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COOPERATION AMONG EGOISTS IN PRISONER'S DILEMMA AND CHICKEN GAMES

by

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I. Introduction.

In "The Emergence of Cooperation among Egoists," Axelrod develops a new approach to answer the question: "under what conditions will cooperation emerge in a world of egoists without central authority?" (Axelrod, 1981.) He uses an evolutionary approach to study the conditions under which cooperation can be part of a "stable" strategy in the iterated Prisoner's Dilemma.

The Prisoner's Dilemma is an interesting game to consider for such a study as it is one in which self-interested behavior seems to lead inexorably to a Pareto inferior outcome. The game is defined by the payoff matrix given in Figure 1. As is well-known (and easily verified), defecting is a dominant strategy in a one-time play of the game and thus mutual defection is a dominant strategy equilibrium. At the same time, though, mutual defection is worse for both players than mutual cooperation.

(INSERT FIGURE 1 ABOUT HERE.)

The core of Axelrod's results is to show that this

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[1]I wish to thank Theodore Bergstrom, the students in his game theory course, the participants in the Industrial Organization seminar at the University of Michigan, and John Ferejohn and others at the panel on evolutionary models at the 1983 meetings of the American Political Science Association for their comments and suggestions. I also wish to thank Kelly McCauley for helping me clarify my writing and thinking. Finally, I want to especially thank Robert Axelrod for his encouragement and advice on this paper. Of course, any remaining errors are my responsibility.

paradox can be eliminated if players interact for many moves and care about what happens to them in the future. This enables cooperators to punish defection severely enough to deter it. If players care "enough" about the future, surprisingly strong forces push them toward cooperation, especially cooperation through reciprocity. (See Kreps, et al., 1982, for a different but related resolution.)[2]

The results are intuitively plausible. But do they carry over to other sorts of situations where the players' incentives lead to results that are socially suboptimal? For an example of such a situation, consider the game of Chicken. If we use the same letters for the payoffs and names for the strategies, then Chicken is the same as the Prisoner's Dilemma except that $S > P$ and we no longer necessarily assume that $2R > T + S$. This seemingly small change in the payoffs leads to a large change in the play of the game.

Snyder (1971), for example, explains the difference between the two games saying "The spirit or leading theme of the prisoner's dilemma is that of frustration of a mutual desire to cooperate. The spirit of a chicken game is that of a contest in which each party is trying to prevail over the other. . . . (I)n the prisoner's dilemma, establishing

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[2]It is important to note that the possibility of cooperation in an infinitely repeated game or a game with an indefinite number of moves has long been known. Axelrod has, however, proposed (and we modify) what one may view as a stability criteria for judging these equilibria which is interestingly motivated and has some surprising ramifications. See Section IV. B. below.

FIGURE 1

	COOPERATE	DEFECT
COOPERATE	R, R	S, T
DEFECT	T, S	P, P

$T > R > P > S$

$2R > T + S$

Row chooser's payoff listed first.

credibility means instilling trust, whereas in chicken, it involves creating fear. . . ." (Emphasis in original.) Why do such major differences come about?

First, there is no longer a dominant strategy. There are, however, two Nash equilibria in pure strategies--column defecting and row cooperating and vice versa. Each player prefers the equilibrium where he defects and the other cooperates, but has no way of forcing the other player to his preferred outcome. The attempt to achieve one's preferred Nash equilibrium is the "contest. . . to prevail" Snyder refers to. There is a Nash equilibrium in mixed strategy, of course, as one must always exist in a game of this sort. (See Luce and Raiffa, 1957.) In this equilibrium, each player cooperates with probability $(S - P)/(T + S - R - P)$.

Though it is difficult to say what outcome we can "expect" to occur in Chicken in the one-play case, it does still appear to be true that self-interested behavior leads to a suboptimal outcome. Mutual cooperation is Pareto superior to the Nash equilibrium in mixed strategies. Admittedly, for each of the Nash equilibria in pure strategies, one player is better off than with mutual cooperation and one is worse off, so that these outcomes are Pareto noncomparable. However, if we consider the two equilibria to be equally likely or go on the basis of total payoffs to the two players, then mutual cooperation is preferred as long as $T + S < 2R$. Thus, at least as long as the inequality holds, it makes sense to say that self-interested behavior may lead to suboptimal outcomes in Chicken as the various Nash equilibria are inferior to mutual cooperation.[3] Can mutual cooperation emerge in Chicken if players care "enough" about the future? Do Axelrod's results generalize to this game?

The question is one of interest in a variety of fields, as numerous situations in the real world appear to be reasonably modelled as games of Chicken. Snyder (1971), for example, gives several historical examples of situations like Chicken in the realm of international politics. Schelling (1960) often seems to see Chicken as _the_ game of superpower confrontation. The evolutionary biologist John Maynard Smith in numerous papers (see Maynard Smith, 1982, for example) has modelled confrontations between animals for a resource--his Hawk-Dove game--as Chicken. Cornell and Roll (1981) analyze equilibrium in information acquisition in financial markets and determine that a necessary condition for "a sensible asset market equilibrium" is that the game be Chicken.[4] Spence (1979) studies entry of firms into newly created markets and analyzes one indeterminate case which is very reminiscent of Chicken. Kreps and Wilson

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[3]The qualification is important. Without it, the game is more a game of coordination ("whose turn is it to get T?") than a game of trying to achieve mutual cooperation. This importance will be especially seen in Section IV.

[4]They derive payoffs corresponding to Chicken, but do not note that the game is Chicken. Maynard Smith (1982) also does not call the game Chicken.

(1982) consider an entry game which they note is quite similar to Chicken. Many bargaining situations where each of the two parties prefer a poor agreement to no agreement can be modelled as Chicken.

The way we shall proceed is to study the game described by the payoff matrix given in Figure 1 with the only restrictions being $T > R > S$ and $R > P$. When $S > P$, we have a game of Chicken. On the other hand, when $S < \min(P, 2R - T)$, we have a Prisoner's Dilemma game.[5]

In the next section, we discuss the evolutionary approach developed by Axelrod which we will use to analyze this game. In Sections III and IV, we analyze the evolution of cooperation in the general game and compare the problems of achieving mutual cooperation in the Prisoner's Dilemma and Chicken. We consider populations where all individuals play the same strategy in Section III and the more complex case of "mixed populations" in Section IV. In Section V, we offer some concluding remarks on our view of the significance of Axelrod's approach and his results.

II. Assumptions and the Approach.

A number of simplifying assumptions governing the interaction between the players will be made. These assumptions are the same as those made in Axelrod (1980,

[5]The second term creates a few problems. It is entirely possible to have $2R - T < S < P$, in which case the game is neither Chicken nor the Prisoner's Dilemma. Our results do apply to this intermediate game, which we might call the Chicken's Dilemma, but we will not discuss it explicitly.

1981, 1984) so readers familiar with any of these works may wish to skim this section.

First, we will analyze pairwise interactions only--i.e., a player interacts with only one other player at a time. He does not know anything about that player prior to their first interaction. Thus he can be thought of as playing an opponent randomly selected from the population. For models which are similar in this regard, see Cornell and Roll (1981) and Rosenthal (1979). One can view the situation as one in which players "wander" through the population encountering other individuals at random. When another individual is encountered, nothing is known about him except one's own past history of play against this individual. The two then play the game some number of times.[6]

The uncertainty regarding the possibility that the current opponent will be faced again on the next play is formulated by using a discount parameter, w. One could view this parameter as discounting future returns in an infinite game. An alternative interpretation, which will be stressed here, is that w represents the probability of the

[6]Numerous formulations yield equivalent payoff calculations (at least up to a scaling constant). We will perform all payoff calculations as though an opponent were selected at random, played against for an uncertain number of moves, and then the game ended. Alternatively, players could meet many other players randomly for some indefinite number of periods. Two equivalent formulations of this approach are possible. A player could meet an opponent at random, play for an uncertain number of moves, and then meet another chosen at random. Or he could change opponents randomly after each play of the game.

interaction continuing for another move. To be consistent with either of these interpretations, we assume that $0 \le w < 1$.

We will let $V(A|B)$ be the payoff to a player using the strategy A against a player using the strategy B. A strategy is defined as a mapping from the past history of the interaction with this opponent into a (possibly random) choice for a move. The total payoff will be taken to be the discounted/expected sum of the payoffs at each move.

We now define a few characteristics a strategy may have, some of which follow Axelrod. A "nice" strategy will be defined as one which is never the first to defect. By contrast, a "mean" strategy is one which is never the first to cooperate. Nice and mean strategies may be either pure or stochastic. A pure strategy, as in the standard use of the term, always specifies its moves with certainty. A stochastic strategy is one which specifies a probability for actions on at least one move rather than specifying the choice with certainty, where the probability may depend on the history of the game to that point. A stochastic strategy may be thought of as making an initial probabilistic choice from among a set (possibly infinite) of pure strategies. That is, it plays pure strategy A the entire game with probability p_a, pure strategy B with probability p_b, and so on.[7] A nice (or mean) strategy

[7]Stochastic strategies are more commonly referred to as behavior strategies. Thus the equivalence being asserted

always cooperates (defects) till the opponent does the opposite, but may follow a stochastic strategy after that point.

We wish to analyze how cooperation may emerge from a Hobbesian "state of nature" where mean strategies hold sway. Following Axelrod, we can divide the evolution of cooperation into three stages. First, we need to end the dominance of mean strategies in the population. Hence, the first question is: given that everyone is playing a mean strategy (and thus no one ever cooperates), can cooperation enter as part of a viable strategy?

In the second stage, cooperation enters and we get some mix of nice, mean, and other strategies. The second question then is: can nice strategies come to dominate the population? A similar--and far more tractable--question is: given that mean strategies cannot monopolize a population, can the population "get stuck" with some mix of strategies, some of which are not nice, rather than have nice strategies take over completely?

In the final stage, everyone plays a nice strategy so that no one ever defects. The third question, then, is: can this situation last?

The first and third questions are the easiest to answer. To consider them, we will focus on the concept of collective stability, due to Axelrod (1981) and strongly

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is just the equivalence between behavior and mixed strategies in games with perfect recall first shown by Kuhn (1953).

related to Maynard Smith's evolutionary stability (1973). A strategy B is said to <u>invade</u> a strategy A if and only if $V(B|A) > V(A|A)$. If there exists no strategy that can invade A, then A is said to be <u>collectively stable</u>.[8] The idea is that if a population is composed solely of individuals playing the strategy A, then the average person in that population receives $V(A|A)$. If someone could do better than this by switching to another strategy, then eventually someone would. Thus we would not expect to see the entire population continuing with the strategy A. In this sense, the alternative strategy invades the population, so that the initial situation was not stable.

We will use the concept of collective stability to rewrite the first and third questions to ask when mean and nice strategies are collectively stable. If no mean strategy is collectively stable, then the state of nature will not persist indefinitely. If some nice strategies are collectively stable, then, if one (or more[9]) of these strategies come to dominate the population, this dominance will endure.

As to the second question, we will work with the concept of a collectively stable mix of strategies. We will

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[8]This is, of course, the same as a symmetric Nash equilibrium. We use Axelrod's terminology to emphasize the interpretation he gives the concept and the way it fits into his view of the evolution of cooperation.

[9]The concept of a collectively stable distribution of strategies will not be defined until Section IV. A., but it is easily shown and should be clear that a mix of collectively stable strategies is collectively stable.

define the concept when it is to be used, but the intuition is the same as that for a collectively stable strategy.

The essence of the evolutionary approach is to view the process by which the distribution of strategies ("genotypes") in the population is determined as a dynamic process of "natural selection." Here the dynamic process is left unspecified and conditions characterizing a "reasonable" notion of a stable steady state are employed. Since the underlying process determining how the population moves to such a state is not explicitly considered, the analysis here cannot be seen as the sole indication of the kind of strategies that develop and survive in any particular environment. In effect, only possible "end points" for the population are considered.

III. The Evolution of Cooperation:

Single Strategy Populations.

A. Breaking the Dominance of Mean Strategies.

Can some mean strategy be collectively stable? Or can a single individual in a state of nature try cooperation and find his efforts sufficiently rewarded? As Snyder (1971) notes, Chicken is generally thought to be a game where toughness and thus, possibly, meanness is a valuable trait. One might suppose, therefore, that it is more difficult to break the monopoly of meanness in Chicken than in the Prisoner's Dilemma. In fact, the opposite is true. If we think of the example that gave Chicken its name--teenage gangs riding their cars at one another to see if either will "chicken out" and swerve (cooperate)--, then the result is not surprising. If one falls in with a gang that never swerves, hopefully, one would eventually intuit that swerving might be a more sensible strategy. In fact,

PROPOSITION 1: A necessary and sufficient condition for the existence of a mean strategy which is collectively stable is $S \le P$. That is, we can find a mean strategy that is collectively stable iff the game is not Chicken.

It is easy to see why this must be true. If players are always defecting, then everyone scores P on every move. But in Chicken, this is the lowest possible payoff on a move. Hence any nonzero probability of cooperating on some move will yield a higher expected payoff on that move. Since a player can always be assured of receiving at least P on every move and, by being "less mean," can receive strictly more than P on some move, he can do better than the population average. In other words, any strategy which is not mean can invade a mean population in Chicken.[10] On the other hand, if $P \ge S$, the best a player can do against someone who always defects is to always defect. Hence if $P \ge S$, then ALL D, the strategy of always defecting, is collectively stable.

We see, then, that the answer to our first question is

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[10]Actually, a strategy that defects on the first move but is still "less mean" than a mean strategy will need moves after the first to matter. That is, such a strategy will invade iff $w > 0$, though w can be made arbitrarily close to zero.

that the state of nature can be overturned in Chicken--and, in fact, much more easily than in the Prisoner's Dilemma.

B. Maintaining the Dominance of Nice Strategies.

Suppose that nice strategies come to dominate the population. Under what conditions can the domination of nice strategies remain unchallengeable? In other words, when can we find a nice strategy which is collectively stable?

PROPOSITION 2: A necessary and sufficient condition for the existence of a nice strategy which is collectively stable is

(1) $$w \geq \frac{T - R}{T - Z}$$

where

(2) $$Z = \max(S, P)$$

Proof: Let A be any nice strategy. Let B be the strategy which defects on the first move and cooperates thereafter if $S > P$ and plays ALL D otherwise. Then

(3) $$V(A|A) = \frac{R}{1 - w}$$

(4) $$V(B|A) \geq T + \frac{wZ}{1 - w}$$

Hence, a necessary condition for B not to be able to invade A is

(5) $$\frac{R}{1 - w} \geq T + \frac{wZ}{1 - w}$$

or

$$(6) \quad w \geq \frac{T - R}{T - Z}$$

so that (1) is a necessary condition. We now show that (1) is also sufficient by showing that if it holds, there exists a particular nice strategy which is collectively stable. This strategy is PR, the "permanent retaliation" strategy, which cooperates until its opponent defects and then always defects thereafter.

To show that PR is collectively stable, we need to show that no other strategy can invade it. We can simplify matters a bit by restricting attention to potential invaders which are pure strategies. It can be shown that if no pure strategy can invade a strategy, then no stochastic strategy can either. Furthermore, for obvious reasons, we can concentrate on the best strategy to play against PR.

Consider first the case where $S > P$. If the strategy of defecting on the first move and cooperating thereafter cannot invade, then no pure strategy can. This can be seen by noting that no nice strategy can invade, so we must consider a strategy which will be the first to defect. Since $S > P$, the best strategy against PR should not defect more than once. Hence the only question remaining is on what move the best strategy to play against PR defects. It is easy to show that a strategy which defects on the n^{th} move and cooperates on all others invades iff a strategy which defects on the first move and cooperates thereafter does.

Similarly, suppose that $S \leq P$. Here the best pure strategy against PR would always defect after its first defection. Again, it is easy to show that a strategy that begins its defections on the n^{th} move invades PR iff ALL D invades.

Clearly, if B as defined above cannot invade PR, then PR is collectively stable. But if A is PR, then equation (4) holds with equality. Hence if (1) holds, PR will be collectively stable. Q.E.D.

A direct implication of the above proof is that if any nice strategy is collectively stable, PR is. This should not be surprising. A collectively stable nice strategy must punish opponents who defect and is more likely to be collectively stable the more severe is that punishment. The punishment imposed by PR is more severe than any other nice strategy can impose. Collective stability does not imply that a strategy is always a good one to play, however, a point we will return to.

Note also that $Z < R$, so that $\frac{T - R}{T - Z} < 1$. Hence another implication of the proposition is that there always exists some $w < 1$ for which some nice strategy is collectively stable.

A strategy figuring prominently in Axelrod (1980, 1981, 1984) is TIT FOR TAT, which we will abbreviate TFT. This is the strategy of cooperating on the first move and, from then on, doing exactly what one's opponent did on the previous move. TFT is, in short, the strategy of reciprocity.

In Proposition 3, originally proved by Axelrod (1981), we show when TFT is collectively stable. The proof for the more general game considered here exactly parallels Axelrod's original proof and so will not be given.

PROPOSITION 3: TFT is collectively stable if and only if

$$(7) \quad w \geq \max\{\frac{T - R}{T - P}, \frac{T - R}{R - S}\}$$

COROLLARY: In the Prisoner's Dilemma, there always exists some $w < 1$ such that TFT is collectively stable. In Chicken, the same can only hold if $2R - T > P$.

Proof: Since $P < R$, the first term on the right-hand side of (7) must be strictly less than 1 in either game. Hence, there exists some value of $w < 1$ such that TFT is collectively stable iff the second term is also strictly less than 1--i.e., iff $T + S < 2R$. This must hold in the Prisoner's Dilemma. But since $S > P$ in Chicken, a necessary condition for it to be able to hold there is $2R - T > P$. Q.E.D.

Note that TFT's possible problems in Chicken arise because TFT requires $2R > T + S$, not because of the relative sizes of S and P. Interestingly, $2R > T + S$ is precisely the equation which we suggested earlier needed to hold for mutual cooperation to be thought of as the social optimum in Chicken.

Another interesting point is that the reversal of the inequality $S < P$ has very different consequences for the collective stability of nice and mean strategies. If the inequality is reversed, then, by Proposition 1, no mean strategy can be collectively stable. But from Proposition 2, we see that the relevant consideration for nice strategies is the magnitude of the larger of the two terms, not which is larger. That is, if we took a Prisoner's Dilemma game for which some nice strategy is collectively stable and create a game of Chicken by switching the values of S and P, then it must still be true that there exists some nice collectively stable strategy. In this sense, there is no difference in these two games in terms of the collective stability of nice strategies.

Note, though, that the magnitude of w is completely irrelevant to mean strategies, while, obviously, it is quite relevant to the collective stability of nice strategies. This should not be surprising. Axelrod refers to w as the "shadow of the future"--that is, the relative value of future payoffs to the players. In both games, if the opponent is expected to cooperate, one wants to defect in order to earn T. But if w is large enough, then a player expecting his opponent to retaliate against defection becomes willing to take R now instead of T in order to get R in the future instead of S or P. The logic is the same regardless of which of S and P is larger.

Z is important for precisely the same reason. Since Z is the larger of S and P, it is the payoff per move that a player seeking to minimize the consequences of retaliation will obtain. If Z is close to R, then a player loses very little per move by incurring retaliation for a defection. Note though that as w approaches 1, the effect of retaliation must outweigh the gain from defection. That is, if payoffs in the future matter enough, then even if Z is very close to R, it is enough less to be an adequate punishment.

IV. The Evolution of Cooperation: Mixed Populations.

A. Collective Stability.

Some of the considerations outlined above suggest that the aspect of the problem left out so far--how strategies fare in a variegated environment--is crucial. For example, though PR is collectively stable for a wider range of parameter values than any other nice strategy, it may be a lousy straegy to play in a variegated environment as it can easily get into debilitating rounds of mutual defection. (See the comments made by Axelrod, 1984, on how poorly PR did in the computer tournament.) Similarly, mean strategies do terribly against each other (and thus are not collectively stable in Chicken), but may thrive in an environment with nice strategies to exploit. This consideration is particularly relevant in Chicken where a player would prefer to be exploited than to engage in mutual defection.

Intuitively, we could envision a scenario like the following. We have a population all following some particular mean strategy. They do very poorly against one another and then some smart person thinks to try a nicer strategy. He does quite well for himself and inspires imitation by others. At the same time, though, he and his imitators enable the people using the mean strategy to do better on average than they did when everyone used the mean strategy. As more people switch to the nice strategy, the expected value of playing the mean strategy rises and, at some point, it no longer pays to switch to the nice strategy. In other words, a "stable" mix of these two strategies is established. Because the mean strategy does well against nice people willing to be exploited, we never move into the third stage where nice strategies dominate the population.

Such a story sounds reasonable but leaves out another possibility. A third strategy might be able to invade this mix. If so, the stable arrangement collapses and we are back to a transition period.

As one might expect, this issue is close to being completely intractable from a theoretical perspective. Axelrod's (1984) computer tournament and his simulation of evolution over time based on it provide some insight into the process. Axelrod (1980) also derives some theoretical results on this issue.

We take a slightly different approach as a primitive further step. We wish to ask: can a population "get stuck" with some mix of strategies? To answer this question, we will take advantage of the analytical equivalence between mixed populations and stochastic strategies.

To see this connection, suppose we have a population where the strategies 1, . . ., n are being played and that the fraction of the population playing each is $p_1, \ldots, p_n$, where $\Sigma\, p_i = 1$. Then the expected payoff of someone playing the i^{th} strategy in this population, which we will call V(i), is

(8) $V(i) = p_1 V(i|1) + p_2 V(i|2) + \ldots + p_n V(i|n)$

Now suppose we create a stochastic strategy, S, which plays the same strategies 1, . . ., n as are played in the population and plays each with the same probability as the proportion of that strategy in the population. Such a stochastic strategy behaves, in many ways, just like the population. In particular, the expected payoff to an individual playing any strategy in the original population would be the same as the expected payoff to an individual playing that strategy in a population where everyone plays S. This is because he faces the same probability distribution over the strategy of the individual he will play against either way. We will write the payoff to playing strategy I when in a population where individuals use this mix of strategies as $V(I|M)$, where M represents this particular mix of strategies with these particular proportions.

The implication is that if S is collectively stable, then M is collectively stable as well. We require of the mix what we require of a collectively stable strategy. That is, a mix M is collectively stable iff there does not exist any strategy I such that $V(I|M) > V(M)$, where

(9) $V(M) = p_1V(1) + . . . + p_nV(n)$

Note that $V(M) = V(S|S)$. Thus if there exists no strategy I such that $V(I|S) > V(S|S)$, then there exists no strategy I that can invade M since $V(I|M) = V(I|S)$ and $V(M) = V(S|S)$. Hence M is collectively stable iff S is.

In short, though the concepts of a stochastic strategy and a population with some mix of strategies are quite different, for our purposes, they are analytically indistinguishable. Since we have already considered mean and nice strategies, stochastic or otherwise, there is no point in discussing mixes of all nice or all mean strategies. Hence in what follows when we refer to a mix, we will always mean a mix which does not contain all nice or all mean strategies.

There are two immediate implications of this similarity. First, Bishop and Cannings (1978) have shown that a necessary condition for S to be collectively stable is that $V(i|S) = V(S|S)$ for all strategies i that S might play.[11] This implies that a necessary condition for the mix M to be collectively stable is $V(i) = V(M)$ for all i in the mix. We will call a mix that satisfies this condition a stable mix.

Second, it is easy to show that there will always exist

[11]This condition should be familiar from standard calculations for mixed Nash equilibria.

a collectively stable stochastic strategy or else ALL C or ALL D is collectively stable, where ALL C is the strategy of always cooperating. To see why this is true, suppose we have a strategy that behaves the same way on each move regardless of what happens. To find a strategy that might invade this one, note that we may as well only consider strategies that behave the same way on every turn also. The reason for this is that the best strategy to play against this strategy would be to repeat the best move against its one move every single time. But this just makes the game into the one-play case. Hence if this strategy is collectively stable in the one-play case, it is always collectively stable.

In the one-play case, in a game of this sort,[12] there will always exist a Nash equilibrium in mixed strategies. Since the payoffs are symmetric, this equilibrium will have the two players choosing the same probabilities. Thus the Nash equilibrium will be symmetric, so that this strategy will be collectively stable by definition.

In the Prisoner's Dilemma, the one-play Nash equilibrium in mixed strategies has both players cooperating with probability zero. Hence, as noted, ALL D is collectively stable in the Prisoner's Dilemma. In Chicken, the one-play Nash equilibrium in mixed strategies has both players cooperating with probability

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[12]We do not specify what "sort" is meant because the class of games for which this is true is very broad.

(S - P)/(T + S - R - P). Thus, in Chicken, the stochastic strategy of cooperating with this probability and defecting otherwise on each move is collectively stable regardless of the value of w. Therefore, there must exist a mix which is collectively stable all w in Chicken.[13] Thus we have shown PROPOSITION 4: A sufficient condition for the existence of a collectively stable mix is that the game be Chicken.

More general necessary and sufficient conditions for the existence of a collectively stable stochastic strategy and thus a collectively stable mix are quite difficult. Instead of following that path, we will narrow our focus somewhat and consider a type of invasion which is "easier" than the kind we have considered up to now. This will enable us to develop a partial answer to our question and to derive some results which are not limited to Chicken or the Prisoner's Dilemma or even to bimatrix games.

B. Invasion in a Minimal p-Cluster.

In the type of invasion considered so far, the invader has been, in effect, a single individual in a large population of individuals playing some other strategy or

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[13]One may be tempted--as was the author--to infer from this that a mix which is collectively stable in Chicken has the fraction x = (S - P)/(T + S - R - P) of the population play ALL C and the rest play ALL D. Note, though, that PR invades this mix. The reason is that the mixed strategy of choosing ALL C with probability x and ALL D otherwise is not equivalent to the behaviorial strategy of cooperating each move with probability x and defecting otherwise. In particular, the former strategy cooperates on the first n moves with probability x while the latter does so with probability x^n.

strategies. Thus the only relevant consideration has been how well that individual's strategy does against the strategy played by the rest of the individuals in the population. If the player could, in effect, bring some friends along, he might be more able to invade as he could interact with them. If they play the same strategy and that strategy is one that does well with itself, then the strategy may now be able to invade. In other words, if there is some correlation between playing the new strategy and encountering others playing it, then this "clustering" effect may be sufficient to enable the new strategy to invade. This is the idea behind Axelrod's notion of invasion in a p-cluster. The invading strategy, say B, is said to <u>invade A in a p-cluster</u> iff

$$(10)\quad pV(B|B) + (1 - p)V(B|A) > V(A|A)$$

for some p strictly between 0 and 1. The number p represents the proportion of its interactions the individuals playing B are able to have with others playing B.

Note that p does not represent the proportion of the people in the population who play B. If it did, then the relevant comparison would be to $pV(A|B) + (1 - p)V(A|A)$. Instead, the proportion of B's in the population is assumed to be trivial, but the B's are "near" one another so that they are not a trivial part of each other's environment. Equivalently, we can view p as the probability of encountering someone playing the strategy B conditional on playing the strategy B oneself. The conditional probability of encountering a B given that one plays a different strategy is approximately zero. So the events "meet a B" and "play B" are not independent.

Collective stability is a necessary but not sufficient condition for a strategy to be able to avoid invasion in a p-cluster. If A is not collectively stable but can be invaded by B, then B can certainly invade A in a p-cluster if p is very close to zero. It is possible, though, that B could invade A in a p-cluster with p arbitrarily close to zero, but B cannot invade A without the p-cluster.

LEMMA: Suppose $V(B|A) = V(A|A)$ and $V(B|B) > V(A|A)$. Then B invades A in a p-cluster for <u>any</u> $p > 0$.

Proof: If $V(B|A) = V(A|A)$, then B invades A in a p-cluster iff

(11) $pV(B|B) + (1 - p)V(A|A) > V(A|A)$:

(12) $pV(B|B) - pV(A|A) > 0$

or

(13) $V(B|B) > V(A|A)$

Note that this does not depend on p. Thus as long as $p > 0$, if (13) holds and $V(B|A) = V(A|A)$, B invades A in a p-cluster. Q.E.D.

When the conditions of the Lemma hold, we will say that B invades A in a minimal p-cluster to convey the idea that p can be chosen as close to zero as we like and the cluster will still invade. This is as close as we can come to regular invasion and still have the invasion be easier.[14]

Note that the Lemma implies that the collectively stable mix referred to in Proposition 4 can be invaded in a minimal p-cluster of ALL C. This is true because in this case, we must have V(M) = V(ALL C|M), which we can write as a convex combination of V(ALL C|ALL C) and V(ALL C|ALL D). Since ALL C does better against itself than it does against ALL D, we see that V(M) < V(ALL C|ALL C). Thus the conditions of the Lemma are met and so the mix is not stable in this slightly stronger sense.

In fact, mixes have a surprising weakness. To show this, let us first offer a definition.

DEFINITION: A mix will be said to be optimal if and only if there is no way to alter the proportions of each strategy in the population to create a population with a higher average payoff for its members.

PROPOSITION 5: A stable mix can always be invaded in a minimal p-cluster unless it is optimal.

Proof: Let M be a stable mix containing strategies

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[14]The possibility of ties--i.e., when V(B|A) = V(A|A)--is troubling generally. If this holds, some people playing B could "wander in" to the population without penalty and change the population from one which is collectively stable to one susceptible to invasion. Invasion in a minimal p-cluster considers one aspect of this problem brought about by the possibility of clustering. Maynard Smith (1982) in his definition of evolutionary stability requires V(A|B) > V(B|B) when ties occur. This condition implies that an inflow of individuals playing B actually helps those playing A more than those playing B. Other stability requirements could also be imposed (see, for example, the "roundabout" stability considerations of Hirshleifer and Riley, 1978).

1, . . ., n. Let I be some stochastic strategy made up solely of strategies from the population where the strategies are played with probabilities $p_1, \ldots, p_n$, some of which may be zero. Then

(14) $V(I|M) = p_1 V(1|M) + p_2 V(2|M) + \ldots + p_n V(n|M) = V(M)$

as $V(i|M) = V(i) = V(M)$ for $i = 1, \ldots, n$. Thus I invades the mixed population M in a minimal p-cluster iff $V(I|I) > V(M)$.

Suppose that the mix M is not optimal as defined above. Then there must exist another mix, say M^*, such that $V(M^*) > V(M)$ where M^* and M are made up of the same strategies but in different proportions in the population. Then let I be the stochastic strategy using each strategy in the mix M^* with the probability associated with the frequency of that strategy in the population. Then $V(I|I) = V(M^*) > V(M)$. Hence I will invade M in a minimal p-cluster. Q.E.D.

Note that the mix M^* does not have to be collectively stable itself. To see why this has some strong implications, suppose for a moment that we are social planners trying to pick the right proportion of each strategy in our population, the n strategies having already been determined for us. Suppose we choose these proportions to try to guarantee that our population will be collectively stable. A necessary condition for this is that the mix be stable--i.e., $V(1) = V(2)$, $V(2) = V(3)$, . . ., $V(n-1) = V(n)$. But this gives us $(n - 1)$ equations and, since the proportions must sum to 1, (n - 1) unknowns. Thus we have no freedom left to seek other goals, at least if the equations are linearly independent.

Suppose, on the other hand, that we decide we don't care if the population could be invaded, but choose the proportions to make the expected payoff of a typical member of the population, V(M), as large as possible. In general, we would be quite surprised if we came to the same answer for both problems.[15] The implication of Proposition 5, however, is that if the answers are not the same, then those n strategies cannot exist together in a mix as an end point for the population. If the answers are different, then the second population will not be collectively stable. The first population may be collectively stable, but even if it is, it can be invaded in a minimal p-cluster.[16] Thus it is

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[15]We have ignored the fact that w and the four payoffs can affect the problem. If we can choose these parameters as well, we may be able to find proportions that make the mix stable and optimal. To see how difficult this is in Chicken, though, suppose we solve the two problems for the optimal frequency and the stable frequency of each strategy. We then have five variables to manipulate subject to five inequalities (ordering of the payoffs and $0 \leq w < 1$) to solve (n - 1) equations. The equations are highly nonlinear in w and the payoffs and so may have numerous roots. However, as n gets large, the existence of a w and set of payoffs which can solve the problem becomes doubtful. The theorem in the Appendix is essentially a simple statement of this problem to derive some very basic implications.

[16]One might wonder if a collectively stable mix of I and 1, . . ., n could evolve after I invades. But the payoff relevant consideration is not the proportion of each strategy as played in the population, but the proportion of each pure strategy played by some part of the population or played with positive probability by some stochastic strategy in the population. That is, the only information required to calculate payoffs is the probability distribution over

a rather exceptional group of strategies that can exist together in a mix that is collectively stable and cannot be invaded in a minimal p-cluster.

There are two further implications that can be drawn from Proposition 5. First, TFT can never be part of a mix that is both collectively stable and safe from invasion by a minimal p-cluster. Thus any population containing TFT and some strategies which are not nice cannot reach a "stable" point until either TFT or the strategies which are not nice die out. This implication is a corollary to a theorem in the Appendix.

Second, as long as $2R > T + S$, any stable mix which contains a nice strategy can be invaded in a minimal p-cluster of that nice strategy. To see this, suppose i is some nice strategy in a mix and that p_i is the proportion of the population using that strategy. Let X_i be the stochastic strategy which uses each of the strategies in the mix <u>except</u> <u>i</u> with probability proportional to the frequency of the strategy in the population. Since $2R > T + S$, the highest total payoff to the two players in any move is 2R. Hence, $V(i|X_i) + V(X_i|i) < \frac{2R}{1 - w}$. The strict inequality is due to the fact that X_i is not a nice strategy because we are not considering any mixes of all nice strategies.

This implies that at least one of the terms on the

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pure strategies induced by stochastic strategies in the population and the distribution of strategies across the population. The introduction of I to the population brings in no new pure strategies but merely alters the proportion of each strategy from those necessary for stability.

left-hand side must be strictly less than $R/(1 - w)$. Suppose that $V(i|X_i) \geq R/(1 - w)$. Since $V(i|i) = R/(1 - w)$, this implies $V(i) > R/(1 - w)$, as $V(i)$ is a convex combination of $V(i|i)$ and $V(i|X_i)$. But since $V(X_i|i)$ and $V(X_i|X_i)$ are both less than $R/(1 - w)$, this implies $V(i) > V(X_i)$, contradicting the assumption that the mix is stable. Hence we must have $V(i|X_i) < R/(1 - w)$. But then

(15) $V(i|i) = R/(1 - w) > V(i) = V(M)$

so that i invades the mix in a minimal p-cluster.[17] Note that $2R > T + S$ is always true in the Prisoner's Dilemma, but not necessarily in Chicken. Again, if this equation does not hold, our assertion that mutual cooperation should be thought of as socially preferrable to the Nash equilibria in Chicken is no longer necessarily valid. Thus mixes may be socially preferrable to a population of all nice

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[17]This proof makes use of a proof in Axelrod (1980). In this paper, he focuses on what he calls strictly stable mixes of two strategies. His main result is that any such mix can be invaded by a p-cluster or any nice strategy for some $p < 1$. The proof is stated for the Prisoner's Dilemma, but relies only on the inequality $2R > T + S$. What we have shown here is that the assumption of two strategies is not necessary for the result and that strict stability, as he defines it, is a necessary condition for a mix to be uninvadable by a minimal p-cluster. Hence, as long as $2R > T + S$, any stable mix containing a nice strategy can be invaded by a minimal p-cluster of that nice strategy. Furthermore, any stable mix not containing a nice strategy can be invaded in a p-cluster of any nice strategy for some $p < 1$. In a way, though, the latter result suggests that invasion in a p-cluster is "too easy" as p approaches 1. It is easy to show that <u>any</u> strategy or mix A such that $V(B|B) > V(A|A)$ for some strategy B can be invaded in a p-cluster of B for some $p < 1$. Requiring a strategy to be safe from invasion by a p-cluster for any $p \in (0, 1)$ may be too strong a requirement. For this reason, we focus on invasion in a minimal p-cluster.

strategies when $2R \leq T + S$. (For an interesting and more general discussion of when mixes are socially preferrable to single-strategy equilibria, see Hallagan and Joerding, 1983.)

In short, Axelrod's results do not carry over completely and directly to mixes in Chicken, but the more general result that rational agents move toward "socially optimal" behavior without central authority still seems to hold.

V. Conclusion.

We see that, broadly speaking, Axelrod's results with the Prisoner's Dilemma carry over to Chicken. There are some differences in the evolution of cooperation in the two games. However, in general, there are still strong forces pushing the players toward mutual cooperation, even though both are self-interested and there is no central authority. Interestingly, the progress toward cooperation appears to be most likely detoured if the detour itself is socially preferrable to mutual cooperation.[18]

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[18]The three "stages" do not constitute a complete cataloging of all possible kinds of populations. Thus even if we knew that no mean and no stochastic strategy were collectively stable and that nice strategies were collectively stable, we would be unable to assert that the population must end up with all nice strategies. An exhaustive classification would be stochastic strategies, pure strategies which cooperate on the first move, and pure strategies which defect on the first move. We have considered the first group. One can show that there exists a strategy in the second group which is collectively stable iff there exists some nice collectively stable strategy. Finally, one can show that there exists a strategy in the third group which is collectively stable iff

We find Axelrod's approach and results interesting for the following reasons. His approach makes explicit use of an evolutionary view of change, a view often used heuristically to justify standard economic models. As Nelson and Winter (1982) have argued, it seems more appropriate to use evolutionary models if we believe that the economic processes being studied do indeed work through a "natural selection" type mechanism. We do not claim that Axelrod's methods are the only way to extend the evolutionary approach to strategic behavior. However, Axelrod and Maynard Smith have given us a starting point.[19]

Axelrod's results and their apparent robustness at least to the small variation considered here have interesting consequences for standard economic arguments on decentralization. Economists have long known that competitive equilibria are Pareto optimal under certain conditions, while game-theoretic equilibria very often are not. The introduction of strategic considerations can easily lead to a Pareto inefficient outcome, as in the two

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$w \geq \max\{\frac{S - P}{R - P}, \frac{T - R}{T - S}\}$ and $S > P$ <u>or</u> $S \leq P$. The fact that this can hold even when no mean strategy is collectively stable emphasizes the point made above. To be able to determine more precisely the circumstances under which we can reasonably expect the population to end up with all nice strategies, we need to analyze this second group of strategies in more detail. Unfortunately, this task is not straightforward.

[19]Axelrod's use of computer simulations, a technique also used by Nelson and Winter, is especially intriguing. An interesting recent paper following this approach is Bergstrom (1984).

games considered here. Axelrod's results and the results derived in this paper suggest that repeated play can act to dampen the negative effect of strategic considerations and lead to Pareto optimal outcomes.

APPENDIX

Let X_i be the stochastic strategy which plays each strategy of a particular mix except for strategy i with probability proportional to the frequency of each strategy in the population.

THEOREM: Consider a stable mix of strategies 1, . . ., n. If $V(i|X_i) \neq V(X_i|i)$ for any $i \in \{1, . . ., n - 1\}$, then the mix can always be invaded by a minimal p-cluster.[20]

Proof: The proof consists of showing that, under the circumstances stated, the mix cannot be optimal as defined in the text. We can write the expected payoff of a strategy I which plays i with probability z and X_i otherwise as

$$\text{(A1)} \quad V(I|I) = z^2V(i|i) + z(1 - z)[V(i|X_i) + V(X_i|i)] + (1 - z)^2V(X_i|X_i)$$

Suppose that the derivative of this expression with respect to z evaluated at $z = p_i$ is not zero. This would imply that there must be some z either smaller or bigger than p_i such that $V(I|I) > V(M)$ as $V(I|I) = V(M)$ if $z = p_i$. The derivative is

$$\text{(A2)} \quad \frac{dV(I|I)}{dz} = 2zV(i|i) + (1 - 2z)[V(i|X_i) + + V(X_i|i)]$$

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[20]One can show that if $V(i|X_i) = V(X_i|i)$ for $i = 1, . . ., n - 1$, it holds for $i = n$ as well. Furthermore, if this condition holds for all pure strategies i in the population, then it holds when we let i be any stochastic strategy using two or more pure strategies in the population.

$$+ 2(1 - z)V(X_i|X_i)$$

or

$$\text{(A3)} \quad = 2[zV(i|i) + (1 - z)V(i|X_i)] - 2[zV(X_i|i) + (1 - z)V(X_i|X_i)] + V(X_i|i) - V(i|X_i)$$

Evaluating the derivative at $z = p_i$:

$$\text{(A4)} \quad = 2V(i) - 2V(X_i) + V(X_i|i) - V(i|X_i)$$

$$= V(X_i|i) - V(i|X_i)$$

which is nonzero by assumption. Hence the mix can be invaded by a minimal p-cluster. Q.E.D.

One can show that TFT does worse against any strategy which is not nice than that strategy does against it. Hence, TFT can never be part of any stable mix which is not invadable by a minimal p-cluster, ignoring, of course, mixes of all nice strategies. This implies that any population including TFT and some strategies which are not nice cannot reach an "equilibrium" until either TFT or the strategies which are not nice die out.

To see why it may be more plausible to expect the latter to die out, notice from (A4) that since TFT does worse against its opponent than its opponent does against it, $V(I|I)$ is increasing in the probability that it plays TFT. That is, if we have a stable mix, then regardless of the proportion of the population playing TFT, a stochastic strategy which plays it with a higher probability than that proportion will invade in a minimal p-cluster.

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