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Convolutions of Harmonic Half-Plane Mappings with Harmonic Vertical Strip Mappings

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Abstract. In the present paper, we prove the convolutions of generalized harmonic right half-plane mappings with harmonic vertical strip mappings are univalent and convex in the horizontal direction. Moreover, some examples of harmonic univalent mappings convex in the horizontal direction are also constructed to illuminate the main results.

1. Introduction

Let $\mathcal{H}$ denote the class of all complex-valued harmonic mappings $f = h + \overline{g}$ in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = 0 = f_z(0) - 1$, where h and g are analytic in $\mathbb{U}$ and have the following power series representation

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n.$$

A function $f \in \mathcal{H}$ is locally univalent and sense-preserving in $\mathbb{U}$ if and only if the dilatation function defined by $\omega(z) = g'(z)/h'(z)$, satisfies $|\omega(z)| < 1$ for all $z \in \mathbb{U}$. Denote by $\mathcal{S}_H$ the class of all sense-preserving harmonic univalent mappings $f = h + \overline{g} \in \mathcal{H}$ and by $\mathcal{S}_H^0$ the subclass of mappings $f \in \mathcal{S}_H$ such that $f_{\overline{z}}(0) = 0$. Further, denote by $\mathcal{K}_H$ (respectively $\mathcal{K}_H^0$) the subclass of S_H (respectively $\mathcal{S}_H^0$) mapping the unit disk $\mathbb{U}$ onto convex domain.

A domain $\Omega \subset \mathbb{C}$ is said to be convex in the direction γ, $0 \leq \gamma < \pi$, if every line parallel to the line joining 0 and $e^{i\gamma}$ has a connected intersection with Ω. In particular, if $\gamma = 0$, we say that Ω is convex in horizontal direction (CHD). The basic theorem of Clunie and Sheil-Small [4] is as follows.

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Theorem A. *Let $f = h + \overline{g}$ be harmonic and locally univalent in the unit disk $\mathbb{U}$. Then f is univalent and its range is* CHD *if and only if $h - g$ has the same properties.*

For two harmonic mappings

$$f = h + \overline{g} = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n}\overline{z}^n$$

and

$$F = H + \overline{G} = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \overline{B_n}\overline{z}^n,$$

define the harmonic convolution as

$$f * F = h * H + \overline{g * G} = z + \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} \overline{b_n B_n}\overline{z}^n.$$

The properties of the harmonic convolutions is not as nice as that of the analytic convolutions. For example, unlike the case of analytic mappings, convolution of two convex harmonic mappings is not necessarily a convex harmonic mappings. But still convolutions of harmonic mappings exhibit some fascinating properties. In the recent years, there has been a considerable interest in the study of planar harmonic convolutions. See, for example [3, 6, 7, 10, 11, 13, 14, 18, 19, 22, 23]. In [1, 5] and [9], explicit descriptions are given for half-plane mappings and strip mappings. If $f = h + \overline{g} \in \mathcal{S}_H^0$ maps $\mathbb{U}$ onto the right half-plane $\mathcal{R} = \{\omega : \mathrm{Re}(\omega) > -1/2\}$, they must satisfy the following condition

$$h(z) + g(z) = \frac{z}{1-z} \quad (z \in \mathbb{U}).$$

Let $f_0 = h_0 + \overline{g_0}$ be the canonical right half-plane mapping with the dilatation $\omega_0 = g_0'/h_0' = -z$, then

$$h_0 = \frac{z - \frac{1}{2}z^2}{(1-z)^2} \quad \text{and} \quad g_0 = \frac{-\frac{1}{2}z^2}{(1-z)^2}.$$

A more general class of harmonic univalent mappings, $L_c = H_c + \overline{G_c}$ was introduced by Muir [20]:

$$\begin{aligned} L_c(z) &= H_c(z) + \overline{G_c(z)} \\ &= \frac{1}{1+c}\left[\frac{z}{1-z} + \frac{cz}{(1-z)^2}\right] + \frac{1}{1+c}\overline{\left[\frac{z}{1-z} - \frac{cz}{(1-z)^2}\right]}, \quad (z \in \mathbb{U}; c > 0). \end{aligned} \tag{1}$$

Clearly, $L_1(z) = f_0(z)$, where $f_0 = h_0 + \overline{g_0}$ with

$$h_0 = \frac{z - \frac{1}{2}z^2}{(1-z)^2} \quad \text{and} \quad g_0 = \frac{-\frac{1}{2}z^2}{(1-z)^2}.$$

It has been proved in [20] that $L_c(z)$ map the unit disk $\mathbb{U}$ onto the generalized right half-plane, $\mathcal{GR} = \{\omega : \mathrm{Re}(\omega) > -1/(1+c)\}$ for each $c > 0$. Then if F is analytic in $\mathbb{U}$ and $F(0) = 0$, we have

$$\begin{aligned} H_c(z) * F(z) &= \frac{1}{1+c}\left[F(z) + czF'(z)\right], \\ G_c(z) * F(z) &= \frac{1}{1+c}\left[F(z) - czF'(z)\right]. \end{aligned} \tag{2}$$

As in the analytic case, results on convolution are useful in deriving properties of sections of harmonic univalent mappings as demonstrated in the recent papers by Li and Ponnusamy [15–17] and also by Boyd and Dorff [2] in which one can find new problems on harmonic convolution.

Let $f_\alpha = h_\alpha + \overline{g_\alpha}$ be the subclass of harmonic mappings obtained by shearing of analytic vertical strip mapping

$$h_\alpha(z) + g_\alpha(z) = \frac{1}{2i\sin\alpha}\log\left(\frac{1+ze^{i\alpha}}{1+ze^{-i\alpha}}\right), \quad (\frac{\pi}{2} \le \alpha < \pi). \tag{3}$$

In recent years, Dorff [6] and Dorff *et al.* [7] obtained the following results, respectively.

Theorem B. *Let $f_1 = h_1 + \overline{g_1} \in \mathcal{K}_H^0$ be a right half-plane mapping and $f_\alpha = h_\alpha + \overline{g_\alpha} \in \mathcal{K}_H^0$ be a vertical strip mapping given by* (3). *If $f_1 * f_\alpha$ is locally univalent and sense-preserving, then $f_1 * f_\alpha \in \mathcal{S}_H^0$ and is convex in the horizonal direction.*

Theorem C. *Let $f_\alpha = h_\alpha + \overline{g_\alpha} \in \mathcal{K}_H^0$ be given by* (3) *with the dilatation $\omega = g'_\alpha/h'_\alpha = e^{i\theta}z^n$. If $n = 1, 2$, then $f_0 * f_\alpha \in \mathcal{S}_H^0$ and is convex in the horizontal direction.*

In this paper, we aim at proving the following results.

Theorem 1.1. *Let $L_c = H_c + \overline{G_c} \in \mathcal{K}_H^0$ be a mapping given by* (1) *and $f_\alpha = h_\alpha + \overline{g_\alpha} \in \mathcal{K}_H^0$ be a harmonic vertical strip mapping obtained from* (3) *with $\alpha \in [\frac{\pi}{2}, \pi)$ and dilatation $\omega(z) = e^{i\theta}z, (\theta \in \mathbb{R})$. Then $L_c * f_\alpha \in \mathcal{S}_H^0$ and is convex in the horizontal direction for $0 < c \le 2$.*

Theorem 1.2. *Let $L_c = H_c + \overline{G_c} \in \mathcal{K}_H^0$ be a mapping given by* (1) *and $f_{\pi/2} = h_{\pi/2} + \overline{g_{\pi/2}}$ be a harmonic vertical strip mapping obtained from* (3) *with $\alpha = \pi/2$ and dilatation $\omega(z) = g'_{\pi/2}/h'_{\pi/2} = e^{i\theta}z^n, (\theta \in \mathbb{R}, n \in \mathbb{N})$. Then $L_c * f_{\pi/2}$ is univalent and convex in the horizontal direction for $0 < c \le \frac{2}{n}$.*

Kumar *et al.* [10] defined the harmonic mappings in the right half-plane $F_a = H_a + \overline{G_a}$ given by $H_a(z) + G_a(z) = z/(1-z)$ with dilatation $\omega_a = (a-z)/(1-az), a \in (-1, 1)$. Then a calculation have the following form

$$H_a(z) = \frac{\frac{1}{1+a}z - \frac{1}{2}z^2}{(1-z)^2} \quad \text{and} \quad G_a(z) = \frac{\frac{a}{1+a}z - \frac{1}{2}z^2}{(1-z)^2}. \tag{4}$$

In [11], Kumar *et al.* proved the following result.

Theorem D. *Let F_a be given by* (4) *and $f_{\pi/2} = h_{\pi/2} + \overline{g_{\pi/2}}$ be the map obtained from* (3) *with $\alpha = \pi/2$ and dilatation $\omega(z) = g'_{\pi/2}/h'_{\pi/2} = e^{i\theta}z^n, (\theta \in \mathbb{R}, n \in \mathbb{N})$. Then $F_a * f_{\pi/2} \in \mathcal{S}_H^0$ and convex in the horizontal direction for $\frac{n-2}{n+2} \le a < 1$.*

In [11], it is proved that $F_a * f_{\pi/2} \in \mathcal{S}_H^0$ and is convex in the horizontal direction by using *Cohn's Rule* and *Schur-Cohn's algorithm* [21]. Unfortunately, the calculation is very complicated and needs to be simplified in some ways. In the present paper, we use *Cohn's Rule* and combine with the *inductive method* to prove it, which greatly simplifies the process of calculation.

2. Preliminaries

The following lemmas will be required in the proof of our main results.

Lemma 2.1. *Let $L_c = H_c + \overline{G_c} \in \mathcal{K}_H^0$ be a mapping given by* (1) *and $f_\alpha = h_\alpha + \overline{g_\alpha} \in \mathcal{K}_H^0$ be a harmonic vertical strip mapping defined by* (3) *with dilatation $\omega = g'_\alpha/f'_\alpha$. Then the dilatation of $L_c * f_\alpha$ is given by*

$$\widetilde{\omega} = \frac{[(1+c)z^2 + 2z\cos\alpha + (1-c)]\omega(1+\omega) - cz(z^2 + 2z\cos\alpha + 1)\omega'}{[(1+c) + 2z\cos\alpha + (1-c)z^2](1+\omega) - cz(z^2 + 2z\cos\alpha + 1)\omega'}. \tag{5}$$

Proof. Since

$$h_\alpha + g_\alpha = \frac{1}{2i\sin\alpha}\log\left(\frac{1+ze^{i\alpha}}{1+ze^{-i\alpha}}\right)$$

and $g'_\alpha = \omega h'_\alpha$, we can solve for h'_α and h''_α in terms of z and ω.

$$\begin{aligned} h'_\alpha &= \frac{1}{(1+\omega)(1+ze^{i\alpha})(1+ze^{-i\alpha})}, \\ h''_\alpha &= -\frac{\omega'(1+2z\cos\alpha+z^2)+2(1+\omega)(\cos\alpha+z))}{(1+\omega)^2(1+ze^{i\alpha})^2(1+ze^{-i\alpha})^2}. \end{aligned}$$

Then from (2), we obtain

$$\begin{aligned} \widetilde{\omega} &= \frac{(G_c * g_\alpha)'}{(H_c * h_\alpha)'} = \frac{(g_\alpha - czg'_\alpha)'}{(h_\alpha + czh'_\alpha)'} \\ &= \frac{(1-c)g'_\alpha - czg''_\alpha}{(1+c)h'_\alpha + czh''_\alpha} = \frac{(1-c)\omega h'_\alpha - cz(\omega' h'_\alpha + \omega h''_\alpha)}{(1+c)h'_\alpha + czh''_\alpha} \\ &= \frac{[(1+c)z^2+2z\cos\alpha+(1-c)]\omega(1+\omega) - cz(z^2+2z\cos\alpha+1)\omega'}{[(1+c)+2z\cos\alpha+(1-c)z^2](1+\omega) - cz(z^2+2z\cos\alpha+1)\omega'}. \end{aligned}$$

This completes the proof. □

By using a similar argument as in the proof of Theorem 7 in [6], we can derive the following result.

Lemma 2.2. *Let L_c be given by* (1) *and $f_\alpha = h_\alpha + \overline{g_\alpha} \in \mathcal{K}_H^0$ be given by* (3). *If $L_c * f_\alpha$ is locally univalent and sense-preserving, then $L_c * f_\alpha \in \mathcal{S}_H^0$ and is convex in the horizontal direction.*

Lemma 2.3. ([21, Cohn's Rule]) *Given a polynomial*

$$p(z) = p_0(z) = a_{n,0}z^n + a_{n-1,0}z^{n-1} + \cdots + a_{1,0}z + a_{0,0} \quad (a_{n,0} \neq 0)$$

of degree n, let

$$p^*(z) = p_0^*(z) = z^n\overline{p(1/\overline{z})} = \overline{a_{n,0}} + \overline{a_{n-1,0}}z + \cdots + \overline{a_{1,0}}z^{n-1} + \overline{a_{0,0}}z^n.$$

Denote by r and s the number of zeros of $p(z)$ inside the unit circle and on it, respectively. If $|a_{0,0}| < |a_{n,0}|$, then

$$p_1(z) = \frac{\overline{a_{n,0}}p(z) - a_{0,0}p^*(z)}{z} \tag{6}$$

is of degree $n-1$ with $r_1 = r-1$ and $s_1 = s$ the number of zeros of $p_1(z)$ inside the unit circle and on it, respectively.

Lemma 2.4. ([11, Lemma 2.3]) *Let $f_\alpha = h_\alpha + \overline{g_\alpha} \in \mathcal{K}_H^0$ be a vertical strip mapping defined by* (3) *with dilatation $\omega = g'_\alpha/h'_\alpha$ and $F_a = H_a + \overline{G_a}$ be a mapping in the right half-plane defined by* (4). *Then ω^*, the dilatation of $F_a * f_\alpha$ is given by*

$$\omega^* = \frac{2[a+(a+1)z\cos\alpha+z^2]\omega(1+\omega) - (1-a)(1+2z\cos\alpha+z^2)z\omega'}{2[1+(a+1)z\cos\alpha+az^2](1+\omega) - (1-a)(1+2z\cos\alpha+z^2)z\omega'}. \tag{7}$$

3. Proof of Theorems

In this section, we give proofs of our main results.

Proof of Theorem 1.1. By Lemma 2.2 and by Lewy's Theorem, we just need to show that $|\widetilde{\omega}| < 1, \forall z \in \mathbb{U}$.

Substituting $\omega = e^{i\theta} z$ in (5) yields

$$\begin{aligned}\widetilde{\omega} &= ze^{2i\theta}\frac{z^3+\frac{2\cos\alpha+e^{-i\theta}}{1+c}z^2+\frac{(1-c)(1+2e^{-i\theta}\cos\alpha)}{1+c}z-\frac{2c-1}{1+c}e^{-i\theta}}{1+\frac{2\cos\alpha+e^{i\theta}}{1+c}z+\frac{(1-c)(1+2e^{i\theta}\cos\alpha)}{1+c}z^2-\frac{2c-1}{1+c}e^{i\theta}z^3}\\ &= ze^{2i\theta}\frac{p(z)}{q(z)} \\ &= ze^{2i\theta}\frac{(z+A)(z+B)(z+C)}{(1+\overline{A}z)(1+\overline{B}z)(1+\overline{C}z)},\end{aligned} \tag{8}$$

where

$$p(z) = z^3 + \frac{(2\cos\alpha + e^{-i\theta})}{1+c}z^2 + \frac{(1-c)(1+2e^{-i\theta}\cos\alpha)}{1+c}z - \frac{(2c-1)}{1+c}e^{-i\theta},$$

and

$$q(z) = z^3\overline{p\,(1/\overline{z})} = 1 + \frac{(2\cos\alpha + e^{i\theta})}{1+c}z + \frac{(1-c)(1+2e^{i\theta}\cos\alpha)}{1+c}z^2 - \frac{(2c-1)}{1+c}e^{i\theta}z^3.$$

We apply *Cohn's Rule* to $p(z)$, note that $|a_{0,0}| = |-\frac{(2c-1)}{1+c}e^{-i\theta}| = |\frac{2c-1}{1+c}| < 1 = |a_{3,0}|$ for $0 < c < 2$, thus we have

$$\begin{aligned}p_1(z) &= \frac{\overline{a_{3,0}}p(z) - a_{0,0}q(z)}{z}\\ &= \frac{3c(2-c)}{(1+c)^2}\left[z^2 + \frac{2(2\cos\alpha + e^{-i\theta})}{3}z + \frac{1+2e^{-i\theta}\cos\alpha}{3}\right]\\ &= \frac{3c(2-c)}{(1+c)^2}q_1(z),\end{aligned}$$

where $q_1(z) = z^2 + \frac{2(2\cos\alpha+e^{-i\theta})}{3}z + \frac{1+2e^{-i\theta}\cos\alpha}{3}$. Since $|\frac{1+2e^{-i\theta}\cos\alpha}{3}| \le \frac{1}{3} + \frac{2}{3}|\cos\alpha| < 1$ (note that $\alpha \neq \pi$), then we use *Cohn's Rule* on $q_1(z)$ again, we get

$$p_2(z) = \frac{2}{9}\left[(4 - 2\cos^2\alpha - 2\cos\alpha\cos\theta)z + (2\cos\alpha - 4e^{-i\theta}\cos^2\alpha - e^{i\theta} + 3e^{-i\theta})\right].$$

Clearly $p_2(z)$ has one zero at

$$z_0 = \frac{2\cos\alpha - 4e^{-i\theta}\cos^2\alpha - e^{i\theta} + 3e^{-i\theta}}{4 - 2\cos^2\alpha - 2\cos\alpha\cos\theta} = \frac{\frac{1}{2}\cos\alpha - e^{-i\theta}\cos^2\alpha - \frac{1}{4}e^{i\theta} + \frac{3}{4}e^{-i\theta}}{1 - \frac{1}{2}\cos^2\alpha - \frac{1}{2}\cos\alpha\cos\theta}.$$

We show that $|z_0| \le 1$, or equivalently,

$$\left|\frac{1}{2}\cos\alpha - e^{-i\theta}\cos^2\alpha - \frac{1}{4}e^{i\theta} + \frac{3}{4}e^{-i\theta}\right|^2 \le \left|1 - \frac{1}{2}\cos^2\alpha - \frac{1}{2}\cos\alpha\cos\theta\right|^2.$$

Then

$$\begin{aligned}&\left|1 - \frac{1}{2}\cos^2\alpha - \frac{1}{2}\cos\alpha\cos\theta\right|^2 - \left|\frac{1}{2}\cos\alpha - e^{-i\theta}\cos^2\alpha - \frac{1}{4}e^{i\theta} + \frac{3}{4}e^{-i\theta}\right|^2\\ &= \left(\frac{1}{4}\cos^4\alpha + \frac{1}{2}\cos\theta\cos^3\alpha - \cos^2\alpha + \frac{1}{4}\cos^2\theta\cos^2\alpha - \cos\theta\cos\alpha + 1\right)\\ &\quad - \left(\cos^4\alpha - \cos\theta\cos^3\alpha - \frac{7}{4}\cos^2\alpha + \cos^2\theta\cos^2\alpha + \frac{1}{2}\cos\theta\cos\alpha - \frac{3}{4}\cos^2\theta + 1\right)\\ &= -\frac{3}{4}\cos^4\alpha + \frac{3}{2}\cos\theta\cos^3\alpha + \frac{3}{4}\cos^2\alpha - \frac{3}{4}\cos^2\theta\cos^2\alpha - \frac{3}{2}\cos\theta\cos\alpha + \frac{3}{4}\cos^2\theta\\ &= -\frac{3}{4}(\cos^2\alpha - 1)(\cos\alpha - \cos\theta)^2 \ge 0.\end{aligned}$$

If $c = 2$, then by (8), we have

$$\begin{aligned}\widetilde{\omega} &= ze^{2i\theta}\frac{z^3 + \frac{2\cos\alpha+e^{-i\theta}}{3}z^2 - \frac{(1+2e^{-i\theta}\cos\alpha)}{3}z - e^{-i\theta}}{1 + \frac{2\cos\alpha+e^{i\theta}}{3}z - \frac{(1+2e^{i\theta}\cos\alpha)}{3}z^2 - e^{i\theta}z^3}\\ &= -ze^{i\theta}.\end{aligned}$$

Hence, $|\widetilde{\omega}(z)| < 1$.

Therefore, by *Cohn's Rule*, $p(z)$ has all its three zeros in $\overline{\mathbb{U}}$, that is $A, B, C \in \overline{\mathbb{U}}$ and so $|\widetilde{\omega}(z)| < 1$ for all $z \in \mathbb{U}$. The proof is now completed. □

If we take $\alpha = \pi/2$ in (3), we prove that $L_c * f_{\pi/2} \in \mathcal{S}_H^0$ and is convex in the horizontal direction for $0 < c \leq 2/n$ and for all $n \in \mathbb{N}$.

Proof of Theorem 1.2. By Lemma 2.2, it suffices to show that the dilatation of $L_c * f_{\pi/2}$ satisfies $|\widetilde{\omega_1}| < 1$ for all $z \in \mathbb{U}$. Substituting $\alpha = \pi/2$ into the equation (5), we have

$$\widetilde{\omega_1} = \frac{[(1+c)z^2 + (1-c)]\omega(1+\omega) - cz(z^2+1)\omega'}{[(1+c) + (1-c)z^2](1+\omega) - cz(z^2+1)\omega'}. \tag{9}$$

Setting $\omega = e^{i\theta}z^n$ in (9), we get

$$\begin{aligned}\widetilde{\omega_1} &= e^{2i\theta}z^n\frac{z^{n+2} + \frac{1-c}{1+c}z^n + \frac{1+(1-n)c}{1+c}e^{-i\theta}z^2 + \frac{1-(1+n)c}{1+c}e^{-i\theta}}{1 + \frac{1-c}{1+c}z^2 + \frac{1+(1-n)c}{1+c}e^{i\theta}z^n + \frac{1-(1+n)c}{1+c}e^{i\theta}z^{n+2}}\\ &= e^{2i\theta}z^n\frac{p(z)}{p^*(z)},\end{aligned} \tag{10}$$

where

$$p(z) = z^{n+2} + \frac{1-c}{1+c}z^n + \frac{1+(1-n)c}{1+c}e^{-i\theta}z^2 + \frac{1-(1+n)c}{1+c}e^{-i\theta} \tag{11}$$

and

$$p^*(z) = z^{n+2}\overline{p(1/\overline{z})} = 1 + \frac{1-c}{1+c}z^2 + \frac{1+(1-n)c}{1+c}e^{i\theta}z^n + \frac{1-(1+n)c}{1+c}e^{i\theta}z^{n+2}.$$

Firstly, we will show that $|\widetilde{\omega_1}| < 1$ for $c = 2/n$. In this case, substituting $c = 2/n$ into the equation (10), yields

$$\begin{aligned}\widetilde{\omega_1} &= e^{2i\theta}z^n\frac{z^{n+2} + \frac{1-\frac{2}{n}}{1+\frac{2}{n}}z^n + \frac{1+(1-n)\frac{2}{n}}{1+\frac{2}{n}}e^{-i\theta}z^2 + \frac{1-(1+n)\frac{2}{n}}{1+\frac{2}{n}}e^{-i\theta}}{1 + \frac{1-\frac{2}{n}}{1+\frac{2}{n}}z^2 + \frac{1+(1-n)\frac{2}{n}}{1+\frac{2}{n}}e^{i\theta}z^n + \frac{1-(1+n)\frac{2}{n}}{1+\frac{2}{n}}e^{i\theta}z^{n+2}}\\ &= e^{2i\theta}z^n\frac{z^{n+2} + \frac{n-2}{n+2}z^n - \frac{n-2}{n+2}e^{-i\theta}z^2 - e^{-i\theta}}{1 + \frac{n-2}{n+2}z^2 - \frac{n-2}{n+2}e^{i\theta}z^n - e^{i\theta}z^{n+2}}\\ &= -e^{i\theta}z^n.\end{aligned}$$

Hence, $|\widetilde{\omega_1}| < 1$.

Next, we will show that $|\widetilde{\omega_1}| < 1$ for $0 < c < 2/n$. Obviously, if z_0 is a zero of $p(z)$, then $1/\overline{z_0}$ is a zero of $p^*(z)$. Then we can write

$$\widetilde{\omega_1} = e^{2i\theta}z^n\frac{(z+A_1)(z+A_2)\cdots(z+A_{n+2})}{(1+\overline{A_1}z)(1+\overline{A_2}z)\cdots(1+\overline{A_{n+2}}z)}.$$

By *Cohn's Rule*, we need to show that all the zeros of (11) lie in $\overline{\mathbb{U}} = \{z \in \mathbb{C} : |z| \le 1\}$ for $0 < c < 2/n$. Since

$$|a_{0,0}| = |\frac{1-(1+n)c}{1+c}e^{-i\theta}| = |\frac{1-(1+n)c}{1+c}| < |a_{n+2,0}| = 1$$

for $0 < c < 2/n$, we have

$$\begin{aligned} p_1(z) &= \frac{\overline{a_{n+2,0}}p(z) - a_{0,0}p^*(z)}{z} \\ &= \frac{(n+2)(2-nc)cz}{(1+c)^2}\left(z^n + \frac{n}{n+2}z^{n-2} + \frac{2}{n+2}e^{-i\theta}\right). \end{aligned}$$

Since $0 < c < 2/n$, we have $(n+2)(2-nc)c/(1+c)^2 > 0$. Let

$$q_1(z) = z^n + \frac{n}{n+2}z^{n-2} + \frac{2}{n+2}e^{-i\theta}, \tag{12}$$

since

$$|a_{0,1}| = |\frac{2}{n+2}e^{-i\theta}| < 1 = |a_{n,1}|,$$

by using (6) on $q_1(z)$ again, we obtain

$$\begin{aligned} p_2(z) &= \frac{\overline{a_{n,1}}q_1(z) - a_{0,1}q_1^*(z)}{z} \\ &= \frac{n(n+4)z}{(n+2)^2}\left(z^{n-2} + \frac{n+2}{n+4}z^{n-4} - \frac{2}{n+4}e^{-i\theta}\right). \end{aligned}$$

Let $q_2(z) = z^{n-2} + \frac{n+2}{n+4}z^{n-4} - \frac{2}{n+4}e^{-i\theta}$. Then $|a_{0,2}| = |\frac{2}{n+4}e^{-i\theta}| < 1 = |a_{n-2,2}|$ and we have

$$\begin{aligned} p_3(z) &= \frac{\overline{a_{n-2,2}}q_2(z) - a_{0,2}q_2^*(z)}{z} \\ &= \frac{(n+2)(n+6)z}{(n+4)^2}\left(z^{n-4} + \frac{n+4}{n+6}z^{n-6} + \frac{2}{n+6}e^{-i\theta}\right). \end{aligned}$$

By means of the mathematical induction, we claim that

$$p_k(z) = \frac{[n+2(k-2)](n+2k)z}{[n+2(k-1)]^2}\left(z^{n-2(k-1)} + \frac{n+2(k-1)}{n+2k}z^{n-2k} + \frac{2(-1)^{k+1}}{n+2k}e^{-i\theta}\right).$$

If we take $n = 2k$, then

$$\begin{aligned} p_k(z) &= \frac{4k(4k-4)z}{(4k-2)^2}\left(z^2 + \frac{2k-1+(-1)^{k+1}e^{-i\theta}}{2k}\right) \\ &= \frac{4k(k-1)z}{(2k-1)^2}\left(z^2 + \frac{2k-1-e^{i(k\pi-\theta)}}{2k}\right). \end{aligned}$$

Obviously, $|\frac{2k-1-e^{i(k\pi-\theta)}}{2k}| \le 1$, so the two zeros of $z^2 + \frac{2k-1-e^{i(k\pi-\theta)}}{2k}$ lie in $\overline{\mathbb{U}}$. Then by *Cohn's Rule*, we know that all zeros of (11) lie in $\overline{\mathbb{U}}$.

If we take $n = 2k+1$, then

$$p_k(z) = \frac{(4k-3)(4k+1)z}{(4k-1)^2}\left(z^3 + \frac{4k-1}{4k+1}z - \frac{2e^{i(k\pi-\theta)}}{4k+1}\right).$$

Let $q_k(z) = z^3 + \frac{4k-1}{4k+1}z - \frac{2e^{i(k\pi-\theta)}}{4k+1}$. Then $|a_{0,k}| = |\frac{2e^{i(k\pi-\theta)}}{4k+1}| < 1 = |a_{3,k}|$, by using *Cohn's Rule* again, we have

$$\begin{aligned} p_{k+1}(z) &= \frac{\overline{a_{3,k}}q_k(z) - a_{0,k}q_k^*(z)}{z} \\ &= \frac{(4k+3)(4k-1)}{(4k+1)^2}\left(z^2 + \frac{2e^{i(k\pi-\theta)}}{4k+3}z + \frac{4k+1}{4k+3}\right). \end{aligned}$$

Let $q_{k+1}(z) = z^2 + \frac{2e^{i(k\pi-\theta)}}{4k+3}z + \frac{4k+1}{4k+3}$, then $|a_{0,k+1}| = \frac{4k+1}{4k+3} < 1 = |a_{2,k+1}|$ and we have

$$\begin{aligned} p_{k+2}(z) &= \frac{\overline{a_{2,k+1}}q_{k+1}(z) - a_{0,k+1}q_{k+1}^*(z)}{z} \\ &= \frac{8(2k+1)}{(4k+3)^2}\left(z + \frac{(4k+3)e^{i(k\pi-\theta)} - (4k+1)e^{-i(k\pi-\theta)}}{4(2k+1)}\right). \end{aligned}$$

Then $z_0 = -\frac{(4k+3)e^{i(k\pi-\theta)} - (4k+1)e^{-i(k\pi-\theta)}}{4(2k+1)}$ is a zero of $p_{k+2}(z)$, and

$$|z_0| = \left|\frac{(4k+3)e^{i(k\pi-\theta)} - (4k+1)e^{-i(k\pi-\theta)}}{4(2k+1)}\right| \le \frac{(4k+3)+(4k+1)}{4(2k+1)} = 1.$$

So z_0 lies inside or on the unit disk $|z| = 1$, by *Cohn's Rule*, we know that all zeros of (11) lie in $\overline{\mathbb{U}}$. The proof is completed. □

Proof of Theorem D. By Theorem B, it suffices to show that the dilatation of $F_a * f$ satisfies $|\omega_1| < 1$ for all $z \in \mathbb{U}$. Substituting $\alpha = \pi/2$ in (7), we derive

$$\omega_1 = \frac{2(a+z^2)\omega(1+\omega) - (1-a)(1+z^2)z\omega'}{2(1+az^2)(1+\omega) - (1-a)(1+z^2)z\omega'} \tag{13}$$

Setting $\omega = e^{i\theta}z^n$ into eq. (13), yield

$$\begin{aligned} \omega_1 &= \frac{2\omega(1+\omega)(a+z^2) - z\omega'(1-a)(1+z^2)}{2(1+az^2)(1+\omega) - z\omega'(1-a)(1+z^2)} \\ &= e^{2i\theta}z^n \frac{z^{n+2} + az^n + \frac{2-n(1-a)}{2}e^{-i\theta}z^2 + \frac{2a-n(1-a)}{2}e^{-i\theta}}{1 + az^2 + \frac{2-n(1-a)}{2}e^{i\theta}z^n + \frac{2a-n(1-a)}{2}e^{i\theta}z^{n+2}} \\ &= e^{2i\theta}z^n \frac{p(z)}{p^*(z)}, \end{aligned} \tag{14}$$

where

$$p(z) = z^{n+2} + az^n + \frac{2-n(1-a)}{2}e^{-i\theta}z^2 + \frac{2a-n(1-a)}{2}e^{-i\theta} \tag{15}$$

and

$$p^*(z) = z^{n+2}\overline{p(1/\overline{z})} = 1 + az^2 + \frac{2-n(1-a)}{2}e^{i\theta}z^n + \frac{2a-n(1-a)}{2}e^{i\theta}z^{n+2}.$$

Now, consider the case in which $a = \frac{n-2}{n+2}$. Then eq. (14) yields

$$\begin{aligned} \omega_1 &= e^{2i\theta}z^n \frac{z^{n+2} + \frac{n-2}{n+2}z^n - \frac{n-2}{n+2}e^{-i\theta}z^2 - e^{-i\theta}}{1 + \frac{n-2}{n+2}z^2 - \frac{n-2}{n+2}e^{i\theta}z^n - e^{i\theta}z^{n+2}} \\ &= -e^{i\theta}z^n. \end{aligned}$$

Hence $|\omega_1| < 1$.

Next, consider the case in which $\frac{n-2}{n+2} < a < 1$. Note that $p^*(z) = z^{n+2}\overline{p(1/\overline{z})}$, if z_0 is a zero of $p(z)$, then $1/\overline{z_0}$ is a zero of $p^*(z)$. Hence

$$\omega_1 = e^{2i\theta} z^n \frac{(z + A_1)(z + A_2)\cdots(z + A_{n+2})}{(1 + \overline{A_1}z)(1 + \overline{A_2}z)\cdots(1 + \overline{A_{n+2}}z)}.$$

By Lemma 2.3, it suffices to show that all zeros of (15) lie inside or on the closed unit disk. Since $|a - \frac{n(1-a)}{2}| = |a_{0,0}| < |a_{n+2,0}| = 1$ for $\frac{n-2}{n+2} < a < 1$, using *Cohn's Rule*, we have

$$\begin{aligned} p_1(z) &= \frac{\overline{a_{n+2,0}}p(z) - a_{0,}p^*(z)}{z} \\ &= \frac{(1-a)(n+2)[(2+n)a - (n-2)]z}{4}\left(z^n + \frac{n}{n+2}z^{n-2} + \frac{2}{n+2}e^{-i\theta}\right). \end{aligned}$$

Since $\frac{n-2}{n+2} < a < 1$, we have $\frac{1}{4}(1-a)(n+2)[(2+n)a - (n-2)] > 0$. Let

$$q_1(z) = z^n + \frac{n}{n+2}z^{n-2} + \frac{2}{n+2}e^{-i\theta},$$

we can use *Cohn's Rule* on $q_1(z)$ again, then the same as proof of Lemma 1.2 eq. (12), we know that all zeros of $q_1(z)$ lie in the closed unit disk. By *Cohn's Rule*, $p(z)$ has all its $n + 2$ zeros in the closed unit disk, and so $|\omega_1| < 1$ for all $z \in \mathbb{U}$. □

4. Some Examples

In this section, three examples are provided to illustrate the obtained results.

Example 4.1. *In Theorem 1.1, if we take $c = 2$, then by* (1), *we have*

$$\begin{aligned} L_2(z) &= H_2 + \overline{G_2} \\ &= \frac{1}{3}\left[\frac{z}{1-z} + \frac{2z}{(1-z)^2}\right] + \frac{1}{3}\overline{\left[\frac{z}{1-z} - \frac{2z}{(1-z)^2}\right]} \\ &= \mathrm{Re}\left\{\frac{2}{3}\frac{z}{1-z}\right\} + i\,\mathrm{Im}\left\{\frac{4}{3}\frac{z}{(1-z)^2}\right\}. \end{aligned}$$

Let $f_1 = h_1 + \overline{g_1}$, where $h_1 + g_1 = \frac{1}{2i\sin\alpha}\log\left(\frac{1+ze^{i\alpha}}{1+ze^{-i\alpha}}\right)$ with $\omega_1 = g_1'/h_1' = z$ and $\alpha = \frac{2\pi}{3}$. By shearing we get

$$h_1' + g_1' = \frac{1}{(1 + ze^{\frac{2\pi}{3}i})(1 + ze^{-\frac{2\pi}{3}i})}, \text{ and}$$

$$\begin{aligned} h_1 &= \frac{1}{3}\log(1+z) - \frac{1}{6}\log(1 - z + z^2) - \frac{i}{2\sqrt{3}}\log\left(\frac{1 + ze^{\frac{2\pi}{3}i}}{1 + ze^{-\frac{2\pi}{3}i}}\right), \\ g_1 &= -\frac{1}{3}\log(1+z) + \frac{1}{6}\log(1 - z + z^2) - \frac{i}{2\sqrt{3}}\log\left(\frac{1 + ze^{\frac{2\pi}{3}i}}{1 + ze^{-\frac{2\pi}{3}i}}\right). \end{aligned}$$

Hence

$$f_1 = \mathrm{Re}\left\{\frac{1}{\sqrt{3}i}\log\left(\frac{1 + ze^{\frac{2\pi}{3}i}}{1 + ze^{-\frac{2\pi}{3}i}}\right)\right\} + i\,\mathrm{Im}\left\{\frac{2}{3}\log(1+z) - \frac{1}{3}\log(1 - z + z^2)\right\}.$$

By (2)*, we have*

$$\begin{aligned} L_2 * f_1 &= H_2 * h_1 + \overline{G_2 * g_1} \\ &= \frac{1}{3}\left[h_1 + 2zh_1'\right] + \frac{1}{3}\overline{\left[g_1 - 2zg_1'\right]} \\ &= \frac{1}{3}\,\mathrm{Re}\left\{\frac{1}{\sqrt{3}i}\log\left(\frac{1+ze^{\frac{2\pi}{3}i}}{1+ze^{-\frac{2\pi}{3}i}}\right) + \frac{2z(1-z)}{(1-z+z^2)(1+z)}\right\} \\ &\quad + \frac{i}{3}\,\mathrm{Im}\left\{\frac{2}{3}\log(1+z) - \frac{1}{3}\log(1-z+z^2) + \frac{2z}{1-z+z^2}\right\}. \end{aligned}$$

By Theorem 1.1, we know that $L_2 * f_1$ is univalent and convex in the horizontal direction. The image of $\mathbb{U}$ under L_2, f_1 and $L_2 * f_1$ are shown in Figure 1, Figure 2 and Figure 3, respectively.

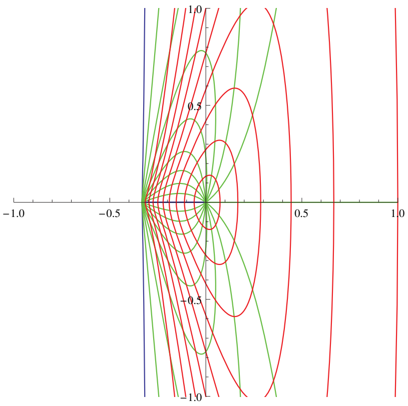

Figure 1: Image of $L_2(z)$.

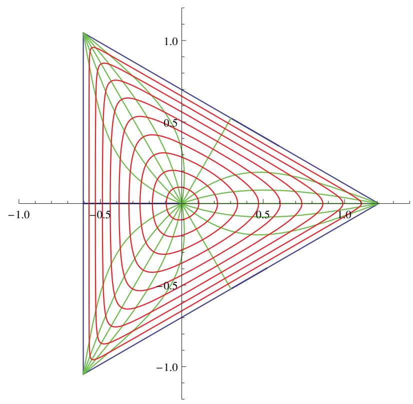

Figure 2: Image of $f_1(z)$.

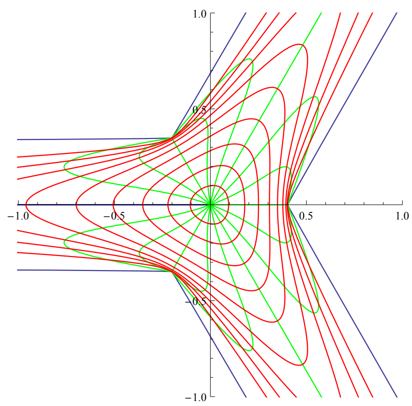

Figure 3: Image of $L_2(z) * f_1(z)$.

Example 4.2. *Let $c = 2/3$ in Theorem 1.2. Then by* (2)*, we have*

$$\begin{aligned} L_{2/3}(z) &= H_{2/3} + \overline{G_{2/3}} \\ &= \frac{3}{5}\left[\frac{z}{1-z} + \frac{2}{3}\frac{z}{(1-z)^2}\right] + \frac{3}{5}\overline{\left[\frac{z}{1-z} - \frac{2}{3}\frac{z}{(1-z)^2}\right]} \\ &= \mathrm{Re}\left\{\frac{6}{5}\frac{z}{1-z}\right\} + i\,\mathrm{Im}\left\{\frac{4}{5}\frac{z}{(1-z)^2}\right\}. \end{aligned}$$

Let $f_2 = h_2 + \overline{g_2}$ *be the harmonic mapping with* $\omega_2 = g_2'/h_2' = -z^3$, *where* $h_2 + g_2 = \frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right)$. *By shearing, we get*

$$\begin{aligned} h_2 &= -\frac{i}{4}\log\left(\frac{1+iz}{1-iz}\right) - \frac{1}{6}\log(1-z) - \frac{1}{4}\log(1+z^2) + \frac{1}{3}\log(1+z+z^2),\\ g_2 &= -\frac{i}{4}\log\left(\frac{1+iz}{1-iz}\right) + \frac{1}{6}\log(1-z) + \frac{1}{4}\log(1+z^2) - \frac{1}{3}\log(1+z+z^2). \end{aligned}$$

Then

$$f_2 = \operatorname{Re}\left\{\frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right)\right\} + i\operatorname{Im}\left\{-\frac{1}{3}\log(1-z) - \frac{1}{2}\log(1+z^2) + \frac{2}{3}\log(1+z+z^2)\right\}.$$

By (2), *we derive*

$$\begin{aligned} L_{2/3} * f_2 &= H_{2/3} * h_2 + \overline{G_{2/3} * g_2}\\ &= \frac{3}{5}\left[h_2 + \frac{2}{3}zh_2'\right] + \overline{\frac{3}{5}\left[g_2 - \frac{2}{3}zg_2'\right]}\\ &= \frac{3}{5}\operatorname{Re}\left\{\frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right) + \frac{2}{3}\frac{z(1+z^3)}{(1+z^2)(1-z^3)}\right\}\\ &\quad + \frac{3i}{5}\operatorname{Im}\left\{-\frac{1}{3}\log(1-z) - \frac{1}{2}\log(1+z^2) + \frac{2}{3}\log(1+z+z^2) + \frac{2}{3}\frac{z}{1+z^2}\right\}. \end{aligned}$$

In view of Theorem 1.2, we know that $L_{2/3} * f_2$ *is univalent and convex in the horizontal direction. The image of* $\mathbb{U}$ *under* $L_{2/3}$, f_2 *and* $L_{2/3} * f_2$ *are shown in Figure 4, Figure 5 and Figure 6, respectively.*

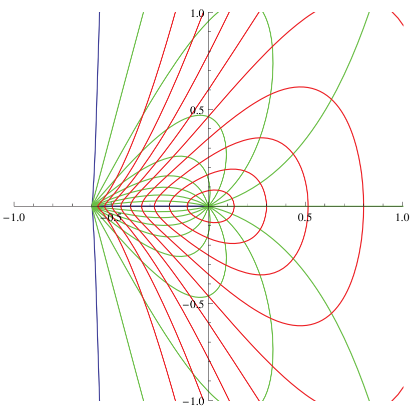

Figure 4: Image of $L_{2/3}(z)$.

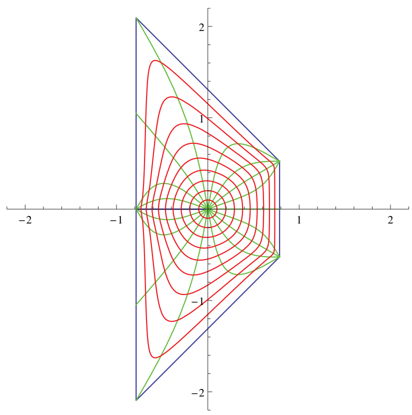

Figure 5: Image of $f_2(z)$.

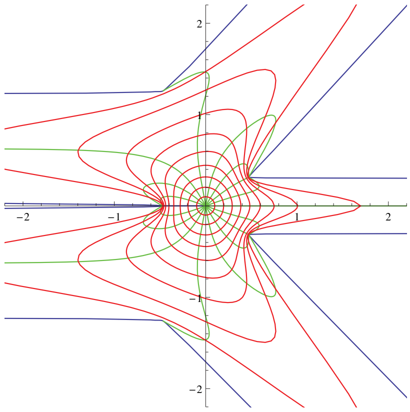

Figure 6: Image of $L_{2/3} * f_2(z)$.

Example 4.3. *Let $c = 1/2$. Then by (2), we have*

$$\begin{aligned}L_{1/2}(z) &= H_{1/2} + \overline{G_{1/2}}\\ &= \frac{2}{3}\left[\frac{z}{1-z} + \frac{1}{2}\frac{z}{(1-z)^2}\right] + \overline{\frac{2}{3}\left[\frac{z}{1-z} - \frac{1}{2}\frac{z}{(1-z)^2}\right]}\\ &= \text{Re}\left\{\frac{4}{3}\frac{z}{1-z}\right\} + i\,\text{Im}\left\{\frac{2}{3}\frac{z}{(1-z)^2}\right\}.\end{aligned}$$

Let $f_3 = h_3 + \overline{g_3}$, where $h_3 + g_3 = \frac{1}{2i\sin\alpha}\log\left(\frac{1+ze^{i\alpha}}{1+ze^{-i\alpha}}\right)$ with $\omega_3 = g_3'/h_3' = -z^4$ and $\alpha = \frac{3\pi}{4}$. We get

$$h_3' + g_3' = \frac{1}{(1+ze^{\frac{3\pi}{4}i})(1+ze^{-\frac{3\pi}{4}i})} = \frac{1}{1-\sqrt{2}z+z^2},$$

and then

$$\begin{aligned}h_3 &= \frac{1}{2\sqrt{2}i}\log\left(\frac{1+ze^{\frac{3\pi}{4}i}}{1+ze^{-\frac{3\pi}{4}i}}\right) - \frac{2+\sqrt{2}}{8}\log(1-z) + \frac{2-\sqrt{2}}{8}\log(1+z) + \frac{\sqrt{2}}{8}\log(1+z^2),\\ g_3 &= \frac{1}{2\sqrt{2}i}\log\left(\frac{1+ze^{\frac{3\pi}{4}i}}{1+ze^{-\frac{3\pi}{4}i}}\right) + \frac{2+\sqrt{2}}{8}\log(1-z) - \frac{2-\sqrt{2}}{8}\log(1+z) - \frac{\sqrt{2}}{8}\log(1+z^2).\end{aligned}$$

So

$$\begin{aligned}f_3 &= h_3 + \overline{g_3}\\ &= \text{Re}\left\{\frac{1}{\sqrt{2}i}\log\left(\frac{1+ze^{\frac{3\pi}{4}i}}{1+ze^{-\frac{3\pi}{4}i}}\right)\right\} + i\,\text{Im}\left\{-\frac{2+\sqrt{2}}{4}\log(1-z) + \frac{2-\sqrt{2}}{4}\log(1+z) + \frac{\sqrt{2}}{4}\log(1+z^2)\right\}.\end{aligned}$$

By (2), we have

$$\begin{aligned}L_{1/2} * f_3 &= H_{1/2} * h_3 + \overline{G_{1/2} * g_3}\\ &= \frac{2}{3}\left[h_3 + \frac{1}{2}zh_3'\right] + \overline{\frac{2}{3}\left[g_3 - \frac{1}{2}zg_3'\right]}\\ &= \frac{2}{3}\,\text{Re}\left\{\frac{1}{\sqrt{2}i}\log\left(\frac{1+ze^{\frac{3\pi}{4}i}}{1+ze^{-\frac{3\pi}{4}i}}\right) + \frac{1}{2}\frac{z(1+z^4)}{(1-z^4)(1-\sqrt{2}z+z^2)}\right\}\\ &\quad + \frac{2i}{3}\,\text{Im}\left\{-\frac{2+\sqrt{2}}{4}\log(1-z) + \frac{2-\sqrt{2}}{4}\log(1+z) + \frac{\sqrt{2}}{4}\log(1+z^2) + \frac{1}{2}\frac{z}{1-\sqrt{2}z+z^2}\right\}.\end{aligned}$$

The image of $\mathbb{U}$ *under* $L_{1/2}$, f_3 *and* $L_{1/2} * f_3$ *are shown in Figure 7, Figure 8 and Figure 9, respectively. As seen in Figure 9,* $L_{1/2} * f_3$ *is univalent and convex in the horizontal direction.*

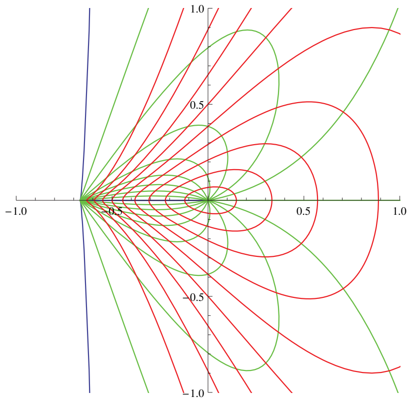

Figure 7: Image of $L_{1/2}(z)$.

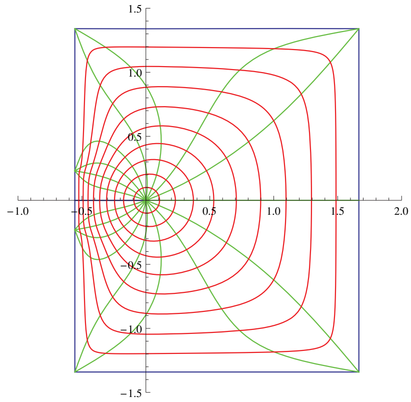

Figure 8: Image of $f_3(z)$.

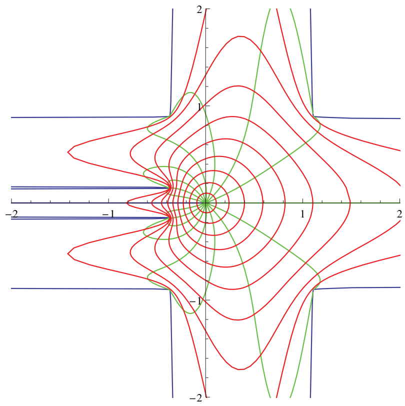

Figure 9: Image of $L_{1/2} * f_3(z)$.

Combining Theorem 1.1, Theorem 1.2 and Example 4.3, we propose the following problem.

Problem 4.4. *Let* $L_c = H_c + \overline{G_c} \in \mathcal{K}_H^0$ *be a mapping given by* (1). *If* $f_\alpha = h_\alpha + \overline{g_\alpha} \in \mathcal{K}_H^0$ *with* $h_\alpha + g_\alpha = \frac{1}{2i\sin\alpha}\log\left(\frac{1+ze^{i\alpha}}{1+ze^{-i\alpha}}\right)(\frac{\pi}{2} \leq \alpha < \pi)$ *and* $\omega(z) = g'_\alpha/h'_\alpha = e^{i\theta}z^n(\theta \in \mathbb{R}, n \in \mathbb{N})$. *Then* $L_c * f_\alpha \in \mathcal{S}_H^0$ *and is convex in the horizontal direction for* $0 < c \leq \frac{2}{n}$.

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