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Convolution, product and Fourier transform of distributions

by

A. KAMIŃSKI (Katowice)

Abstract. By modifying the Mikusiński general scheme of defining operations on distributions, we obtain particularly a sequential definition of the convolution of distributions in several versions. Some of them are equivalent to the known definitions of the convolution given in [2], [13], [14], [16], [3]. Relations between definitions of the convolution and the product of distributions by using various classes of unit-sequences and delta-sequences are examined. As a consequence, we explain mutual relations of the exchange formulae $\mathcal{F}(f*g) = \mathcal{F}(f)\mathcal{F}(g)$ and $\mathcal{F}(f\cdot g) = \mathcal{F}(f)*\mathcal{F}(g)$. In particular, it occurs that the second formula does not hold in the sense of Itano [5].

1. Introduction. In the sequential approach to the theory of distributions, it is natural to define for distributions (in some general way) such operations which are given for smooth functions (see [10], [1]). Some of operations can be defined for all distributions (regular operations), others—only for particular distributions (irregular operations). The defining operations in [10] and [1] are based on a concept of a delta-sequence $\{\delta_n\}$. In many cases, this method allows one to define in a simple way various operations for sufficiently large classes of distributions.

It is possible to adopt various versions of the definition. For instance, the product of distributions f, g can be defined in the four ways:

(1) $$[f\cdot g] = \lim_{i\to\infty}(f*\delta_i)\cdot(g*\delta_i),$$

(2) $$[f]\cdot[g] = \lim_{i\to\infty}(f*\delta_i)\cdot(g*\tilde{\delta}_i),$$

(3) $$[f]\cdot g = \lim_{i\to\infty}\lim_{j\to\infty}(f*\delta_i)\cdot(g*\tilde{\delta}_j) = \lim_{i\to\infty}(f*\delta_i)\cdot g,$$

(4) $$f\cdot[g] = \lim_{i\to\infty}\lim_{j\to\infty}(f*\delta_j)\cdot(g*\tilde{\delta}_i) = \lim_{i\to\infty} f\cdot(g*\tilde{\delta}_i),$$

whenever the limits (distributional) exist for any delta-sequences $\{\delta_n\}$ and $\{\tilde{\delta}_n\}$ (see [12], [1], [11], [4], [15]).

Definitions (2)–(4) are equivalent (see Section 2); definition (1) is more general.

However, this method is less adequate for operations which are not defined for all smooth functions as the convolution, for instance. Some additional conditions are then needed (for the convolution see [1], pp. 153 and 131).

In this paper we propose to define such operations for distributions by using so-called unit-sequences $\{\eta_n\}$ (see [7], [16] and [3]). The method will be shown in the case of locally regular operations (see Section 2), but it is possible to generalize it for any operation (see [8]).

In the case of the convolution of distributions our method leads to the definitions:

(5) $$[f*g] = \lim_{i\to\infty}(\eta_i f)*(\eta_i g),$$

(6) $$[f]*[g] = \lim_{i\to\infty}(\eta_i f)*(\tilde{\eta}_i g),$$

(7) $$[f]*g = \lim_{i\to\infty}\lim_{j\to\infty}(\eta_i f)*(\tilde{\eta}_j g) = \lim_{i\to\infty}(\eta_i f)*g,$$

(8) $$f*[g] = \lim_{i\to\infty}\lim_{j\to\infty}(\eta_j f)*(\tilde{\eta}_i g) = \lim_{i\to\infty} f*(\tilde{\eta}_i g),$$

where the limits (distributional) are supposed to exist for any unit-sequences $\{\eta_n\}$, $\{\tilde{\eta}_n\}$.

By taking various classes of delta-sequences and unit-sequences, we can obtain various definitions (1)–(8). Relations between them will be discussed in the paper.

Also connections with known definitions will be considered. In particular, we shall show that definitions (6)–(8) of the convolution by using the classes E of unit-sequences and $\bar{E}$ of special unit-sequences (E-convolution, $\bar{E}$-convolution) are equivalent to each other and to the definitions of C. Chevalley [2], L. Schwartz [13], R. Shiraishi [14], V. S. Vladimirov [16] and P. Dierolf–J. Voigt [3].

A similar result is obtained for convolutions in $\mathcal{S}'$.

Definition (5) is essentially more general than definitions (6)–(8) (in $\mathcal{D}'$ and in $\mathcal{S}'$).

A generalization of another type can be obtained by restricting classes of unit- and delta-sequences. This leads to a concept of model unit- and delta-sequences (see Section 4). The classes of model sequences are minimal in some sense (the sequences are generated by single functions). The model sequences turn out to be very convenient in considering relations between the convolution and the product of distributions with respect to the Fourier transform:

$$\mathcal{F}(f)(t) = \int_{R^q}\exp(2\pi itx)f(x)\,dx \qquad (t\in R^q).$$

Namely, the two exchange formulae:

(9) $$\mathcal{F}(f*g) = \mathcal{F}(f)\cdot\mathcal{F}(g)$$

and

(10) $$\mathcal{F}(f\cdot g) = \mathcal{F}(f)*\mathcal{F}(g)$$

hold for tempered distributions f, g, provided the convolution and the product are defined by formulae (1)–(4) and (5)–(8), respectively, and by using model delta- and unit-sequences (see Section 4; also [7]).

Since definitions (2)–(4) of the product by using delta-sequences, introduced in [1], and by model delta-sequences are equivalent (see Section 5), we obtain immediately a generalization of the classical theorem on the exchange formula (9).

On the other hand, the exchange formula (10) is not true for E-convolutions (5)–(8), which gives the negative answer to the problem of M. Itano (see [5]).

Several unsolved problems are posed in Section 7.

2. Δ-operations and E-operations. We shall use the notation from [1].

By *a delta-sequence* we mean a sequence of smooth functions $\delta_n(x)$, $x\in R^q$, for which there exists a sequence of positive numbers α_n such that $\alpha_n\to 0$ and

(a) $\delta_n(x) = 0$ for $|x|\geqslant \alpha_n$;

(b) $\int \delta_n(x)\,dx = 1$;

(c) for every $k\in P^q$ there is a constant $M_k>0$ such that

$$\alpha_n^k\int|\delta_n^{(k)}(x)|\,dx < M_k$$

for $n = 1, 2, \ldots$ (see [1], p. 116).

The above class of delta-sequences will be denoted by Δ.

It is well known that for any distributions $f\in\mathcal{D}'$ and $g\in\mathcal{S}'$ and for every $\{\delta_n\}\in\Delta$, the convolutions $f*\delta_n$, $g*\delta_n$ are smooth functions and

$$f*\delta_n\to f \text{ in } \mathcal{D}', \qquad g*\delta_n\to g \text{ in } \mathcal{S}'$$

as $n\to\infty$.

Now, we remind the Mikusiński definition of operations on distributions (see [10], [1]). For simplicity, we consider operations of two arguments. Let $A(\cdot,\cdot)$ be some operation defined for smooth functions φ, ψ, results of which are smooth functions again (or numbers which can be interpreted as constant functions), and let f, g be distributions. If for every delta-sequence $\{\delta_n\}\in\Delta$ the functions (numbers)

(11) $$\Phi_n = A(f*\delta_n, g*\delta_n)$$

are defined for almost all $n = 1, 2, \ldots$ and the sequence $\{\Phi_n\}$ is fundamental, i.e. distributionally convergent (in the case of numbers, this means the usual convergence of numbers), then we define

$$A(f, g) = \lim_{n\to\infty} \Phi_n. \tag{12}$$

It is clear that limit (12) does not depend on the choice of a delta-sequence $\{\delta_n\} \in \Delta$.

We shall say that $A(f, g)$ exists as *Δ-operation (in $\mathscr{D}'$)*. For instance, formula (1) for $\{\delta_n\} \in \Delta$ defines the *Δ-product* $[f\cdot g]$ *(in $\mathscr{D}'$)*.

It is possible to give some modifications of the above definition. For example, we can take in (11) different delta-sequences $\{\delta_n\}, \{\tilde{\delta}_n\} \in \Delta$ for f and g. Then the definition of the *Δ-product* $[f]\cdot[g]$ is obtained as a special case.

Sometimes an operation $A(\cdot,\cdot)$ is regular with respect to each arguments, i.e. $A(f, g)$ exists for every pair (f, g) of distributions, one of which being a smooth function. Then we can define $A(f, g)$ for any $f, g \in \mathscr{D}'$ as $\lim\limits_{n\to\infty} A(f * \delta_n, g)$ or $\lim\limits_{n\to\infty} A(f, g * \delta_n)$. In particular, the definitions of the *Δ-products* $[f]\cdot g$ and $f\cdot[g]$ are obtained.

If $f, g \in \mathscr{S}'$ and the respective limit is tempered, then we say about *Δ-operation in $\mathscr{S}'$*.

Note that the Δ-product $[f\cdot g]$ exists for a wider class of distributions than the Δ-products $[f]\cdot[g]$, $[f]\cdot g$ and $f\cdot[g]$. For example, the product $x^{-1}\delta(x)$ exists in the sense of (1), but does not in the sense of (2)–(4) (cf. [7]).

On the other hand, we have

THEOREM (R. Shiraishi, M. Itano [15]). *Let $f, g \in \mathscr{D}'$. If one of limits (2)–(4) exists in $\mathscr{D}'$ for any delta-sequenes $\{\delta_n\}, \{\tilde{\delta}_n\} \in \Delta$, then the other limits also exist for any $\{\delta_n\}, \{\tilde{\delta}_n\} \in \Delta$ and*

$$[f]\cdot g = f\cdot[g] = [f]\cdot[g]. \tag{13}$$

In other words, Δ-products (2)–(4) are equivalent.

As a matter of fact, Shiraishi and Itano proved in [15] the above theorem for a wider class of delta-sequences than Δ, but it is possible to do it also for Δ (see [8]).

Definition (12) is less useful for such operations as the convolution, the integral, the Fourier transform. For those operations and, more generally, for locally regular operations, we are going to present a definition based on a concept of a unit-sequence. We restrict ourselves to operations of two arguments.

We say that a given operation $A(\cdot,\cdot)$ is *locally regular* if for any intervals $I_1, I_2 \subset \mathbf{R}^q$ and for any fundamental sequences of smooth functions $\{\varphi_{1n}\}$, $\{\varphi_{2n}\}$ such that the supports $\operatorname{supp}\varphi_{in}$ are included in I_i $(i = 1, 2;\ n = 1, 2, \ldots)$ the sequence

$$\{A(\varphi_{1n}, \varphi_{2n})\} \quad (n = 1, 2, \ldots)$$

is fundamental.

Obviously, locally regular operations can be defined by formulae (11) and (12) for arbitrary distributions f, g with compact supports.

A sequence $\{\eta_n(x)\}$, $x \in \mathbf{R}^q$, of functions of the class $\mathscr{D}$ is said to be *a unit-sequence* if

(α) for every $k \in \mathbf{P}^q$

$$(\eta_n(x) - 1)^{(k)} \to 0$$

almost uniformly in $\mathbf{R}^q$;

(β) for every $k \in \mathbf{P}^q$ there exists $M_k > 0$ such that

$$|\eta_n^{(k)}(x)| < M_k$$

for all $x \in \mathbf{R}^q$ and $n = 1, 2, \ldots$ (see [3]).

A sequence $\{\eta_n(x)\}$, $x \in \mathbf{R}^q$ of functions of the class $\mathscr{D}$ is said to be a *special unit-sequence* if it satisfies (β) and the following (stronger than (1)) condition:

(γ) for every interval $I \subset \mathbf{R}^q$ there exists n_0 such that $\eta_n(x) = 1$ for $x \in I$ and $n > n_0$ (see [16] and [3]).

The classes of all special unit-sequences will be denoted by E and $\bar{E}$, respectively.

If A is a locally regular operation, then $A(\eta f, \eta g)$ exists for any $f, g \in \mathscr{D}'$ and $\eta \in \mathscr{D}$, so we can adopt the following definition for given $f, g \in \mathscr{D}'$:

$$A(f, g) = \lim_{n\to\infty} A(\eta_n f, \eta_n g), \tag{14}$$

whenever the limit exists in $\mathscr{D}'$ for any $\{\eta_n\} \in E$ or $\{\eta_n\} \in \bar{E}$. We shall say then that $A(f, g)$ exists as *E-operation* or *$\bar{E}$-operation*, respectively.

Note that limit (14) does not depend on the choice of a unit-sequence $\{\eta_n\} \in E$ ($\{\eta_n\} \in \bar{E}$).

Since the operation of convolution is locally regular, we obtain especially definition (5) of the convolution (in $\mathscr{D}'$ and in $\mathscr{S}'$). We can modify definition (14) similarly as in the case of Δ-operations, which leads to definitions (6)–(8) of the convolution (in $\mathscr{D}'$ and in $\mathscr{S}'$).

Note that the E-convolution and $\bar{E}$-convolution $[f * g]$ exist for more pairs of distributions than the E-convolutions and $\bar{E}$-convolutions $[f]*[g]$, $[f]*g$ and $f*[g]$. In fact, the E-convolution ($\bar{E}$-convolution) $1 * \operatorname{sgn} x$ exists in the sense of (5), but not in the sense of (6)–(8) (cf. [7]).

3. Convolution. Now we are going to show relations between various definitions of the convolution for distributions. Applying results from [3] and [14], we shall obtain:

THEOREM 1. *Let $f, g \in \mathcal{D}'$. If one of limits* (6)–(8) *exists in $\mathcal{D}'$ for any unit-sequences $\{\eta_n\}, \{\tilde{\eta}_n\} \in \bar{E}$, then all limits exist in $\mathcal{D}'$ for any $\{\eta_n\}, \{\tilde{\eta}_n\} \in E$ and*

$$[f]*g = f*[g] = [f]*[g]. \tag{15}$$

In other words, E-convolutions and $\bar{E}$-convolutions (6)–(8) *in $\mathcal{D}'$ are all equivalent.*

Proof. Suppose that $f, g \in \mathcal{D}'(\boldsymbol{R}^q)$ and that

$$[f]*g = \lim_{n\to\infty} (\eta_n f)*g$$

exists in $\mathcal{D}'$ for any $\{\eta_n\} \in \bar{E}(\boldsymbol{R}^q)$. Since for any $\{\eta_n\} \in \bar{E}(\boldsymbol{R}^q)$ we have

$$\langle(\eta_n f)*g, \varphi\rangle_q = \langle f(g^- * \varphi), \eta_n\rangle_q \quad (n = 1, 2, \ldots),$$

where $g^-(x) = g(-x)$, the assumption implies that the distribution $f(g^- * \varphi)$ is integrable in $\boldsymbol{R}^q$, by virtue of (1.1) Proposition in [3]. Hence $f(x)g(y)\varphi(x+y)$ is an integrable distribution in $\boldsymbol{R}^{2q}$ $(x, y \in \boldsymbol{R}^q)$ by Theorem 2 in [14] (see also (1.3) Theorem in [3]). Applying (1.1) Proposition from [3] once again, we see that the limit

$$\lim_{n\to\infty} \langle f(x)g(y)\varphi(x+y), \eta_n(x)\tilde{\eta}_n(y)\rangle_{2q} = \lim_{n\to\infty} \langle(\eta_n f)*(\tilde{\eta}_n g), \varphi\rangle_q$$

exists for any $\{\eta_n\}, \{\tilde{\eta}_n\} \in E(\boldsymbol{R}^q)$ and $\varphi \in \mathcal{D}'(\boldsymbol{R}^q)$. This means that $[f]*[g]$ exists in $\mathcal{D}'$ and

$$\langle[f]*[g], \varphi\rangle_q = \langle f(x)g(y)\varphi(x+y), 1\rangle_{2q} = \langle f(g^- * \varphi), 1\rangle_q = \langle[f]*g, \varphi\rangle_q.$$

Similarly, if the $\bar{E}$-convolution $f*[g]$ exists in $\mathcal{D}'$, then the E-convolution $[f]*[g]$ exists in $\mathcal{D}'$ and $f*[g] = [f]*[g]$.

It remains to show that if the E-convolution $[f]*[g]$ exists, then also the E-convolutions $[f]*g$ and $f*[g]$ exist, but this is obvious, because

$$[f]*[g] = \lim_{i,j\to\infty} (\eta_i f)*(\tilde{\eta}_j g)$$

for any $\{\eta_n\}, \{\tilde{\eta}_n\} \in E$, where the limit is double.

Remark 1. Theorem 1 together with the result of P. Dierolf and J. Voigt ([3], (1.3) Theorem) shows that definitions (6)–(8) of the convolution in $\mathcal{D}'$ are equivalent to the definitions of C. Chevalley [2], L. Schwartz [13], R. Shiraishi [14], V. S. Vladimirov [16] and P. Dierolf–J. Voigt [3].

Applying (1.1) Proposition in [3] and Theorem 3 in [14] (see also (2.3) Theorem in [3]), we can obtain in a similar way as Theorem 1 the following result:

THEOREM 2. *Let $f, g \in \mathcal{S}'$. If one of limits* (6)–(8) *exists in $\mathcal{S}'$ for any $\{\eta_n\}, \{\tilde{\eta}_n\} \in \bar{E}$, then the all limits exist in $\mathcal{S}'$ for any $\{\eta_n\}, \{\tilde{\eta}_n\} \in E$ and* (15) *holds.*

In other words, E-convolutions and $\bar{E}$-convolutions (6)–(8) *in $\mathcal{S}'$ are all equivalent.*

Remark 2. Theorem 2 together with (2.3) Theorem in [3] give us the equivalence of the convolutions (6)–(8) in $\mathcal{S}'$ and the convolutions of C. Chevalley [2] (see also [4]), L. Schwartz [13], R. Shiraishi [14], V. S. Vladimirov [16] and P. Dierolf – J. Voigt [3], applied for tempered distributions.

In limits (6)–(8) in $\mathcal{S}'$ unit-sequences belonging to the classes E and $\bar{E}$ involve. However, it seems to be natural to consider in this case also the following unit-sequences:

A sequence $\eta_n(x)$, $x \in \boldsymbol{R}^q$ of smooth functions is said to be a *unit-sequence* (*a special unit-sequence*) *in $\mathcal{S}$* if $\eta_n \in \mathcal{S}$ for $n = 1, 2, \ldots$ and conditions (α) and (β) ((β) and (γ)) hold.

The classes of all unit-sequences in $\mathcal{S}$ will be denoted by E^s and $\bar{E}^s$, respectively.

Remark 3. By a little modification of the proof of (1.1) Proposition in [3], it can be shown that a distribution f is integrable in $\boldsymbol{R}^q$ if and only if the sequence $\langle f, \eta_n\rangle_q$ is convergent for every $\{\eta_n\} \in E^s$ or, equivalently, for every $\{\eta_n\} \in \bar{E}^s$ (see [8]). This implies that unit-sequences in (6)–(8) in Theorem 2 can be replaced equivalently by unit-sequences of the class E^s or the class $\bar{E}^s$.

This remark can be written as

COROLLARY 1. *Let $f, g \in \mathcal{S}'$. The E, $\bar{E}$, E^s and $\bar{E}^s$-convolutions $[f]*[g]$, $[f]*g$, $f*[g]$ in $\mathcal{S}'$ are all equivalent.*

4. Model sequences. The class Δ of delta-sequences contains obviously the sequences of the form:

$$\delta_n(x) = \beta_n^q \sigma(\beta_n x) \quad (x \in \boldsymbol{R}^q), \tag{16}$$

where

$$\sigma \in \mathcal{D}, \tag{17}$$

$\int \sigma = 1$ and $\beta_n \to \infty$.

Similarly, the class E of unit-sequences contains the sequences:

$$\eta_n(x) = \eta(x/\beta_n) \quad (x \in \boldsymbol{R}^q), \tag{18}$$

where

$$\eta \in \mathcal{D}, \tag{19}$$

$\eta(0) = 1$ and $\beta_n \to \infty$.

In general, operations defined by delta-sequences and unit-sequences of the form (16) and (18), respectively, can depend on the choice of the functions σ and η. Therefore we define the classes Δ_m and E_m of so-called *model delta-* and *unit-sequences*, which are composed of a finite number of sequences of the form (16) and (18), respectively (see [7]).

In the case where conditions (16) and (18) are replaced respectively by the conditions:

(20) $$\sigma \in \mathcal{S},$$

(21) $$\eta \in \mathcal{S},$$

we obtain the definitions of *model delta-* and *unit-sequences* in $\mathcal{S}$. The classes of such sequences are denoted by Δ_m^s and E_m^s, respectively.

Model sequences are convenient in calculations. In particular, the theorems on the exchange formulae for the Fourier transform of tempered distributions are easy to obtain. Namely, we have

THEOREM 3. *Let $f, g \in \mathcal{S}'$. If the E_m^s-convolution $[f*g]$ exists in $\mathcal{S}'$, then the Δ_m^s-product $[\mathcal{F}(f)\cdot\mathcal{F}(g)]$ exists in $\mathcal{S}'$ and*

$$\mathcal{F}([f*g]) = [\mathcal{F}(f)\cdot\mathcal{F}(g)].$$

If the Δ_m^s-product $[f\cdot g]$ exists in $\mathcal{S}'$, then the E_m^s-convolution $[\mathcal{F}(f)*\mathcal{F}(g)]$ exists in $\mathcal{S}'$ and

$$\mathcal{F}([f\cdot g]) = [\mathcal{F}(f)*\mathcal{F}(g)].$$

THEOREM 4. *Let $f, g \in \mathcal{S}'$. If the E_m^s-convolution $[f]*[g]$ exists in $\mathcal{S}'$, then the Δ_m^s-product $[\mathcal{F}(f)]\cdot[\mathcal{F}(g)]$ exists in $\mathcal{S}'$ and*

(22) $$\mathcal{F}([f]*[g]) = [\mathcal{F}(f)]\cdot[\mathcal{F}(g)].$$

If the Δ_m^s-product $[f]\cdot[g]$ exists in $\mathcal{S}'$, then the E_m^s-convolution $[\mathcal{F}(f)]*[\mathcal{F}(g)]$ exists in $\mathcal{S}'$ and

$$\mathcal{F}([f]\cdot[g]) = [\mathcal{F}(f)]*[\mathcal{F}(g)].$$

THEOREM 5. *Let $f, g \in \mathcal{S}'$. If the E_m^s-convolution $[f]*g$ exists in $\mathcal{S}'$, then the Δ_m^s-product $[\mathcal{F}(f)]\cdot\mathcal{F}(g)$ exists in $\mathcal{S}'$ and*

(23) $$\mathcal{F}([f]*g) = [\mathcal{F}(f)]\cdot\mathcal{F}(g).$$

If the Δ_m^s-product $[f]\cdot g$ exists in $\mathcal{S}'$, then the E_m^s-convolution $[\mathcal{F}(f)]*\mathcal{F}(g)$ exists in $\mathcal{S}'$ and

$$\mathcal{F}([f]\cdot g) = [\mathcal{F}(f)]*\mathcal{F}(g).$$

Proof. Theorems 3 and 4 are proved in [7] (cf. Theorems 1, 2 and the remark after them; in the exchange formulae in [7] constants $(2\sqrt{\pi})^q$ and $(2\sqrt{\pi})^{-q}$ appear, which is connected with a different definition of the Fourier transform adopted there). The proof of Theorem 5 is similar. Namely we have

(24) $$\mathcal{F}\big((\eta f)*g\big) = \mathcal{F}(\eta f)\cdot\mathcal{F}(g) = \big(\mathcal{F}(f)*\mathcal{F}(\eta)\big)\cdot\mathcal{F}(g)$$

and

(25) $$\mathcal{F}\big((f*\sigma)\cdot g\big) = \big(\mathcal{F}(\sigma)\mathcal{F}(f)\big)*\mathcal{F}(g)$$

for any $f, g \in \mathcal{S}'$ and $\eta \in \mathcal{S}$.

Equation (24) holds, because ηf is a rapidly decreasing distribution (see [6]) and (25) follows from (24). In fact, let $f_1 = \mathcal{F}(f)$, $g_1 = \mathcal{F}(g)$ and $\sigma_1 = \mathcal{F}(\sigma)$. Then (24) yields

$$\mathcal{F}\big((\sigma_1 f_1)*g_1\big) = \big(\mathcal{F}(f_1)*\mathcal{F}(\sigma_1)\big)\cdot\mathcal{F}(g_1) = \big((f*\sigma)g\big)^-,$$

where $h^-(x) = h(-x)$. Hence

$$\big(\mathcal{F}(\sigma)\cdot\mathcal{F}(f)\big)*\mathcal{F}(g) = (\sigma_1 f_1)*g_1 = \mathcal{F}\big((f*\sigma)g\big),$$

as desired.

From (24) and (25), we get

(26) $$\mathcal{F}\big((\eta_n f)*g\big) = \big(\mathcal{F}(f)*\delta_n\big)\cdot\mathcal{F}(g)$$

and

(27) $$\mathcal{F}\big((f*\delta_n)\cdot g\big) = \big(\eta_n\cdot\mathcal{F}(f)\big)*\mathcal{F}(g)$$

for any unit- and delta-sequences of the form (16) and (18) (with assumptions (20), (21), respectively), connected by the identities $\mathcal{F}^{-1}(\sigma) = \eta$ and $\mathcal{F}^{-1}(\eta) = \sigma$, respectively. Passing to the limit in (26) and (27), we obtain our assertion.

5. Product. Now we are going to show that definitions of the product of distributions by using various classes of delta-sequences are equivalent.

Beside the classes Δ, Δ_m and Δ_m^s, we shall also consider the class Δ^s of delta-sequences in $\mathcal{S}$.

We say that a sequence $\{\varphi_n\}$, $\varphi_n \in \mathcal{S}$ *tends to 0 in $\mathcal{S}$ in an open set* $Q \subset \mathbf{R}^q$ (and we write $\varphi_n \xrightarrow{\mathcal{S}} 0$ in Q) if $p\varphi_n^{(k)} \to 0$ in Q for every polynomial p and $k \in \mathbf{P}^q$.

A sequence $\{\delta_n\}$ of smooth functions is said to be a *delta-sequence in $\mathcal{S}$* if $\delta_n \in \mathcal{S}$ $(n = 1, 2, \ldots)$ and

(a′) $\delta_n \xrightarrow{\mathcal{S}} 0$ in $\mathbf{R}^q \setminus I$ for every closed interval I containing 0;

(b′) $\int \delta_n = 1$ $(n = 1, 2, \ldots)$;

(c′) for every $k \in \mathbf{P}^q$ there exists $M_k > 0$ such that

$$\int |x^k \delta_n^{(k)}(x)|\,dx < M_k \qquad (n = 1, 2, \ldots)$$

(cf. [6]).

LEMMA. *If $f \in \mathcal{D}'$ and*

(28) $$\lim_{n\to\infty}(f*\delta_n)(0)$$

exists for all $\{\delta_n\} \in \Delta_m$, then (28) exists for all $\{\delta_n\} \in \Delta$.

If $f \in \mathcal{S}'$ and (28) exists for all $\{\delta_n\} \in \Delta_m$, then (28) exists for all $\{\delta_n\} \in \Delta^s$.

Proof. We shall prove only the second part of Lemma. The first part can be shown analogously (see the proof of Theorem 12.2.1 and Remark in [1], pp. 240–241).

Suppose that $f \in \mathscr{S}'$, i.e. $f = F^{(k)}$ for some $k \in \boldsymbol{P}^q$ and continuous function F, bounded by a polynomial, and let (28) exist for all $\{\delta_n\} \in \Delta_m$. That means

$$\lim_{n\to\infty} (f * \delta_n^0)(0) = c \tag{29}$$

for some constant c and all $\{\delta_n^0\}$ of the form (16) with assumption (17).

We have

$$(f*\delta_n^0)(0) - c = \langle f(-\beta_n^{-1}\cdot t) - c, \sigma(t)\rangle$$

and thus (29) implies that

$$f(at) - c \to 0 \tag{30}$$

in $\mathscr{D}'$ as $a \to 0$, i.e. f has the value c at 0 (see [9]; [1], p. 38).

By the Łojasiewicz theorem (see [9] or [1], p. 40), there exist a continuous function G and $l \in \boldsymbol{P}^q$ such that $G^{(l)} = f$ in some neighbourhood of 0 and

$$\lim_{x\to 0} x^{-l} G(x) = c\cdot (l!)^{-1}. \tag{31}$$

Of course, we can assume that $k = l$ and thus the function G is bounded by some polynomial.

In view of (31), for any $\varepsilon > 0$ there is a number $\gamma > 0$ such that

$$|x^{-k}G(x) - c(k!)^{-1}| < \varepsilon \quad \text{for} \quad |x| < \gamma.$$

If $\{\delta_n\} \in \Delta^s$, then

$$|(f*\delta_n)(0) - c| \leqslant \int |G(-x) - c(-x)^k (k!)^{-1}|\cdot|\delta_n^{(k)}(x)|\,dx$$

$$\leqslant \varepsilon\cdot \int_{-\gamma}^{\gamma} |x|^k |\delta_n^{(k)}(x)|\,dx + \varepsilon \leqslant \varepsilon(M_k + 1)$$

for sufficiently large n, in view of condition (a′).

This means that $(f*\delta_n)(0) \to c$ for any delta-sequence $\{\delta_n\} \in \Delta^s$ and the proof is complete.

THEOREM 6. *Let $f, g \in \mathscr{D}'$. The Δ-products and Δ_m-products $[f]\cdot[g]$, $[f]\cdot g$, $f\cdot[g]$ are equivalent in $\mathscr{D}'$.*

Proof. If one of limits (2)–(4) exists for all delta-sequences of the class Δ, then all limits exist for delta-sequences of the classes Δ, Δ_m and (13) holds, by the theorem of Shiraishi and Itano [15] (see Section 2).

Now suppose that limit (3) exists in $\mathscr{D}'$ for all $\{\delta_n\} \in \Delta_m$. Since

$$\langle (f*\delta_n) g, \varphi\rangle = \langle \delta_n, f^- * (g\varphi)\rangle = \big((f*(g\varphi)^-) * \delta_n\big)(0)$$

for any $\varphi \in \mathscr{D}$, limit (3) exists for all $\{\delta_n\} \in \Delta$, in view of the first part of Lemma. Now, the existence of the remain limits and (13) follow from the previous remark.

The proof is similar if (4) exists in $\mathscr{D}'$ for all $\{\delta_n\} \in \Delta_m$.

Finally, assume that limit (2) exists for $\{\delta_n\}, \{\tilde\delta_n\} \in \Delta_m$. Then the double limit $\lim_{i,j\to\infty} (f*\delta_i)\cdot(g*\tilde\delta_j)$ exists in $\mathscr{D}'$. Consequently, $\lim_{i\to\infty}(f*\delta_i)g$ exists in $\mathscr{D}'$ for any $\{\delta_i\} \in \Delta_m$, which reduces the question to the case considered earlier.

COROLLARY 2. *Let $f, g \in \mathscr{D}'$. If one of limits (2)–(4) exists in $\mathscr{D}'$ for any $\{\delta_n\}, \{\tilde\delta_n\} \in \Delta_m$, then the other limits also exist in $\mathscr{D}'$ for any $\{\delta_n\}, \{\tilde\delta_n\} \in \Delta_m$ and (13) holds.*

THEOREM 7. *Let $f, g \in \mathscr{S}'$. The products $[f]\cdot g$ in $\mathscr{S}'$ defined by the classes Δ_m, Δ_m^s, Δ and Δ^s, respectively, are equivalent.*

Proof. It suffices to prove that if limit (3) exists in $\mathscr{S}'$ for delta-sequences $\{\delta_n\} \in \Delta_m$, then this limit exists for all $\{\delta_n\} \in \Delta^s$. But this follows from the identity

$$\langle (f*\delta_n) g, \varphi\rangle = \big((f*(g\varphi)^-) * \delta_n\big)(0),$$

by virtue of the second part of Lemma.

In [8] it is proved the following result:

THEOREM 8. *Let $f, g \in \mathscr{S}'$. The products $[f]\cdot[g]$ in $\mathscr{S}'$ defined by the classes Δ_m and Δ are equivalent.*

6. The exchange formulae. Theorems 7 and 8 allow to formulate the first parts of Theorems 4 and 5 on the exchange formulae in the following way:

THEOREM 9. *Let $f, g \in \mathscr{S}'$. If the E_m^s-convolution $[f]*[g]$ exists in $\mathscr{S}'$, then the Δ-product $[\mathscr{F}(f)]\cdot[\mathscr{F}(g)]$ exists in $\mathscr{S}'$ and (22) holds.*

THEOREM 10. *Let $f, g \in \mathscr{S}'$. If the E_m^s-convolution $[f]*g$ exists in $\mathscr{S}'$, then the Δ^s-product $[\mathscr{F}(f)]\cdot\mathscr{F}(g)$ exists in $\mathscr{S}'$ and (23) holds.*

In [4] and [15] it has been proved that if $f, g \in \mathscr{S}'$ and $\mathscr{S}'$-convolution $f*g$ (in the sense of C. Chevalley [2] and Y. Hirata–H. Ogata [4]) exists, then the Δ-product $[\mathscr{F}(f)]\cdot\mathscr{F}(g) = \mathscr{F}(f)\cdot[\mathscr{F}(g)] = [\mathscr{F}(f)]\cdot[\mathscr{F}(g)]$ exists in $\mathscr{D}'$ ([4] and [15]) and the Δ-product $[\mathscr{F}(f)]\cdot[\mathscr{F}(g)]$ exists in $\mathscr{S}'$ ([15]) and the respective formula holds. Since the definition of the convolution $f*g$ is equivalent to the E^s-convolutions $[f]*g = f*[g] = [f]*[g]$ in $\mathscr{S}'$ (cf. Theorem 2 and Remark 3), we see that Theorems 9 and 10 are generalizations of the above result.

These generalizations are essential as the below example shows. The example shows, too, that E_m-convolutions (5)–(8) are essentially more general than E-convolutions (5)–(8) (in $\mathscr{D}'$ and in $\mathscr{S}'$) and that the second parts of Theorems 3, 4 and 5 are not true, in general. In particular, this gives the negative answer for one of the two problems posed by M. Itano in [5] (p. 65). The second one remains open.

EXAMPLE. Let

$$\theta(t) = \begin{cases} 1 & \text{for} \quad |x| \leqslant 1/2 \\ 0 & \text{for} \quad |x| > 1/2 \end{cases} \quad (x \in \boldsymbol{R}).$$

It is easy to see that the Δ^s-products $[\theta\cdot\delta]$, $[\theta]\cdot[\delta]$, $[\theta]\cdot\delta$, $\theta\cdot[\delta]$ exist in $\mathscr{S}'$ and

$$[\theta\cdot\delta] = [\theta]\cdot[\delta] = [\theta]\cdot\delta = \theta\cdot[\delta] = \delta.$$

Note that

$$\mathscr{F}(\delta) = 1, \quad \mathscr{F}(\theta) = h,$$

where $h(x) = \sin\pi x/\pi x$ $(x \in \boldsymbol{R})$.

In [3] it is noticed that the distribution h is not integrable, i.e. the convolution $1*[h]$ does not exist in $\mathscr{D}'$ (with respect to the class $\bar{E}$). We shall show that even the convolution $[1*h]$ does not exist in $\mathscr{D}'$ (with respect to $\bar{E}$). Let p_n $(n = 1, 2, \ldots)$ be integers such that

$$\frac{1}{2n+1} + \frac{1}{2n+3} + \ldots + \frac{1}{2p_n+1} \to \infty. \tag{32}$$

For any set $A \subset \boldsymbol{R}$, we denote by $A_{-\varepsilon}$ $(\varepsilon > 0)$ the set

$$A_{-\varepsilon} = \{x\colon x+y \in A \text{ for every } y,\ |y| < \varepsilon\},$$

Let $A^n = B^n \cup C^n$, where

$$B^n = \{x\colon |x| < 2n-1\}$$

and

$$C^n = \bigcup_{i=n}^{p_n} \{x\colon 2i < |x| < 2i+1\}.$$

Let ε be fixed positive number less than 1/16. Note that

$$\int_{B^n_{-8\varepsilon}} h(x)\,dx \geqslant 0. \tag{33}$$

In fact, this is obvious that for $n = 1$ and for $n \geqslant 2$ we have

$$\int_{B^n_{-8\varepsilon}} h(x)\,dx \geqslant 2\int_0^{2(n-1)} \frac{\sin\pi x}{\pi x}\,dx = \frac{2}{\pi}\sum_{k=0}^{2n-3}(-1)^k\int_0^{\pi}\frac{\sin y}{y+k\pi}\,dy \geqslant 0.$$

Take an arbitrary non-negative smooth and even function φ such that

$$\varphi(x) = \begin{cases} 1 & \text{for} \quad |x| \leqslant \varepsilon, \\ 0 & \text{for} \quad |x| > 2\varepsilon \end{cases}$$

and define a sequence of functions $\theta_n(x)$ as follows:

$$\theta_n(x) = \begin{cases} 1 & \text{for} \quad x \in A^n_{-4\varepsilon}, \\ 0 & \text{otherwise.} \end{cases}$$

Putting

$$\eta_n = \theta_n * \varphi,$$

we see that $\{\eta_n\}$ is a special unit-sequence. Moreover,

$$\eta_n(x) = (\eta_n*\varphi)(x) = 1 \quad \text{for} \quad x \in A^n_{-8\varepsilon},$$

$$\eta_n(x) = (\eta_n*\varphi)(x) = 0 \quad \text{for} \quad x \notin A^n$$

and

$$\eta_n(x) \geqslant 0, \quad (\eta^n*\varphi)(x) \geqslant 0 \quad \text{for} \quad x \in A^n \setminus A^n_{-8\varepsilon}.$$

Since $h(x) \geqslant 0$ for $x \in C^n$ and $h(x) \geqslant 0$ for $x \in A^n\setminus A^n_{-8\varepsilon}$, we have

$$\langle \eta_n*(\eta_n h), \varphi\rangle = \langle h, (\eta_n*\varphi)\cdot\eta_n\rangle$$

$$\geqslant \int_{A^n_{-8\varepsilon}} h(x)\,dx \geqslant \frac{2}{\pi}\sum_{k=n}^{p_n}\int_{8\varepsilon}^{\pi-8\varepsilon}\frac{\sin x}{x+2k\pi}\,dx,$$

in view of (33). Hence

$$\langle \eta_n*(\eta_n h), \varphi\rangle \geqslant \frac{2\sin 8\varepsilon}{\pi^2}\sum_{k=n}^{p_n}\frac{1}{2k+1},$$

i.e.

$$\langle \eta_n*(\eta_n h), \varphi\rangle \to \infty,$$

according to (32). Consequently, the convolution $[1*h]$ does not exist in $\mathscr{D}'$ with respect to the class $\bar{E}$, which was announced.

7. Problems. Finally, we would like to present a list of some of unsolved problems.

PROBLEM 1. Let $f, g \in \mathscr{S}'$ and let the Δ-product (or, equivalently, Δ_m-product) $[f]\cdot g$ exists in $\mathscr{S}'$. Does exist then in $\mathscr{S}'$ the Δ-product (or, equivalently, Δ_m-product) $[f]\cdot[g]$?

PROBLEM 2. Let

(a) $f, g \in \mathscr{D}'$ and the E_m-convolution $[f]*g$ exist in $\mathscr{D}'$,
(b) $f, g \in \mathscr{S}'$ and the E^s_m-convolution $[f]*g$ exist in $\mathscr{S}'$.

Does exist then

(a) the E_m-convolution $[f]*[g]$ in $\mathscr{D}'$,
(b) the E^s_m-convolution $[f]*[g]$ in $\mathscr{S}'$?

Problem 3. Let
(a) $f, g \in \mathscr{D}'$ and the Δ_m-product $[f\cdot g]$ exist in $\mathscr{D}'$,
(b) $f, g \in \mathscr{S}'$ and the Δ_m-product $[f\cdot g]$ exist in $\mathscr{S}'$.
Does exist then
(a) the Δ-product $[f\cdot g]$ in $\mathscr{D}'$,
(b) the Δ^s-product (or Δ-product, or Δ_m^s-product) $[f\cdot g]$ in $\mathscr{S}'$?

Problem 4. Let $f, g \in \mathscr{S}'$ and let the E_m^s-convolution $[f*g]$ exist in $\mathscr{S}'$. Does exist then the Δ^s-product (or Δ-product) $[\mathscr{F}(f)\cdot\mathscr{F}(g)]$ in $\mathscr{S}'$ or in $\mathscr{D}'$?

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Non-removable ideals in commutative Banach algebras

by

V. MÜLLER (Praha)

Abstract. We show that an ideal I in a commutative Banach algebra with unit is non-removable if and only if it consists of joint topological divisors of zero. This gives the positive answer to the conjecture of Arens and Żelazko. From this it follows also that any finite family of removable ideals is removable.

Introduction. All algebras considered in this paper are assumed to be commutative complex Banach algebras with unit. However, some of these properties (complexity and completeness) are not essential.

We say that an ideal I in a commutative Banach algebra A is *removable* if there exists a superalgebra $B \supset A$ (i.e. B is a commutative Banach algebra and there is an isometric isomorphism $f: A \to B$ preserving the unit) such that I is not contained in a proper ideal in B. A family $\{I_j\}_{j\in J}$ of ideals in A is called *removable* if there is a superalgebra $B \supset A$ such that, for each $j \in J$, I_j is not contained in a proper ideal in B. An ideal which is not removable is said to be *non-removable*.

These notions were introduced by Arens in [1] where the following question was also presented: Is every (every finite) family of removable ideals removable?

Removability of ideals was further studied by Arens [2], Żelazko [8], [9], [10] and Bollobás [3]. Bollobás exhibited an example of a non-countable family of removable ideals which is not removable.

W. Żelazko introduced the following definition: We say that an ideal $I \subset A$ *consists of joint topological divisors of zero* if $\inf_{\substack{x\in A\\ \|x\|=1}} \sum_{i=1}^{n} |s_i x| = 0$ for every finite family $s_1, \ldots, s_n \in I$. We denote this shortly $I \in \mathfrak{l}(A)$.

It is easy to see that if $I \in \mathfrak{l}(A)$ then it is non-removable. Żelazko [7], [9] conjectured that the converse statement is also true, i.e. that I is non-removable if and only if $I \in \mathfrak{l}(A)$. However, the same question was presented (in an equivalent formulation) in the original paper of Arens [1].

The answer has been known in some special cases. In the case of principal ideals the conjecture turns into the theorem of Arens: An element (in a commutative Banach algebra) is permanently singular if and only