of $\Delta$ and $\Delta * K(1-p)$. The $\frac{1}{2}$-median or median triangle is for example $\Delta * \left(-\frac{1}{2}, 0, \frac{1}{2}\right)$ with centroid $0$.

## 5. Isosceles ears

The added ears are isosceles if and only if the shape of the convolving triangle is real or $\infty$ and different from $-1$. The convolving triangle is then $aK_{\text{iso}}(\theta) + be_0$, $a \neq 0$, for some $K_{\text{iso}}(\theta) = K\left(\frac{1+i\tan\theta}{2}\right)$, $|\theta| < \frac{\pi}{2}$, with shape $\frac{1-\sqrt{3}\tan\theta}{1+\sqrt{3}\tan\theta} = \frac{2}{1+\sqrt{3}\tan\theta} - 1$. Since a product of real shapes is real, a composition of convolutions with $a_1K_{\text{iso}}(\theta_1) + b_1e_0$ and $a_2K_{\text{iso}}(\theta_2) + b_2e_0$ is again a convolution with some $aK_{\text{iso}}(\theta) + be_0$, or with $a(0,1,-1) + be_0$, or with a trivial triangle. Since $\sigma_{K_{\text{iso}}(\theta)}$

is real and nonzero for $\theta \neq \pm\frac{\pi}{6}$, the triangles $\Delta$ and $\Delta * K_{\text{iso}}(\theta)$, $\theta \neq \pm\frac{\pi}{6}$, are by Figure 2 always simultaneously isosceles or automedian, respectively.

If $\Delta'(z_0)$ is a normalized triangle with finite $z_0 \neq e^{\pm i\pi/3}$, the shapes of the classical Kiepert triangles $\Delta'(z_0)*K_{\text{iso}}(\theta)$, $|\theta| < \frac{\pi}{2}$, and of $\Delta'(z_0)*(0,1,-1)$ form the extended line $\lambda\sigma_{\Delta'(z_0)}$ and are the shapes of the normalized triangles $\Delta'(z)$ with vertex $z$ on the circle $\mathcal{C}'$ through $e^{i\pi/3}$, $e^{-i\pi/3}$, and $z_0$ (Figure 4). As $\theta$ grows on $\left[0, \frac{\pi}{2}\right[$, the vertex $z$ of the triangle $\Delta'(z)$ with the shape of $\Delta'(z_0) * K_{\text{iso}}(\theta)$ moves from $z_0$ to $Z_0$ (excepted) on the arc of $\mathcal{C}'$ that contains $e^{i\pi/3}$.

Each proper nonequilateral $\Delta$ of shape $s$ has exactly two degenerate classical Kiepert triangles $\Delta * K_{\text{iso}}(\theta)$: for $\theta = \arctan\frac{|s|-1}{\sqrt{3}(1+|s|)}$ and $\theta = \arctan\frac{1+|s|}{\sqrt{3}(|s|-1)}$; these are inward Kiepert triangles, *i.e.*, the ears intersect $\Delta$'s interior, their position does not depend on the vertices' order in $\Delta$, they correspond to the real points of the circle $\mathcal{C}'$ and mark the transition between the positively and negatively oriented $\Delta * K_{\text{iso}}(\theta)$. These two perpendicular [20] degenerate triangles intersect at the centroid of $\Delta$ (which is their common centroid) and they are parallel to the asymptotes of $\Delta$'s Kiepert hyperbola (Figure 5). The degenerate classical Kiepert triangles of a nontrivial degenerate $\Delta$ are the medial triangle and the almost Kiepert $\Delta * (0,1,-1)$.

We determine now the conditions under which the triangles $\Delta$ and $\Delta * K_{\text{iso}}(\theta)$ are similar. We suppose that $\Delta$ is proper, not equilateral, and positively oriented, and we exclude the evident case $K_{\text{iso}}(0) = K(\frac{1}{2})$. We set $\sigma_\Delta = s$ with $0 < |s| < 1$ and denote the shape $\frac{1-\sqrt{3}\tan\theta}{1+\sqrt{3}\tan\theta}$ of $K_{\text{iso}}(\theta)$ by $\mu \in \mathbf{R} \setminus \{-1, 1\}$. $\mu s$ is the shape of a triangle inversely similar to $\Delta$ if and only if $\mu s = \frac{1}{\bar{s}}$, *i.e.*, if and only if $\mu = \frac{1}{|s|^2}(> 1)$: this corresponds to inward ears. $\mu s$ is the shape of a triangle directly similar to $\Delta$ if and only if $\mu s \in \left\{\frac{1}{s}, \zeta\frac{1}{s}, \zeta^2\frac{1}{s}\right\}$, and this is possible in two cases:

(1) $s = \lambda\zeta^k$, $\lambda \in \left]-1, 1\right[$, $k = 0, 1, 2$, and $\mu = \frac{1}{|s|^2}(> 1)$: $\Delta$ is then isosceles and $K_{\text{iso}}(\theta)$ is the same as in the inversely similar case.
(2) $s = \lambda i\zeta^k$, $\lambda \in \left]-1, 1\right[$, $k = 0, 1, 2$, and $\mu = \frac{-1}{|s|^2}(< -1)$: $\Delta$ is then automedian.

Writing $|s| = R$ and dropping now the condition that $\Delta$ is positively oriented, one sees that the proper nonequilateral $\Delta$ and $\Delta * K_{\text{iso}}(\theta)$, $\theta \neq 0$, are inversely similar (and inversely oriented) exactly when $\theta = \Theta_1 = \arctan\frac{R^2-1}{\sqrt{3}(R^2+1)}$, and directly similar (and inversely oriented) exactly when $\Delta$ is automedian and $\theta = \Theta_2 = \arctan\frac{R^2+1}{\sqrt{3}(R^2-1)}$. Note that $0 < |\Theta_1| < \frac{\pi}{6} < |\Theta_2| < \frac{\pi}{2}$ and that these are inward ears.

By (3), $\Delta * K_{\text{iso}}(\Theta_1) = \Delta * K_{\text{iso}}(\mp\omega)$ is the first Brocard triangle (the base angles of the ears are the Brocard angle of $\Delta$): it is thus immediate that the first Brocard triangle has the centroid of $\Delta$. Figure 11 shows two equibrocardal isosceles triangles with the same base (the apex angle of the right triangle is 30°), their first Brocard triangles and their Brocard points. Equibrocardal isosceles triangles $\Delta_{1,2}$ with the same orientation have base angles $\theta_1$, $\theta_2$ given by (4), and

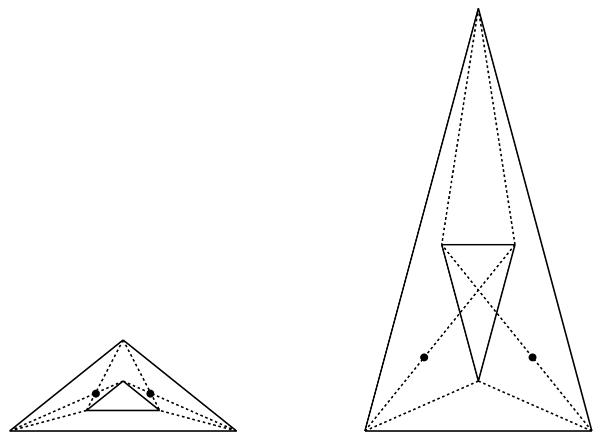

Figure 11

$\tan\Theta_1 = \frac{-2\tan\theta_{1,2}}{3+\tan^2\theta_{1,2}}$; if one neglects the order of the vertices, the first Brocard triangle is obtained from $\Delta_{1,2}$ by a homothety of ratio $-\frac{1}{2} + \frac{3}{3+\tan^2\theta_{1,2}} \in \left]-\frac{1}{2}, \frac{1}{2}\right[$ with respect to the centroid of $\Delta_{1,2}$ (the homothety ratio can be computed directly by considering the normalized $\Delta = \left(0, 1, \frac{1+i\tan\theta_{1,2}}{2}\right)$); the sign of the homothety ratio changes at $\theta = \frac{\pi}{3}$, and the homothety ratios corresponding to $\theta_1$ and to $\theta_2$ differ only by their sign (Figure 11).

Suppose now that $\Delta$ is proper and automedian with Brocard angle $\omega$. One has $3\tan|\Theta_2| = \cot\omega$ by (3). On the other side, $\cot\omega = 3\cot\gamma$ when $\gamma$ is the middle angle of any nontrivial automedian triangle [5, p. 17]: thus $|\Theta_2| = \frac{\pi}{2} - \gamma$, the ears' apex angle is $2\gamma$, the apex of the ear over the middle side is the circumcenter of $\Delta$ by the inscribed angle theorem, and the sides of $\Delta * K_{\mathrm{iso}}(\Theta_2)$ are perpendicular to the sides of $\Delta$ with middle side opposite to the circumcenter. $\Delta * K_{\mathrm{iso}}(\Theta_2)$ is thus obtained from $\Delta$ by a quarter-turn about the centroid of $\Delta$ followed by a homothety of ratio $\frac{|\cot\varphi|}{2}$ about this centroid, where $\varphi$ (positive or negative as the orientation of $\Delta$) is the angle between the middle side of $\Delta$ and its median (the homothety ratio can be computed directly by considering the normalized automedian triangle $\left(0, 1, \frac{1}{2} + \frac{\sqrt{3}}{2}e^{i\varphi}\right)$). Notice that one has also $\tan\Theta_2 = \frac{-1}{\sqrt{3}\sin\varphi}$, $\tan\Theta_1 = \tan(\mp\omega) = \frac{-1}{\sqrt{3}}\sin\varphi$, and that the similarity ratio of the first Brocard triangle to $\Delta$ is $\frac{1}{4}\left|1 + e^{i2\varphi}\right| \in \left[0, \frac{1}{2}\right[$ (0 when $\Delta$ is equilateral).

The following theorem is proven.

**Theorem 3.** (1) *A proper nonequilateral triangle $\Delta$ is similar to exactly three or two of its classical Kiepert triangles according as it is automedian or not: it is directly similar to its medial triangle, inversely similar to its first Brocard triangle, and, if automedian, directly similar to the triangle constructed from inward isosceles ears with apex angle twice the middle angle of $\Delta$ (the apex of the ear over the middle side is then the circumcenter of $\Delta$ and the triangles are perpendicular to*

*each other). If $\Delta$ is automedian, the two cases $\lim_{\theta\to\frac{\pi}{2}-}\Delta * K_{\text{iso}}(\pm\theta)$ are asymptotically inversely similar to $\Delta$.*
(2) *A proper nonequilateral triangle $\Delta$ is isosceles or automedian if and only if it is directly similar to one of its classical Kiepert triangles other than the medial triangle: the corresponding sides are then parallel in the isosceles and perpendicular in the automedian case.*

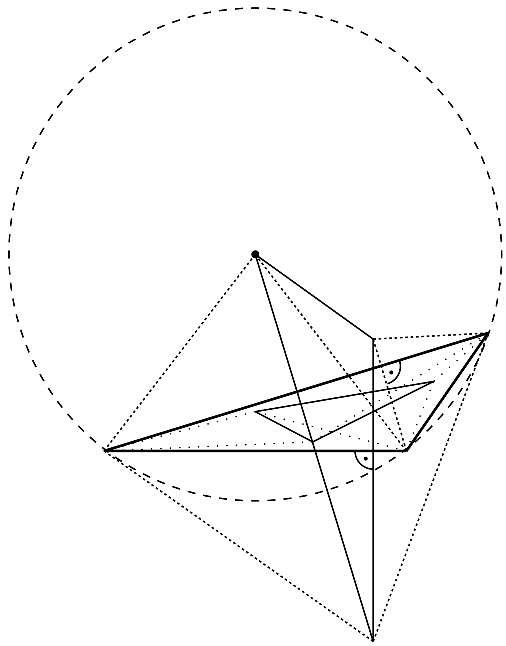

Figure 12. The (up to similarity) only nontrivial triangle with a congruent classical Kiepert triangle, together with its first Brocard triangle

The automedian triangle with sides $1$, $\sqrt{1-\sqrt{\frac{3}{5}}}$ and $\sqrt{1+\sqrt{\frac{3}{5}}}$ (Figure 12) is up to similarity the only nontrivial triangle with a congruent classical Kiepert triangle: the cotangent of the angle formed by the side $1$ and its median is $2$. The base angles of the inward ears are $\arctan\sqrt{\frac{5}{3}}$ for the directly congruent classical Kiepert triangle and $\omega=\arctan\frac{1}{\sqrt{15}}$ for the first Brocard triangle, whose similarity ratio to the initial triangle is $\frac{1}{\sqrt{5}}$.

## 6. Sequences of outward and inward Kiepert triangles

If the triangle $\Delta=(w_0,w_1,w_2)$ is positively oriented or degenerate and if $\operatorname{Im} z\geq 0$, the *outward* and *inward* Kiepert triangles of $\Delta$ corresponding to outward and inward ears directly similar to the triangle $(0,1,z)$ are defined by $\Delta^{\text{out}}(z)=\Delta * K(z)$ and $\Delta^{\text{in}}(z)=\Delta * K(1-z)$, respectively. If $\Delta$ is negatively oriented and proper, $\Delta^{\text{out}}(z)=\Delta * K(1-z)$ and $\Delta^{\text{in}}(z)=\Delta * K(z)$. The outward ears added to

$\Delta$ do not intersect $\Delta$'s interior. We set further $\Delta^{\text{out}}(\nabla) = \Delta^{\text{in}}(\nabla) = \Delta*(0,1,-1)$. Given a sequence $(z_n)_{n\in\mathbf{N}}$ with values in $\{z \in \mathbf{C} \mid \operatorname{Im} z \geq 0\} \cup \{\nabla\}$, and starting from $\Delta = \Delta_0 = \Delta_0^{\text{out}} = \Delta_0^{\text{in}}$, we define inductively $\Delta_{n+1}^{\text{out}} = \Delta_n^{\text{out}}(z_n)$ and $\Delta_{n+1}^{\text{in}} = \Delta_n^{\text{in}}(z_n)$. The centroid remains the centroid of $\Delta_0$ until the first convolution with $(0,1,-1)$, if any, moves it to the origin.

**Theorem 4.** *Let $(z_n)_{n\geq 0}$ be a sequence of symbols $\nabla$ and of complex numbers with nonnegative imaginary part; set $\xi = \frac{1}{\sqrt{3}}e^{i\pi/6}$ and $\frac{\nabla-\xi}{\nabla-\overline{\xi}} = -\sigma(0,1,-1) = 1$. If $z_n \in \mathbf{C}$, denote by $\tilde{z}_n = \frac{1+i\tan\theta_n}{2}$, with*

$$\theta_n = \arctan\frac{|z_n-\overline{\xi}| - |z_n-\xi|}{\sqrt{3}\big(|z_n-\overline{\xi}| + |z_n-\xi|\big)} \in [0, \tfrac{\pi}{6}],$$

*the south pole of the Apollonius circle $\left|\frac{z-\xi}{z-\overline{\xi}}\right| = R$ containing $z_n$, and set $\theta_n = 0$ if $z_n = \nabla$. Let $\Delta_0$ be a nontrivial and nonequilateral triangle.*

*The following properties are equivalent.*

(1) *The sequence $(\Delta_n^{\text{out}})_{n\geq 0}$ constructed from $\Delta_0$ and $(z_n)$ converges in shape to an equilateral limit.*
(2) $\lim_{n\to\infty}\prod_{k=0}^{n}\frac{z_k-\xi}{z_k-\overline{\xi}} = 0$.
(3) *The sequence of classical Kiepert triangles $(\widetilde{\Delta}_n^{\text{out}})_{n\geq 0}$ constructed from $\Delta_0$ and $(\tilde{z}_n)$ converges in shape to an equilateral limit.*
(4) $\lim_{n\to\infty}\prod_{k=0}^{n}\frac{1-\sqrt{3}\tan\theta_k}{1+\sqrt{3}\tan\theta_k} = 0$.
(5) $\theta_n = \frac{\pi}{6}$ *for some $n$ or* $\sum_{n=0}^{\infty}\theta_n = \infty$.

*The existence of the equilateral limit does not depend on the choice of the nonequilateral $\Delta_0$.*

One can also allow to choose each $\tilde{z}_n$ freely as the north or south pole of the Apollonius circle (in order to always leave $\tilde{z}_n = z_n$ when $\operatorname{Re} z_n = \frac{1}{2}$ for example). One has then to take $\theta_n = \arctan\frac{|z_n-\overline{\xi}|+|z_n-\xi|}{\sqrt{3}(|z_n-\overline{\xi}|-|z_n-\xi|)}$ when $\theta_n \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right[$, and the condition $\sum_{n=0}^{\infty}\theta_n = \infty$ has to be replaced by $\sum_{n=0}^{\infty}\min\big(\theta_n, \frac{\pi}{2}-\theta_n\big) = \infty$, as we showed in [23].

Theorem 4 generalizes [34], where the iterated convolution with a constant triangle $K_{\text{iso}}(\theta)$ is analyzed, and [23], where only classical iterated Kiepert triangles are considered.

*Proof.* If $\Delta_0$ is positively oriented or degenerate, so are all $\Delta_n^{\text{out}}$, and $\sigma_{\Delta_n^{\text{out}}} = \sigma_{\Delta_0}\prod_{k=0}^{n-1}\frac{\xi-z_k}{z_k-\overline{\xi}}$. $\lim_{n\to\infty}\sigma_{\Delta_n^{\text{out}}} = 0$ means $\lim_{n\to\infty}\prod_{k=0}^{n}\left|\frac{z_k-\xi}{z_k-\overline{\xi}}\right| = 0$ since $\Delta_0$ is not equilateral, and each factor in this product is constant on the corresponding Apollonius circle. We proved the equivalence of (4) and (5) in [23].

If $\Delta_0$ is negatively oriented and proper, so are all $\Delta_n^{\text{out}}$. Since $\sigma_{K(1-z)} = 1/\sigma_{K(z)}$, one has then $\sigma_{\Delta_n^{\text{out}}} = \sigma_{\Delta_0}\prod_{k=0}^{n-1}\frac{z_k-\overline{\xi}}{\xi-z_k}$: every factor of this product has a modulus $\geq 1$, and one has $\lim_{n\to\infty}\sigma_{\Delta_n^{\text{out}}} = \infty$ under the same conditions as in the first case. $\square$

A nonequilateral limit shape for $(\Delta_n^{\text{out}})$ is only possible when $\frac{\xi - z_n}{z_n - \bar{\xi}}$ converges to 1, *i.e.*, when $\lim_{n\to\infty} z_n = \frac{1}{2}$. But this condition is not sufficient: if $z_n = \frac{1}{2} + \frac{1}{n}$, $n \geq 1$, for example, $\arg\bigl(\sigma_{\Delta_n^{\text{out}}}\bigr)$ diverges like the harmonic series.

Suppose that the sequence $(\Delta_n^{\text{out}})$ has no equilateral limit shape: the infinite product $\prod_{n=0}^{\infty} \left|\frac{\xi - z_n}{z_n - \bar{\xi}}\right|$, whose factors lie in $]0,1]$, has then a limit $L \in\, ]0,1]$ (and any such $L$ can be obtained by an appropriate choice of the $z_n$'s). The accumulation points of $\bigl(\sigma_{\Delta_n^{\text{out}}}\bigr)$ lie on the circle $|s| = L|\sigma_{\Delta_0}|$ $(\leq |\sigma_{\Delta_0}| \leq 1)$ if $\Delta_0$ is positively oriented or degenerate, and on the circle $|s| = \frac{1}{L}|\sigma_{\Delta_0}|$ $(\geq |\sigma_{\Delta_0}| > 1)$ if $\Delta_0$ is negatively oriented and proper: these accumulation shapes correspond to equibrocardal normalized triangles $(0,1,z)$ with vertex $z$ on the Neuberg circle $\left|\frac{z - e^{i\pi/3}}{z - e^{-i\pi/3}}\right| = L^{\pm 1}|\sigma_{\Delta_0}|$, respectively (Figure 4). The sequence $\bigl(\sigma_{\Delta_n^{\text{out}}}\bigr)$ can tend to the accumulation circle with any behavior since the argument of each factor $\frac{\xi - z_n}{z_n - \bar{\xi}}$ can be changed arbitrarily by replacing $z_n$ by an appropriate number of the Apollonius circle $\left|\frac{z-\xi}{z-\bar{\xi}}\right| = R$ containing $z_n$.

Suppose that a sequence of classical iterated Kiepert triangles is given by the successive convolutions with $K_{\text{iso}}(\theta_n)$, $0 \leq \theta_n < \frac{\pi}{2}$, $n \geq 0$, and that this sequence $(\Delta_n)$ starts from a positively oriented nonequilateral $\Delta_0$ and has no equilateral limit, *i.e.*, $\sum_{n=0}^{\infty} \min\bigl(\theta_n, \frac{\pi}{2} - \theta_n\bigr) < \infty$: the one or two accumulation points of the sequence $(\theta_n)$ belong to $\bigl\{0, \frac{\pi}{2}\bigr\}$, and the corresponding subsequences converge rapidly to the accumulation points, since the sum of the corresponding $\theta_n$ or $\frac{\pi}{2} - \theta_n$ is finite; the convolution with $K_{\text{iso}}(\theta_n)$ multiplies the shape by $\lambda_n = \sigma_{K_{\text{iso}}(\theta_n)} = \frac{1-\sqrt{3}\tan\theta_n}{1+\sqrt{3}\tan\theta_n} \in\, ]-1,1]$, and $\lambda_n$ is about 1 or $-1$ when $\theta_n$ is near to 0 or to $\frac{\pi}{2}$, respectively. As above, $\prod_{n=0}^{\infty}|\lambda_n| = L \in\, ]0,1]$, but the $\lambda_n$ are now real: $(\sigma_{\Delta_n})$ has thus the nonzero limit $\sigma_{\Delta_0}\prod_{n=0}^{\infty}\frac{1-\sqrt{3}\tan\theta_n}{1+\sqrt{3}\tan\theta_n}$ $(= \pm L\sigma_{\Delta_0})$ if all $\lambda_n$ are eventually positive, *i.e.*, if $\lim_{n\to\infty}\theta_n = 0$. Otherwise, the infinite subsequences of the $\sigma_{\Delta_n}$ with positive and negative $\prod_{k=0}^{n-1}\lambda_k$ have nonzero limits $L\sigma_{\Delta_0}$ and $-L\sigma_{\Delta_0}$, respectively, and the sequence $(\sigma_{\Delta_n})$ has exactly two accumulation points given by $\pm L\sigma_{\Delta_0}$ with $0 < L < 1$. The limit or accumulation shapes and the $\sigma_{\Delta_n}$ are shapes of classical Kiepert triangles of $\Delta_0$ of the form $\Delta_0 * K_{\text{iso}}(\theta)$, $0 \leq \theta < \frac{\pi}{2}$. If existing, the two accumulation shapes $\pm L\sigma_{\Delta_0}$ are also the shapes of two normalized equibrocardal triangles $\Delta'(z_\pm)$ that are directly similar to the median triangle of each other. If the normalized $\Delta'(z_0)$ has the shape of $\Delta_0$, *i.e.*, if $z_0 = \frac{\zeta\sigma_{\Delta_0} - 1}{\zeta - \sigma_{\Delta_0}}$, the vertices $z_\pm$ are given by the intersections of the Neuberg circle $\left|\frac{z - e^{i\pi/3}}{z - e^{-i\pi/3}}\right| = L|\sigma_{\Delta_0}|$ with (the upper half of) the circle $\mathcal{C}'$ through $e^{i\pi/3}$, $e^{-i\pi/3}$, and $z_0$, on both sides of $e^{i\pi/3}$ (Figure 4): $z_+$ lies on the arc between $e^{i\pi/3}$ and $z_0$, $z_-$ between $e^{i\pi/3}$ and $Z_0 = \frac{z_0 - 2}{2z_0 - 1}$, which corresponds to the shape $-\sigma_{\Delta_0}$. $z_0$ and $Z_0$ have a strictly positive imaginary part if $\Delta_0$ is proper and are real if $\Delta_0$ is degenerate.

Since $\Delta$ and an outward Kiepert triangle of $\Delta$ are always simultaneously in the category "positively oriented or degenerate" or in the category "negatively oriented and proper", an iterated *outward* Kiepert triangle remains unchanged if one modifies the order of the successive convolutions. This is not the case for an iterated inward triangle, because the orientation may change after a convolution, leading to the next convolution with $K(1-z)$ instead of $K(z)$ for example, and these orientation changes may depend on the order of the convolutions. Note that the inward Kiepert triangle of an outward Kiepert triangle given by the same ears has the shape of the initial triangle or is trivial, since the shapes of $K(z) * K(1-z)$, $K(1-z) * K(z)$, and $(0,1,-1) * (0,1,-1)$ are all 1 for $z \neq \xi, \overline{\xi}$. A nontrivial outward Kiepert triangle of an inward Kiepert triangle of $\Delta_0$ given by the same ears is in general not even similar to $\Delta_0$: if $\Delta_0$ is proper, nonequilateral, and positively oriented with shape $s_0$, for example, and if $\sigma_{K(z)} = s$ with $0 < |s| < |s_0|$, $\Delta_0 * K(1-z)$ is negatively oriented and the end triangle $\Delta_0 * K(1-z) * K(1-z)$ is also negatively oriented with shape $\frac{s_0}{s^2}$ of modulus $> \frac{1}{|s_0|}$; this end triangle is never similar to $\Delta_0$.

Except when the sequence has been stopped before by $0 \cdot \infty$ or $\infty \cdot 0$, the shape of $\Delta^{\text{in}}_{n+1}$, $n \geq 0$, is given recursively by $\sigma_{\Delta^{\text{in}}_{n+1}} = \frac{z_n - \overline{\xi}}{\xi - z_n} \cdot \sigma_{\Delta^{\text{in}}_n}$ if $\left|\sigma_{\Delta^{\text{in}}_n}\right| \leq 1$ (then $\left|\sigma_{\Delta^{\text{in}}_{n+1}}\right| \geq \left|\sigma_{\Delta^{\text{in}}_n}\right|$) and by $\sigma_{\Delta^{\text{in}}_{n+1}} = \frac{\xi - z_n}{z_n - \overline{\xi}} \cdot \sigma_{\Delta^{\text{in}}_n}$ if $\left|\sigma_{\Delta^{\text{in}}_n}\right| > 1$ (then $\left|\sigma_{\Delta^{\text{in}}_{n+1}}\right| \leq \left|\sigma_{\Delta^{\text{in}}_n}\right|$). The turning points between stretching factors $\left(\frac{\xi - z_n}{z_n - \overline{\xi}}\right)^{\pm 1}$ of modulus $\leq 1$ or $\geq 1$, respectively, depend also on $\Delta_0$. By choosing $\Delta_0$ and the $z_n$ appropriately, the sequence $(\Delta^{\text{in}}_n)$ can thus have any behavior in shape within these constraints. The iterated first Brocard triangles of a proper nonequilateral $\Delta_0$ are for example alternately inversely and directly similar to $\Delta_0$ when $\Delta_0$ is not isosceles, and all directly similar to $\Delta_0$ when $\Delta_0$ is isosceles. If $\Delta_0$ is automedian and if the ears are isosceles with constant apex angle twice the middle angle of $\Delta_0$, the iterated inward Kiepert triangles are all directly similar to $\Delta_0$: the automedian triangle of Figure 12 initiates in particular a 4-periodic sequence given by quarter-turns about the centroid.

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