Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2015, Article ID 973613, 6 pages
http://dx.doi.org/10.1155/2015/973613

Research Article

Convolution Properties for Some Subclasses of Meromorphic Functions of Complex Order

M. K. Aouf,[1] A. O. Mostafa,[1] and H. M. Zayed[2]

[1] *Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt*
[2] *Department of Mathematics, Faculty of Science, Menofia University, Shebin El Kom 32511, Egypt*

Correspondence should be addressed to H. M. Zayed; hanaa_zayed42@yahoo.com

Received 3 June 2015; Accepted 5 July 2015

Academic Editor: Allan Peterson

Making use of the operator $\mathscr{L}_\upsilon$ for functions of the form $f(z) = 1/z + \sum_{k=1}^{\infty} a_k z^{k-1}$, which are analytic in the punctured unit disc $\mathbb{U}^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = \mathbb{U} \setminus \{0\}$, we introduce two subclasses of meromorphic functions and investigate convolution properties, coefficient estimates, and containment properties for these subclasses.

1. Introduction

Let Σ denote the class of meromorphic functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^{k-1}, \quad (1)$$

which are analytic in the punctured unit disc $\mathbb{U}^* = \{z : z \in \mathbb{C}$ and $0 < |z| < 1\} = \mathbb{U} \setminus \{0\}$. Let $g(z) \in \Sigma$ be given by

$$g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^{k-1}; \quad (2)$$

then, the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k b_k z^k = (g * f)(z). \quad (3)$$

We recall some definitions which will be used in our paper.

Definition 1. For two functions $f(z)$ and $g(z)$, analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$ and written $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ $(z \in \mathbb{U})$. Furthermore, if the function $g(z)$ is univalent in $\mathbb{U}$, then we have the following equivalence (see [1]):

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0), \quad f(\mathbb{U}) \subset g(\mathbb{U}). \quad (4)$$

Now, consider Bessel's function of the first kind of order υ where υ is an unrestricted (real or complex) number, defined by (see Watson [2, page 40]) (see also Baricz [3, page 7])

$$J_\upsilon(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\upsilon+1)} \left(\frac{z}{2}\right)^{2k+\upsilon}, \quad (5)$$

which is a particular solution of the second order linear homogenous Bessel differential equation (see, e.g., Watson [2, page 38]) (see also Baricz [3, page 7])

$$z^2 w''(z) + z w'(z) + (z^2 - \upsilon^2) w(z) = 0. \quad (6)$$

Also, let us define

$$\mathscr{L}_\upsilon(z) = \frac{2^\upsilon \Gamma(\upsilon+1)}{z^{\upsilon/2+1}} J_\upsilon(z^{1/2}) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(-1)^k \Gamma(\upsilon+1)}{4^k \Gamma(k+1)\Gamma(k+\upsilon+1)} z^{k-1} \quad (z \in \mathbb{U}^*). \quad (7)$$

The operator $\mathscr{L}_v$ is a modification of the operator introduced by Szász and Kupán [4] for analytic functions.

By using the Hadamard product (or convolution), we define the operator $\mathscr{L}_v$ as follows:

$$\begin{aligned}(\mathscr{L}_v f)(z) &= \mathscr{L}_v(z) * f(z)\\ &= \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(-1)^k \Gamma(v+1)}{4^k \Gamma(k+1)\Gamma(k+v+1)} a_k z^{k-1}.\end{aligned} \tag{8}$$

It is easy to verify from (8) that

$$\begin{aligned}z\left(\left(\mathscr{L}_{v+1} f\right)(z)\right)' &= (v+1)\left(\mathscr{L}_v f\right)(z)\\ &\quad - (v+2)\left(\mathscr{L}_{v+1} f\right)(z).\end{aligned} \tag{9}$$

Definition 2. For $0 \le \lambda < 1$, $-1 \le B < A \le 1$, and $b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, let $\Sigma\mathcal{S}_\lambda^*[b; A, B]$ be the subclass of Σ consisting of function $f(z)$ of the form (1) and satisfying the analytic criterion

$$1 + \frac{1}{b}\left[\frac{-zf'(z)}{(1-\lambda) f(z) - \lambda z f'(z)} - 1\right] \prec \frac{1+Az}{1+Bz}. \tag{10}$$

Also, let $\Sigma\mathcal{K}_\lambda[b; A, B]$ be the subclass of Σ consisting of function $f(z)$ of the form (1) and satisfying the analytic criterion

$$\begin{aligned}1 + \frac{1}{b}&\left[\frac{-z\left(zf'(z)\right)'}{(1-\lambda) zf'(z) - \lambda z\left(zf'(z)\right)'} - 1\right]\\ &\prec \frac{1+Az}{1+Bz}.\end{aligned} \tag{11}$$

It is easy to verify from (10) and (11) that

$$\begin{aligned}f(z) &\in \Sigma\mathcal{K}_\lambda[b; A, B]\\ &\Longleftrightarrow -zf'(z) \in \Sigma\mathcal{S}_\lambda^*[b; A, B].\end{aligned} \tag{12}$$

We note that

(i) $\Sigma\mathcal{S}_0^*[b; A, B] = \Sigma\mathcal{S}^*[b; A, B]$ and $\Sigma\mathcal{K}_0[b; A, B] = \Sigma\mathcal{K}[b; A, B]$ (see Bulboacă et al. [5]);

(ii) $\Sigma\mathcal{S}_0^*[b; 1, -1] = \Sigma\mathcal{S}(b)$ and $\Sigma\mathcal{K}_0[b; 1, -1] = \Sigma\mathcal{K}(b)$ (see Aouf [6]);

(iii) $\Sigma\mathcal{S}_0^*[(1-\alpha)e^{-i\mu}\cos\mu; 1, -1] = \Sigma\mathcal{S}^\mu(\alpha)$ and $\Sigma\mathcal{K}_0[(1-\alpha)e^{-i\mu}\cos\mu; 1, -1] = \Sigma\mathcal{K}^\mu(\alpha)$ ($\mu \in \mathbb{R}$, $|\mu| \le \pi/2$, $0 \le \alpha < 1$) (see Ravichandran et al. [7, with $p = 1$]).

Definition 3. For $0 \le \lambda < 1$, $-1 \le B < A \le 1$, $b \in \mathbb{C}^*$ and v is an unrestricted (real or complex) number, let

$$\begin{aligned}&\Sigma\mathcal{S}_{\lambda,v}^*[b; A, B]\\ &\quad = \left\{f(z) \in \Sigma : \left(\mathscr{L}_v f\right)(z) \in \Sigma\mathcal{S}_\lambda^*[b; A, B]\right\},\\ &\Sigma\mathcal{K}_{\lambda,v}[b; A, B]\\ &\quad = \left\{f(z) \in \Sigma : \left(\mathscr{L}_v f\right)(z) \in \Sigma\mathcal{K}_\lambda[b; A, B]\right\}.\end{aligned} \tag{13}$$

It is easy to show that

$$\begin{aligned}f(z) &\in \Sigma\mathcal{K}_{\lambda,v}[b; A, B]\\ &\Longleftrightarrow -zf'(z) \in \Sigma\mathcal{S}_{\lambda,v}^*[b; A, B].\end{aligned} \tag{14}$$

The object of the present paper is to investigate some convolution properties, coefficient estimates, and containment properties for the subclasses $\Sigma\mathcal{S}_{\lambda,v}^*[b; A, B]$ and $\Sigma\mathcal{K}_{\lambda,v}[b; A, B]$.

2. Main Results

Unless otherwise mentioned, we assume throughout this paper that $0 \le \lambda < 1$, $-1 \le B < A \le 1$, $b \in \mathbb{C}^*$ and v is an unrestricted (real or complex) number.

Theorem 4. *If $f(z) \in \Sigma$, then $f(z) \in \Sigma\mathcal{S}_\lambda^*[b; A, B]$ if and only if*

$$z\left[f(z) * \frac{1 - [(\lambda-1)M + (\lambda+1)]z}{z(1-z)^2}\right] \ne 0 \quad \text{for } z \in \mathbb{U}, \tag{15}$$

where $M = M_\theta = (e^{-i\theta} + B)/(A-B)b$, $\theta \in [0, 2\pi)$, and also $M = 0$.

Proof. It is easy to verify that

$$\begin{aligned}f(z) * \frac{1}{z(1-z)} &= f(z),\\ f(z) * \left[\frac{1}{z(1-z)^2} - \frac{2}{(1-z)^2}\right] &= -zf'(z)\\ &\forall z \in \mathbb{U}^*;\ f \in \Sigma.\end{aligned} \tag{16}$$

(i) In view of (10), $f(z) \in \Sigma\mathcal{S}_\lambda^*[b; A, B]$ if and only if (10) holds. Since the function $(1 + [B + (A-B)b]z)/(1+Bz)$ is analytic in $\mathbb{U}$, it follows that $(1-\lambda)f(z) - \lambda zf'(z) \ne 0$ for $z \in \mathbb{U}^*$ or $z[(1-\lambda)f(z) - \lambda zf'(z)] \ne 0$ for $z \in \mathbb{U}$; this is equivalent to (15) holdding for $M = 0$. To prove (15) for all $M \ne 0$, we write (10) by using the principle of subordination as

$$\frac{-zf'(z)}{(1-\lambda) f(z) - \lambda zf'(z)} = \frac{1 + [B + (A-B)b]w(z)}{1 + Bw(z)}, \tag{17}$$

where $w(z)$ is Schwarz function, analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$; hence,

$$\begin{aligned}z\Big[-zf'(z)\left(1 + Be^{i\theta}\right) &- \left[(1-\lambda) f(z) - \lambda zf'(z)\right]\\ &\cdot \left[1 + [B + (A-B)b]e^{i\theta}\right]\Big] \ne 0\\ &\text{for } z \in \mathbb{U},\ \theta \in [0, 2\pi).\end{aligned} \tag{18}$$

Using (16), (18) may be written as

$$z\left[f(z) * \frac{1-\left[(\lambda-1)\left(\left(e^{-i\theta}+B\right)/(A-B)b\right)+(\lambda+1)\right]z}{z(1-z)^2}\right] \neq 0 \quad \text{for } z \in \mathbb{U}. \tag{19}$$

Thus, the first part of Theorem 4 was proved.

(ii) Reversely, because assumption (15) holds for $M = 0$, it follows that $z[(1-\lambda)f(z) - \lambda z f'(z)] \neq 0$ for $z \in \mathbb{U}$. This implies that $\varphi(z) = -zf'(z)/((1-\lambda)f(z) - \lambda z f'(z))$ is analytic in $\mathbb{U}$ (i.e., it is regular in $z = 0$, with $\varphi(0) = 1$).

Since it was shown in the first part of the proof that assumption (18) is equivalent to (15), we obtain that

$$\frac{-zf'(z)}{(1-\lambda)f(z) - \lambda z f'(z)} \neq \frac{1 + [B + (A-B)b]e^{i\theta}}{1 + Be^{i\theta}} \quad \text{for } z \in \mathbb{U},\ \theta \in [0, 2\pi). \tag{20}$$

Assume that

$$\psi(z) = \frac{1 + [B + (A-B)b]e^{i\theta}}{1 + Be^{i\theta}}. \tag{21}$$

Relation (20) means that $\varphi(\mathbb{U}) \cap \psi(\partial\mathbb{U}) = \emptyset$. Thus, the simply connected domain is included in a connected component of $\mathbb{C} \setminus \psi(\partial\mathbb{U})$. From this, using the fact that $\varphi(0) = \psi(0)$ and the univalence of the function ψ, it follows that $\varphi(z) \prec \psi(z)$; this implies that $f(z) \in \Sigma\mathcal{S}^*_{\lambda,v}[b; A, B]$. Thus, the proof of Theorem 4 is completed. □

Remark 5. (i) Putting $\lambda = 0$ in Theorem 4, we obtain the result obtained by Bulboacă et al. [5, Theorem 1].

(ii) Putting $\lambda = 0$, $b = 1$, and $e^{i\theta} = x$ in Theorem 4, we obtain the result obtained by Ponnusamy [8, Theorem 2.1].

(iii) Putting $\lambda = 0$, $b = (1-\alpha)e^{-i\mu}\cos\mu$ ($\mu \in \mathbb{R}$, $|\mu| \leq \pi/2$, $0 \leq \alpha < 1$), $A = 1$, $B = -1$, and $e^{i\theta} = x$ in Theorem 4, we obtain the result obtained by Ravichandran et al. [7, Theorem 1.2 with $p = 1$].

Theorem 6. *If $f(z) \in \Sigma$, then $f(z) \in \Sigma\mathcal{K}_\lambda[b; A, B]$ if and only if*

$$z\left[f(z) * \frac{1 - 3z + 2[(\lambda-1)M + (\lambda+1)]z^2}{z(1-z)^3}\right] \neq 0 \quad \text{for } z \in \mathbb{U}, \tag{22}$$

where $M = M_\theta = (e^{-i\theta} + B)/(A-B)b$, $\theta \in [0, 2\pi)$, and also $M = 0$.

Proof. Putting

$$g(z) = \frac{1 - [(\lambda-1)M + (\lambda+1)]z}{z(1-z)^2}, \tag{23}$$

then

$$-zg'(z) = \frac{1 - 3z + 2[(\lambda-1)M + (\lambda+1)]z^2}{z(1-z)^3}. \tag{24}$$

From (12) and using the identity

$$\left[-zf'(z)\right] * g(z) = f(z) * \left[-zg'(z)\right], \tag{25}$$

we obtain the required result from Theorem 4. □

Remark 7. (i) Putting $\lambda = 0$ in Theorem 6, we obtain the result obtained by Bulboacă et al. [5, Theorem 2].

(ii) Putting $\lambda = 0$, $b = 1$, and $e^{i\theta} = x$ in Theorem 4, we obtain the result obtained by Ponnusamy [8, Theorem 2.2].

Theorem 8. *If $f(z) \in \Sigma$, then $f(z) \in \Sigma\mathcal{S}^*_{\lambda,v}[b; A, B]$ if and only if*

$$1 + \sum_{k=1}^{\infty} \frac{(-1)^k (1 - k\lambda)\Gamma(v+1)}{4^k \Gamma(k+1)\Gamma(k+v+1)} a_k z^k \neq 0, \tag{26}$$

$$1 + \sum_{k=1}^{\infty} (-1)^k \Gamma(v+1) \cdot \left[\frac{(1-k\lambda)(A-B)b}{4^k (A-B)b\Gamma(k+1)\Gamma(k+v+1)} - \frac{-k(\lambda-1)\left(e^{-i\theta}+B\right)}{4^k (A-B)b\Gamma(k+1)\Gamma(k+v+1)}\right] a_k z^k \neq 0, \tag{27}$$

for all $\theta \in [0, 2\pi)$.

Proof. If $f(z) \in \Sigma$, from Theorem 4, we have $f(z) \in \Sigma\mathcal{S}^*_{\lambda,v}[b; A, B]$ if and only if

$$z\left[(\mathcal{L}_v f)(z) * \frac{1 - [(\lambda-1)M + (\lambda+1)]z}{z(1-z)^2}\right] \neq 0 \quad \text{for } z \in \mathbb{U}, \tag{28}$$

where $M = M_\theta = (e^{-i\theta} + B)/(A-B)b$, $\theta \in [0, 2\pi)$, and also $M = 0$. Since

$$\frac{1 - (\lambda+1)z}{z(1-z)^2} = \frac{1}{z} + \sum_{k=1}^{\infty} (1 - k\lambda) z^{k-1}, \quad z \in \mathbb{U}^*, \tag{29}$$

it is easy to show that (28) holds for $M = 0$ if and only if (26) holds. Also,

$$\frac{1 - [(\lambda-1)M + (\lambda+1)]z}{z(1-z)^2} = \frac{1}{z} + \sum_{k=1}^{\infty} [(1 - k\lambda) - k(\lambda-1)M] z^{k-1}, \quad z \in \mathbb{U}^*; \tag{30}$$

we may easily check that (28) is equivalent to (27). This completes the proof of Theorem 8. □

Theorem 9. *If $f(z) \in \Sigma$, then $f(z) \in \Sigma\mathscr{K}_{\lambda,v}[b; A, B]$ if and only if*

$$1+\sum_{k=1}^{\infty}\frac{(-1)^k(k\lambda-1)(k-1)\Gamma(v+1)}{4^k\Gamma(k+1)\Gamma(k+v+1)}a_k z^k \neq 0, \tag{31}$$

$$\begin{aligned}1+\sum_{k=1}^{\infty}(-1)^k(k-1)\Gamma(v+1)\\ \cdot\left[\frac{(k\lambda-1)(A-B)b}{4^k(A-B)b\Gamma(k+1)\Gamma(k+v+1)}\right.\\ \left.+\frac{k(\lambda-1)\left(e^{-i\theta}+B\right)}{4^k(A-B)b\Gamma(k+1)\Gamma(k+v+1)}\right]a_k z^k \neq 0,\end{aligned} \tag{32}$$

for all $\theta \in [0, 2\pi)$.

Proof. If $f(z) \in \Sigma$, from Theorem 6, we have $f(z) \in \Sigma\mathscr{K}_{\lambda,v}[b; A, B]$ if and only if

$$\begin{aligned} z\left[\left(\mathscr{L}_v f\right)(z) * \frac{1-3z+2[(\lambda-1)M+(\lambda+1)]z^2}{z(1-z)^3}\right]\\ \neq 0 \quad \text{for } z \in \mathbb{U},\end{aligned} \tag{33}$$

where $M = M_\theta = (e^{-i\theta}+B)/(A-B)b$, $\theta \in [0, 2\pi)$, and also $M = 0$. Since

$$\begin{aligned}\frac{1-3z+2(\lambda+1)z^2}{z(1-z)^3} = \frac{1}{z}+\sum_{k=1}^{\infty}(k\lambda-1)(k-1)z^{k-1},\\ z \in \mathbb{U}^*,\end{aligned} \tag{34}$$

it is easy to show that (33) holds for $M = 0$ if and only if (31) holds. Also,

$$\begin{aligned}\frac{1-3z+2[(\lambda-1)M+(\lambda+1)]z^2}{z(1-z)^3}\\ = \frac{1}{z}+\sum_{k=1}^{\infty}(k-1)[k(\lambda-1)M+(k\lambda-1)]z^{k-1};\end{aligned} \tag{35}$$

for $z \in \mathbb{U}^*$, we may easily check that (33) is equivalent to (32). This completes the proof of Theorem 9. □

Unless otherwise mentioned, we assume throughout the remainder of this section that v is a real number ($v > -1$).

Theorem 10. *If $f(z) \in \Sigma$ satisfies inequalities*

$$\sum_{k=1}^{\infty}|k\lambda-1|\frac{\Gamma(v+1)}{4^k\Gamma(k+1)\Gamma(k+v+1)}|a_k| < 1, \tag{36}$$

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{[(|1-k\lambda|)(A-B)|b|+k(1-\lambda)(1+|B|)]\Gamma(v+1)}{4^k\Gamma(k+1)\Gamma(k+v+1)}|a_k|\\ < (A-B)|b|,\end{aligned} \tag{37}$$

then $f(z) \in \Sigma\mathcal{S}^*_{\lambda,v}[b; A, B]$.

Proof. We have

$$\begin{aligned}&\left|1-\sum_{k=1}^{\infty}\frac{(-1)^k(k\lambda-1)\Gamma(v+1)}{4^k\Gamma(k+1)\Gamma(k+v+1)}a_k z^k\right|\\ &\geq 1-\left|\sum_{k=1}^{\infty}\frac{(-1)^k(k\lambda-1)\Gamma(v+1)}{4^k\Gamma(k+1)\Gamma(k+v+1)}a_k z^k\right|\\ &\geq 1-\sum_{k=1}^{\infty}|k\lambda-1|\frac{\Gamma(v+1)}{4^k\Gamma(k+1)\Gamma(k+v+1)}|a_k||z^k|\\ &\geq 1-\sum_{k=1}^{\infty}|k\lambda-1|\frac{\Gamma(v+1)}{4^k\Gamma(k+1)\Gamma(k+v+1)}|a_k|\\ &> 0, \quad \text{for } z \in \mathbb{U},\end{aligned} \tag{38}$$

which implies inequality (36). Also,

$$\begin{aligned}&\left|1+\sum_{k=1}^{\infty}(-1)^k\Gamma(v+1)\right.\\ &\cdot\left[\frac{(1-k\lambda)(A-B)b}{4^k(A-B)b\Gamma(k+1)\Gamma(k+v+1)}\right.\\ &\left.\left.-\frac{k(\lambda-1)\left(e^{-i\theta}+B\right)}{4^k(A-B)b\Gamma(k+1)\Gamma(k+v+1)}\right]a_k z^k\right| \geq 1\\ &-\left|\sum_{k=1}^{\infty}(-1)^k\Gamma(v+1)\right.\\ &\cdot\left[\frac{(1-k\lambda)(A-B)b}{4^k(A-B)b\Gamma(k+1)\Gamma(k+v+1)}\right.\\ &\left.\left.-\frac{k(\lambda-1)\left(e^{-i\theta}+B\right)}{4^k(A-B)b\Gamma(k+1)\Gamma(k+v+1)}\right]a_k z^k\right| \geq 1\\ &-\sum_{k=1}^{\infty}\Gamma(v+1)\\ &\cdot\left[\frac{|1-k\lambda|(A-B)|b|}{4^k(A-B)|b|\Gamma(k+1)\Gamma(k+v+1)}\right.\\ &\left.+\frac{k(1-\lambda)(1+|B|)}{4^k(A-B)|b|\Gamma(k+1)\Gamma(k+v+1)}\right]|a_k||z^k|\\ &\geq 1-\sum_{k=1}^{\infty}\Gamma(v+1)\\ &\cdot\left[\frac{|1-k\lambda|(A-B)|b|}{4^k(A-B)|b|\Gamma(k+1)\Gamma(k+v+1)}\right.\\ &\left.+\frac{k(1-\lambda)(1+|B|)}{4^k(A-B)|b|\Gamma(k+1)\Gamma(k+v+1)}\right]|a_k| > 0,\\ &\qquad\qquad \text{for } z \in \mathbb{U},\end{aligned} \tag{39}$$

which implies inequality (37). Thus, the proof of Theorem 10 is completed. □

Using similar arguments to those in the proof of Theorem 10, we obtain the following theorem.

Theorem 11. *If $f(z) \in \Sigma$ satisfies inequalities*

$$\sum_{k=1}^{\infty} |k\lambda - 1| \frac{(k-1)\Gamma(\upsilon+1)}{4^k\Gamma(k+1)\Gamma(k+\upsilon+1)} |a_k| < 1,$$

$$\sum_{k=1}^{\infty} (k-1)\Gamma(\upsilon+1) \left[\frac{(|k\lambda - 1|)(A-B)|b|}{4^k\Gamma(k+1)\Gamma(k+\upsilon+1)} + \frac{k(1-\lambda)(1+|B|)}{4^k\Gamma(k+1)\Gamma(k+\upsilon+1)} \right] |a_k| < (A-B)|b|, \tag{40}$$

then $f(z) \in \Sigma\mathcal{K}_{\lambda,\upsilon}[b; A, B]$.

Now, using the method due to Ahuja [9], we will prove the containment relations for the subclasses $\Sigma\mathcal{S}^*_{\lambda,\upsilon}[b; A, B]$ and $\Sigma\mathcal{K}_{\lambda,\upsilon}[b; A, B]$.

Theorem 12. *For $\upsilon > -1$, we have $\Sigma\mathcal{S}^*_{\lambda,\upsilon+1}[b; A, B] \subset \Sigma\mathcal{S}^*_{\lambda,\upsilon}[b; A, B]$.*

Proof. Since $f(z) \in \Sigma\mathcal{S}^*_{\lambda,\upsilon+1}[b; A, B]$, we see from Theorem 8 that

$$1 + \sum_{k=1}^{\infty} \frac{(-1)^k (1-k\lambda)\Gamma(\upsilon+2)}{4^k\Gamma(k+1)\Gamma(k+\upsilon+2)} a_k z^k \neq 0,$$

$$1 + \sum_{k=1}^{\infty} (-1)^k \Gamma(\upsilon+2) \cdot \left[\frac{(1-k\lambda)(A-B)b}{4^k(A-B)b\Gamma(k+1)\Gamma(k+\upsilon+2)} - \frac{k(\lambda-1)\left(e^{-i\theta}+B\right)}{4^k(A-B)b\Gamma(k+1)\Gamma(k+\upsilon+2)} \right] a_k z^k \neq 0. \tag{41}$$

We can write (41) as

$$\left[1 + \sum_{k=1}^{\infty} \frac{\upsilon+1}{k+\upsilon+1} z^k \right] * \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k (1-k\lambda)\Gamma(\upsilon+1)}{4^k\Gamma(k+1)\Gamma(k+\upsilon+1)} a_k z^k \right] \neq 0,$$

$$\left[1 + \sum_{k=1}^{\infty} \frac{\upsilon+1}{k+\upsilon+1} z^k \right] * \left[1 + \sum_{k=1}^{\infty} (-1)^k \Gamma(\upsilon+1) \cdot \left[\frac{(1-k\lambda)(A-B)b}{4^k(A-B)b\Gamma(k+1)\Gamma(k+\upsilon+1)} - \frac{k(\lambda-1)\left(e^{-i\theta}+B\right)}{4^k(A-B)b\Gamma(k+1)\Gamma(k+\upsilon+1)} \right] a_k z^k \right] \neq 0, \tag{42}$$

since

$$\left[1 + \sum_{k=1}^{\infty} \frac{\upsilon+1}{k+\upsilon+1} z^k \right] * \left[1 + \sum_{k=1}^{\infty} \frac{k+\upsilon+1}{\upsilon+1} z^k \right] = 1 + \sum_{k=1}^{\infty} z^k. \tag{43}$$

By using the property, if $f \neq 0$ and $g*h \neq 0$, then $f*(g*h) \neq 0$; (42) can be written as

$$1 + \sum_{k=1}^{\infty} \frac{(-1)^k (1-k\lambda)\Gamma(\upsilon+1)}{4^k\Gamma(k+1)\Gamma(k+\upsilon+1)} a_k z^k \neq 0,$$

$$1 + \sum_{k=1}^{\infty} (-1)^k \Gamma(\upsilon+1) \cdot \left[\frac{(1-k\lambda)(A-B)b}{4^k(A-B)b\Gamma(k+1)\Gamma(k+\upsilon+1)} - \frac{k(\lambda-1)\left(e^{-i\theta}+B\right)}{4^k(A-B)b\Gamma(k+1)\Gamma(k+\upsilon+1)} \right] a_k z^k \neq 0, \tag{44}$$

which means that $f(z) \in \Sigma\mathcal{S}^*_{\lambda,\upsilon}[b; A, B]$. This completes the proof of Theorem 12. □

Using the same arguments as in the proof of Theorem 12, we obtain the following theorem.

Theorem 13. *For $\upsilon > -1$, we have $\Sigma\mathcal{K}_{\lambda,\upsilon+1}[b; A, B] \subset \Sigma\mathcal{K}_{\lambda,\upsilon}[b; A, B]$.*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] S. S. Miller and P. T. Mocanu, *Differential Subordinations: Theory and Applications*, vol. 225 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 2000.

[2] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, UK, 1944.

[3] A. Baricz, *Generalized Bessel Functions of the First Kind*, vol. 1994 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 2010.

[4] R. Szász and P. A. Kupán, “About the univalence of the Bessel functions,” *Studia Universitatis Babeş-Bolyai—Series Mathematica*, vol. 54, no. 1, pp. 127–132, 2009.

[5] T. Bulboacă, M. K. Aouf, and R. M. El-Ashwah, “Convolution properties for subclasses of meromorphic univalent functions of complex order,” *Filomat*, vol. 26, no. 1, pp. 153–163, 2012.

[6] M. K. Aouf, “Coefficient results for some classes of meromorphic functions,” *The Journal of Natural Sciences and Mathematics*, vol. 27, no. 2, pp. 81–97, 1987.

[7] V. Ravichandran, S. S. Kumar, and K. G. Subramanian, “Convolution conditions for spirallikeness and convex spirallikenesss of certain p-valent meromorphic functions,” *Journal of Inequalities in Pure and Applied Mathematics*, vol. 5, no. 1, pp. 1–7, 2004.

[8] S. Ponnusamy, “Convolution properties of some classes of meromorphic univalent functions,” *Proceedings of the Indian Academy of Sciences—Mathematical Sciences*, vol. 103, no. 1, pp. 73–89, 1993.

[9] O. P. Ahuja, “Families of analytic functions related to Ruscheweyh derivatives and subordinate to convex functions,” *Yokohama Mathematical Journal*, vol. 41, no. 1, pp. 39–50, 1993.