

Research Article

Convolution Properties for Certain Classes of Analytic Functions Defined by q -Derivative Operator

T. M. Seoudy^{1,2} and M. K. Aouf³

¹ Department of Mathematics, Faculty of Science, Fayoum University, Fayoum 63514, Egypt

² The University College at Al-Jamoom, Umm Al-Qura University, Makkah 21421, Saudi Arabia

³ Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

Correspondence should be addressed to T. M. Seoudy; tms00@fayoum.edu.eg

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We investigate convolution properties and coefficients estimates for two classes of analytic functions involving the q -derivative operator defined in the open unit disc. Some of our results improve previously known results.

1. Introduction

Simply, h -calculus or q -calculus is ordinary classical calculus without the notion of limits. Here h ostensibly stands for Planck's constant, while q stands for quantum. Recently, the area of q -calculus has attracted the serious attention of researchers. This great interest is due to its application in various branches of mathematics and physics. The application of q -calculus was initiated by Jackson [1, 2]. He was the first to develop q -integral and q -derivative in a systematic way. Later, geometrical interpretation of q -analysis has been recognized through studies on quantum groups. It also suggests a relation between integrable systems and q -analysis. Aral and Gupta [3–5] defined and studied the q -analogue of Baskakov Durrmeyer operator which is based on q -analogue of beta function. Another important q -generalization of complex operators is q -Picard and q -Gauss-Weierstrass singular integral operators discussed in [6–8]. Mohammed and Darus [9] studied approximation and geometric properties of these q -operators in some subclasses of analytic functions in compact disk. These q -operators are defined by using convolution of normalized analytic functions and q -hypergeometric functions, where several interesting results are obtained (see also [10, 11]). A comprehensive study on applications of q -calculus in operator theory may be found in [12].

Let \mathcal{A} denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (1)$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}(\alpha)$ and $\mathcal{K}(\alpha)$ ($0 \leq \alpha < 1$) denote the subclasses of \mathcal{A} that consists, respectively, of starlike of order α and convex of order α in \mathbb{U} (see [13]). If $f(z)$ and $g(z)$ are analytic in \mathbb{U} , we say that $f(z)$ is subordinate to $g(z)$, written $f(z) \prec g(z)$ if there exists a Schwarz function ω , which (by definition) is analytic in \mathbb{U} with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathbb{U}$, such that $f(z) = g(\omega(z))$, $z \in \mathbb{U}$. Furthermore, if the function g is univalent in \mathbb{U} , then we have the following equivalence (see [14–16]):

$$f(z) \prec g(z) \iff f(0) = g(0), f(\mathbb{U}) \subset g(\mathbb{U}). \quad (2)$$

For functions f given by (1) and g given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \quad (3)$$

the Hadamard product or convolution of f and g is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \quad (4)$$

Let $\mathcal{S}[A, B]$ and $\mathcal{K}[A, B]$ denote the subclasses of the class \mathcal{A} for $-1 \leq B < A \leq 1$ which are defined by (see [17–22])

$$\begin{aligned} \mathcal{S}[A, B] &= \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} < \frac{1 + Az}{1 + Bz}, z \in \mathbb{U} \right\}, \\ \mathcal{K}[A, B] &= \left\{ f \in \mathcal{A} : \frac{(zf'(z))'}{f'(z)} < \frac{1 + Az}{1 + Bz}, z \in \mathbb{U} \right\}. \end{aligned} \quad (5)$$

We note that

$$\mathcal{S}[1 - 2\alpha, -1] = \mathcal{S}(\alpha), \quad \mathcal{K}[1 - 2\alpha, -1] = \mathcal{K}(\alpha) \quad (6)$$

$(0 \leq \alpha < 1).$

For function $f \in \mathcal{A}$ given by (1) and $0 < q < 1$, the q -derivative of a function f is defined by (see [1])

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z} \quad (z \neq 0), \quad (7)$$

and $D_q f(0) = f'(0)$. From (7), we deduce that

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}, \quad z \neq 0, \quad (8)$$

where

$$[k]_q = \frac{1 - q^k}{1 - q}. \quad (9)$$

As $q \rightarrow 1$, $[k]_q \rightarrow k$. For a function $h(z) = z^k$, we observe that

$$\begin{aligned} D_q h(z) &= D_q(z^k) = \frac{1 - q^k}{1 - q} z^{k-1} = [k]_q z^{k-1}, \\ \lim_{q \rightarrow 1} D_q h(z) &= \lim_{q \rightarrow 1} [k]_q z^{k-1} = kz^{k-1} = h'(z), \end{aligned} \quad (10)$$

where h' is the ordinary derivative.

Making use of the q -derivative $D_q f(z)$, we introduce the subclasses $\mathcal{S}_q[A, B]$ and $\mathcal{K}_q[A, B]$ of \mathcal{A} for $0 < q < 1$ and $-1 \leq B < A \leq 1$ as follows:

$$\begin{aligned} \mathcal{S}_q[A, B] &= \left\{ f \in \mathcal{A} : \frac{zD_q f(z)}{f(z)} < \frac{1 + Az}{1 + Bz}, z \in \mathbb{U} \right\}, \\ \mathcal{K}_q[A, B] &= \left\{ f \in \mathcal{A} : \frac{D_q(zD_q f(z))}{D_q f(z)} < \frac{1 + Az}{1 + Bz}, z \in \mathbb{U} \right\}. \end{aligned} \quad (11)$$

We note that

$$\begin{aligned} \text{(i)} \quad \mathcal{S}_q[1 - 2\alpha, -1] &= \mathcal{S}_q(\alpha) \quad (0 \leq \alpha < 1) \\ \mathcal{S}_q(\alpha) &= \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{zD_q f(z)}{f(z)} > \alpha, z \in \mathbb{U} \right\}; \end{aligned} \quad (12)$$

$$\begin{aligned} \text{(ii)} \quad \mathcal{K}_q[1 - 2\alpha, -1] &= \mathcal{K}_q(\alpha) \quad (0 \leq \alpha < 1) \\ \mathcal{K}_q(\alpha) &= \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{D_q(zD_q f(z))}{D_q f(z)} > \alpha, z \in \mathbb{U} \right\}; \end{aligned} \quad (13)$$

$$\begin{aligned} \text{(iii)} \quad \mathcal{S}_q[(1 - 2\alpha)\beta, -\beta] &= \mathcal{S}_q(\alpha, \beta) \quad (0 \leq \alpha < 1, 0 < \beta \leq 1) \\ \mathcal{S}_q(\alpha, \beta) &= \left\{ f \in \mathcal{A} : \left| \frac{(zD_q f(z)/f(z)) - 1}{(zD_q f(z)/f(z)) + 1 - 2\alpha} \right| < \beta, z \in \mathbb{U} \right\}, \end{aligned} \quad (14)$$

$$\begin{aligned} \text{(iv)} \quad \mathcal{K}_q[(1 - 2\alpha)\beta, -\beta] &= \mathcal{K}_q(\alpha, \beta) \quad (0 \leq \alpha < 1, 0 < \beta \leq 1) \\ \mathcal{K}_q(\alpha, \beta) &= \left\{ f \in \mathcal{A} : \left| \frac{(D_q(zD_q f(z))/D_q f(z)) - 1}{(D_q(zD_q f(z))/D_q f(z)) + 1 - 2\alpha} \right| < \beta, \right. \\ &\quad \left. z \in \mathbb{U} \right\}, \end{aligned} \quad (15)$$

$$\begin{aligned} \text{(v)} \quad \lim_{q \rightarrow 1} \mathcal{S}_q[A, B] &= \left\{ f \in \mathcal{A} : \lim_{q \rightarrow 1} \frac{zD_q f(z)}{f(z)} < \frac{1 + Az}{1 + Bz} \right\} \\ &= \mathcal{S}[A, B], \\ \lim_{q \rightarrow 1} \mathcal{K}_q[A, B] &= \left\{ f \in \mathcal{A} : \lim_{q \rightarrow 1} \frac{D_q(zD_q f(z))}{D_q f(z)} < \frac{1 + Az}{1 + Bz} \right\} \\ &= \mathcal{K}[A, B]. \end{aligned} \quad (16)$$

From (11), we have

$$f \in \mathcal{K}_q[A, B] \iff zD_q f \in \mathcal{S}_q[A, B]. \quad (17)$$

In this paper, we investigate convolution properties, the necessary and sufficient condition and coefficient estimates for the classes $\mathcal{S}_q[A, B]$ and $\mathcal{K}_q[A, B]$ associated with the q -derivative $D_q f(z)$. The motivation of this paper is to improve and generalize previously known results.

2. Convolution Properties

Unless otherwise mentioned, we assume throughout this section that $\theta \in [0, 2\pi)$, $0 < q < 1$ and $-1 \leq B < A \leq 1$.

Theorem 1. *The function f defined by (1) is in the class $\mathcal{S}_q[A, B]$ if and only if*

$$\frac{1}{z} \left[f(z) * \frac{z - Lqz^2}{(1-z)(1-qz)} \right] \neq 0 \quad (z \in \mathbb{U}) \quad (18)$$

for all $L = L_\theta = (e^{-i\theta} + A)/(A - B)$ and also $L = 1$.

Proof. First suppose f defined by (1) is in the class $\mathcal{S}_q[A, B]$; we have

$$\frac{zD_q f(z)}{f(z)} < \frac{1 + Az}{1 + Bz}. \quad (19)$$

Since the function from the left-hand side of the subordination is analytic in \mathbb{U} , it follows $f(z) \neq 0, z \in \mathbb{U}^* = \mathbb{U} \setminus \{0\}$; that is, $(1/z)f(z) \neq 0, z \in \mathbb{U}$, and this is equivalent to the fact that (18) holds for $L = 1$. From (19) according to the subordination of two analytic functions we say that there exists a function $w(z)$ analytic in \mathbb{U} with $w(0) = 0, |w(z)| < 1$ such that

$$\frac{zD_q f(z)}{f(z)} = \frac{1 + Aw(z)}{1 + Bw(z)} \quad (z \in \mathbb{U}) \quad (20)$$

which is equivalent to

$$\frac{zD_q f(z)}{f(z)} \neq \frac{1 + Ae^{i\theta}}{1 + Be^{i\theta}} \quad (z \in \mathbb{U}; 0 \leq \theta < 2\pi), \quad (21)$$

or

$$\frac{1}{z} \left[(1 + Be^{i\theta})zD_q f(z) - (1 + Ae^{i\theta})f(z) \right] \neq 0 \quad (z \in \mathbb{U}; 0 \leq \theta < 2\pi). \quad (22)$$

Since

$$\begin{aligned} f(z) * \frac{z}{1-z} &= f(z), \\ f(z) * \frac{z}{(1-z)(1-qz)} &= zD_q f(z). \end{aligned} \quad (23)$$

Now from (23), we may write (22) as

$$\begin{aligned} &\frac{1}{z} \left[f(z) * \left(\frac{(1 + Be^{i\theta})z}{(1-z)(1-qz)} - \frac{(1 + Ae^{i\theta})z}{1-z} \right) \right] \\ &= \frac{(B - A)e^{i\theta}}{z} \\ &\times \left[f(z) * \frac{z - ((e^{-i\theta} + A)/(A - B))qz^2}{(1-z)(1-qz)} \right] \neq 0 \quad (z \in \mathbb{U}; 0 \leq \theta < 2\pi), \end{aligned} \quad (24)$$

which leads to (18), which proves the necessary part of Theorem 1.

Reversely, because assumption (18) holds for $L = 1$, it follows that $(1/z)f(z) \neq 0$ for all $z \in \mathbb{U}$; hence, the function $\varphi(z) = zD_q f(z)/f(z)$ is analytic in \mathbb{U} (i.e., it is regular at $z_0 = 0$, with $\varphi(0) = 0$). Since it was shown in the first part of the proof that assumption (18) is equivalent to (21), we obtain that

$$\frac{zD_q f(z)}{f(z)} \neq \frac{1 + Ae^{i\theta}}{1 + Be^{i\theta}} \quad (z \in \mathbb{U}; 0 \leq \theta < 2\pi), \quad (25)$$

and if we denote

$$\psi(z) = \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}), \quad (26)$$

relation (25) shows that $\varphi(\mathbb{U}) \cap \psi(\mathbb{U}) = \emptyset$. Thus, the simply connected domain $\varphi(\mathbb{U})$ is included in a connected component of $\mathbb{C} \setminus \psi(\partial\mathbb{U})$. From here, using the fact that $\varphi(0) = \psi(0)$ together with the univalence of the function ψ , it follows that $\varphi(z) < \psi(z)$, which represents in fact subordination (19); that is, $f \in \mathcal{S}_q[A, B]$. This completes the proof of Theorem 1. \square

Taking $q \rightarrow 1^-$ in Theorem 1, we obtain the following result which improves the convolution result of Aouf and Seoudy [23, Theorem 1] and also the result of Silverman and Silvia [21, Theorem 7].

Corollary 2. *The function f defined by (1) is in the class $\mathcal{S}[A, B]$ if and only if*

$$\frac{1}{z} \left[f(z) * \frac{z - Lz^2}{(1-z)^2} \right] \neq 0 \quad (z \in \mathbb{U}) \quad (27)$$

for all $L = L_\theta = (e^{-i\theta} + A)/(A - B)$ and also $L = 1$.

Putting $A = 1 - 2\alpha$ ($0 \leq \alpha < 1$) and $B = -1$ in Theorem 1, we obtain the following corollary.

Corollary 3. *The function f defined by (1) is in the class $\mathcal{S}_q(\alpha)$ ($0 \leq \alpha < 1$) if and only if*

$$\frac{1}{z} \left[f(z) * \frac{z - Mqz^2}{(1-z)(1-qz)} \right] \neq 0 \quad (z \in \mathbb{U}) \quad (28)$$

for all $M = M_\theta = (e^{-i\theta} + 1 - 2\alpha)/2(1 - \alpha)$, $0 \leq \alpha < 1$, and also $M = 1$.

Taking $q \rightarrow 1^-$ in Corollary 3, we obtain the following result which improves the convolution result of Silverman et al. [22, Theorems 1].

Corollary 4. *The function f defined by (1) is in the class $\mathcal{S}(\alpha)$ ($0 \leq \alpha < 1$) if and only if*

$$\frac{1}{z} \left[f(z) * \frac{z - Mz^2}{(1-z)^2} \right] \neq 0 \quad (z \in \mathbb{U}) \quad (29)$$

for all $M = M_\theta = (e^{-i\theta} + 1 - 2\alpha)/2(1 - \alpha)$, $0 \leq \alpha < 1$, and also $M = 1$.

Theorem 5. The function f defined by (1) is in the class $\mathcal{K}_q[A, B]$ if and only if

$$\frac{1}{z} \left[f(z) * \frac{z + [1 - (q + 1)L]qz^2}{(1 - z)(1 - qz)(1 - q^2z)} \right] \neq 0 \quad (z \in \mathbb{U}) \quad (30)$$

for all $L = L_\theta = (e^{-i\theta} + A)/(A - B)$ and also $L = 1$.

Proof. Set

$$g(z) = \frac{z - Lqz^2}{(1 - z)(1 - qz)}, \quad (31)$$

and we note that

$$zD_q g(z) = \frac{z + [1 - (q + 1)L]qz^2}{(1 - z)(1 - qz)(1 - q^2z)}. \quad (32)$$

From the identity $zD_q f(z) * g(z) = f(z) * zD_q g(z)$ ($f, g \in \mathcal{A}$) and the fact that

$$f \in \mathcal{K}_q[A, B] \iff zD_q f(z) \in \mathcal{S}_q[A, B] \quad (33)$$

the result follows from Theorem 1. \square

Taking $q \rightarrow 1^-$ in Theorem 1, we obtain the following result which improves the result of Aouf and Seoudy [23, Theorem 2].

Corollary 6. The function f defined by (1) is in the class $\mathcal{K}[A, B]$ if and only if

$$\frac{1}{z} \left[f(z) * \frac{z + [1 - 2L]z^2}{(1 - z)^3} \right] \neq 0 \quad (z \in \mathbb{U}) \quad (34)$$

for all $L = L_\theta = (e^{-i\theta} + A)/(A - B)$ and also $L = 1$.

Putting $A = 1 - 2\alpha$ ($0 \leq \alpha < 1$) and $B = -1$ in Theorem 5, we obtain the following corollary.

Corollary 7. The function f defined by (1) is in the class $\mathcal{K}_q(\alpha)$ ($0 \leq \alpha < 1$) if and only if

$$\frac{1}{z} \left[f(z) * \frac{z + [1 - (q + 1)L]qz^2}{(1 - z)(1 - qz)(1 - q^2z)} \right] \neq 0 \quad (z \in \mathbb{U}) \quad (35)$$

for all $M = M_\theta = (e^{-i\theta} + 1 - 2\alpha)/2(1 - \alpha)$, $0 \leq \alpha < 1$, and also $L = 1$.

Taking $q \rightarrow 1^-$ in Corollary 7, we obtain the following result which improves the convolution result of Silverman et al. [22, Theorem 2].

Corollary 8. The function f defined by (1) is in the class $\mathcal{K}(\alpha)$ ($0 \leq \alpha < 1$) if and only if

$$\frac{1}{z} \left[f(z) * \frac{z + [1 - 2L]qz^2}{(1 - z)^3} \right] \neq 0 \quad (z \in \mathbb{U}) \quad (36)$$

for all $M = M_\theta = (e^{-i\theta} + 1 - 2\alpha)/2(1 - \alpha)$, $0 \leq \alpha < 1$, and also $L = 1$.

Theorem 9. A necessary and sufficient condition for the function f defined by (1) to be in the class $\mathcal{S}_q[A, B]$ is that

$$1 - \sum_{k=2}^{\infty} \frac{[k]_q (e^{-i\theta} + B) - e^{-i\theta} - A}{A - B} a_k z^{k-1} \neq 0 \quad (37)$$

($z \in \mathbb{U}$).

Proof. From Theorem 1, we find that $f \in \mathcal{S}_q[A, B]$ if and only if

$$\frac{1}{z} \left[f(z) * \frac{z - Lqz^2}{(1 - z)(1 - qz)} \right] \neq 0 \quad (z \in \mathbb{U}) \quad (38)$$

for all $L = L_\theta = (e^{-i\theta} + A)/(A - B)$ and also for $L = 1$. The left-hand side of (38) can be written as

$$\begin{aligned} & \frac{1}{z} \left[f(z) * \left(\frac{z}{(1 - z)(1 - qz)} - \frac{Lqz^2}{(1 - z)(1 - qz)} \right) \right] \\ &= \frac{1}{z} \left\{ zD_q f(z) - L \left[zD_q f(z) - f(z) \right] \right\} \\ &= 1 - \sum_{k=2}^{\infty} ([k]_q (L - 1) - L) a_k z^{k-1}. \end{aligned} \quad (39)$$

Thus, the proof of Theorem 9 is completed. \square

Taking $q \rightarrow 1^-$ in Theorem 9, we obtain the following result.

Corollary 10. A necessary and sufficient condition for the function f defined by (1) to be in the class $\mathcal{S}[A, B]$ is that

$$1 - \sum_{k=2}^{\infty} \frac{k (e^{-i\theta} + B) - e^{-i\theta} - A}{A - B} a_k z^{k-1} \neq 0 \quad (z \in \mathbb{U}). \quad (40)$$

Putting $A = 1 - 2\alpha$ ($0 \leq \alpha < 1$) and $B = -1$ in Theorem 9, we obtain the following corollary.

Corollary 11. A necessary and sufficient condition for the function f defined by (1) to be in the class $\mathcal{S}_q(\alpha)$ is that

$$1 - \sum_{k=2}^{\infty} \frac{[k]_q (e^{-i\theta} - 1) - e^{-i\theta} - 1 + 2\alpha}{2(1 - \alpha)} a_k z^{k-1} \neq 0 \quad (z \in \mathbb{U}). \quad (41)$$

Taking $q \rightarrow 1^-$ in Corollary 11, we obtain the following corollary which improves the result of Ahuja [17, Corollary 1 when $n = 0$].

Corollary 12. A necessary and sufficient condition for the function f defined by (1) to be in the class $\mathcal{S}(\alpha)$ is that

$$1 - \sum_{k=2}^{\infty} \frac{k (e^{-i\theta} - 1) - e^{-i\theta} - 1 + 2\alpha}{2(1 - \alpha)} a_k z^{k-1} \neq 0 \quad (z \in \mathbb{U}). \quad (42)$$

Theorem 13. A necessary and sufficient condition for the function $f(z)$ defined by (1) to be in the class $\mathcal{K}_q[A, B]$ is that

$$1 - \sum_{k=2}^{\infty} [k]_q \frac{[k]_q (e^{-i\theta} + B) - e^{-i\theta} - A}{A - B} a_k z^{k-1} \neq 0 \quad (z \in \mathbb{U}). \tag{43}$$

Proof. From Theorem 5, we find that $f \in \mathcal{K}_q[A, B]$ if and only if

$$\frac{1}{z} \left\{ f(z) * \frac{z + [1 - (q + 1)L] qz^2}{(1 - z)(1 - qz)(1 - q^2z)} \right\} \neq 0 \quad (z \in \mathbb{U}), \tag{44}$$

for all $L = L_\theta = (e^{-i\theta} + A)/(A - B)$ and also for $L = 1$. The left-hand side of (44) can be written as

$$\begin{aligned} & \frac{1}{z} \left\{ f(z) * \left(\frac{z}{(1 - z)(1 - qz)(1 - q^2z)} \right. \right. \\ & \quad \left. \left. + \frac{[1 - (q + 1)L] qz^2}{(1 - z)(1 - qz)(1 - q^2z)} \right) \right\} \\ &= \frac{1}{z} \left\{ qz^2 D_q (D_q f(z)) + z D_q f(z) \right. \\ & \quad \left. - L [qz^2 D_q (D_q f(z))] \right\} \\ &= 1 - \sum_{k=2}^{\infty} [k]_q \frac{[k - 1]_q q e^{-i\theta} - A + [k]_q B}{A - B} a_k z^{k-1}, \end{aligned} \tag{45}$$

and this proves Theorem 13. □

Taking $q \rightarrow 1^-$ in Theorem 13, we obtain the following result.

Corollary 14. A necessary and sufficient condition for the function $f(z)$ defined by (1) to be in the class $\mathcal{K}[A, B]$ is that

$$1 - \sum_{k=2}^{\infty} k \frac{k (e^{-i\theta} + B) - e^{-i\theta} - A}{A - B} a_k z^{k-1} \neq 0 \quad (z \in \mathbb{U}). \tag{46}$$

Putting $A = 1 - 2\alpha$ ($0 \leq \alpha < 1$) and $B = -1$ in Theorem 13, we obtain the following corollary.

Corollary 15. A necessary and sufficient condition for the function f defined by (1) to be in the class $\mathcal{K}_q(\alpha)$ ($0 \leq \alpha < 1$) is that

$$1 - \sum_{k=2}^{\infty} [k]_q \frac{[k]_q (e^{-i\theta} - 1) - e^{-i\theta} - 1 + 2\alpha}{2(1 - \alpha)} a_k z^{k-1} \neq 0 \tag{47}$$

$(z \in \mathbb{U}).$

Taking $q \rightarrow 1^-$ in Corollary 15, we obtain the following corollary which improves the result of Ahuja [17, Corollary 1 when $n = 1$].

Corollary 16. A necessary and sufficient condition for the function f defined by (1) to be in the class $\mathcal{K}(\alpha)$ ($0 \leq \alpha < 1$) is that

$$1 - \sum_{k=2}^{\infty} k \frac{k (e^{-i\theta} - 1) - e^{-i\theta} - 1 + 2\alpha}{2(1 - \alpha)} a_k z^{k-1} \neq 0 \quad (z \in \mathbb{U}). \tag{48}$$

3. Coefficient Estimates

As an application of Theorems 9 and 13, we next determine coefficient estimate and inclusion property for a function of form (1) to be in the classes $\mathcal{S}_q[A, B]$ and $\mathcal{K}_q[A, B]$.

Theorem 17. If the function f defined by (1) satisfies the following inequality:

$$\sum_{k=2}^{\infty} \{ [k]_q (1 - B) - 1 + A \} |a_k| \leq A - B, \tag{49}$$

then $f \in \mathcal{S}_q[A, B]$.

Proof. Since

$$\begin{aligned} & \left| 1 - \sum_{k=2}^{\infty} [k]_q \frac{(e^{-i\theta} + B) - e^{-i\theta} - A}{A - B} a_k z^{k-1} \right| \\ & > 1 - \sum_{k=2}^{\infty} \left| \frac{[k]_q (e^{-i\theta} + B) - e^{-i\theta} - A}{A - B} \right| |a_k| \\ &= 1 - \sum_{k=2}^{\infty} \left| \frac{[k]_q (e^{-i\theta} + B) - e^{-i\theta} - A}{A - B} \right| |a_k| \\ & > 1 - \sum_{k=2}^{\infty} \frac{[k]_q (1 - B) - 1 + A}{A - B} |a_k| > 0 \end{aligned} \tag{50}$$

the result follows from Theorem 9. □

Taking $q \rightarrow 1^-$ in Theorem 17, we obtain the result of Ahuja [17, Theorem 3 when $n = 0$].

Corollary 18. If the function f defined by (1) satisfies the following inequality:

$$\sum_{k=2}^{\infty} [k(1 - B) - 1 + A] |a_k| \leq A - B, \tag{51}$$

then $f \in \mathcal{S}[A, B]$.

Putting $A = 1 - 2\alpha$ ($0 \leq \alpha < 1$) and $B = -1$ in Theorem 21, we obtain the following corollary.

Corollary 19. If the function f defined by (1) satisfies the following inequality:

$$\sum_{k=2}^{\infty} ([k]_q - \alpha) |a_k| \leq 1 - \alpha, \tag{52}$$

then $f \in \mathcal{S}_q(\alpha)$.

Taking $q \rightarrow 1^-$ in Corollary 19, we obtain the following corollary obtained by Silverman [24].

Corollary 20. *If the function f defined by (1) satisfies the following inequality:*

$$\sum_{k=2}^{\infty} (k - \alpha) |a_k| \leq 1 - \alpha, \quad (53)$$

then $f \in \mathcal{S}(\alpha)$.

Similarly, we can prove the following theorem.

Theorem 21. *If the function f defined by (1) satisfies the following inequality:*

$$\sum_{k=2}^{\infty} [k]_q \{ [k]_q (1 - B) - 1 + A \} |a_k| \leq A - B, \quad (54)$$

then $f \in \mathcal{K}_q[A, B]$.

Taking $q \rightarrow 1^-$ in Theorem 21, we obtain the result of Ahuja [17, Theorem 3 when $n = 1$].

Corollary 22. *If the function f defined by (1) satisfies the following inequality:*

$$\sum_{k=2}^{\infty} k [k(1 - B) - 1 + A] |a_k| \leq A - B, \quad (55)$$

then $f \in \mathcal{K}[A, B]$.

Putting $A = 1 - 2\alpha$ ($0 \leq \alpha < 1$) and $B = -1$ in Theorem 21, we obtain the following corollary.

Corollary 23. *The function f defined by (1) belongs to the class $\mathcal{K}_q(\alpha)$ ($0 \leq \alpha < 1$) if*

$$\sum_{k=2}^{\infty} [k]_q ([k]_q - \alpha) |a_k| \leq 1 - \alpha. \quad (56)$$

Taking $q \rightarrow 1^-$ in Corollary 23, we obtain the following corollary obtained by Silverman [24].

Corollary 24. *The function f defined by (1) belongs to the class $\mathcal{K}(\alpha)$ ($0 \leq \alpha < 1$) if*

$$\sum_{k=2}^{\infty} k(k - \alpha) |a_k| \leq 1 - \alpha. \quad (57)$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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