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Convolution operators in $A^{-\infty}$ for convex domains

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Abstract. We consider the convolution operators in spaces of functions which are holomorphic in a bounded convex domain in $\mathbb{C}^n$ and have a polynomial growth near its boundary. A characterization of the surjectivity of such operators on the class of all domains is given in terms of low bounds of the Laplace transformation of analytic functionals defining the operators.

1. Introduction

By $\mathcal{O}(\Omega)$ denote the space of functions holomorphic in a domain $\Omega\subset\mathbb{C}^n$. If $z,\zeta\in\mathbb{C}^n$, then $|z|=(z_1\bar{z}_1+\ldots+z_n\bar{z}_n)^{1/2}$ and $\langle z,\zeta\rangle=z_1\zeta_1+\ldots+z_n\zeta_n$. By $B(z;R)$ denote the open ball in $\mathbb{C}^n$ centered at $z\in\mathbb{C}^n$ of radius R. The supporting function of a convex set M in $\mathbb{C}^n$ is $H_M(\xi):=\sup_{z\in M}\operatorname{Re}\langle z,\xi\rangle$. Also put $R_M:=\sup_{z\in M}|z|$.

Let Ω be a convex bounded domain in $\mathbb{C}^n$ and $d_\Omega(z):=\inf_{\zeta\in\partial\Omega}|z-\zeta|$, $z\in\Omega$. The space $A^{-\infty}(\Omega)$ of holomorphic functions in Ω with polynomial growth near the boundary $\partial\Omega$ is defined as

$$A^{-\infty}(\Omega):=\Big\{f\in\mathcal{O}(\Omega):\|f\|_{p,\Omega}:=\sup_{z\in\Omega}|f(z)|d_\Omega(z)^p<\infty \text{ for some } p>0\Big\}$$

and equipped with its natural inductive limit topology.

The main goal of this note is to establish surjectivity criteria for convolution operator $\mu*\colon A^{-\infty}(\Omega+K)\to A^{-\infty}(\Omega)$, where K is a convex compact set in $\mathbb{C}^n$. It should be noted that the surjectivity of convolution operators for the spaces $\mathcal{O}(\Omega)$ of holomorphic functions in convex domains of $\mathbb{C}^n$ have been understood quite well (see, e.g., [17], [19] and [23] and references therein), whereas it is known less for the spaces of holomorphic functions with prescribed growth near the boundary of Ω (see [21]). Moreover, for the spaces of type $A^{-\infty}(\Omega)$, as far as we know, this problem is not yet treated, although the spaces of such a type have been studied in various directions by many authors (we refer the reader to [7], [8] and [24], as well as [4]–[6] and [9]).

The structure of this paper is as follows. Section 2 is concerned with the acting of convolution operators in spaces of type $A^{-\infty}(\Omega)$ and their conjugates which are actually multiplication operators. The main result in Section 2 is a surjectivity functional criterion for convolution operators (see Theorem 2.9). In Section 3 we study a surjectivity problem for the class of all convex domains. We introduce a condition (S^a) for the Laplace transformation of analytic functionals in $\mathbb{C}^n$ and in terms of this condition prove a criterion for surjectivity for convolution operators (see Theorem 3.10). In the last Section 4 we give some examples, discuss the problem of existence of functions satisfying the condition (S^a), and get an explicit representation of solutions for convolution equations in a form of Dirichlet series.

We note that some of our results were announced in [3].

2. Convolution operators

2.1. Analytic functionals carried by a compact convex set

Let μ be an analytic functional on $\mathbb{C}^n$, carried by a compact convex set K, and Ω be a bounded convex domain in $\mathbb{C}^n$. Consider the convolution operator

$$\mu * f(z) := \langle \mu_w, f(z+w)\rangle,$$

which maps $\mathcal{O}(\Omega+K)$ into $\mathcal{O}(\Omega)$ continuously (see, e.g., [18, Chapter 9] and [23]). Notice that the Laplace (or Fourier–Borel) transformation

$$\hat{\mu}(\zeta) := \langle \mu_z, e^{\langle z,\zeta\rangle}\rangle, \quad \zeta\in\mathbb{C}^n,$$

of the functional μ is an entire function in $\mathbb{C}^n$ of exponential type that belongs to the space

$$P_K := \left\{ f\in\mathcal{O}(\mathbb{C}^n) : \sup_{\zeta\in\mathbb{C}^n} \frac{|f(\zeta)|}{e^{H_K(\zeta)+\varepsilon|\zeta|}} < \infty \text{ for all } \varepsilon>0 \right\}.$$

Conversely, each $f\in P_K$ defines an analytic functional μ, carried by K, with $\hat{\mu}=f$.

Our nearest aim is to find out conditions on μ (or on $\hat{\mu}$) under which μ acts from $A^{-\infty}(\Omega+K)$ into $A^{-\infty}(\Omega)$. Recall that $A^{-\infty}(\Omega)$ is a dual Fréchet–Schwartz space for any Ω. Furthermore, as was announced in [4, Theorem 2.1], if either $n=1$, or $n>1$ and Ω has C^2 boundary, the strong dual $(A^{-\infty}(\Omega))'_b$ of $A^{-\infty}(\Omega)$ can be identified, via the Laplace transformation of functionals, with the Fréchet–Schwartz space of entire functions

$$A_\Omega^{-\infty} = \left\{ f\in\mathcal{O}(\mathbb{C}^n) : |f|_{p,\Omega} = \sup_{\zeta\in\mathbb{C}^n} \frac{|f(\zeta)|(1+|\zeta|)^p}{e^{H_\Omega(\zeta)}} < \infty \text{ for all } p\in\mathbb{N} \right\}.$$

Denote by $\mathcal{D}^{-\infty,n}$ the family of all bounded convex domains Ω in $\mathbb{C}^n$ for which the same isomorphism $(A^{-\infty}(\Omega))'_b \simeq A_\Omega^{-\infty}$ is valid. In what follows we will identify

$(A^{-\infty}(\Omega))'_b$ with $A_\Omega^{-\infty}$ for $\Omega\in\mathcal{D}^{-\infty,n}$. We strongly believe that $\mathcal{D}^{-\infty,n}$ coincides with the family of all bounded convex domains in $\mathbb{C}^n$, $n>1$, and, if it is so, we can omit our further condition that Ω and $\Omega+K$ belong to $\mathcal{D}^{-\infty,n}$.

Put

$$A_K^\infty := \left\{\varphi\in\mathcal{O}(\mathbb{C}^n) : \sup_{\zeta\in\mathbb{C}^n}\frac{|\varphi(\zeta)|}{(1+|\zeta|)^p e^{H_K(\zeta)}}<\infty \text{ for some } p\in\mathbb{N}\right\}.$$

Proposition 2.1. *Let Ω and $\Omega+K$ be in $\mathcal{D}^{-\infty,n}$. Then $\mu*A^{-\infty}(\Omega+K)\subseteq A^{-\infty}(\Omega)$ if and only if $\hat{\mu}\in A_K^\infty$. In addition, for each nontrivial μ with $\hat{\mu}\in A_K^\infty$ the following statements hold*:

(i) *The convolution operator $\mu*\colon A^{-\infty}(\Omega+K)\to A^{-\infty}(\Omega)$ is continuous and has a dense range*;

(ii) *The conjugate operator to $\mu*$ is the multiplication operator*

$$\Lambda_{\hat{\mu}} : A_\Omega^{-\infty} \longrightarrow A_{\Omega+K}^{-\infty},$$
$$f \longmapsto \hat{\mu}f.$$

Proof. Let $\mu*A^{-\infty}(\Omega+K)\subseteq A^{-\infty}(\Omega)$ and suppose that a net $(f_\beta,\mu*f_\beta)_{\beta\in B}$ converges in $A^{-\infty}(\Omega+K)\times A^{-\infty}(\Omega)$ to (f,g). Obviously, $A^{-\infty}(\Omega+K)\hookrightarrow\mathcal{O}(\Omega+K)$ and $A^{-\infty}(\Omega)\hookrightarrow\mathcal{O}(\Omega)$. Here $\hookrightarrow$ is the symbol of continuous embedding. From this and continuity of $\mu*\colon\mathcal{O}(\Omega+K)\to\mathcal{O}(\Omega)$ it follows that $(f_\beta,\mu*f_\beta)_{\beta\in B}$ converges in $\mathcal{O}(\Omega+K)\times\mathcal{O}(\Omega)$ to $(f,\mu*f)$. Thus, $g=\mu*f$ and consequently the operator $\mu*\colon A^{-\infty}(\Omega+K)\to A^{-\infty}(\Omega)$ has a closed graph. By the Grothendieck closed graph theorem [13] this operator is continuous. By virtue of the fact that $A^{-\infty}(\Omega+K)$ and $A^{-\infty}(\Omega)$ are dual Fréchet–Schwartz spaces, we then get that there exist $m\in\mathbb{N}$ and $C>0$ such that

$$\|\mu*f\|_{m,\Omega}\le C\|f\|_{1,\Omega+K} \quad \text{for all } f\in A^{-\infty}(\Omega+K).$$

In particular, for $f_\zeta(\,\cdot\,):=e^{\langle\zeta,\,\cdot\,\rangle}$, $\zeta\in\mathbb{C}$, using the relation $\mu*e^{\langle\zeta,z+\,\cdot\,\rangle}=\hat{\mu}(\zeta)e^{\langle\zeta,z\rangle}$ for all $z\in\Omega$ and $\zeta\in\mathbb{C}^n$ and applying the fact [6, Lemma 2.2] that there exist constants $A_1>0$ and $a_m>0$ such that for all $\zeta\in\mathbb{C}^n$,

$$\left\|e^{\langle\zeta,\,\cdot\,\rangle}\right\|_{1,\Omega+K}\le A_1\frac{e^{H_\Omega(\zeta)+H_K(\zeta)}}{1+|\zeta|}$$

and

$$\left\|e^{\langle\zeta,\,\cdot\,\rangle}\right\|_{m,\Omega}\ge a_m\frac{e^{H_\Omega(\zeta)}}{(1+|\zeta|)^m},$$

we then have

$$|\hat{\mu}(\zeta)|\le\frac{CA_1}{a_m}(1+|\zeta|)^{m-1}e^{H_K(\zeta)} \quad \text{for all } \zeta\in\mathbb{C}^n.$$

Hence, $\hat{\mu}\in A_K^{\infty}$.

Conversely, let $\hat{\mu}\in A_K^{\infty}$. It is easy to see that $\Lambda_{\hat{\mu}}$ maps $A_{\Omega}^{-\infty}$ into $A_{\Omega+K}^{-\infty}$ continuously. Then the conjugate operator $\Lambda_{\hat{\mu}}^{\tau}:A^{-\infty}(\Omega+K)\to A^{-\infty}(\Omega)$ is continuous and $\Lambda_{\hat{\mu}}^{\tau}(e^{\langle\zeta,z+w\rangle})=\hat{\mu}(\zeta)e^{\langle\zeta,z\rangle}$ for all $z\in\Omega$ and $\zeta\in\mathbb{C}^n$.

Since $\hat{\mu}\in A_K^{\infty}\subset P_K$, the convolution operator $\mu*$ maps $\mathcal{O}(\Omega+K)$ into $\mathcal{O}(\Omega)$ continuously and

$$\mu*e^{\langle\zeta,z+\,\cdot\,\rangle}=\hat{\mu}(\zeta)e^{\langle\zeta,z\rangle}\quad\text{for all } z\in\Omega \text{ and } \zeta\in\mathbb{C}^n. \tag{1}$$

Thus,

$$\mu*e^{\langle\zeta,\,\cdot\,\rangle}=\Lambda_{\hat{\mu}}^{\tau}(e^{\langle\zeta,\,\cdot\,\rangle})\quad\text{for all } \zeta\in\mathbb{C}^n. \tag{2}$$

Take any $f\in A^{-\infty}(\Omega+K)$. From [4, Theorem 2.1] it follows that the system of exponential functions $E:=\{e^{\langle\zeta,\,\cdot\,\rangle}:\zeta\in\mathbb{C}^n\}$ is complete in $A^{-\infty}(\Omega+K)$. Then there exists a sequence $\{e_k\}_{k=1}^{\infty}$ from Span(E) which converges to f in $A^{-\infty}(\Omega+K)$. Moreover, $\{e_k\}_{k=1}^{\infty}$ converges to f in $\mathcal{O}(\Omega+K)$. Using this, (2) and the continuity of the operators $\mu*$: $\mathcal{O}(\Omega+K)\to\mathcal{O}(\Omega)$ and $\Lambda_{\hat{\mu}}^{\tau}:A^{-\infty}(\Omega+K)\to A^{-\infty}(\Omega)$, we find that $\mu*f=\Lambda_{\hat{\mu}}^{\tau}(f)$ for all $f\in A^{-\infty}(\Omega+K)$. Consequently, $\mu*$ maps $A^{-\infty}(\Omega+K)$ into $A^{-\infty}(\Omega)$ and is continuous.

Since the statement (ii) follows immediately from the equality (1) and the notion of conjugate operator, it remains to check that if μ is nontrivial, then $\mu*A^{-\infty}(\Omega+K)$ is dense in $A^{-\infty}(\Omega)$. Taking into account (1) again, to do this it is enough to get that the system $\{e^{\langle\zeta,\,\cdot\,\rangle}:\hat{\mu}(\zeta)\neq 0\}$ is complete in $A^{-\infty}(\Omega)$. By [4, Theorem 2.1], the system $\{e^{\langle\zeta,\,\cdot\,\rangle}:\zeta\in\mathbb{C}^n\}$ is complete in $A^{-\infty}(\Omega)$. Next, by the uniqueness theorem for holomorphic functions, for each ζ with $\hat{\mu}(\zeta)=0$ there exists a sequence $\{\zeta^{(k)}\}_{k=1}^{\infty}$ with $\hat{\mu}(\zeta^{(k)})\neq 0$ such that $\zeta^{(k)}\to\zeta$ as $k\to\infty$. Thus, the only the thing we have to prove is that $e^{\langle\zeta,\,\cdot\,\rangle}$ converges in $A^{-\infty}(\Omega)$ to $e^{\langle\zeta^0,\,\cdot\,\rangle}$ as $\zeta\to\zeta^0$, where $\zeta^0\in\mathbb{C}^n$ is arbitrary.

Recall that $R_\Omega=\sup_{z\in\Omega}|z|$. For all $z\in\Omega$ and $|\zeta-\zeta^0|\le 1/R_\Omega$,

$$\begin{aligned}|e^{\langle\zeta,z\rangle}-e^{\langle\zeta^0,z\rangle}|&=|e^{\langle\zeta^0,z\rangle}|\,|e^{\langle\zeta-\zeta^0,z\rangle}-1|\\&\le e|e^{\langle\zeta^0,z\rangle}|\,|\langle\zeta-\zeta^0,z\rangle|\le eR_\Omega|e^{\langle\zeta^0,z\rangle}|\,|\zeta-\zeta^0|.\end{aligned}$$

Consequently, for each $n\in\mathbb{N}$,

$$\|e^{\langle\zeta,z\rangle}-e^{\langle\zeta^0,z\rangle}\|_{n,\Omega}\le eR_\Omega\|e^{\langle\zeta^0,z\rangle}\|_n|\zeta-\zeta^0|\to 0\quad\text{as } \zeta\to\zeta^0.$$

This completes the proof. □

By standard arguments from the theory of duality, Proposition 2.1 implies the following functional criterion of surjectivity for convolution operators.

Proposition 2.2. *Let Ω and $\Omega+K$ be in $\mathcal{D}^{-\infty,n}$ and μ be a nontrivial analytic functional with $\hat{\mu}\in A_K^{\infty}$. The convolution operator $\mu*\colon A^{-\infty}(\Omega+K)\to A^{-\infty}(\Omega)$ is surjective if and only if $\Lambda_{\hat{\mu}}(A_{\Omega}^{-\infty})$ is closed in $A_{\Omega+K}^{-\infty}$.*

Proof. Applying [10, Corollary 8.6.4 and Theorem 8.6.8], we have that if the operator $\mu*\colon A^{-\infty}(\Omega+K)\to A^{-\infty}(\Omega)$ is surjective, then the set $(\mu*)'(A^{-\infty}(\Omega)')$ is closed in the strong dual $(A^{-\infty}(\Omega+K))_b'$ of $A^{-\infty}(\Omega+K)$.

Let now $(\mu*)'(A^{-\infty}(\Omega)')$ be closed in the strong dual $(A^{-\infty}(\Omega+K))_b'$. Since $A^{-\infty}(\Omega)$ and $A^{-\infty}(\Omega+K)$ are dual Fréchet–Schwartz spaces, their duals are Fréchet spaces (and moreover, Fréchet–Schwartz spaces). Hence, by [10, Theorem 8.6.13], the set $(\mu*)''(A^{-\infty}(\Omega+K)'')$ is closed in the strong dual $(A^{-\infty}(\Omega))_b''$ of $(A^{-\infty}(\Omega))_b'$. Using that $A^{-\infty}(\Omega)$ and $A^{-\infty}(\Omega+K)$ are reflexive (as dual Fréchet–Schwartz spaces), we get that $\mu*(A^{-\infty}(\Omega+K))$ is closed in $A^{-\infty}(\Omega)$. Applying the statement (i) of Proposition 2.1, we obtain that $\mu*\colon A^{-\infty}(\Omega+K)\to A^{-\infty}(\Omega)$ is surjective.

Thus, $\mu*\colon A^{-\infty}(\Omega+K)\to A^{-\infty}(\Omega)$ is surjective if and only if $(\mu*)'(A^{-\infty}(\Omega)')$ is closed in $(A^{-\infty}(\Omega+K))_b'$. It remains to use [4, Theorem 2.1] and Proposition 2.1(ii), to finish the proof. □

2.2. Multiplicators from $A_{\Omega}^{-\infty}$ into $A_{\Omega+K}^{-\infty}$

In view of Proposition 2.2, recall that an entire function φ in $\mathbb{C}^n$ is a *multiplicator* from $A_{\Omega}^{-\infty}$ into $A_{\Omega+K}^{-\infty}$ if $\varphi g\in A_{\Omega+K}^{-\infty}$, whenever $g\in A_{\Omega}^{-\infty}$. As we will see later, a description of all multiplicators from $A_{\Omega}^{-\infty}$ into $A_{\Omega+K}^{-\infty}$ plays an important role in the study of surjectivity of convolution operators. For doing this we recall some definitions and results from [2] in a particular case sufficient for our purposes.

Denote by V^n the family of all functions $v\colon\mathbb{C}^n\to\mathbb{R}$ which are bounded above on each compact set in $\mathbb{C}^n$. We associate with $v\in V^n$ the Banach space

$$E(v):=\left\{f\in\mathcal{O}(\mathbb{C}^n): |f|_v:=\sup_{\zeta\in\mathbb{C}^n}\frac{|f(\zeta)|}{e^{v(\zeta)}}<\infty\right\}.$$

Let $\Phi=\{\varphi_k\}_{k=1}^{\infty}$ be a sequence of functions from V^n such that there exists some constant A_k for which $\varphi_{k+1}(\zeta)\le\varphi_k(\zeta)+A_k$ for all $\zeta\in\mathbb{C}^n$, $k=1,2,\dots$. Define the Fréchet space

$$P(\Phi):=\bigcap_{k=1}^{\infty}E(\varphi_k).$$

Let $P(\Psi)$ be another space of the same type. We then say that an entire function g is a *multiplicator* from $P(\Phi)$ into $P(\Psi)$ if $gP(\Phi)\subseteq P(\Psi)$.

Denote by $\mathcal{M}(\Phi,\Psi)$ the set of all multiplicators from $P(\Phi)$ into $P(\Psi)$. Each $g\in\mathcal{M}(\Phi,\Psi)$ generates a linear operator $\Lambda_g\colon f\in P(\Phi)\mapsto gf\in P(\Psi)$. It is easy to see that this operator is continuous. Indeed, since the topologies in $P(\Phi)$ and $P(\Psi)$ are finer than the topology of pointwise convergence in $\mathbb{C}^n$, the graph of $\Lambda_g\colon P(\Phi)\to P(\Psi)$ is closed. Then, by the Banach closed graph theorem, this operator is continuous.

It is clear that the set $\bigcap_{m=1}^{\infty}\bigcup_{k=1}^{\infty}E(\psi_m-\varphi_k)$ is always contained in $\mathcal{M}(\Phi,\Psi)$. The following result is an immediate consequence of [2, Propositions 1 and 5].

Proposition 2.3. *Let Φ satisfy the following conditions*:

(a) *Φ consists of plurisubharmonic (psh) functions in $\mathbb{C}^n$*;

(b) *For each $k\in\mathbb{N}$ there exists $m\in\mathbb{N}$ such that $P(\Phi)$ is dense in $E(\varphi_m)$ with respect to the norm $|\cdot|_{\varphi_k}$*;

(c) *For each $k\in\mathbb{N}$ there exist $m\in\mathbb{N}$ and $M>0$ such that*

$$\sup_{|w|\le 1}\varphi_m(\zeta+w)+\log(1+|\zeta|)\le\varphi_k(\zeta)+M\quad\text{for all } \zeta\in\mathbb{C}^n.$$

Then

$$\mathcal{M}(\Phi,\Psi)=\bigcap_{m=1}^{\infty}\bigcup_{k=1}^{\infty}E(\psi_m-\varphi_k).$$

Remark 2.4. In [2, Proposition 1] $\bigcap_{m=1}^{\infty}\bigcup_{k=1}^{\infty}E(\psi_m-\varphi_k)$ was misprinted as $\bigcap_{m=1}^{\infty}\bigcap_{k=1}^{\infty}E(\psi_m-\varphi_k)$.

In the sequel we write $\mathcal{M}_{\Omega,\Omega+K}^{-\infty}$ instead of $\mathcal{M}(A_\Omega^{-\infty},A_{\Omega+K}^{-\infty})$. Applying Proposition 2.3 to the spaces $A_\Omega^{-\infty}$ and $A_{\Omega+K}^{-\infty}$, we have the following result.

Proposition 2.5. *For any bounded convex domain Ω and convex compact set K,*

$$\mathcal{M}_{\Omega,\Omega+K}^{-\infty}=A_K^{\infty}.$$

Proof. Without loss of generality we can assume that $0\in\Omega$. For each $k\in\mathbb{N}$ define

$$H_{\Omega,k}(\zeta):=\sup_{z\in\Omega}(\operatorname{Re}\langle z,\zeta\rangle+k\log d_\Omega(z)),\quad \zeta\in\mathbb{C}^n,$$

where, as above, $d_\Omega(z)$ is the distance between $z\in\Omega$ and $\partial\Omega$. Clearly, $H_{\Omega,k}$ is psh in $\mathbb{C}^n$ and

$$|H_{\Omega,k}(w)-H_{\Omega,k}(\zeta)|\le R_\Omega|w-\zeta|\quad\text{for all } w,\zeta\in\mathbb{C}^n.$$

Thus, $H_{\Omega,k}$ are psh functions in $\mathbb{C}^n$ satisfying Lipschitz conditions. Next, by the proof of [6, Lemma 2.2],

$$c_k\le H_{\Omega,k}(\zeta)-H_\Omega(\zeta)+k\log(1+|\zeta|)\le C_k\quad\text{for all } \zeta\in\mathbb{C}^n \text{ and } k\in\mathbb{N}, \tag{3}$$

where $c_k := k\log\min\{r_\Omega/e, kr_\Omega/eR_\Omega\}$, and $r_\Omega := \inf_{z\in\partial\Omega}|z|$. From this it follows that, for every $k\in\mathbb{N}$,

$$A_\Omega^{-k} := \left\{ f\in\mathcal{O}(\mathbb{C}^n) : |f|_{k,\Omega} = \sup_{\zeta\in\mathbb{C}^n} \frac{|f(\zeta)|(1+|\zeta|)^k}{e^{H_\Omega(\zeta)}} < \infty \right\} = E(H_{\Omega,k})$$

and, consequently, $A_\Omega^{-\infty} = P(\Phi)$, where $\Phi = \{H_{\Omega,k}\}_{k=1}^\infty$. To finish the proof it is sufficient only to check that Φ satisfies conditions (b) and (c) of Proposition 2.3.

(b) Let

$$P_\Omega := \left\{ f\in H(\mathbb{C}^n) : \sup_{\zeta\in\mathbb{C}^n} \frac{|f(\zeta)|}{e^{H_\Omega(\zeta)-\varepsilon|z|}} < \infty \text{ for some } \varepsilon > 0 \right\}.$$

Fix any $k\in\mathbb{N}$ and function $f\in A_\Omega^{-k-1}$. Evidently, $f_\gamma(\cdot) := f(\gamma\cdot)$ belongs to P_Ω for every $\gamma\in(0,1)$. By the proof of [5, Lemma 2.11] $|f_\gamma - f|_{k,\Omega}\to 0$ as $\gamma\to 1-$. Therefore, P_Ω is dense in A_Ω^{-k-1} with respect to the norm $|\cdot|_{H_{\Omega,k}}$. By [5, Lemma 2.10] $P_\Omega\subset A_\Omega^{-\infty}$. Thus, $A_\Omega^{-\infty}$ is dense in A_Ω^{-k-1} with respect to the norm $|\cdot|_{H_{\Omega,k}}$, and (b) holds with $m=k+1$.

(c) From (3) and the Lipschitz condition for $H_{\Omega,k}$ it follows that

$$\begin{aligned}\sup_{|w|\le 1} H_{\Omega,k+1}(\zeta+w) + \log(1+|\zeta|) &\le H_\Omega(\zeta) + R_\Omega - k\log(1+|\zeta|) + k\log 2 + C_{k+1}\\ &\le H_{\Omega,k}(\zeta) + R_\Omega + k\log 2 + C_{k+1} - c_k.\end{aligned}$$

This means that (c) holds with $m=k+1$ and $M = R_\Omega + k\log 2 + C_{k+1} - c_k$. □

Corollary 2.6. *Let Ω be a bounded convex domain in $\mathbb{C}^n$. Then the set $\mathcal{M}_{\Omega,\Omega}^{-\infty}$ of all multiplicators from $A_\Omega^{-\infty}$ into $A_\Omega^{-\infty}$ coincides with the family of all polynomials.*

Proof. This is a direct consequence of Proposition 2.5. □

2.3. Functional criterion for surjectivity

Definition 2.7. A nontrivial function $\varphi\in A_K^\infty$ is a *divisor* from $A_{\Omega+K}^{-\infty}$ into $A_\Omega^{-\infty}$ if the theorem of division is valid for φ, i.e. the following implication is fulfilled:

$$f\in A_{\Omega+K}^{-\infty} \text{ and } \frac{f}{\varphi}\in\mathcal{O}(\mathbb{C}^N) \implies \frac{f}{\varphi}\in A_\Omega^{-\infty}.$$

Denote by $\mathcal{D}_{\Omega+K,\Omega}^{-\infty}$ the set of all divisors from $A_{\Omega+K}^{-\infty}$ into $A_\Omega^{-\infty}$.

Proposition 2.8. *Let* $\varphi\in A_K^\infty$. *Consider the following assertions*:

(i) $\Lambda_\varphi(A_\Omega^{-\infty})$ *is closed in* $A_{\Omega+K}^{-\infty}$;

(ii) *For each* $p\in\mathbb{N}$ *there exist* $m\in\mathbb{N}$ *and* $C>0$ *such that*

$$\sup_{\zeta\in\mathbb{C}^n}\frac{|f(\zeta)|(1+|\zeta|)^p}{e^{H_\Omega(\zeta)}}\le C\sup_{\zeta\in\mathbb{C}^n}\frac{|\varphi(\zeta)|\,|f(\zeta)|(1+|\zeta|)^m}{e^{H_\Omega(\zeta)+H_K(\zeta)}}\quad \textit{for all } f\in A_\Omega^{-\infty};$$

(iii) $\varphi\in\mathcal{D}_{\Omega+K,\Omega}^{-\infty}$.

Then (iii)$\Rightarrow$(ii)$\Leftrightarrow$(i).

Proof. (i)$\Leftrightarrow$(ii) As was noted above, the operator $\Lambda_\varphi\colon A_\Omega^{-\infty}\to A_{\Omega+K}^{-\infty}$ is continuous. Additionally, by the uniqueness theorem for holomorphic functions it is injective. Then (ii) means that Λ_φ is a topological isomorphism from $A_\Omega^{-\infty}$ onto $\Lambda_\mu(A_\Omega^{-\infty})$ endowed with the topology induced from $A_{\Omega+K}^{-\infty}$. Since $A_\Omega^{-\infty}$ and $A_{\Omega+K}^{-\infty}$ are Fréchet spaces, this is equivalent to (i).

(iii)$\Rightarrow$(i) This follows, by standard arguments, from the fact that the original topology in $A_{\Omega+K}^{-\infty}$ is finer than the topology of uniform convergence on compact sets in $\mathbb{C}^n$. □

We have the following functional criterion for surjectivity.

Theorem 2.9. *Let* Ω *and* $\Omega+K$ *be in* $\mathcal{D}^{-\infty,n}$ *and* μ *be an analytic functional with* $\hat\mu\in A_K^\infty$. *Consider the following assertions*:

(i) $\mu*\colon A^{-\infty}(\Omega+K)\to A^{-\infty}(\Omega)$ *is surjective*;

(ii) *For each* $p\in\mathbb{N}$ *there exist* $m\in\mathbb{N}$ *and* $C>0$ *such that*

$$\sup_{\zeta\in\mathbb{C}^n}\frac{|f(\zeta)|(1+|\zeta|)^p}{e^{H_\Omega(\zeta)}}\le C\sup_{\zeta\in\mathbb{C}^n}\frac{|\hat\mu(\zeta)|\,|f(\zeta)|(1+|\zeta|)^m}{e^{H_\Omega(\zeta)+H_K(\zeta)}}\quad \textit{for all } f\in A_\Omega^{-\infty};$$

(iii) $\hat\mu\in\mathcal{D}_{\Omega+K,\Omega}^{-\infty}$.

Then (iii)$\Rightarrow$(ii)$\Leftrightarrow$(i).

Proof. This is a direct consequence of Propositions 2.2 and 2.8. □

Remark 2.10. Note that for various function spaces (see, e.g., Ehrenpreis [11], Epifanov [12], Krivosheev [17], Momm [21], Sigurdsson [23] and Tkachenko [25]) (ii)$\Leftrightarrow$(iii), and that the proof of the implication (ii)$\Rightarrow$(iii) is based on the description of all divisors. In the next section we give a description of all $\varphi\in A_K^\infty$ that belong to $\mathcal{D}_{\Omega+K,\Omega}^{-\infty}$ for any Ω. As a consequence, those φ and only they satisfy the condition (ii) of Proposition 2.8 for any Ω. Thus the conditions (i)–(iii) of

Proposition 2.8 are equivalent when applied to all bounded convex domains Ω simultaneously. Moreover we prove that if one of the conditions (i)–(iii) is valid for every open bounded convex polyhedron Ω in $\mathbb{C}^n$, then all three conditions are valid for any bounded convex domain Ω in $\mathbb{C}^n$. The equivalence of three conditions (i)–(iii) for an individual domain Ω is still open. We strongly believe that the answer depends on the smoothness of the boundary of Ω.

3. Surjectivity on the class of all domains

3.1. Condition (S^a)

Let $\varphi(\zeta)$ be an entire function of exponential type. Its *regularized radial indicator* $h^*_\varphi(\zeta)$ is defined as follows:

$$h^*_\varphi(\zeta) := \limsup_{\zeta'\to\zeta} \limsup_{r\to\infty} \frac{\log|\varphi(r\zeta')|}{r}, \quad \zeta\in\mathbb{C}^n.$$

We recall the condition (S), originally due to T. Kawai [16], that was introduced in [15].

Definition 3.1. An entire function $\varphi\in\mathcal{O}(\mathbb{C}^n)$ of exponential type is said to satisfy the *condition* (S) at direction $\zeta_0\in\mathbb{C}^n\setminus\{0\}$, if for each $\varepsilon>0$ there exists $N>0$ such that for all $r>N$ and $\zeta\in\mathbb{C}^n$ with $|\zeta-\zeta_0|<\varepsilon r$ we have

$$\frac{\log|\varphi(r\zeta)|}{r} \ge h^*_\varphi(\zeta_0)-\varepsilon.$$

Remark 3.2. It was showed in [15] that condition (S) is nothing but the condition of regular growth, the classical notion in the theory of entire functions.

As above, let μ be an analytic functional with $\hat{\mu}\in A^\infty_K$. Then $h^*_{\hat{\mu}}(\zeta)\le H_K(\zeta)$ in $\mathbb{C}^n$. Throughout this section we assume that the assumption $h^*_{\hat{\mu}}(\zeta)=H_K(\zeta)$ is always satisfied. Note that for spaces of holomorphic functions in convex domains, this last condition with the condition (S) is, in a sense, necessary and sufficient for the solvability of the nonhomogeneous convolution equation $\mu*f=g$. We refer the reader to [17] for the more precise statement (see also Theorem 9.35 in [18]).

We now define another condition, similar to the complete regular growth condition (S), but stronger than (S) and more appropriate for spaces with polynomial growth near the boundary.

Definition 3.3. An entire function $\varphi\in\mathcal{O}(\mathbb{C}^n)$ of exponential type is said to satisfy the *condition* (S^a), if there exist $s,N>0$ such that for each $\zeta\in\mathbb{C}^n$ with $|\zeta|>N$ there is $\zeta'\in\mathbb{C}^n$ with $|\zeta'-\zeta|<\log(1+|\zeta|)$ satisfying

$$\log|\varphi(\zeta')|\geq h^*_\varphi(\zeta)-s\log|\zeta|.$$

3.2. Sufficient conditions

The following result shows that the condition (S^a) is sufficient for the division theorem in the classes $A^{-\infty}_\Omega$.

Proposition 3.4. *Let $\varphi\in A^\infty_K$ be such that $h^*_\varphi=H_K$. If φ satisfies (S^a), then $\varphi\in\mathcal{D}^{-\infty}_{\Omega+K,\Omega}$.*

We recall a lemma due to Harnack, Malgrange and Hörmander ([14, Lemma 3.1]).

Lemma 3.5. *Let Φ, F and $G=F/\Phi$ be three holomorphic functions in the open ball $B(0;R)$. If the inequalities $|\Phi(w)|\leq A$ and $|F(w)|\leq B$ hold on $B(0;R)$, then we have*

$$|G(w)|\leq BA^{2|w|/(R-|w|)}|\Phi(0)|^{-(R+|w|)/(R-|w|)},\quad w\in B(0;R).$$

Proof of Proposition 3.4. Let $s,N>0$ be as in the condition (S^a) for φ. We can assume without loss of generality that $\log(1+t)\leq\frac{1}{6}(1+t)$ for all $t\geq N$. In the sequel, we will write $\ell(w):=\log(1+|w|)$, $w\in\mathbb{C}^n$, for simplicity.

Since $\varphi\in A^\infty_K$, there exist $A>0$ and $p\in\mathbb{N}$ such that

$$\log|\varphi(w)|\leq A+H_K(w)+p\ell(w),\quad w\in\mathbb{C}^n. \tag{4}$$

Consider any function $f\in A^{-\infty}_{\Omega+K}$ with $f/\varphi\in\mathcal{O}(\mathbb{C}^n)$. Since $f\in A^{-\infty}_{\Omega+K}$, for each $m\in\mathbb{N}$ there is $B>0$ such that

$$\log|f(w)|\leq B+H_{\Omega+K}(w)-m\ell(w),\quad w\in\mathbb{C}^n. \tag{5}$$

Given $\zeta\in\mathbb{C}^n$ with $|\zeta|>N$, take ζ' as in the condition (S^a). Noting that $|\zeta''-\zeta|\leq 3\ell(\zeta)$ for all $\zeta''\in B(\zeta';2\ell(\zeta))$ and using the choice of N, we get

$$\ell(\zeta'')\geq\log(1+|\zeta|-3\ell(\zeta))=\log(1+|\zeta|)+\log\left(1-\frac{3\ell(\zeta)}{1+|\zeta|}\right)\geq\ell(\zeta)-1.$$

From (5) it then follows that

$$\sup_{\zeta''\in B(\zeta';2\ell(\zeta))}\log|f(\zeta'')|\le B+\sup_{|\zeta''-\zeta'|\le 2\ell(\zeta)}(H_{\Omega+K}(\zeta'')-m\ell(\zeta''))$$
$$\le B+m+H_{\Omega+K}(\zeta)-(m-3R_{\Omega+K})\ell(\zeta).$$

In the same way, using (4) we have that

$$\sup_{\zeta''\in B(\zeta';2\ell(\zeta))}\log|\varphi(\zeta'')|\le A+p+H_K(\zeta)+(p+3R_K)\ell(\zeta).$$

Applying Lemma 3.5 with $R{:=}2\ell(\zeta)$, $\Phi(w)=\varphi(\zeta'+w)$, $F(w)=f(\zeta'+w)$ and $w=\zeta-\zeta'$, and using the condition (S^a) for φ, we get that

$$\begin{aligned}\log\left|\frac{f(\zeta)}{\varphi(\zeta)}\right| &\le B+m+H_{\Omega+K}(\zeta)-(m-3R_{\Omega+K})\ell(\zeta)\\ &\quad+\frac{2|\zeta-\zeta'|}{2\ell(\zeta)-|\zeta-\zeta'|}(A+p+H_K(\zeta)+(p+3R_K)\ell(\zeta))\\ &\quad-\frac{2\ell(\zeta)+|\zeta-\zeta'|}{2\ell(\zeta)-|\zeta-\zeta'|}(H_K(\zeta)-s\ell(\zeta))\\ &\le B+m+2(A+p)+H_\Omega(\zeta)-(m-3R_{\Omega+K}-6R_K-2p-3s)\ell(\zeta).\end{aligned}$$

Since m is arbitrary, we have that $f/\varphi\in A_\Omega^{-\infty}$. This completes the proof. □

As a consequence of Theorem 2.9 and Proposition 3.4 we have the following sufficient conditions for the surjectivity of convolution operators.

Proposition 3.6. *Let Ω and $\Omega+K$ be in $\mathcal{D}^{-\infty,n}$ and μ be an analytic functional with $\hat\mu\in A_K^\infty$. If $h_{\hat\mu}^*=H_K$ and $\hat\mu$ satisfies (S^a), then the convolution operator $\mu*\colon A^{-\infty}(\Omega+K)\to A^{-\infty}(\Omega)$ is surjective.*

3.3. Necessary conditions

In this section we prove that the condition (S^a) is necessary for the convolution operator to be surjective from $A^{-\infty}(\Omega+K)$ onto $A^{-\infty}(\Omega)$ for each convex bounded domain Ω. Below K is a fixed convex compact set and S^n is the unit sphere in $\mathbb{C}^n$.

Lemma 3.7. *A function $g\in A_K^\infty$ with radial indicator H_K satisfies (S^a) if and only if there are s, j and N such that*

$$\sup_{|w-a|\le j\rho(t)}|g(tw)|\ge tH_K(a)-s\log(1+t),\quad a\in S^n \text{ and } t\ge N, \tag{6}$$

where $\rho(t){:=}\log(1+t)/t$.

Proof. It is trivial that (S^a) implies (6). Assume that g satisfies (6). Consider any $z\in\mathbb{C}^n\setminus\{0\}$, put $a:=z/|z|$ and $t:=|z|$, and let

$$M_g(z;r):=\max\{|g(w)|:|w_1-z_1|=\ldots=|w_n-z_n|=r\},\quad r>0.$$

Clearly,

$$\sup_{|w-a|\le j\rho(t)}|g(tw)|\le M_g(z;j\log(1+|z|)),\quad \text{if } |z|\ge N.$$

Applying (6) we then have

$$\log M_g(z;j\log(1+|z|))\ge H_K(z)-s\log(1+|z|),\quad \text{if } |z|\ge N. \tag{7}$$

Since $g\in A_K^\infty$, there exist $q,M>0$ such that

$$\log|g(\zeta)|\le H_K(\zeta)+q\log(1+|\zeta|)\quad \text{for all } |\zeta|\ge M. \tag{8}$$

Take L so large that $L-2j\log L\ge M+2j$, and remember that H_K, as a support function of a compact set, satisfies the Lipschitz condition

$$|H(\zeta^1)-H(\zeta^2)|\le A\|\zeta^1-\zeta^2\|\quad \text{for some } A>0 \text{ and all } \zeta^1,\zeta^2\in\mathbb{C}^n,$$

where $\|\zeta\|:=\max_{1\le k\le n}|\zeta_k|$. It then follows from (8) that

$$\log M_g(z;2j\log(1+|z|))\le H_K(z)+2j(A+q)\log(1+|z|)\quad \text{for all } |z|\ge L. \tag{9}$$

As $\log M_g(z;r)$ is convex with respect to $\log r$, we get that

$$\log M_g(z;j\log(1+|z|))\le \frac{j}{2j-1/\sqrt{n}}\log M_g\left(z;\frac{\log(1+|z|)}{\sqrt{n}}\right) + \frac{j-1/\sqrt{n}}{2j-1/\sqrt{n}}\log M_g(z;2j\log(1+|z|)).$$

Hence, using (7) and (9), we find that, for $|z|\ge M+L$,

$$\begin{aligned}\sup_{|w-z|\le\log(1+|z|)}\log|g(w)| &\ge \log M_g\left(z;\frac{\log(1+|z|)}{\sqrt{n}}\right)\\ &\ge \frac{2j-1/\sqrt{n}}{j}\log M_g(z;j\log(1+|z|))\\ &\quad -\frac{j-1/\sqrt{n}}{j}\log M_g(z;2j\log(1+|z|))\\ &\ge \frac{2j-1/\sqrt{n}}{j}H_K(z)-2s\log(1+|z|)-\frac{j-1/\sqrt{n}}{j}H_K(z)\\ &\quad -2j(A+q)\log(1+|z|)\\ &= H_K(z)-p\log(1+|z|),\end{aligned}$$

where $p:=2s+2j(A+q)$. Thus, g satisfies (S^a). □

Lemma 3.8. *Let $g\in A_K^\infty$ satisfy condition* (ii) *of Proposition* 2.8 *for every open bounded convex polyhedron $\Omega\subset\mathbb{C}^n$. Then the radial indicator of g coincides with H_K and g satisfies (S^a).*

Proof. The condition (ii) of Proposition 2.8 implies that there exist $m\in\mathbb{N}$ and $M>0$ such that

$$\sup_{z\in\mathbb{C}^n}\frac{|f(z)|}{e^{H_\Omega(z)}}\le M\sup_{z\in\mathbb{C}^n}\frac{|g(z)|\,|f(z)|(1+|z|)^m}{e^{H_\Omega(z)+H_K(z)}}\quad\text{for all } f\in A_\Omega^{-\infty}. \tag{10}$$

Certainly, m and M depend on Ω but not on $f\in A_\Omega^{-\infty}$.

Suppose that the radial indicator h_g^* of g does not coincide with H_K or $h_g^*=H_K$ but g does not satisfy (S^a). Using Lemma 3.7 we have that in both cases the condition (6) does not hold. Then there exist $a\in S^n$ and $t_j\uparrow\infty$ such that

$$\sup_{|w-a|\le j\rho(t_j)}|g(t_jw)|\le t_jH_K(a)-j^2\log(1+t_j)\quad\text{for each } j\in\mathbb{N}.$$

Without loss of generality we can assume that $t_j\ge 2j\log(1+t_j)+2$ for all $j\in\mathbb{N}$. Put

$$\Delta_K:=\max_{w\in K}|w|,\quad z_j:=t_ja\quad\text{and}\quad R_j:=j\log(1+t_j)=j\log(1+|z_j|).$$

Notice that for $|w-z_j|\le R_j$ we have

$$\tfrac{1}{2}\log(1+|w|)\le\log\bigl(1+\tfrac{2}{3}|w|\bigr)\le\log(1+t_j)\le\log(1+2|w|)\le 2\log(1+|w|).$$

Then, for such w and $j\ge 8\Delta_K$,

$$\begin{aligned}\log|g(w)|&\le H_K(z_j)-j^2\log(1+t_j)\\&\le H_K(w)+\Delta_KR_j-j^2\log(1+t_j)\\&\le H_K(w)+2\Delta_Kj\log(1+|w|)-\frac{j^2}{2}\log(1+|w|)\\&\le H_K(w)-\frac{j^2}{4}\log(1+|w|).\end{aligned}$$

Thus,

$$\log|g(w)|\le H_K(w)-\frac{j^2}{4}\log(1+|w|)\quad\text{for all } |w-z_j|\le R_j \text{ and } j\ge 8\Delta_K. \tag{11}$$

Since $g\in A_K^\infty$, there exists $p>0$ such that

$$\log|g(w)|\le H_K(w)+p\log(1+|w|)+p\quad\text{for all } w\in\mathbb{C}^n. \tag{12}$$

For any bounded convex polyhedron Ω in $\mathbb{C}^n$, any fixed point $z\in\mathbb{C}^n$, and any number $R>0$ consider the function $h_\Omega(z,R)(\zeta)$ which coincides with $H_\Omega(\zeta)$ for $|\zeta-z|\geq R$ and equals

$$\sup\Big\{u(w): u \text{ is psh in } B(z,R), \text{ and } \limsup_{w\to\zeta} u(w)\leq H_\Omega(\zeta) \text{ for } \zeta\in\partial B(z,R)\Big\}$$

for $\zeta\in B(z,R):=\{\zeta\in\mathbb{C}^n: |\zeta-z|<R\}$. By [20, Lemma 2] $h_\Omega(z,R)$ is psh and continuous in $\mathbb{C}^n$. Next, put

$$S_\Omega^*:=\{z\in S^n: h_\Omega(z,R)(z)>H_\Omega(z) \text{ for all } R>0\}$$

and note that $S_\Omega^*\neq\varnothing$ for all Ω (see, for example, [21, the remark after Lemma 3.3]).

Let now Ω be an open bounded convex polyhedron in $\mathbb{C}^n$ with $0\in\Omega$ and $a\in S_\Omega^*$. By [21, Lemma 3.1] there exists $\varepsilon_0>0$ such that

$$\sup_{\zeta\in B(ta,R)}(h_\Omega(ta,R)(\zeta)-H_\Omega(\zeta))\geq\varepsilon_0 R \quad \text{for all } R>0 \text{ and } t>0.$$

On the other hand,

$$h_\Omega(z,R)(w)\leq\sup_{|\zeta-z|\leq R}H_\Omega(\zeta)\leq H_\Omega(w)+2\Delta_\Omega R \quad \text{for all } z,w\in\mathbb{C}^n \text{ with } |w-z|\leq R,$$

where $\Delta_\Omega:=\sup_{\zeta\in\Omega}|\zeta|$. Taking $t=t_j$, $z=z_j$, and $R=R_j/2$, we find ζ_j with $|\zeta_j-z_j|\leq R_j/2$ so that, for any $j\in\mathbb{N}$,

$$h_\Omega\left(z_j,\frac{R_j}{2}\right)(\zeta_j)\geq H_\Omega(\zeta_j)+\frac{\varepsilon_0}{2}R_j \tag{13}$$

and

$$h_\Omega\left(z_j,\frac{R_j}{2}\right)(w)\leq H_\Omega(w)+\Delta_\Omega R_j \quad \text{for all } |w-z_j|\leq R_j/2. \tag{14}$$

Fix a sequence $\{q_j\}_{j=1}^\infty$ with $\frac{1}{2}\leq q_j\uparrow 1$. By [1, Lemma 4] (see also [2, Lemma 3]) there exist an absolute constant $A=A(n)$ and a family $\{f_j: j\in\mathbb{N}\}$ of entire functions in $\mathbb{C}^n$ so that, for each $j\in\mathbb{N}$,

$$\log|f_j(\zeta_j)|=q_j h_\Omega(z_j,R_j/2)(\zeta_j), \tag{15}$$

$$\log|f_j(z)|\leq q_j\sup_{|w-z|\leq 1}h_\Omega(z_j,R_j/2)(w)+2n\log(1+|z|)+A \quad \text{for all } z\in\mathbb{C}^n. \tag{16}$$

Notice that from the definition of $h_\Omega(z_j,R_j/2)$ and (14) it follows that for $|z-z_j|\le \frac{1}{2}R_j+1$,

$$\sup_{|w-z|\le 1} h_\Omega(z_j,R_j/2)(w)\le \sup_{|w-z|\le 1} H_\Omega(w)+\Delta_\Omega R_j$$
$$\le H_\Omega(z)+\Delta_\Omega R_j+\Delta_\Omega \le H_\Omega(z)+2\Delta_\Omega j\log(1+|z|)+\Delta_\Omega,$$

while for $|z-z_j|>\frac{1}{2}R_j+1$,

$$\sup_{|w-z|\le 1} h_\Omega(z_j,R_j/2)(w)=\sup_{|w-z|\le 1} H_\Omega(w)\le H_\Omega(z)+\Delta_\Omega.$$

Therefore, from (13), (15), and (16) it follows that, for each $j\in\mathbb{N}$,

$$\log|f_j(\zeta_j)|\ge q_j H_\Omega(\zeta_j)+\frac{\varepsilon_0}{4}j\log(1+|z_j|)\ge q_j H_\Omega(\zeta_j)+\frac{\varepsilon_0}{8}j\log(1+|\zeta_j|), \tag{17}$$

$$\log|f_j(z)|\le q_j H_\Omega(z)+2(\Delta_\Omega j+n)\log(1+|z|)+A+\Delta_\Omega,\quad |z-z_j|\le\frac{R_j}{2}, \tag{18}$$

and

$$\log|f_j(z)|\le q_j H_\Omega(z)+2n\log(1+|z|)+A+\Delta_\Omega,\quad |z-z_j|>\frac{R_j}{2}. \tag{19}$$

Next, estimate

$$A_j:=\sup_{z\in\mathbb{C}^n}\frac{|g(z)|\,|f_j(z)|(1+|z|)^m}{e^{H_\Omega(z)+H_K(z)}}.$$

If $|z-z_j|\le R_j/2+1$, then, applying (11) and (18), we have, for $j\ge 16(\Delta_K+\Delta_\Omega+n)$,

$$\log|g(z)f_j(z)|\le H_K(z)-\frac{j^2}{4}\log(1+|z|)+q_jH_\Omega(z)+2(\Delta_\Omega j+n)\log(1+|z|)+A+\Delta_\Omega$$
$$\le H_\Omega(z)+H_K(z)-\frac{j^2}{8}\log(1+|z|)+A+\Delta_\Omega.$$

Hence,

$$\sup_{|z-z_j|\le R_j/2+1}\frac{|g(z)|\,|f_j(z)|(1+|z|)^m}{e^{H_\Omega(z)+H_K(z)}}\le e^{A+\Delta_\Omega},\quad \text{when } j^2\ge 8m.$$

Further, from (12) and (19) it follows that

$$\log|g(z)f_j(z)|\le H_K(z)+p\log(1+|z|)+p+q_jH_\Omega(z)+2n\log(1+|z|)+A+\Delta_\Omega$$
$$\le H_\Omega(z)+H_K(z)+(p+2n)\log(1+|z|)-(1-q_j)\delta_\Omega|z|+A+\Delta_\Omega,$$

where $\delta_\Omega:=\inf_{|\zeta|=1} H_\Omega(\zeta)$. Then, for all $j\in\mathbb{N}$,

$$\sup_{|z-z_j|>R_j/2+1} \frac{|g(z)|\,|f_j(z)|(1+|z|)^m}{e^{H_\Omega(z)+H_K(z)}} \le e^{A+\Delta_\Omega+\delta_\Omega} \sup_{s\ge 0} \frac{s^{m+p+2n}}{e^{(1-q_j)\delta_\Omega s}}$$
$$= e^{A+\Delta_\Omega+\delta_\Omega}\left(\frac{m+p+2n}{(1-q_j)\delta_\Omega e}\right)^{m+p+2n}.$$

Without loss of generality we can assume that $m+p+2n\ge\delta_\Omega e$. Then we finally have that

$$A_j \le e^{A+\Delta_\Omega+\delta_\Omega}\left(\frac{m+p+2n}{(1-q_j)\delta_\Omega e}\right)^{m+p+2n}. \tag{20}$$

From (18) and (19) it easily follows that $f_j\in A_\Omega^{-\infty}$ for every $j\in\mathbb{N}$. At the same time, (17) implies that

$$B_j := \sup_{z\in\mathbb{C}^n} \frac{|f_j(z)|}{e^{H_\Omega(z)}} \ge \frac{|f_j(\zeta_j)|}{e^{H_\Omega(\zeta_j)}} \ge \frac{(1+|\zeta_j|)^{\varepsilon_0 j/8}}{e^{(1-q_j)\Delta_\Omega(1+|\zeta_j|)}}.$$

Taking $q_j=1-\varepsilon_0 j/8(1+|\zeta_j|)\Delta_\Omega$, we find that

$$B_j \ge \left(\frac{\varepsilon_0 j}{8(1-q_j)\Delta_\Omega e}\right)^{\varepsilon_0 j/8}. \tag{21}$$

From (20) and (21) we have that $B_j/A_j\to\infty$ as $j\to\infty$. This contradicts (10) and completes the proof. □

Remark 3.9. As follows from the proof, in Lemma 3.8 it is enough to require that $g\in A_K^\infty$ satisfies condition (ii) of Proposition 2.8 for every polyhedron Ω from some subclass $\mathcal{D}$ having the following property: *for each $a\in S^n$ there is $\Omega\in\mathcal{D}$ such that $a\in S_\Omega^*$.*

3.4. Criterion for surjectivity

Now we can state a criterion for the convolution operator to be surjective on the class of all convex bounded domains in $\mathbb{C}^n$.

Theorem 3.10. *Let μ be an analytic functional on $\mathbb{C}^n$, carried by a compact convex set K, such that $\mu * A^{-\infty}(\Omega+K)\subset A^{-\infty}(\Omega)$ for each convex bounded domain $\Omega\subset\mathbb{C}^n$. Then $\mu*\colon A^{-\infty}(\Omega+K)\to A^{-\infty}(\Omega)$ is surjective for every Ω if and only if the radial indicator of $\hat\mu$ coincides with H_K and $\hat\mu$ satisfies (S^a).*

Proof. This result is a direct consequence of Proposition 3.6, Theorem 2.9 and Lemma 3.8. □

In the next section we give some additional results for surjectivity of convolution operators. In particular, we prove that each differential operator of finite order maps $A^{-\infty}(\Omega)$ onto $A^{-\infty}(\Omega)$.

4. Examples and applications

4.1. Examples

In this section we consider examples of functions satisfying the condition (S^a) and discuss the question about existence of such functions for a given convex compact set K.

Example 4.1. Let $n\geq 1$ and $\zeta=(\zeta_1,...,\zeta_n)$. Then any $P(\zeta)\in\mathbb{C}[\zeta]$ satisfies (S^a) and we also have that $h^*_{\hat{\mu}}(\zeta)=H_0(\zeta)=0$.

More generally, we have the following statement.

Proposition 4.2. *Let $\lambda_1,...,\lambda_N\in\mathbb{C}^n$ and $P_j(\zeta)\in\mathbb{C}[\zeta]$, $1\leq j\leq N$. Consider the exponential-polynomial*

$$f(\zeta):=\sum_{j=1}^{N} P_j(\zeta)e^{\langle\lambda_j,\zeta\rangle}$$

corresponding to the differential-difference operator. Set $\Lambda:=\{\lambda_1,...,\lambda_N\}$ and $K:=\operatorname{conv}\Lambda$, the convex hull of Λ. Then $f(\zeta)$ satisfies (S^a) and $h^*_f(\zeta)=H_K(\zeta)$.

Proof. The case $N=1$ is trivial. Let $N\geq 2$. For any $\zeta_0\in\mathbb{C}^n$, we may suppose that

$$H_K(\zeta_0)=\operatorname{Re}\langle\lambda_1,\zeta_0\rangle\geq\operatorname{Re}\langle\lambda_k,\zeta_0\rangle,\quad 2\leq k\leq N.$$

Let $m_j:=\deg P_j$ and $m:=\max_j m_j$. Set

$$g_N(\zeta):=P_1(\zeta)e^{\langle\lambda_1-\lambda_N,\zeta\rangle}+...+P_{N-1}(\zeta)e^{\langle\lambda_{N-1}-\lambda_N,\zeta\rangle}+P_N(\zeta).$$

We then have

$$f(\zeta)=e^{\langle\lambda_N,\zeta\rangle}g_N(\zeta).$$

If we take any $\alpha(N)$ with $|\alpha(N)|=m_N+1$, we have $D_\zeta^{\alpha(N)}P_N(\zeta)=0$ and then with another polynomial $Q_k^N(\zeta)$ of degree m_k, $1\le k\le N-1$, we have

$$D_\zeta^{\alpha(N)}g_N(\zeta)=Q_1^N(\zeta)e^{\langle\lambda_1-\lambda_N,\zeta\rangle}+\ldots+Q_{N-1}^N(\zeta)e^{\langle\lambda_{N-1}-\lambda_N,\zeta\rangle}.$$

Next we set

$$g_{N-1}(\zeta):=Q_1^N(\zeta)e^{\langle\lambda_1-\lambda_{N-1},\zeta\rangle}+\ldots+Q_{N-2}^N(\zeta)e^{\langle\lambda_{N-2}-\lambda_{N-1},\zeta\rangle}+Q_{N-1}^N(\zeta)$$

and then

$$D_\zeta^{\alpha(N)}g_N(\zeta)=e^{\langle\lambda_{N-1}-\lambda_N,\zeta\rangle}g_{N-1}(\zeta).$$

Taking any $\alpha(N-1)$ with $|\alpha(N-1)|=m_{N-1}+1$, we have $D_\zeta^{\alpha(N-1)}Q_N^{N-1}(\zeta)=0$ and then with another polynomial $Q_k^{N-1}(\zeta)$ of degree m_k, $1\le k\le N-2$, we have

$$D_\zeta^{\alpha(N-1)}g_{N-1}(\zeta)=Q_1^{N-1}(\zeta)e^{\langle\lambda_1-\lambda_{N-1},\zeta\rangle}+\ldots+Q_{N-2}^{N-1}(\zeta)e^{\langle\lambda_{N-2}-\lambda_{N-1},\zeta\rangle}.$$

Repeating these procedure, we finally have $\alpha(2)$ with $|\alpha(2)|=m_2+1$ and a polynomial $Q_1^2(\zeta)$ of degree m_1 such that

$$D_\zeta^{\alpha(2)}g_2(\zeta)=e^{\langle\lambda_1-\lambda_2,\zeta\rangle}Q_1^2(\zeta).$$

By the preceding example, we have a constant $c_1>0$ such that there exists $\zeta^{(1)}\in B(\zeta_0;1/N\ell(|\zeta_0|))$ satisfying

$$|Q_1^2(\zeta^{(1)})|>c_1.$$

Then by the Cauchy estimate, there exists $\zeta^{(2)}\in B(\zeta^{(1)};1/N\ell(|\zeta_0|))$ such that

$$|g_2(\zeta^{(2)})|\ge c_2' e^{\mathrm{Re}\langle\lambda_1-\lambda_2,\zeta^{(1)}\rangle}$$

with a constant $c_2'>0$.

Note that $|\zeta^{(1)}|\ge|\zeta_0|-|\zeta^{(1)}-\zeta^{(0)}|\ge|\zeta^{(0)}|-1/N\ell(|\zeta_0|)$. Setting

$$a_2:=\frac{1}{N}|\lambda_1-\lambda_2|\ge 0,$$

we have with $c_2>0$,

$$|g_2(\zeta^{(2)})|\ge c_2e^{\mathrm{Re}\langle\lambda_1-\lambda_2,\zeta_0\rangle}(1+|\zeta_0|)^{-|\lambda_1-\lambda_2|/N}=c_2e^{\mathrm{Re}\langle\lambda_1-\lambda_2,\zeta_0\rangle}(1+|\zeta_0|)^{-a_2}.$$

In the same way, we find $\zeta^{(3)}\in B(\zeta^{(2)};1/N\ell(|\zeta_0|))$ and $a_3\ge 0$ such that

$$|g_3(\zeta^{(3)})|\ge c_3' e^{\mathrm{Re}\langle\lambda_2-\lambda_3,\zeta^{(2)}\rangle}|g_2(\zeta^{(2)})|\ge c_3e^{\mathrm{Re}\langle\lambda_1-\lambda_3,\zeta_0\rangle}(1+|\zeta_0|)^{-a_3}$$

with constants $c_3,c_3'>0$.

Repeating this procedure, we finally have a point $\zeta^{(N)}\in B(\zeta^{(N-1)};1/N\ell(|\zeta_0|))\subset B(\zeta_0;\ell(|\zeta_0|))$, and constants $a_N,a_N'\ge 0$ and $c_N,c_N'>0$ such that

$$|g_N(\zeta^{(N)})|\ge c_Ne^{\mathrm{Re}\langle\lambda_1-\lambda_N,\zeta_0\rangle}(1+|\zeta_0|)^{-a_N}$$

and so

$$|f(\zeta^{(N)})| = e^{\mathrm{Re}\langle\lambda_N,\zeta^{(N)}\rangle}|g_N(\zeta^{(N)})| \geq c'_N e^{\mathrm{Re}\langle\lambda_1,\zeta_0\rangle}(1+|\zeta_0|)^{-a_N}$$
$$\geq c'_N e^{H_K(\zeta_0)}(1+|\zeta_0|)^{-a'_N}.$$

We note that all the constants c_k, c'_k, a_k, and a'_k are independent of ζ_0 and the first assertion follows.

The property $h^*_f(\zeta)=H_K(\zeta)$ is proved in [22, Theorem 6.1.1]. □

Proposition 4.3. *Let* Λ, K *and* f *be as in Proposition 4.2 and* μ_f* *be the differential-difference operator generated by* f. *Then* μ_f* *maps* $A^{-\infty}(\Omega+K)$ *onto* $A^{-\infty}(\Omega)$ *for any* $\Omega,\Omega+K\in\mathcal{D}^{-\infty,n}$. *In particular, for each polynomial* P *in* $\mathbb{C}^n$ *the differential operator* μ_P* *maps* $A^{-\infty}(\Omega)$ *onto* $A^{-\infty}(\Omega)$ *for any* $\Omega\in\mathcal{D}^{-\infty,n}$.

Proof. This result is an immediate consequence of Theorem 3.10 and Proposition 4.2. □

Let us discuss the question about existence of functions which belong to A^∞_K and satisfy (S^a). For $n=1$ the answer is always affirmative and can be obtained rather simply.

Proposition 4.4. *For each convex compact set* K *in* $\mathbb{C}$ *there exists a function* g *in* A^∞_K *which satisfies* (S^a).

Proof. If K is a singleton, say $\{a\}$, then A^∞_K coincides with the family $\{p(z)e^{az}: p \text{ is a polynomial}\}$. By Proposition 4.2, each function of the type $p(z)e^{az}$ with nontrivial polynomial p satisfies (S^a).

Now consider K with more than one point. Applying [26, Theorem 5] to a subharmonic function H_K in $\mathbb{C}$ we can find an entire function g such that

$$\big|\log|g(z)|-H_K(z)\big| \leq C\log(1+|z|), \quad z\notin E_0, \tag{22}$$

C being a constant and E_0 being an exceptional set in $\mathbb{C}$ which can be covered by a sequence of rings $\{z:|z-z_k|\leq r_k\}$ with $\sum_{k=1}^\infty r_k<\infty$. Clearly, g belongs to A^∞_K and satisfies (S^a). □

Corollary 4.5. *Let* $K=K_1\times\ldots\times K_n$, *where* K_j *are convex compact sets in* $\mathbb{C}$, $1\leq j\leq n$. *Then there exists a function* $g\in A^\infty_K$ *which satisfies* (S^a).

Proof. It is sufficient to take a function $g(z):=\prod_{j=1}^n g_j(z_j)$, where g_j satisfies (22) with $K=K_j$, $1\leq j\leq n$. □

Remark 4.6. In [27, Theorem 4] the following analog of (22) for $n>1$ was established

$$\big|\log|g(z)|-H_K(z)\big|\le(1+|z|)^{2/3},\quad z\notin E_0,$$

where K is a convex compact set in $\mathbb{C}^n$ and E_0 is some exceptional set. Evidently, this inequality cannot guarantee the existence of functions belonging to A_K^∞ and satisfying (S^a) for arbitrary K and $n>1$. So we have the following question.

Open problem. Let K be a convex compact set in $\mathbb{C}^n$, $n>1$. Does there exist a function g in A_K^∞ which satisfies (S^a)?

4.2. Explicit form for solutions

In this section we discuss the problem of explicit representation of solutions for convolution operators in the space $A^{-\infty}(\Omega)$. Throughout what follows we assume that either $n=1$, or $n>1$ and Ω and $\Omega+K$ have C^2 boundaries. We have the following representation result.

Proposition 4.7. *Let $\mu*\colon A^{-\infty}(\Omega+K)\to A^{-\infty}(\Omega)$ be a surjective convolution operator. Then there exists a sequence $\Lambda=\{\lambda_k\}_{k=1}^\infty$ in $\mathbb{C}^n$ with $|\lambda_k|\to\infty$ such that each function $g\in A^{-\infty}(\Omega)$ can be represented in the form*

$$g(z)=\sum_{k=1}^\infty c_k e^{\langle\lambda_k,z\rangle},\quad z\in\Omega,\tag{23}$$

and the function

$$f(w)=\sum_{k=1}^\infty \frac{c_k}{\hat\mu(\lambda_k)}e^{\langle\lambda_k,w\rangle},\quad w\in\Omega+K,\tag{24}$$

belongs to $A^{-\infty}(\Omega+K)$ and is a solution of the equation $\mu*f=g$.

Proof. In fact, by [6, Theorem 4.3] there exists a sequence $\Lambda=\{\lambda_k\}_{k=1}^\infty$ in $\mathbb{C}^n$ with $|\lambda_k|\to\infty$ so that the system of exponential functions $E_\Lambda:=\{e^{\langle\lambda_k,z\rangle}:k\in\mathbb{N}\}$ is an absolutely representing system in $A^{-\infty}(\Omega+K)$. Then, given $g\in A^{-\infty}(\Omega)$, we find $f\in A^{-\infty}(\Omega+K)$ with $\mu*f=g$ and expand f into the series

$$f(z)=\sum_{k=1}^\infty a_k e^{\langle\lambda_k,w\rangle},\quad w\in\Omega+K,\tag{25}$$

which converges to f absolutely in $A^{-\infty}(\Omega+K)$. Since $\mu*\colon A^{-\infty}(\Omega+K)\to A^{-\infty}(\Omega)$ is continuous, the series $\sum_{k=1}^\infty a_k\hat\mu(\lambda_k)e^{\langle\lambda_k,z\rangle}$ converges to g in $A^{-\infty}(\Omega)$. It remains to put $c_k:=a_k\hat\mu(\lambda_k)$, $k\in\mathbb{N}$. □

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