

CONVOLUTION SUMS ARISING FROM DIVISOR FUNCTIONS

AERAN KIM, DAEYEOL KIM, AND LI YAN

ABSTRACT. Let $\sigma_s(N)$ denote the sum of the s th powers of the positive divisors of a positive integer N and let $\tilde{\sigma}_s(N) = \sum_{d|N} (-1)^{d-1} d^s$ with d , N , and s positive integers. Hahn [12] proved that

$$16 \sum_{k < N} \tilde{\sigma}_1(k) \tilde{\sigma}_3(N-k) = -\tilde{\sigma}_5(N) + 2(N-1)\tilde{\sigma}_3(N) + \tilde{\sigma}_1(N).$$

In this paper, we give a generalization of Hahn's result. Furthermore, we find the formula $\sum_{k=1}^{N-1} \tilde{\sigma}_1(2^{n-m}k) \tilde{\sigma}_3(2^n N - 2^m k)$ for m ($0 \leq m \leq n$).

1. Introduction

For $N, m, d \in \mathbb{N}$ with $r, s \in \mathbb{N} \cup \{0\}$, we define some necessary divisor functions and infinite products for later use, which also appear in many areas of number theory:

$$\begin{aligned} \sigma_s(N) &= \sum_{d|N} d^s, & \sigma_{s,r}(N; m) &= \sum_{\substack{d|N \\ d \equiv r \pmod{m}}} d^s, \\ \tilde{\sigma}_s(N) &= \sum_{d|N} (-1)^{d-1} d^s, & S_1 &:= \sum_{N \text{ odd}} \sigma_{1,1}(N; 2) q^N, \\ S_2 &:= \sum_{\substack{N \geq 2 \text{ even}}} \sigma_{1,1}(N; 2) q^N, & (a; q)_\infty &:= (a)_\infty := \prod_{N \geq 0} (1 - aq^N). \end{aligned}$$

We also make use of the following convention:

$$\sigma_s(N) = 0 \text{ if } N \notin \mathbb{Z} \text{ or } N < 0, \quad \sigma(N) := \sigma_1(N) = \sum_{d|N} d.$$

The history of the convolution sums involving the divisor functions $\sigma_s(N)$ goes back to Glaisher [9, 10, 11]. Many recent works on convolution formulas for divisor functions can be found in B. C. Berndt [3]; H. Hahn [12]; J. G. Huard, Z. M. Ou, B. K. Spearman, and K. S. Williams [13]; G. Melfi [18]; B. Cho, D. Kim, and J.-K. Koo [4, 5]; and A. Alaca, S. Alaca, and K. S. Williams [1, 2].

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In 1997, Melfi [18] considered among others the convolution sums

$$\sum_{k < N/2} \sigma_1(k)\sigma_3(N - 2k) \quad \text{and} \quad \sum_{k < N/2} \sigma_3(k)\sigma_1(N - 2k),$$

for the case when N is odd, and proved that

$$\begin{aligned} \sum_{k < N/2} \sigma_1(k)\sigma_3(N - 2k) &= \frac{1}{48}\sigma_5(N) + \frac{(2 - 3N)}{48}\sigma_3(N), \quad N \equiv 1 \pmod{2}, \\ \sum_{k < N/2} \sigma_3(k)\sigma_1(N - 2k) &= \frac{1}{240}\sigma_5(N) - \frac{1}{240}\sigma_1(N), \quad N \equiv 1 \pmod{2}. \end{aligned}$$

In 2002, Huard, Ou, Spearman, and Williams [13] extended Melfi's result to

$$\begin{aligned} \sum_{k < N/2} \sigma_1(k)\sigma_3(N - 2k) &= \frac{1}{48}\sigma_5(N) + \frac{1}{15}\sigma_5\left(\frac{N}{2}\right) + \frac{(2 - 3N)}{48}\sigma_3(N) \\ &\quad - \frac{1}{240}\sigma_1\left(\frac{N}{2}\right), \\ (1) \quad \sum_{k < N/2} \sigma_3(k)\sigma_1(N - 2k) &= \frac{1}{240}\sigma_5(N) + \frac{1}{12}\sigma_5\left(\frac{N}{2}\right) + \frac{(1 - 3N)}{24}\sigma_3\left(\frac{N}{2}\right) \\ &\quad - \frac{1}{240}\sigma_1(N), \end{aligned}$$

where N is an arbitrary positive integer. In this paper, we give a generalization of (1).

The paper is organized as follows. In Section 2, we derive some basic comments on $\sum_{k=1}^{N-1} \sigma_{1,1}(k; 2)\sigma_{1,1}(N - k; 2) = \frac{1}{24}[11\sigma_3(N) - \sigma_3(2N) - 2\sigma_{1,1}(N; 2)]$ from $E_4(\tau) = 1 + 240 \sum_{N \geq 1} \sigma_3(N)q^N$. In Section 3, we derive some identities involving $\sum_{k=1}^{N-1} \sigma_{1,1}(N; 2)\sigma_{1,1}(N - k; 2)$ for certain N . In Section 4, we obtain

$$\begin{aligned} &\sum_{k=1}^{N-1} \sigma_1(2^{n-m}k)\sigma_3(2^n(N - k)) \\ &= \frac{1}{1680} \{ (3 - 2^{n-m+4} - 5 \cdot 2^{3n+4} + 15 \cdot 2^{4n-m+4})\sigma_5(N) \\ &\quad + 2^{-m+4}(9 \cdot 2^m + 2^n + 5 \cdot 2^{3n+m} - 15 \cdot 16^n)\sigma_5\left(\frac{N}{2}\right) \\ &\quad - 10(8^{n+1} - 1)(-1 + 3 \cdot 2^{n-m}N)\sigma_3(N) \\ &\quad + 80(8^n - 1)(-1 + 3 \cdot 2^{n-m}N)\sigma_3\left(\frac{N}{2}\right) - 7(2^{n-m+1} - 1)\sigma_1(N) \\ &\quad + 14(2^{n-m} - 1)\sigma_1\left(\frac{N}{2}\right) \} \end{aligned}$$

for any positive integer N and m ($0 \leq m \leq n$) with $n \in \mathbb{N} \cup \{0\}$ (Theorem 4.6). This is a generalization of (1). In the last section, we also find

$$\begin{aligned} & \sum_{k=1}^{N-1} \tilde{\sigma}_1(2^{n-m}k) \tilde{\sigma}_3(2^n(N-k)) \\ &= \frac{1}{112 \cdot 2^m} [(16^{n+1} - 2^{3n+m+4} - 2^{n+4} + 9 \cdot 2^m) \sigma_5(N) \\ & \quad + 16(27 \cdot 2^m + 2^n + 2^{3n+m} - 16^n) \sigma_5(\frac{N}{2}) \\ & \quad + \{2^{m+1}(2^{3n+3} - 15) - 2(8 \cdot 16^n - 15 \cdot 2^n)N\} \sigma_3(N) + 16\{2^m(15 - 2^{3n}) \\ & \quad + (16^n - 15 \cdot 2^n)N\} \sigma_3(\frac{N}{2}) \\ & \quad + 7(3 \cdot 2^m - 2^{n+1}) \sigma_1(N) - 14(3 \cdot 2^m - 2^n) \sigma_1(\frac{N}{2})] \end{aligned}$$

for any positive integer N and m ($0 \leq m \leq n$) with $n \in \mathbb{N} \cup \{0\}$ (Theorem 5.1). This is a generalization of Hanh's result [12],

$$(2) \quad 16 \sum_{k < N} \tilde{\sigma}_1(k) \tilde{\sigma}_3(N-k) = -\tilde{\sigma}_5(N) + 2(N-1)\tilde{\sigma}_3(N) + \tilde{\sigma}_1(N).$$

2. Preliminaries

Let $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$ ($\tau \in \mathcal{H}$, the complex upper-half plane) be a lattice and $z \in \mathbb{C}$. The Weierstrass \wp function relative to Λ_τ is defined by the series

$$\wp(z; \Lambda_\tau) = \frac{1}{z^2} + \sum_{\substack{\omega \in \Lambda_\tau \\ \omega \neq 0}} \left\{ \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right\}.$$

The Eisenstein series of weight $2k$ for Λ_τ with $k > 1$ is defined by

$$G_{2k}(\Lambda_\tau) = \sum_{\substack{\omega \in \Lambda_\tau \\ \omega \neq 0}} \omega^{-2k}$$

and the normalized Eisenstein series of weight $2k$ with $k > 1$ is given by

$$E_{2k}(\tau) = -\frac{(2k)!B_{2k}}{(2\pi i)^{2k}} G_{2k}(\Lambda_\tau) = 1 - \frac{4k}{B_{2k}} \sum_{N=1}^{\infty} \sigma_{2k-1}(N) q^N, \quad \tau \in \mathcal{H},$$

with B_N the N th Bernoulli number and $q = e^{2\pi i\tau}$. We use notations $\wp(z)$ and G_{2k} instead of $\wp(z; \Lambda_\tau)$ and $G_{2k}(\Lambda_\tau)$, respectively, when lattice Λ_τ has been fixed. Now, the Laurent series for $\wp(z)$ about $z = 0$ is given by

$$\wp(z) = z^{-2} + \sum_{k=1}^{\infty} (2k+1) G_{2k+2} z^{2k}.$$

As is customary, by setting

$$g_2(\tau) = g_2(\Lambda_\tau) = 60G_4 \quad \text{and} \quad g_3(\tau) = g_3(\Lambda_\tau) = 140G_6,$$

the algebraic relation between $\wp(z)$ and $\wp'(z)$ becomes

$$\wp'(z)^2 = 4\wp(z)^3 - g_2(\tau)\wp(z) - g_3(\tau).$$

Let

$$e_1 = \wp\left(\frac{\tau}{2}\right), \quad e_2 = \wp\left(\frac{1}{2}\right), \quad \text{and} \quad e_3 = \wp\left(\frac{\tau+1}{2}\right).$$

From [17, p. 251], we get

$$\begin{aligned} e_2 - e_1 &= \pi^2 (q^2; q^2)_\infty^4 \frac{1}{(q; q^2)_\infty^8 (-q^2; q^2)_\infty^8}, \\ e_2 - e_3 &= \pi^2 (q^2; q^2)_\infty^4 (q; q^2)_\infty^8, \end{aligned}$$

and

$$e_3 - e_1 = 2^4 \pi^2 q (q^2; q^2)_\infty^4 \frac{(q^4; q^4)_\infty^8}{(q^2; q^2)_\infty^8}$$

with $q = \exp(\pi i\tau)$. Next, we state two identities which appear in [8, pp. 78 and 79], [4]:

$$(3) \quad \frac{(q^2; q^2)_\infty^{20}}{(q)_\infty^8 (q^4; q^4)_\infty^8} = 1 + 8 \sum_{N=1}^{\infty} q^N (2 + (-1)^N) \sum_{\substack{\omega|N \\ \omega \text{ odd}}} \omega,$$

$$(4) \quad \frac{q(q^4; q^4)_\infty^8}{(q^2; q^2)_\infty^4} = \sum_{N \text{ odd}} \sigma(N) q^N.$$

Using (3) and (4), we obtain the following identities for $\wp(z)$ (for details, see [14]):

$$\begin{aligned} (5) \quad \wp\left(\frac{\tau}{2}\right) &= -\frac{\pi^2}{3} \left(\frac{(q^2; q^2)_\infty^{20}}{(q)_\infty^8 (q^4; q^4)_\infty^8} + 16 \frac{q(q^4; q^4)_\infty^8}{(q^2; q^2)_\infty^4} \right) \\ &= -\frac{\pi^2}{3} \left(1 + 8 \sum_{\substack{N \text{ odd} \\ \omega \text{ odd}}} q^N \sum_{\substack{\omega|N \\ \omega \text{ odd}}} \omega + 24 \sum_{N \text{ even}} q^N \sum_{\substack{\omega|N \\ \omega \text{ odd}}} \omega + 16 \sum_{N \text{ odd}} \sigma(N) q^N \right) \\ &= -\frac{\pi^2}{3} \left(1 + 24 \sum_{N=1}^{\infty} \sigma_{1,1}(N; 2) q^N \right) \\ &= -\frac{\pi^2}{3} (1 + 24S_1 + 24S_2). \end{aligned}$$

Similarly, Eqs. (3) and (4) yield the following arithmetic results:

$$\begin{aligned} (6) \quad \wp\left(\frac{\tau+1}{2}\right) &= -\frac{\pi^2}{3} \left(\frac{(q^2; q^2)_\infty^{20}}{(q)_\infty^8 (q^4; q^4)_\infty^8} - 32 \frac{q(q^4; q^4)_\infty^8}{(q^2; q^2)_\infty^4} \right) \\ &= -\frac{\pi^2}{3} (1 - 24S_1 + 24S_2), \end{aligned}$$

$$(7) \quad \begin{aligned} \wp\left(\frac{1}{2}\right) &= \frac{2\pi^2}{3} \left(\frac{(q^2; q^2)_\infty^{20}}{(q)_\infty^8 (q^4; q^4)_\infty^8} - 8 \frac{q(q^4; q^4)_\infty^8}{(q^2; q^2)_\infty^4} \right) \\ &= \frac{2\pi^2}{3} (1 + 24S_2). \end{aligned}$$

Thus we deduce the following result [19, p. 59]:

$$\begin{aligned} E_4(\tau) &= \frac{2^2 \cdot 3}{(2\pi)^4} g_2(\tau) \\ &= \frac{2^2 \cdot 3}{(2\pi)^4} (-4(e_1e_2 + e_2e_3 + e_3e_1)) \\ &= \frac{2^2 \cdot 3}{(2\pi)^4} \cdot \frac{4\pi^4}{9} [3(1 + 24S_2)^2 + 24^2 S_1^2]. \end{aligned}$$

Note that the right-hand side of the above equation is a power series of q^2 . So we change variable q^2 to q , that is, from now on, we always assume $q = \exp(2\pi i\tau)$ unless otherwise specified. Therefore,

$$(8) \quad \begin{aligned} E_4(\tau) &= 1 + 240q + \sum_{N=2}^{\infty} [48\sigma_{1,1}(2N; 2) + 576 \sum_{\substack{k=1 \\ k+l=N}}^{N-1} \sigma_{1,1}(2k; 2)\sigma_{1,1}(2l; 2) \\ &\quad + 192 \sum_{\substack{k=1 \\ k+l-1=N}}^N \sigma_{1,1}(2k-1; 2)\sigma_{1,1}(2l-1; 2)]q^N. \end{aligned}$$

From [19, p. 59], we already know that

$$(9) \quad E_4(\tau) = 1 + 240 \sum_{N \geq 1} \sigma_3(N)q^N.$$

3. Some convolution sums of $\sigma_{1,1}(k; 2)$

In [7, p. 300], Glaisher proved that

$$(10) \quad \sigma(1)\sigma(2N-1) + \sigma(3)\sigma(2N-3) + \cdots + \sigma(2N-1)\sigma(1) = \frac{1}{8}[\sigma_3(2N) - \sigma_3(N)].$$

Combining (8), (9), and (10), we can obtain

$$(11) \quad \sum_{k=1}^{N-1} \sigma_{1,1}(k; 2)\sigma_{1,1}(N-k; 2) = \frac{1}{24}[11\sigma_3(N) - \sigma_3(2N) - 2\sigma_{1,1}(N; 2)].$$

In this section, we discuss some new convolution sums derived from the existing ones. By using Eqs. (10) and (11), we can obtain the following lemma.

Lemma 3.1. (a) If $N \geq 1$ is a positive integer, then

$$U(N) := \sum_{k=1}^N \sigma_{1,1}(2k-1; 2)\sigma_{1,1}(2(N-k+1); 2) = \frac{1}{24}[\sigma_3(2N+1) - \sigma_1(2N+1)].$$

(b) If $N \geq 1$ is a positive integer, then

$$V(N) := \sum_{k=1}^N \sigma_{1,1}(2k-1; 2)\sigma_{1,1}(2(N-k)+1; 2) = \frac{1}{8}[\sigma_3(2N) - \sigma_3(N)].$$

Proof. (a) Note that

$$\sum_{k=1}^{2N} \sigma_{1,1}(k; 2)\sigma_{1,1}(2N+1-k; 2) = \sum_{k=1}^N 2\sigma_{1,1}(2k-1; 2)\sigma_{1,1}(2(N-k+1); 2).$$

Hence, by (11), we obtain

$$\begin{aligned} & \frac{1}{24}[11\sigma_3(2N+1) - \sigma_3(2(2N+1)) - 2\sigma_{1,1}(2N+1; 2)] \\ &= 2 \sum_{k=1}^N \sigma_{1,1}(2k-1; 2)\sigma_{1,1}(2(N-k+1); 2). \end{aligned}$$

Since $\sigma_3(2(2N+1)) = 9\sigma_3(2N+1)$, we get the desired result.

(b) It is deduced directly from (10). \square

Example 3.2. The first twelve values of $U(N)$ and $V(N)$ are given in Table 1.

Table 1. Examples for $U(N)$ and $V(N)$.

N	2	3	4	5	6	7	8	9	10	11	12	13
$U(N)$	5	14	31	55	91	146	204	285	400	506	655	850
$V(N)$	8	28	64	126	224	344	512	757	1008	1332	1792	2198

Remark 3.3. If $N(\geq 3)$ is an odd integer, then

$$\begin{aligned} U\left(\frac{N-1}{2}\right) &:= \sum_{k=1}^{\frac{N-1}{2}} \sigma_{1,1}(2k-1; 2)\sigma_{1,1}(N-2k+1; 2) \\ &= \sum_{k=1}^{\frac{N-1}{2}} \sigma_{1,1}(2k; 2)\sigma_{1,1}(N-2k; 2) \end{aligned}$$

by Lemma 3.1(a) and (b). Let $S(N) := \sum_{k=1}^N k^2$.

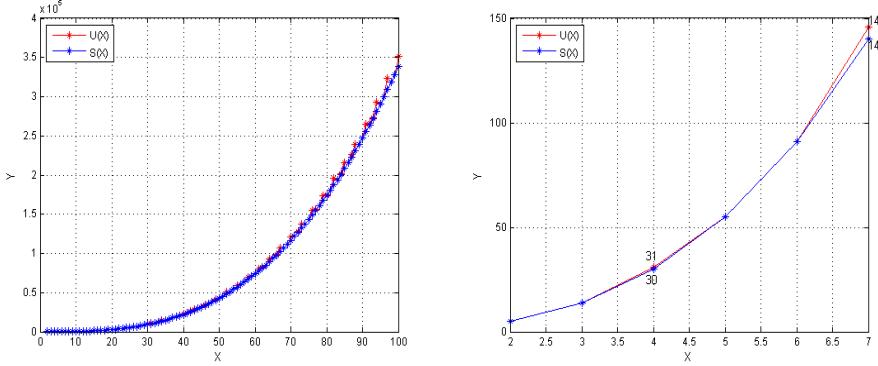
In particular, if p is an odd prime integer, then

$$(12) \quad U\left(\frac{p-1}{2}\right) = S\left(\frac{p-1}{2}\right) = \sum_{k=1}^{\frac{p-1}{2}} k^2.$$

Thus we can ask a similar question regarding convolution formulas as follows:

(Question) Can one find $r_1, r_2, s_1, s_2, m, \alpha_1, \beta_1, \beta$ in \mathbb{Z} satisfying

$$\sum_{k < \beta p / \beta_1} \sigma_{r_1, s_1}(\alpha_1 k; m)\sigma_{r_2, s_2}(\beta p - \beta_1 k; m) = \sum_{k=1}^{\frac{p-1}{2}} k^u$$

FIGURE 1. $U(X)$ and $S(X)$.

for a fixed u and a fixed odd prime p ?

We feel that this sort of problem is generally not easy to solve. Equation (12) is a special case for this question with $u = 2$.

In the following proposition, we state a property of divisor functions, which will be used frequently in our proofs.

Proposition 3.4 ([20, p. 26]). *Let p be a prime. Let $k, N \in \mathbb{N}$. Then*

$$\sigma_k(pN) - (p^k + 1)\sigma_k(N) + p^k\sigma_k\left(\frac{N}{p}\right) = 0.$$

Lemma 3.5. *If $N \geq 2$ is a positive integer, then*

$$\sum_{k=1}^{N-1} k\sigma_{1,1}(k; 2)\sigma_{1,1}(N-k; 2) = \frac{N}{48}[11\sigma_3(N) - \sigma_3(2N) - 2\sigma_{1,1}(N; 2)].$$

Proof. It is obvious from [13, p. 8]. □

Theorem 3.6. *Let N be an odd integer greater than 3. Then*

$$(13) \quad \begin{aligned} & \sum_{L=1}^{N-1} \sum_{l=1}^L \sigma_{1,1}(2N - (2L+1); 2)\sigma_{1,1}(2l-1; 2)\sigma_{1,1}(2(L-l+1); 2) \\ &= \frac{1}{24}[\sigma_5(N) - \sigma_3(N)]. \end{aligned}$$

Proof. The coefficients of q^{2N} in

$$\sum_{k,l,m=1}^{\infty} \sigma_{1,1}(2k-1; 2)\sigma_{1,1}(2l-1; 2)\sigma_{1,1}(2m; 2)q^{2(k+l+m-1)}$$

with $k + l + m - 1 = N$ can be expanded as

$$\begin{aligned}
 A &:= \sum_{L=1}^{N-1} \sigma_{1,1}(2N - (2L+1); 2) \sum_{l=1}^L \sigma_{1,1}(2l-1; 2) \sigma_{1,1}(2(L-l+1); 2) \\
 &= \sum_{L=1}^{N-1} \sigma_1(2N - (2L+1)) \frac{1}{24} [\sigma_3(2L+1) - \sigma_1(2L+1)] \\
 (14) \quad &= \frac{1}{24} \sum_{L=1}^{N-1} \sigma_1(2(N-L-1)+1) \sigma_3(2N-2(N-L-1)-1) \\
 &\quad - \frac{1}{24} \sum_{L=1}^{N-1} \sigma_1(2N-(2L+1)) \sigma_1(2L+1).
 \end{aligned}$$

In [13, p. 24], Eq. (15) is described, for an odd N , as

$$(15) \quad \sum_{k=0}^{N-1} \sigma_1(2k+1) \sigma_3(2N-2k-1) = \sigma_5(N).$$

Substituting (15) and (10) into (14), we get

$$\begin{aligned}
 A &= \frac{1}{24} [\sigma_5(N) - \sigma_1(2N-1)] - \frac{1}{24} \left\{ \frac{1}{8} [\sigma_3(2N) - \sigma_3(N)] - \sigma_1(2N-1) \right\} \\
 &= \frac{1}{24} [\sigma_5(N) - \frac{1}{8} \sigma_3(2N) + \frac{1}{8} \sigma_3(N)] \\
 &= \frac{1}{24} [\sigma_5(N) - \sigma_3(N)]. \quad \square
 \end{aligned}$$

Using modular forms, we can give a generalization of Theorem 3.6. Indeed, from pp. 18–19 of [6], we know

$$G_{2,2}(\tau) := -\frac{\pi^2}{3} (1 + 24 \sum_{N=1}^{\infty} \sigma_{1,1}(N; 2) q^N) \in M_2(\Gamma_0(2)).$$

Let

$$(16) \quad g(\tau) = \frac{1}{24} + \sum_{N=1}^{\infty} \sigma_{1,1}(N; 2) q^N \in M_2(\Gamma_0(2)).$$

From p. 107 of [6], we know

$$d(\Gamma_0(2)) = 3, \quad \epsilon_2(\Gamma_0(2)) = 1, \quad \epsilon_3(\Gamma_0(2)) = 0, \quad \text{and } \epsilon_{\infty}(\Gamma_0(2)) = 2.$$

From Theorem 3.1.1 of [6], the genus of modular curve $X(\Gamma_0(2))$ is

$$g = 1 + \frac{d}{12} - \frac{\epsilon_2}{4} - \frac{\epsilon_3}{3} - \frac{\epsilon_{\infty}}{2} = 0.$$

From Theorem 3.5.1 of [6], we know, for even $k \geq 4$,

$$(17) \quad \dim M_k(\Gamma_0(2)) = (k-1)(g-1) + \left\lfloor \frac{k}{4} \right\rfloor \epsilon_2 + \left\lfloor \frac{k}{3} \right\rfloor \epsilon_3 + \frac{k}{2} \epsilon_{\infty} = 1 + \left\lfloor \frac{k}{4} \right\rfloor.$$

Thus $\dim M_6(\Gamma_0(2)) = 2$. Let

$$\widetilde{E}_6(\tau) = -\frac{1}{504} + \sum_{N=1}^{\infty} \sigma_5(N)q^N \in M_6(SL_2(\mathbb{Z})) \subset M_6(\Gamma_0(2))$$

(see [16, p. 111]). By Proposition 17 of [16],

$$\widetilde{E}_6(2\tau) = -\frac{1}{504} + \sum_{N=1}^{\infty} \sigma_5\left(\frac{N}{2}\right)q^N \in M_6(\Gamma_0(2)),$$

where $\sigma_5\left(\frac{N}{2}\right) = 0$ if N is odd. Obviously, $\widetilde{E}_6(\tau)$ and $\widetilde{E}_6(2\tau)$ are linearly independent. Hence

$$(18) \quad M_6(\Gamma_0(2)) = \mathbb{C}\widetilde{E}_6(\tau) \oplus \mathbb{C}\widetilde{E}_6(2\tau).$$

Computing the first few Fourier coefficients, we get

$$(19) \quad g(\tau)^3 = \frac{1}{192}\widetilde{E}_6(\tau) - \frac{1}{24}\widetilde{E}_6(2\tau).$$

Comparing the coefficients of the Fourier expansion of (19), we get, for $N \geq 3$,

$$\begin{aligned} & \sum_{\substack{k+l+h=N \\ k,l,h>0}} \sigma_{1,1}(k;2)\sigma_{1,1}(l;2)\sigma_{1,1}(h;2) \\ & + 3 \cdot \frac{1}{24} \sum_{\substack{k+l=N \\ k,l>0}} \sigma_{1,1}(k;2)\sigma_{1,1}(l;2) + 3\left(\frac{1}{24}\right)^2 \sigma_{1,1}(N;2) \\ & = \frac{1}{192}\sigma_5(N) - \frac{1}{24}\sigma_5\left(\frac{N}{2}\right). \end{aligned}$$

Combining this with Eq. (11), we get

$$\begin{aligned} & \sum_{\substack{k+l+h=N \\ k,l,h>0}} \sigma_{1,1}(k;2)\sigma_{1,1}(l;2)\sigma_{1,1}(h;2) \\ & = \frac{1}{192}\sigma_5(N) - \frac{1}{24}\sigma_5\left(\frac{N}{2}\right) - \frac{1}{8} \left[\frac{1}{24} \{11\sigma_3(N) - \sigma_3(2N) - 2\sigma_{1,1}(N;2)\} \right] \\ & \quad - \frac{1}{192}\sigma_{1,1}(N;2) \\ & = -\frac{29}{768}\sigma_5(N) + \frac{1}{768}\sigma_5(2N) - \frac{11}{192}\sigma_3(N) + \frac{1}{192}\sigma_3(2N) + \frac{1}{192}\sigma_{1,1}(N;2). \end{aligned}$$

The last equality is derived by using Proposition 3.4.

This yields the following theorem.

Theorem 3.7. *For $N \geq 3$,*

$$\sum_{\substack{k+l+h=N \\ k,l,h>0}} \sigma_{1,1}(k;2)\sigma_{1,1}(l;2)\sigma_{1,1}(h;2)$$

$$= -\frac{29}{768}\sigma_5(N) + \frac{1}{768}\sigma_5(2N) - \frac{11}{192}\sigma_3(N) + \frac{1}{192}\sigma_3(2N) + \frac{1}{192}\sigma_{1,1}(N; 2).$$

Remark 3.8. By Theorem 3.7, for $N \geq 3$,

$$\begin{aligned} & \sum_{\substack{k+l+h=N \\ k,l,h>0}} k\sigma_{1,1}(k; 2)\sigma_{1,1}(l; 2)\sigma_{1,1}(h; 2) \\ &= \frac{1}{3} \sum_{\substack{k+l+h=N \\ k,l,h>0}} (k+l+h)\sigma_{1,1}(k; 2)\sigma_{1,1}(l; 2)\sigma_{1,1}(h; 2) \\ &= \frac{N}{3} \left[-\frac{29}{768}\sigma_5(N) + \frac{1}{768}\sigma_5(2N) - \frac{11}{192}\sigma_3(N) + \frac{1}{192}\sigma_3(2N) + \frac{1}{192}\sigma_{1,1}(N; 2) \right]. \end{aligned}$$

This reproves the formula in [20, p. 133].

Corollary 3.9. *Let $p = 2q + 1$ be an odd prime integer.*

- (a) $\sum_{k=1}^{p-1} \sigma_{1,1}(k; 2)\sigma_{1,1}(p-k; 2) = \sum_{k=1}^q 2k^2$.
- (b) $\sum_{k=1}^{p-1} k\sigma_{1,1}(k; 2)\sigma_{1,1}(p-k; 2) = p \sum_{k=1}^q k^2$.
- (c) $\sum_{k+l+h=p} \sigma_{1,1}(k; 2)\sigma_{1,1}(l; 2)\sigma_{1,1}(h; 2) = (\sum_{k=1}^q k)(\sum_{k=1}^q k^2)$.
- (d) $\sum_{k+l+h=p} k\sigma_{1,1}(k; 2)\sigma_{1,1}(l; 2)\sigma_{1,1}(h; 2) = (\sum_{k=1}^q k^2)^2$.

Proof. From (11), Lemma 3.5, Theorem 3.7 and Remark 3.8, we can deduce the proof. \square

Example 3.10. The first thirteen values of $\alpha(X) := \sum_{k=1}^{2X} \sigma_{1,1}(k; 2)\sigma_{1,1}(2X+1-k; 2)$ and $\beta(X) := \sum_{k=1}^X 2k^2$ are given in Table 2.

Table 2. Examples for $\alpha(X)$ and $\beta(X)$.

X	1	2	3	4	5	6	7	8	9	10	11	12	13
$\alpha(X)$	2	10	28	62	110	182	292	408	570	800	1012	1310	1700
$\beta(X)$	2	10	28	60	110	182	280	408	570	770	1012	1300	1638

We can see when $2X+1$ is prime, $\sum_{k=1}^{2X} \sigma_{1,1}(k; 2)\sigma_{1,1}(2X+1-k; 2)$ coincides with $\sum_{k=1}^X 2k^2$ in Figure 2. A similar result for consecutive integers can be found in [15, (2.10)].

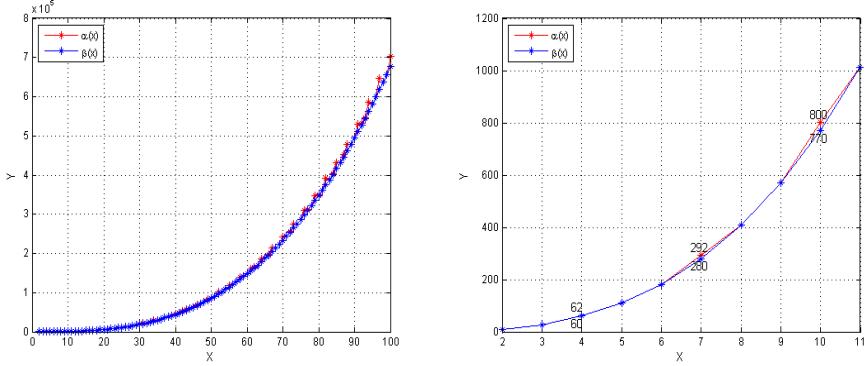
Considering Theorem 3.6 from another point of view, we get the following formula.

Proposition 3.11. *For an odd $N \geq 3$, we have*

$$\sum_{k=1}^{N-1} [\sigma_3(2k) - \sigma_3(k)]\sigma_{1,1}(N-k; 2) = \frac{1}{3}[\sigma_5(N) - \sigma_3(N)].$$

Proof. Precisely, the coefficients of

$$\sum_{k,l,m=1}^{\infty} \sigma_{1,1}(2k-1; 2)\sigma_{1,1}(2l-1; 2)\sigma_{1,1}(2m; 2)q^{2(k+l+m-1)}$$

FIGURE 2. $\alpha(X)$ and $\beta(X)$.

can be written as

$$(20) \quad \begin{aligned} & \sum_{k=1}^{N-1} \sum_{n=1}^k \sigma_{1,1}(2n-1; 2) \sigma_{1,1}(2k-(2n-1); 2) \sigma_{1,1}(2N-2k; 2) \\ & = \sum_{k=1}^{N-1} \frac{1}{8} [\sigma_3(2k) - \sigma_3(k)] \sigma_{1,1}(2N-2k; 2). \end{aligned}$$

Since Eq. (20) equates with Theorem 13 for an odd N , we have

$$\sum_{k=1}^{N-1} \frac{1}{8} [\sigma_3(2k) - \sigma_3(k)] \sigma_{1,1}(2N-2k; 2) = \frac{1}{24} [\sigma_5(N) - \sigma_3(N)],$$

which concludes the proof. \square

Theorem 3.12. *If $N (\geq 3)$ is an odd integer, then*

$$\sum_{k=1}^{N-1} \sigma_3(k) \sigma_{1,1}(N-k; 2) = \frac{1}{240} \{11\sigma_5(N) - 10\sigma_3(N) - \sigma_1(N)\}.$$

Proof. Now let us look at the formula

$$\wp'(z)^2 = 4\wp(z)^3 - g_2(\tau)\wp(z) - g_3(\tau), \quad \tau \in \mathcal{H},$$

more closely. Since $\wp'(\frac{\tau}{2}) = 0$, we get

$$(21) \quad 4\wp(\frac{\tau}{2})^3 = \frac{(2\pi)^4}{2^2 \cdot 3} E_4(\tau) \wp(\frac{\tau}{2}) + \frac{(2\pi)^6}{2^3 \cdot 3^3} E_6(\tau).$$

Substituting

$$\wp(\frac{\tau}{2}) = -\frac{\pi^2}{3} (1 + 24 \sum_{N=1}^{\infty} \sigma_{1,1}(N; 2) q^N),$$

$$\begin{aligned} E_4(\tau) &= 1 + 240 \sum_{N=1}^{\infty} \sigma_3(N) q^{2N}, \\ E_6(\tau) &= 1 - 504 \sum_{N=1}^{\infty} \sigma_5(N) q^{2N} \end{aligned}$$

into Eq. (21), we obtain

$$\begin{aligned} (22) \quad B &:= (1 + 24 \sum_{N=1}^{\infty} \sigma_{1,1}(N; 2) q^N)^3 \\ &= 3(1 + 240 \sum_{N=1}^{\infty} \sigma_3(N/2) q^N)(1 + 24 \sum_{N=1}^{\infty} \sigma_{1,1}(N; 2) q^N) \\ &\quad - 2(1 - 504 \sum_{N=1}^{\infty} \sigma_5(N/2) q^N). \end{aligned}$$

After applying the usual computation, we will compare the coefficient of q^{2N} and q^{2N-1} on the left-hand side and right-hand side of Eq. (22), separately.

For q^{2N-1} , we can deduce that

$$\begin{aligned} &17280 \sum_{k=1}^N \sigma_3(k+1) \sigma_1(2(N-k)+1) \\ &= -72\sigma_1(2N+3) - 17280\sigma_1(2N+1) + 720\sigma_5(2N+3). \end{aligned}$$

For q^{2N} , we have

$$\begin{aligned} (23) \quad &240 \sum_{k=1}^{N-1} \sigma_3(N) \sigma_{1,1}(N-k; 2) \\ &= -\sigma_{1,1}(N; 2) - 10\sigma_3(N) - 21\sigma_5(N) + 32\sigma_1(2N-1) \\ &\quad + 32 \sum_{k=1}^{N-1} \sigma_1(2(N-k)-1) \sigma_3(2k+1), \end{aligned}$$

where N is any positive integer. From (15),

$$\sum_{k=1}^{N-1} \sigma_1(2(N-k)-1) \sigma_3(2k+1) = \sigma_5(N) - \sigma_1(2N-1).$$

Therefore, (23) is further recalculated and the proof is complete. \square

Remark 3.13. Theorem 3.12 can be proved directly by modular forms. Moreover, the proof given below also generalizes Theorem 3.12 to the case of N being even.

From pp. 18–19 of [6], we know that

$$(24) \quad g(\tau) = \frac{1}{24} + \sum_{N=1}^{\infty} \sigma_{1,1}(N; 2) q^N \in M_2(\Gamma_0(2)).$$

Let

$$(25) \quad \widetilde{E}_4(\tau) = \frac{1}{240} + \sum_{N=1}^{\infty} \sigma_3(N)q^N \in M_4(SL_2(\mathbb{Z})) \subset M_4(\Gamma_0(2)).$$

It is easy to see that

$$g(\tau)\widetilde{E}_4(\tau) \in M_6(\Gamma_0(2)).$$

From (18), $g\widetilde{E}_4$ is a linear combination of $\widetilde{E}_6(\tau)$ and $\widetilde{E}_6(2\tau)$. A direct computation shows that

$$(26) \quad g\widetilde{E}_4 = \frac{11}{240}\widetilde{E}_6(\tau) - \frac{2}{15}\widetilde{E}_6(2\tau).$$

Comparing the coefficients of the Fourier expansion in (26), we get

$$(27) \quad \sum_{k=1}^{N-1} \sigma_{1,1}(k; 2)\sigma_3(N-k) = \frac{11}{240}\sigma_5(N) - \frac{2}{15}\sigma_5\left(\frac{N}{2}\right) - \frac{1}{24}\sigma_3(N) - \frac{1}{240}\sigma_{1,1}(N; 2)$$

for $N \geq 2$.

4. Special convolution sums

Lemma 4.1. *Let N be any positive integer. Then we have the following:*

(a)

$$\begin{aligned} & \sum_{k=1}^{N-1} \sigma_{1,1}(k; 2)\sigma_3(N-k) \\ &= \frac{1}{240}(11\sigma_5(N) - 32\sigma_5\left(\frac{N}{2}\right) - 10\sigma_3(N) - \sigma_1(N) + 2\sigma_1\left(\frac{N}{2}\right)). \end{aligned}$$

(b)

$$\begin{aligned} & \sum_{k=1}^{N-1} \sigma_{1,1}(k; 2)\sigma_{3,0}(N-k; 2) \\ &= \frac{1}{30}\sigma_5(N) - \frac{11}{15}\sigma_5\left(\frac{N}{2}\right) - \frac{1}{3}\sigma_3\left(\frac{N}{2}\right) - \frac{1}{30}\sigma_1(N) + \frac{1}{15}\sigma_1\left(\frac{N}{2}\right). \end{aligned}$$

(c)

$$\begin{aligned} & \sum_{k=1}^{N-1} \sigma_{1,1}(k; 2)\sigma_{3,1}(N-k; 2) \\ &= \frac{1}{80}\sigma_5(N) + \frac{3}{5}\sigma_5\left(\frac{N}{2}\right) - \frac{1}{24}\sigma_3(N) + \frac{1}{3}\sigma_3\left(\frac{N}{2}\right) + \frac{7}{240}\sigma_1(N) \\ & \quad - \frac{7}{120}\sigma_1\left(\frac{N}{2}\right). \end{aligned}$$

Proof. (a) We refer to Eq. (27).

(b) We consider the convolution sum $\sum_{k=1}^{N-1} \sigma_{1,1}(k; 2) \sigma_{3,0}(N-k; 2)$ when N is odd or even. Let N be even. Given the fact that $\sigma_{3,0}(\text{odd}; 2) = 0$ and $\sigma_{3,0}(2N-2k; 2) = 8\sigma_3(N-k)$, we can obtain

$$\begin{aligned} & \sum_{k=1}^{2N-1} \sigma_{1,1}(k; 2) \sigma_{3,0}(2N-k; 2) \\ &= \sum_{k=1}^{N-1} \sigma_{1,1}(2k; 2) \sigma_{3,0}(2N-2k; 2) \\ &= 8 \left\{ \sum_{k=1}^{N-1} \sigma_1(k) \sigma_3(N-k) - \sum_{k=1}^{N-1} \sigma_{1,0}(k; 2) \sigma_3(N-k) \right\}. \end{aligned}$$

Then

$$\begin{aligned} \sum_{k=1}^{N-1} \sigma_{1,0}(k; 2) \sigma_3(N-k) &= \sum_{k=1}^{\frac{N}{2}-1} \sigma_{1,0}(2k; 2) \sigma_3(N-2k) \\ &= 2 \sum_{k=1}^{\frac{N}{2}-1} \sigma_1(k) \sigma_3(N-2k) \end{aligned}$$

by $\sigma_{1,0}(2k; 2) = 2\sigma_1(k)$, and therefore, we refer to

$$\begin{aligned} & \sum_{k<\frac{N}{2}} \sigma_1(k) \sigma_3(N-2k) \\ &= \frac{1}{240} \left\{ 5\sigma_5(N) + (10-15N)\sigma_3(N) + 16\sigma_5\left(\frac{N}{2}\right) - \sigma_1\left(\frac{N}{2}\right) \right\} \end{aligned}$$

in [13, Theorem 6]. Lastly, we use $\sigma_5(2N) = 33\sigma_5(N) - 32\sigma_5\left(\frac{N}{2}\right)$ and $\sigma_1(2N) = 3\sigma_1(N) - 2\sigma_1\left(\frac{N}{2}\right)$. Then

$$\begin{aligned} (28) \quad & \sum_{k=1}^{2N-1} \sigma_{1,1}(k; 2) \sigma_{3,0}(2N-k; 2) \\ &= \frac{1}{30}\sigma_5(2N) - \frac{11}{15}\sigma_5(N) - \frac{1}{3}\sigma_3(N) - \frac{1}{30}\sigma_1(2N) + \frac{1}{15}\sigma_1(N). \end{aligned}$$

To obtain the formula for an odd N , we use Hahn's proof

$$(29) \quad 16 \sum_{k<N} \tilde{\sigma}_1(k) \tilde{\sigma}_3(N-k) = -\tilde{\sigma}_5(N) + 2(N-1)\tilde{\sigma}_3(N) + \tilde{\sigma}_1(N)$$

in [12, p. 12]. In (29), let us consider the left-hand side

$$\begin{aligned}
 & \sum_{k=1}^{N-1} \tilde{\sigma}_1(k) \tilde{\sigma}_3(N-k) \\
 &= \sum_{k=1}^{N-1} [\sigma_{1,1}(k; 2) - \sigma_{1,0}(k; 2)][\sigma_{3,1}(N-k; 2) - \sigma_{3,0}(N-k; 2)] \\
 (30) \quad &= \sum_{k=1}^{N-1} [\sigma_{1,1}(k; 2) - \sigma_{1,0}(k; 2)][-\sigma_3(N-k) + 2\sigma_{3,1}(N-k; 2)] \\
 &= - \sum_{k=1}^{N-1} \sigma_{1,1}(k; 2) \sigma_3(N-k) + 2 \sum_{k=1}^{N-1} \sigma_{1,1}(k; 2) \sigma_{3,1}(N-k; 2) \\
 &\quad + \sum_{k=1}^{N-1} \sigma_{1,0}(k; 2) \sigma_3(N-k) - 2 \sum_{k=1}^{N-1} \sigma_{1,0}(k; 2) \sigma_{3,1}(N-k; 2).
 \end{aligned}$$

Then

$$\begin{aligned}
 & - \sum_{k=1}^{N-1} \sigma_{1,1}(k; 2) \sigma_3(N-k) \\
 &= - \sum_{k=1}^{N-1} [\sigma_1(k) - \sigma_{1,0}(k; 2)] \sigma_3(N-k) \\
 &= - \sum_{k=1}^{N-1} \sigma_1(k) \sigma_3(N-k) + \sum_{k=1}^{\frac{N-1}{2}} \sigma_{1,0}(2k; 2) \sigma_3(N-2k) \\
 &= - \frac{1}{240} [21\sigma_5(N) + (10 - 30N)\sigma_3(N) - \sigma_1(N)] \\
 &\quad + \sum_{k=1}^{\frac{N-1}{2}} \sigma_{1,0}(2k; 2) \sigma_3(N-2k), \\
 \sum_{k=1}^{N-1} \sigma_{1,0}(k; 2) \sigma_3(N-k) &= \sum_{k=1}^{\frac{N-1}{2}} \sigma_{1,0}(2k; 2) \sigma_3(N-2k),
 \end{aligned}$$

and

$$\sum_{k=1}^{N-1} \sigma_{1,0}(k; 2) \sigma_{3,1}(N-k; 2) = \sum_{k=1}^{\frac{N-1}{2}} \sigma_{1,0}(2k; 2) \sigma_{3,1}(N-2k; 2).$$

Hence we can rewrite (30) as

$$\sum_{k=1}^{N-1} \tilde{\sigma}_1(k) \tilde{\sigma}_3(N-k) = - \frac{1}{240} [21\sigma_5(N) + (10 - 30M)\sigma_3(N) - \sigma_1(N)]$$

$$+ 2 \sum_{k=1}^{N-1} \sigma_{1,1}(k; 2) \sigma_{3,1}(N-k; 2).$$

From (29), we can get the exact value of (30); therefore,

$$\begin{aligned} & \frac{1}{16} [-\sigma_5(N) + 2(N-1)\sigma_3(N) + \sigma_1(N)] \\ &= -\frac{1}{240} [21\sigma_5(N) + (10 - 30N)\sigma_3(N) - \sigma_1(N)] \\ & \quad + 2 \sum_{k=1}^{N-1} \sigma_{1,1}(k; 2) \sigma_{3,1}(N-k; 2). \end{aligned}$$

Thus

$$(31) \quad \sum_{k=1}^{N-1} \sigma_{1,1}(k; 2) \sigma_{3,1}(N-k; 2) = \frac{1}{80} \sigma_5(N) - \frac{1}{24} \sigma_3(N) + \frac{7}{240} \sigma_1(N).$$

Comparing Eq. (27) with Eq. (31), we obtain the formula

$$\sum_{k=1}^{N-1} \sigma_{1,1}(k; 2) \sigma_{3,0}(N-k; 2).$$

So the proof is complete.

(c) This follows readily from (a) and (b) that

$$\begin{aligned} & \sum_{k=1}^{N-1} \sigma_{1,1}(k; 2) \sigma_{3,1}(N-k; 2) \\ &= \sum_{k=1}^{N-1} \sigma_{1,1}(k; 2) \sigma_3(N-k) - \sum_{k=1}^{N-1} \sigma_{1,1}(k; 2) \sigma_{3,0}(N-k; 2) \\ &= \left\{ \frac{1}{240} (11\sigma_5(N) - 32\sigma_5(\frac{N}{2}) - 10\sigma_3(N) - \sigma_1(N) + 2\sigma_1(\frac{N}{2})) \right\} \\ & \quad - \left\{ \frac{1}{30}\sigma_5(N) - \frac{11}{15}\sigma_5(\frac{N}{2}) - \frac{1}{3}\sigma_3(\frac{N}{2}) - \frac{1}{30}\sigma_1(N) + \frac{1}{15}\sigma_1(\frac{N}{2}) \right\}. \quad \square \end{aligned}$$

Corollary 4.2. Let $N \geq 3$ be odd. Then

- (a) $\sum_{k=1}^{N-1} \sigma_{1,1}(2k; 2) \sigma_{3,0}(N-k; 2) = \frac{1}{30} [\sigma_5(N) - \sigma_1(N)].$
- (b) $\sum_{k=1}^{\frac{N-1}{2}} \sigma_{1,1}(k; 2) \sigma_{3,0}(N-2k; 2) = 0.$
- (c) $\sum_{k=1}^{N-1} \sigma_{3,0}(2k; 2) \sigma_{1,1}(N-k; 2) = \frac{11}{30} \sigma_5(N) - \frac{1}{3} \sigma_3(N) - \frac{1}{30} \sigma_1(N).$
- (d) $\sum_{k=1}^{N-1} \sigma_{1,1}(N-k; 2) \sigma_3(2k) = \frac{91}{240} \sigma_5(N) - \frac{3}{8} \sigma_3(N) - \frac{1}{240} \sigma_1(N).$

Proof. (a) Using $\sigma_{1,1}(2k; 2) = \sigma_{1,1}(k; 2)$ in Lemma 4.1(a), (a) is obtained.

(b) This follows from $\sigma_{3,0}(odd; 2) = 0$.

(c) By $\sigma_{3,0}(2k; 2) = 8\sigma_3(k)$, we can get

$$\begin{aligned} \sum_{k=1}^{N-1} \sigma_{3,0}(2k; 2)\sigma_{1,1}(N-k; 2) &= 8 \sum_{k=1}^{N-1} \sigma_3(k)\sigma_{1,1}(N-k; 2) \\ &= 8 \sum_{k=1}^{N-1} \sigma_{1,1}(k; 2)\sigma_3(N-k). \end{aligned}$$

Now, by Lemma 4.1(b), we can obtain

$$\sum_{k=1}^{N-1} \sigma_{3,0}(2k; 2)\sigma_{1,1}(N-k; 2) = \frac{11}{30}\sigma_5(N) - \frac{1}{3}\sigma_3(N) - \frac{1}{30}\sigma_1(N).$$

(d) In Proposition 3.11, for an odd $N \geq 3$, we found that

$$\sum_{k=1}^{N-1} [\sigma_3(2k) - \sigma_3(k)]\sigma_{1,1}(N-k; 2) = \frac{1}{3}[\sigma_5(N) - \sigma_3(N)].$$

From this equation, we can deduce

$$\sum_{k=1}^{N-1} \sigma_3(2k)\sigma_{1,1}(N-k; 2) = \frac{1}{3}[\sigma_5(N) - \sigma_3(N)] + \sum_{k=1}^{N-1} \sigma_{1,1}(k; 2)\sigma_3(N-k).$$

The last term is obtained by applying Lemma 4.1(b). \square

Lemma 4.3. *Let N be any positive integer. Then we have the following:*

(a)

$$\begin{aligned} &\sum_{k=1}^{N-1} \sigma_{1,0}(k; 2)\sigma_3(N-k) \\ &= \frac{1}{120}[5\sigma_5(N) + (10 - 15N)\sigma_3(N) + 16\sigma_5(\frac{N}{2}) - \sigma_1(\frac{N}{2})]. \end{aligned}$$

(b)

$$\begin{aligned} &\sum_{k=1}^{N-1} \sigma_{1,0}(k; 2)\sigma_{3,0}(N-k; 2) \\ &= \frac{1}{15}[21\sigma_5(\frac{N}{2}) + (10 - 15N)\sigma_3(\frac{N}{2}) - \sigma_1(\frac{N}{2})]. \end{aligned}$$

(c)

$$\begin{aligned} &\sum_{k=1}^{N-1} \sigma_{1,0}(k; 2)\sigma_{3,1}(N-k; 2) \\ &= \frac{1}{24}\sigma_5(N) - \frac{19}{15}\sigma_5(\frac{N}{2}) + \frac{2-3N}{24}\sigma_3(N) + \frac{3N-2}{3}\sigma_3(\frac{N}{2}) \\ &\quad + \frac{7}{120}\sigma_1(\frac{N}{2}). \end{aligned}$$

Proof. (a) We know that

$$\begin{aligned} \sum_{k=1}^{N-1} \sigma_{1,0}(k; 2) \sigma_3(N-k) &= \sum_{k<N/2} \sigma_{1,0}(2k; 2) \sigma_3(N-2k) \\ &= 2 \sum_{k<N/2} \sigma_1(k) \sigma_3(N-2k). \end{aligned}$$

From (1), we obtain the desired result.

(b) If N is an odd integer, then

$$\sigma_{1,0}(k; 2) \sigma_{3,0}(N-k; 2) = \sigma_{1,0}(2k; 2) \sigma_{3,0}(N-2k; 2)$$

is zero since $\sigma_{3,0}(\text{odd}; 2) = 0$. And if $N = 2L$ is an even integer, then

$$\begin{aligned} \sum_{k=1}^{2L-1} \sigma_{1,0}(k; 2) \sigma_{3,0}(2L-k; 2) &= \sum_{k=1}^{L-1} \sigma_{1,0}(2k; 2) \sigma_{3,0}(2L-2k; 2) \\ &= 16 \sum_{k=1}^{L-1} \sigma_1(k) \sigma_3(L-k) \end{aligned}$$

since $\sigma_{1,0}(2k; 2) = 2\sigma_1(k)$ and $\sigma_{3,0}(2k; 2) = 8\sigma_3(k)$. Next we refer to

$$(32) \quad \sum_{k=1}^{N-1} \sigma_1(k) \sigma_3(N-k) = \frac{1}{240} [21\sigma_5(N) + (10 - 30N)\sigma_3(N) - \sigma_1(N)]$$

in [13] and obtain

$$\sum_{k=1}^{2L-1} \sigma_{1,0}(k; 2) \sigma_{3,0}(2L-k; 2) = \frac{1}{15} [21\sigma_5(L) + (10 - 30L)\sigma_3(L) - \sigma_1(L)].$$

(c) This is directly derived from (a) and (b). \square

Corollary 4.4. Let $N \geq 3$ be odd. Then

- (a) $\sum_{k=1}^{N-1} k \sigma_{1,0}(2k; 2) \sigma_3(N-k) = \frac{1}{120} [7N\sigma_5(N) - 6N^2\sigma_3(N) - N\sigma_1(N)]$.
- (b) $\sum_{k=1}^{N-1} k^2 \sigma_{1,0}(2k; 2) \sigma_3(N-k) = \frac{1}{40} N^2 \sigma_5(N) - \frac{1}{60} N^3 \sigma_3(N) - \frac{1}{120} N^2 \sigma_1(N)$.

Proof. (a) Using $\sigma_{1,0}(2k; 2) = 2\sigma_1(k)$, we deduce that

$$\sum_{k=1}^{N-1} k \sigma_{1,0}(2k; 2) \sigma_3(N-k) = 2 \sum_{k=1}^{N-1} k \sigma_1(k) \sigma_3(N-k).$$

Then we can use

$$\sum_{k=1}^{N-1} k \sigma_1(k) \sigma_3(N-k) = \frac{1}{240} [7N\sigma_5(N) - 6N^2\sigma_3(N) - N\sigma_1(N)]$$

in [13, (3.13)].

(b) The proof is similar to that of (a), except that we refer to

$$\sum_{k=1}^{N-1} k^2 \sigma_1(k) \sigma_3(N-k) = \frac{1}{80} N^2 \sigma_5(N) - \frac{1}{120} N^3 \sigma_3(N) - \frac{1}{240} N^2 \sigma_1(N),$$

in [20, p. 156]. \square

We give the following proposition in order to grasp Theorem 4.6.

Proposition 4.5. *Let N be any positive integer. Then we have the following:*

(a)

$$\begin{aligned} & \sum_{k=1}^{N-1} \sigma_1(k) \sigma_3\left(\frac{N-k}{2}\right) \\ &= \frac{1}{240} \left\{ \sigma_5(N) + 20\sigma_5\left(\frac{N}{2}\right) + (10 - 30N)\sigma_3\left(\frac{N}{2}\right) - \sigma_1(N) \right\}. \end{aligned}$$

(b)

$$\begin{aligned} & \sum_{k=1}^{N-1} \sigma_1\left(\frac{k}{2}\right) \sigma_3(N-k) \\ &= \frac{1}{48} \sigma_5(N) + \frac{1}{15} \sigma_5\left(\frac{N}{2}\right) + \frac{(2-3N)}{48} \sigma_3(N) - \frac{1}{240} \sigma_1\left(\frac{N}{2}\right). \end{aligned}$$

(c)

$$\begin{aligned} & \sum_{k=1}^{N-1} \sigma_1\left(\frac{k}{2}\right) \sigma_3\left(\frac{N-k}{2}\right) \\ &= \frac{1}{240} [21\sigma_5\left(\frac{N}{2}\right) + (10 - 15N)\sigma_3\left(\frac{N}{2}\right) - \sigma_1\left(\frac{N}{2}\right)]. \end{aligned}$$

Proof. (a) From [13, p. 25], we see that an odd N satisfies

$$(33) \quad \sum_{k=1}^{N-1} \sigma_1(k) \sigma_3\left(\frac{N-k}{2}\right) = \frac{1}{240} \sigma_5(N) - \frac{1}{240} \sigma_1(N)$$

and that an even N , i.e., $N = 2L$, yields

$$\begin{aligned} (34) \quad & \sum_{k=1}^{2L-1} \sigma_1(k) \sigma_3\left(\frac{2L-k}{2}\right) = \sum_{k=1}^{L-1} \sigma_{1,1}(k; 2) \sigma_3(L-k) + 2 \sum_{k=1}^{L-1} \sigma_1(k) \sigma_3(L-k) \\ &= \frac{1}{240} \{20\sigma_5(L) + \sigma_5(2L) + (10 - 60L)\sigma_3(L) - \sigma_1(2L)\} \end{aligned}$$

by Lemma 4.1(b) and (32). Thus we combine (33) and (34) to complete the proof.

(b) This is written as

$$\sum_{k=1}^{N-1} \sigma_1\left(\frac{k}{2}\right) \sigma_3(N-k) = \sum_{k < N/2} \sigma_1(k) \sigma_3(N-2k),$$

so refer to (1).

(c) We can obtain the desired result using (32). \square

Theorem 4.6. *Let N be any positive integer. If m ($0 \leq m \leq n$) is any nonnegative integer with $n \in \mathbb{N} \cup \{0\}$, then the following assertions hold:*

(a)

$$\begin{aligned} & \sum_{k=1}^{N-1} \sigma_1(2^{n-m}k) \sigma_3(2^n(N-k)) \\ &= \frac{1}{1680} \{(3 - 2^{n-m+4} - 5 \cdot 2^{3n+4} + 15 \cdot 2^{4n-m+4}) \sigma_5(N) \\ & \quad + 2^{-m+4} (9 \cdot 2^m + 2^n + 5 \cdot 2^{3n+m} - 15 \cdot 16^n) \sigma_5\left(\frac{N}{2}\right) \\ & \quad - 10(8^{n+1} - 1)(-1 + 3 \cdot 2^{n-m}N) \sigma_3(N) \\ & \quad + 80(8^n - 1)(-1 + 3 \cdot 2^{n-m}N) \sigma_3\left(\frac{N}{2}\right) \\ & \quad - 7(2^{n-m+1} - 1) \sigma_1(N) + 14(2^{n-m} - 1) \sigma_1\left(\frac{N}{2}\right)\}. \end{aligned}$$

(b)

$$\begin{aligned} & \sum_{k=1}^{N-1} \sigma_3(2^{n-m}k) \sigma_1(2^n(N-k)) \\ &= \frac{1}{1680} \{(15 \cdot 2^{4n-3m+4} - 5 \cdot 2^{3n-3m+4} - 2^{n+4} + 3) \sigma_5(N) \\ & \quad + 16(5 \cdot 2^{3(n-m)} - 15 \cdot 2^{4n-3m} + 2^n + 9) \sigma_5\left(\frac{N}{2}\right) \\ & \quad + 10(2^{3(n-m+1)} - 1)(1 - 3 \cdot 2^n N) \sigma_3(N) \\ & \quad + 80(2^{3(n-m)} - 1)(-1 + 3 \cdot 2^n N) \sigma_3\left(\frac{N}{2}\right) \\ & \quad - 7(2^{n+1} - 1) \sigma_1(N) + 14(2^n - 1) \sigma_1\left(\frac{N}{2}\right)\}. \end{aligned}$$

Proof. (a) We use induction on m to obtain the general formula for $\sigma_1(2^m k)$. If $m = 1$, then

$$\sigma_1(2k) = 3\sigma_1(k) - 2\sigma_1\left(\frac{k}{2}\right).$$

If $m = 2$, then

$$\sigma_1(2^2 k) = (3^2 - 2)\sigma_1(k) - 3 \cdot 2\sigma_1\left(\frac{k}{2}\right).$$

Continuing this process, we get

$$\sigma_1(2^m k) = (2^{m+1} - 1)\sigma_1(k) + (2 - 2^{m+1})\sigma_1\left(\frac{k}{2}\right).$$

Similarly, we obtain

$$\sigma_3(2^n(N - k)) = \frac{8^{n+1} - 1}{7}\sigma_3(N - k) + \frac{8 - 8^{n+1}}{7}\sigma_3\left(\frac{N - k}{2}\right).$$

Therefore, we know that

$$\begin{aligned} & \sum_{k=1}^{N-1} \sigma_1(2^m k) \sigma_3(2^n(N - k)) \\ &= \sum_{k=1}^{N-1} \{(2^{m+1} - 1)\sigma_1(k) + (2 - 2^{m+1})\sigma_1\left(\frac{k}{2}\right)\} \\ & \quad \left\{ \frac{8^{n+1} - 1}{7}\sigma_3(N - k) + \frac{8 - 8^{n+1}}{7}\sigma_3\left(\frac{N - k}{2}\right) \right\} \\ (35) \quad &= (2^{m+1} - 1) \cdot \frac{8^{n+1} - 1}{7} \sum_{k=1}^{N-1} \sigma_1(k) \sigma_3(N - k) \\ & \quad + (2^{m+1} - 1) \cdot \frac{8 - 8^{n+1}}{7} \sum_{k=1}^{N-1} \sigma_1(k) \sigma_3\left(\frac{N - k}{2}\right) \\ & \quad + (2 - 2^{m+1}) \cdot \frac{8^{n+1} - 1}{7} \sum_{k=1}^{N-1} \sigma_1\left(\frac{k}{2}\right) \sigma_3(N - k) \\ & \quad + (2 - 2^{m+1}) \cdot \frac{8 - 8^{n+1}}{7} \sum_{k=1}^{N-1} \sigma_1\left(\frac{k}{2}\right) \sigma_3\left(\frac{N - k}{2}\right). \end{aligned}$$

Then we use Proposition 4.5 and change $m \rightarrow n - m$.

(b) This is similar to Theorem 4.6(a), but we refer to

$$\begin{aligned} \sum_{k=1}^{N-1} \sigma_3(k) \sigma_1(N - k) &= \sum_{t=1}^{N-1} \sigma_3(N - t) \sigma_1(t), \\ \sum_{k=1}^{N-1} \sigma_3(k) \sigma_1\left(\frac{N - k}{2}\right) &= \sum_{t=1}^{N-1} \sigma_3(N - 2t) \sigma_1(t), \\ \sum_{k=1}^{N-1} \sigma_3\left(\frac{k}{2}\right) \sigma_1(N - k) &= \sum_{t=1}^{N-1} \sigma_3(t) \sigma_1(N - 2t), \end{aligned}$$

$$\sum_{k=1}^{N-1} \sigma_3\left(\frac{k}{2}\right) \sigma_1\left(\frac{N-k}{2}\right) = \sum_{t=1}^{N-1} \sigma_3\left(\frac{N}{2}-t\right) \sigma_1(t).$$

□

Corollary 4.7. *Let N be any positive integer. Then*

$$\sum_{k=1}^N \sigma_1(2k-1) \sigma_3(2N-2k+1) = \frac{1}{32} (\sigma_5(2N) - \sigma_5(N)).$$

In particular, if N is odd, then

$$\sum_{k=1}^N \sigma_1(2k-1) \sigma_3(2N-2k+1) = \sigma_5(N),$$

which is also the result of Huard, Ou, Spearman, and Williams [13, Corollary 3].

Proof. Consider

$$\begin{aligned} & \sum_{k=1}^{2N-1} \sigma_1(k) \sigma_3(2N-k) \\ &= \sum_{k=1}^N \sigma_1(2k-1) \sigma_3(2N-2k+1) + \sum_{k=1}^{N-1} \sigma_1(2k) \sigma_3(2N-2k). \end{aligned}$$

Then by replacing N with $2N$ in [13, (3.12)] and $n = 1, m = 0$ in

$$\sum_{k=1}^{N-1} \sigma_1(2^{n-m}k) \sigma_3(2^n(N-k)),$$

we get

$$\begin{aligned} & \sum_{k=1}^N \sigma_1(2k-1) \sigma_3(2N-2k+1) \\ &= \frac{1}{240} \{21\sigma_5(2N) + (10 - 60N)\sigma_3(2N) - \sigma_1(2N)\} \\ &\quad - \left\{ \frac{151}{80}\sigma_5(N) - \frac{9}{5}\sigma_5\left(\frac{N}{2}\right) + \frac{3}{8}\sigma_3(N) - \frac{9N}{4}\sigma_3(N) - \frac{1}{3}\sigma_3\left(\frac{N}{2}\right) + 2N\sigma_3\left(\frac{N}{2}\right) \right. \\ &\quad \left. - \frac{1}{80}\sigma_1(N) + \frac{1}{120}\sigma_1\left(\frac{N}{2}\right) \right\}. \end{aligned}$$

Then we use $\sigma_5(2N) = 33\sigma_5(N) - 32\sigma_5\left(\frac{N}{2}\right)$, $\sigma_3(2N) = 9\sigma_3(N) - 8\sigma_3\left(\frac{N}{2}\right)$, and $\sigma_1(2N) = 3\sigma_1(N) - 2\sigma_1\left(\frac{N}{2}\right)$. □

Corollary 4.8. *Let N be any positive integer with $n \in \mathbb{N} \cup \{0\}$. Then we have the following:*

(a)

$$\begin{aligned}
& \sum_{k=1}^{N-1} \sigma_{1,0}(2^{n-m}k; 2) \sigma_3(2^n(N-k)) \\
&= \frac{1}{840 \cdot 2^m} [(3 \cdot 2^m - 2^{n+3} - 5 \cdot 2^{3n+m+4} + 15 \cdot 2^{4n+3}) \sigma_5(N) \\
&\quad + 8(9 \cdot 2^{m+1} + 2^n + 5 \cdot 2^{3n+m+1} - 15 \cdot 2^{4n}) \sigma_5(\frac{N}{2}) \\
&\quad + 5\{2^m(2^{3n+4} - 2) + 3 \cdot 2^n N(1 - 2^{3n+3})\} \sigma_3(N) \\
&\quad + 40(2^{3n} - 1)(3 \cdot 2^n N - 2^{m+1}) \sigma_3(\frac{N}{2}) \\
&\quad - 7(2^n - 2^m) \sigma_1(N) + 7(2^n - 2^{m+1}) \sigma_1(\frac{N}{2})].
\end{aligned}$$

(b)

$$\begin{aligned}
& \sum_{k=1}^{N-1} \sigma_{1,1}(2^{n-m}k; 2) \sigma_3(2^n(N-k)) \\
&= \frac{1}{1680} \{(5 \cdot 2^{3n+4} - 3) \sigma_5(N) - 16(5 \cdot 2^{3n} + 9) \sigma_5(\frac{N}{2}) \\
&\quad + 10(1 - 2^{3n+3}) \sigma_3(N) + 80(2^{3n} - 1) \sigma_3(\frac{N}{2}) - 7\sigma_1(N) + 14\sigma_1(\frac{N}{2})\}.
\end{aligned}$$

(c)

$$\begin{aligned}
& \sum_{k=1}^{N-1} \sigma_3(2^{n-m}k) \sigma_{1,0}(2^n(N-k); 2) \\
&= \frac{1}{840} [(3 - 2^{n+3} - 5 \cdot 2^{3n-3m+4} + 15 \cdot 2^{4n-3m+3}) \sigma_5(N) \\
&\quad + 8(18 + 2^n + 5 \cdot 2^{3n-3m+1} - 15 \cdot 2^{4n-3m}) \sigma_5(\frac{N}{2}) \\
&\quad + 5\{-2 + 2^{3n-3m+4} - 3 \cdot 2^n N(2^{3(n-m+1)} - 1)\} \sigma_3(N) \\
&\quad + 40\{2 - 2^{3n-3m+1} + 3 \cdot 2^n N(2^{3(n-m)} - 1)\} \sigma_3(\frac{N}{2}) \\
&\quad - 7(2^n - 1) \sigma_1(N) + 7(2^n - 2) \sigma_1(\frac{N}{2})].
\end{aligned}$$

(d)

$$\begin{aligned}
& \sum_{k=1}^{N-1} \sigma_3(2^{n-m}k) \sigma_{1,1}(2^n(N-k); 2) \\
&= \frac{1}{1680} \{(5 \cdot 2^{3n-3m+4} - 3) \sigma_5(N) - 16(5 \cdot 2^{3(n-m)} + 9) \sigma_5(\frac{N}{2})
\end{aligned}$$

$$\begin{aligned} & -10(2^{3(n-m+1)} - 1)\sigma_3(N) + 80(2^{3(n-m)} - 1)\sigma_3\left(\frac{N}{2}\right) \\ & - 7\sigma_1(N) + 14\sigma_1\left(\frac{N}{2}\right)\}. \end{aligned}$$

(e)

$$\begin{aligned} & \sum_{k=1}^{N-1} \sigma_{1,1}(2^{n-m}k; 2)\sigma_{3,0}(2^n(N-k); 2) \\ & = \frac{1}{210}\{(5 \cdot 2^{3n+1} - 3)\sigma_5(N) - 2(5 \cdot 8^n + 72)\sigma_5\left(\frac{N}{2}\right) - 10(8^n - 1)\sigma_3(N) \\ & + 10(8^n - 8)\sigma_3\left(\frac{N}{2}\right) - 7\sigma_1(N) + 14\sigma_1\left(\frac{N}{2}\right)\}. \end{aligned}$$

Proof. (a) From the fact that

$$\begin{aligned} \sum_{k=1}^{N-1} \sigma_{1,0}(2^{n-m}k; 2)\sigma_3(2^n(N-k)) & = \sum_{k=1}^{N-1} 2\sigma_1(2^{n-m-1}k)\sigma_3(2^n(N-k)) \\ & = 2 \sum_{k=1}^{N-1} \sigma_1(2^{n-(m+1)}k)\sigma_3(2^n(N-k)), \end{aligned}$$

we put $m \rightarrow m+1$ in Theorem 4.6 when $m < n$. If $m = n$, then we use $2\sigma_1\left(\frac{k}{2}\right) = 3\sigma_1(k) - \sigma_1(2k)$.

(b)-(e) This is similar to Corollary 4.8(a). \square

Remark 4.9. Corollary 4.8(b) can be proved directly via Hecke operators. Let

$$g(\tau) = \frac{1}{24} + \sum_{N=1}^{\infty} \sigma_{1,1}(N; 2) \in M_2(\Gamma_0(2)),$$

where $q = e^{2\pi i\tau}$ and $\tau \in \mathcal{H}$. From (17), we know that, for even $k \geq 4$,

$$\dim M_k(\Gamma_0(2)) = 1 + \left\lfloor \frac{k}{4} \right\rfloor.$$

In particular,

$$\dim M_4(\Gamma_0(2)) = \dim M_6(\Gamma_0(2)) = 2.$$

Let

$$\begin{aligned} \widetilde{E}_4 & = \frac{1}{240} + \sum_{N=1}^{\infty} \sigma_3(N)q^N \in M_4(SL_2(\mathbb{Z})) \subset M_4(\Gamma_0(2)), \\ \widetilde{E}_6 & = -\frac{1}{504} + \sum_{N=1}^{\infty} \sigma_5(N)q^N \in M_6(SL_2(\mathbb{Z})) \subset M_6(\Gamma_0(2)) \end{aligned}$$

(see [16, p. 111]). By Proposition 17 of [16],

$$\begin{aligned}\widetilde{E}_4(2\tau) &= \frac{1}{240} + \sum_{N=1}^{\infty} \sigma_3\left(\frac{N}{2}\right)q^N \in M_4(\Gamma_0(2)), \\ \widetilde{E}_6(2\tau) &= -\frac{1}{504} + \sum_{N=1}^{\infty} \sigma_5\left(\frac{N}{2}\right)q^N \in M_6(\Gamma_0(2)),\end{aligned}$$

where $\sigma_3\left(\frac{N}{2}\right) = \sigma_5\left(\frac{N}{2}\right) = 0$ if $2 \nmid N$. Therefore,

$$\begin{aligned}M_4(\Gamma_0(2)) &= \mathbb{C}\widetilde{E}_4(\tau) \oplus \mathbb{C}\widetilde{E}_4(2\tau), \\ M_6(\Gamma_0(2)) &= \mathbb{C}\widetilde{E}_6(\tau) \oplus \mathbb{C}\widetilde{E}_6(2\tau).\end{aligned}$$

The Hecke operator,

$$T_2 : M_k(\Gamma_0(2)) \rightarrow M_k(\Gamma_0(2)),$$

is a linear map such that, for any $\sum a_N q^N \in M_k(\Gamma_0(2))$,

$$T_2\left(\sum a_N q^N\right) = \sum a_{2N} q^{2N}$$

(see Proposition 37 of Chapter 3 of [16]). It is easy to see that $\widetilde{E}_4(\tau)$ and $T_2\widetilde{E}_4(\tau)$ are linearly independent in $M_4(\Gamma_0(2))$. Taking them as a basis, we get

$$T_2(\widetilde{E}_4, T_2\widetilde{E}_4) = (\widetilde{E}_4, T_2\widetilde{E}_4) \begin{pmatrix} 0 & -8 \\ 1 & 9 \end{pmatrix}.$$

Since

$$\begin{pmatrix} 0 & -8 \\ 1 & 9 \end{pmatrix}^n = \frac{1}{7} \begin{pmatrix} 8 - 8^n & 8 - 8^{n+1} \\ -1 + 8^n & -1 + 8^{n+1} \end{pmatrix},$$

we get

$$(36) \quad T_2^n \widetilde{E}_4 = \frac{1}{7}(8 - 8^n)\widetilde{E}_4 + \frac{1}{7}(8^n - 1)T_2\widetilde{E}_4.$$

Since $g\widetilde{E}_4, gT_2\widetilde{E}_4 \in M_6(\Gamma_0(2))$, they are linear combinations of $\widetilde{E}_6(\tau)$ and $\widetilde{E}_6(2\tau)$. A direct computation shows that

$$(37) \quad g\widetilde{E}_4(\tau) = \frac{11}{240}\widetilde{E}_6(\tau) - \frac{2}{15}\widetilde{E}_6(2\tau),$$

$$(38) \quad gT_2\widetilde{E}_4(\tau) = \frac{91}{240}\widetilde{E}_6(\tau) - \frac{7}{15}\widetilde{E}_6(2\tau).$$

Combining (36), (37), and (38), we obtain

$$\begin{aligned}(39) \quad &gT_2^n \widetilde{E}_4 \\ &= \frac{1}{7}(8 - 8^n)\left(\frac{11}{240}\widetilde{E}_6(\tau) - \frac{2}{15}\widetilde{E}_6(2\tau)\right) + \frac{1}{7}(8^n - 1)T_2\left(\frac{91}{240}\widetilde{E}_6(\tau) - \frac{7}{15}\widetilde{E}_6(2\tau)\right) \\ &= \left(-\frac{1}{560} + \frac{8^n}{21}\right)\widetilde{E}_6(\tau) + \left(-\frac{9}{105} - \frac{8^n}{21}\right)\widetilde{E}_6(2\tau).\end{aligned}$$

Comparing the coefficients of q^N ($N \geq 2$) on both sides of (39), we get

$$\begin{aligned} & \sum_{\substack{k+l=N \\ k,l>0}} \sigma_{1,1}(k; 2)\sigma_3(2^n l) + \frac{1}{24}\sigma_3(2^n N) + \frac{1}{240}\sigma_{1,1}(N; 2) \\ &= \left(-\frac{1}{560} + \frac{8^n}{21}\right)\sigma_5(N) + \left(-\frac{3}{35} - \frac{8^n}{21}\right)\sigma_5\left(\frac{N}{2}\right). \end{aligned}$$

This yields Corollary 4.8(b).

5. Formulation of $\sum_{k=1}^{M-1} \tilde{\sigma}_1(2^{n-m}k)\tilde{\sigma}_3(2^n(M-k))$

In Hahn's paper, we see that

$$(40) \quad 16 \sum_{k < N} \tilde{\sigma}_1(k)\tilde{\sigma}_3(N-k) = -\tilde{\sigma}_5(N) + 2(N-1)\tilde{\sigma}_3(N) + \tilde{\sigma}_1(N)$$

by [12, Theorem 4.3]. Now we generalize (40) so that we can use Theorem 4.6 and find the convolution formula for the summation

$$\sum_{k=1}^{N-1} \tilde{\sigma}_1(2^{n-m}k)\tilde{\sigma}_3(2^n(N-k))$$

with N being odd.

Theorem 5.1. *Let N be any positive integer. If m ($0 \leq m \leq n$) is any positive integer with $n \in \mathbb{N} \cup \{0\}$, then*

(a)

$$\begin{aligned} & \sum_{k=1}^{N-1} \tilde{\sigma}_1(2^{n-m}k)\tilde{\sigma}_3(2^n(N-k)) \\ &= \frac{1}{112 \cdot 2^m} \{(16^{n+1} - 2^{3n+m+4} - 2^{n+4} + 9 \cdot 2^m)\sigma_5(N) \\ & \quad + 16(27 \cdot 2^m + 2^n + 2^{3n+m} - 16^n)\sigma_5\left(\frac{N}{2}\right) \\ & \quad + 2(2^{3(n+1)} - 15)(2^m - 2^n N)\sigma_3(N) + 16(2^{3n} - 15)(-2^m + 2^n N)\sigma_3\left(\frac{N}{2}\right) \\ & \quad + 7(3 \cdot 2^m - 2^{n+1})\sigma_1(N) - 14(3 \cdot 2^m - 2^n)\sigma_1\left(\frac{N}{2}\right)\}. \end{aligned}$$

(b)

$$\begin{aligned} & \sum_{k=1}^{N-1} \tilde{\sigma}_3(2^{n-m}k)\tilde{\sigma}_1(2^n(N-k)) \\ &= \frac{1}{112 \cdot 2^{3m}} \{(16^{n+1} - 2^{3n+4} - 2^{n+3m+4} + 9 \cdot 2^{3m})\sigma_5(N) \\ & \quad + 16(27 \cdot 2^{3m} + 2^{3n} + 2^{3m+n} - 16^n)\sigma_5\left(\frac{N}{2}\right) \\ & \quad + 2(2^{3(n+1)} - 15 \cdot 2^{3m})(1 - 2^n N)\sigma_3(N) \end{aligned}$$

$$\begin{aligned}
& + 16(15 \cdot 2^{3m} - 2^{3n})(1 - 2^n N)\sigma_3\left(\frac{N}{2}\right) \\
& - 7 \cdot 8^m(2^{n+1} - 3)\sigma_1(N) + 7 \cdot 2^{3m+1}(2^n - 3)\sigma_1\left(\frac{N}{2}\right)\}.
\end{aligned}$$

Proof. (a) If $n = 0$ in Theorem 5.1(a), then Theorem 5.1(a) is equivalent to Eq. (40). So we assume $n \geq 1$. We can expand the convolution sum as

$$\begin{aligned}
(41) \quad & \sum_{k=1}^{N-1} \tilde{\sigma}_1(2^{n-m}k)\tilde{\sigma}_3(2^n(N-k)) \\
& = \sum_{k=1}^{N-1} \{\sigma_1(2^{n-m}k) - 4\sigma_1(2^{n-m-1}k)\}\{\sigma_3(2^n(N-k)) - 16\sigma_3(2^{n-1}(N-k))\}
\end{aligned}$$

by using $\tilde{\sigma}_s(N) = \sigma_s(N) - 2^{s+1}\sigma_s(N/2)$, where $N \in \mathbb{N}$ (see [12, (1.12)]). Then the right-hand side of (41) can be written as

$$\begin{aligned}
(42) \quad & \sum_{k=1}^{N-1} \sigma_1(2^{n-m}k)\sigma_3(2^n(N-k)) - 16 \sum_{k=1}^{N-1} \sigma_1(2^{n-m}k)\sigma_3(2^{n-1}(N-k)) \\
& - 4 \sum_{k=1}^{N-1} \sigma_1(2^{n-m-1}k)\sigma_3(2^n(N-k)) + 64 \sum_{k=1}^{N-1} \sigma_1(2^{n-m-1}k)\sigma_3(2^{n-1}(N-k)) \\
& = \sum_{k=1}^{N-1} \sigma_1(2^{n-m}k)\sigma_3(2^n(N-k)) - 16 \sum_{k=1}^{N-1} \sigma_1(2^{n-m}k)\sigma_3(2^{n-1}(N-k)) \\
& + \sum_{k=1}^{N-1} \{2\sigma_1(2^{n-m+1}k) - 6\sigma_1(2^{n-m}k)\}\sigma_3(2^n(N-k)) \\
& + \sum_{k=1}^{N-1} \{-32\sigma_1(2^{n-m+1}k) + 96\sigma_1(2^{n-m}k)\}\sigma_3(2^{n-1}(N-k)),
\end{aligned}$$

where an elementary formula $\sigma_1(2N) = 3\sigma_1(N) - 2\sigma_1(N/2)$ is used with $N \in \mathbb{N}$. Now, Eq. (42) becomes

$$\begin{aligned}
(43) \quad & \sum_{k=1}^{N-1} \sigma_1(2^{n-m}k)\sigma_3(2^n(N-k)) - 16 \sum_{k=1}^{N-1} \sigma_1(2^{(n-1)-(m-1)}k)\sigma_3(2^{n-1}(N-k)) \\
& + 2 \sum_{k=1}^{N-1} \sigma_1(2^{n-(m-1)}k)\sigma_3(2^n(N-k)) - 6 \sum_{k=1}^{N-1} \sigma_1(2^{n-m}k)\sigma_3(2^n(N-k)) \\
& - 32 \sum_{k=1}^{N-1} \sigma_1(2^{(n-1)-(m-2)}k)\sigma_3(2^{n-1}(N-k))
\end{aligned}$$

$$\begin{aligned}
& + 96 \sum_{k=1}^{N-1} \sigma_1(2^{(n-1)-(m-1)} k) \sigma_3(2^{n-1}(N-k)) \\
& = -5 \sum_{k=1}^{N-1} \sigma_1(2^{n-m} k) \sigma_3(2^n(N-k)) \\
& + 80 \sum_{k=1}^{N-1} \sigma_1(2^{(n-1)-(m-1)} k) \sigma_3(2^{n-1}(N-k)) \\
& + 2 \sum_{k=1}^{N-1} \sigma_1(2^{n-(m-1)} k) \sigma_3(2^n(N-k)) \\
& - 32 \sum_{k=1}^{N-1} \sigma_1(2^{(n-1)-(m-2)} k) \sigma_3(2^{n-1}(N-k)).
\end{aligned}$$

Finally, we apply Theorem 4.6(a) to get the result.

(b) This is similar to Theorem 5.1(a). \square

Remark 5.2. Let $\sigma_s^*(N) := \sum_{\substack{d|N \\ \frac{N}{d} \text{ odd}}} d^s$. For instance, the function $\sigma_s^*(N)$ has the formula [20, p. 27]

$$(44) \quad \sigma_s^*(N) = \sigma_s(N) - \sigma_s\left(\frac{N}{2}\right).$$

Using (44), we can rewrite Theorem 5.1 as

$$\begin{aligned}
& \sum_{k=1}^{N-1} \tilde{\sigma}_1(2^{n-m} k) \tilde{\sigma}_3(2^n(N-k)) \\
& = \frac{1}{16} \{-\tilde{\sigma}_5(N) + 2(N-1)\tilde{\sigma}_3(N) + \tilde{\sigma}_1(N)\} + \frac{1}{56} (2^{n-m} - 1) \{8(8^n - 1)\sigma_5^*(N) \\
& \quad + 15N\sigma_{3,1}(N; 2) - 8^{n+1}N\sigma_3^*(N) - 7\sigma_1^*(N)\} - \frac{1}{7} (8^n - 1)(N-1)\sigma_3^*(N), \\
& \sum_{k=1}^{N-1} \tilde{\sigma}_3(2^{n-m} k) \tilde{\sigma}_1(2^n(N-k)) \\
& = \frac{1}{16} \{-\tilde{\sigma}_5(N) + 2(N-1)\tilde{\sigma}_3(N) + \tilde{\sigma}_1(N)\} \\
& \quad + \frac{1}{7} (8^{n-m} - 1) \{(2^n - 1)\sigma_5^*(N) - (2^n N - 1)\sigma_3^*(N)\} \\
& \quad + \frac{1}{8} (2^n - 1) \{N\tilde{\sigma}_3(N) - \sigma_1^*(N)\}.
\end{aligned}$$

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AERAN KIM

DEPARTMENT OF MATHEMATICS AND INSTITUTE OF PURE AND APPLIED MATHEMATICS

CHONBUK NATIONAL UNIVERSITY

CHONJU 561-756, KOREA

E-mail address: ae_ran_kim@hotmail.com

DAEYEOL KIM

NATIONAL INSTITUTE FOR MATHEMATICAL SCIENCES

YUSEONG-DAERO 1689-GIL

YUSEONG-GU, DAEJEON 305-811, KOREA

E-mail address: daeyeoul@nims.re.kr

LI YAN
DEPARTMENT OF APPLIED MATHEMATICS
CHINA AGRICULTURE UNIVERSITY
BEIJING 100083 P. R. CHINA
E-mail address: liyan_00@mails.tsinghua.edu.cn