Cooperation and control in multiplayer social dilemmas

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Direct reciprocity and conditional cooperation are important mechanisms to prevent free riding in social dilemmas. However, in large groups, these mechanisms may become ineffective because they require single individuals to have a substantial influence on their peers. However, the recent discovery of zero-determinant strategies in the iterated prisoner's dilemma suggests that we may have underestimated the degree of control that a single player can exert. Here, we develop a theory for zero-determinant strategies for iterated multiplayer social dilemmas, with any number of involved players. We distinguish several particularly interesting subclasses of strategies: fair strategies ensure that the own payoff matches the average payoff of the group; extortionate strategies allow a player to perform above average; and generous strategies let a player perform below average. We use this theory to describe strategies that sustain cooperation, including generalized variants of Tit-for-Tat and Win-Stay Lose-Shift. Moreover, we explore two models that show how individuals can further enhance their strategic options by coordinating their play with others. Our results highlight the importance of individual control and coordination to succeed in large groups.

evolutionary game theory | alliances | public goods game | volunteer's dilemma | cooperation

Cooperation among self-interested individuals is generally difficult to achieve (1–3), but typically the free rider problem is aggravated even further when groups become large (4–9). In small communities, cooperation can often be stabilized by forms of direct and indirect reciprocity (10–17). For large groups, however, it has been suggested that these mechanisms may turn out to be ineffective, as it becomes more difficult to keep track of the reputation of others and because the individual influence on others diminishes (4–8). To prevent the tragedy of the commons and to compensate for the lack of individual control, many successful communities have thus established central institutions that enforce mutual cooperation (18–22).

However, a recent discovery suggests that we may have underestimated the amount of control that single players can exert in repeated games. For the repeated prisoner's dilemma, Press and Dyson (23) have shown the existence of zero-determinant strategies (or ZD strategies), which allow a player to unilaterally enforce a linear relationship between the own payoff and the coplayer's payoff, irrespective of the coplayer's actual strategy. The class of zero-determinant strategies is surprisingly rich: for example, a player who wants to ensure that the own payoff will always match the coplayer's payoff can do so by applying a fair ZD strategy, like Tit-for-Tat. On the other hand, a player who wants to outperform the respective opponent can do so by slightly tweaking the Tit-for-Tat strategy to the own advantage, thereby giving rise to extortionate ZD strategies. The discovery of such strategies has prompted several theoretical studies, exploring how different ZD strategies evolve under various evolutionary conditions (24–30).

ZD strategies are not confined to the repeated prisoner's dilemma. Recently published studies have shown that ZD strategies also exist in other repeated two player games (29) or in repeated public goods games (31). Herein, we will show that such strategies exist for all symmetric social dilemmas, with an arbitrary number of participants. We use this theory to describe which ZD strategies can be used to enforce fair outcomes or to prevent free riders from taking over. Our results, however, are not restricted to the space of ZD strategies. By extending the techniques introduced by Press and Dyson (23) and Akin (27), we also derive exact conditions when generalized versions of Grim, Tit-for-Tat, and Win-Stay Lose-Shift allow for stable cooperation. In this way, we find that most of the theoretical solutions for the repeated prisoner's dilemma can be directly transferred to repeated dilemmas with an arbitrary number of involved players.

In addition, we also propose two models to explore how individuals can further enhance their strategic options by coordinating their play with others. To this end, we extend the notion of ZD strategies for single players to subgroups of players (to which we refer as ZD alliances). We analyze two models of ZD alliances, depending on the degree of coordination between the players. When players form a strategy alliance, they only agree on the set of alliance members, and on a common strategy that each alliance member independently applies during the repeated game. When players form a synchronized alliance, on the other hand, they agree to act as a single entity, with all alliance members playing the same action in a given round. We show that the strategic power of ZD alliances depends on the size of the alliance, the applied strategy of the allies, and on the properties of the underlying social dilemma. Surprisingly, the degree of coordination only plays a role as alliances become large (in which case a synchronized alliance has more strategic options than a strategy alliance).

To obtain these results, we consider a repeated social dilemma between n players. In each round of the game, players can decide whether to cooperate (C) or to defect (D). A player's payoff depends on the player's own decision and on the decisions of all other group members (Fig. 1*A*): in a group in which j of the other group members cooperate, a cooperator receives the payoff a_j, whereas a defector obtains b_j. We assume that payoffs satisfy the following three properties that are characteristic for social dilemmas (corresponding to the individual-centered interpretation

Significance

Many of the world's most pressing problems, like the prevention of climate change, have the form of a large-scale social dilemma with numerous involved players. Previous results in evolutionary game theory suggest that multiplayer dilemmas make it particularly difficult to achieve mutual cooperation because of the lack of individual control in large groups. Herein, we extend the theory of zero-determinant strategies to multiplayer games to describe which strategies maintain cooperation. Moreover, we propose two simple models of alliances in multiplayer dilemmas. The effect of these alliances is determined by their size, the strategy of the allies, and the properties of the social dilemma. When a single individual's strategic options are limited, forming an alliance can result in a drastic leverage.

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of altruism in ref. 32): (*i*) irrespective of the own strategy, players prefer the other group members to cooperate ($a_{j+1} \geq a_j$ and $b_{j+1} \geq b_j$ for all j); (*ii*) within any mixed group, defectors obtain strictly higher payoffs than cooperators ($b_{j+1} > a_j$ for all j); and (*iii*) mutual cooperation is favored over mutual defection ($a_{n-1} > b_0$). To illustrate our results, we will discuss two particular examples of multiplayer games (Fig. 1*B*). In the first example, the public goods game (33), cooperators contribute an amount $c>0$ to a common pool, knowing that total contributions are multiplied by r (with $1<r<n$) and evenly shared among all group members. Thus, a cooperator's payoff is $a_j = rc(j+1)/n - c$, whereas defectors yield $b_j = rcj/n$. In the second example, the volunteer's dilemma (34), at least one group member has to volunteer to bear a cost $c>0$ in order for all group members to derive a benefit $b>c$. Therefore, cooperators obtain $a_j = b-c$ (irrespective of j), whereas defectors yield $b_j = b$ if $j \geq 1$ and $b_0 = 0$. Both examples (and many more, such as the collective risk dilemma) (7, 8, 35) are simple instances of multiplayer social dilemmas.

We assume that the social dilemma is repeated, such that individuals can react to their coplayers' past actions (for simplicity, we will focus here on the case of an infinitely repeated game). As usual, payoffs for the repeated game are defined as the average payoff that players obtain over all rounds. In general, strategies for such repeated games can become arbitrarily complex, as subjects may condition their behavior on past events and on the round number in nontrivial ways. Nevertheless, as in pairwise games, ZD strategies turn out to be surprisingly simple.

A

Number of cooperators among co-players	n-1	n-2	....	2	1	0
Cooperator's payoff	a_{n-1}	a_{n-2}	...	a_2	a_1	a_0
Defector's payoff	b_{n-1}	b_{n-2}	...	b_2	b_1	b_0

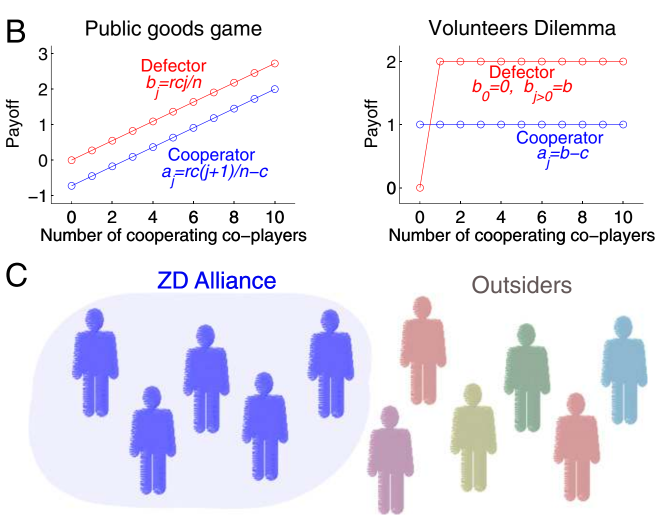

Fig. 1. Illustration of the model assumptions for repeated social dilemmas. (*A*) We consider symmetric n–player social dilemmas in which each player can either cooperate or defect. The player's payoff depends on its own decision and on the number of other group members who decide to cooperate. (*B*) We will discuss two particular examples: the public goods game (in which payoffs are proportional to the number of cooperators) and the volunteer's dilemma (as the most simple example of a nonlinear social dilemma). (*C*) In addition to individual strategies, we will also explore how subjects can enhance their strategic options by coordinating their play with other group members. We refer to the members of such a ZD alliance as allies, and we call group members that are not part of the ZD alliance outsiders. Outsiders are not restricted to any particular strategy. Some or all of the outsiders may even form their own alliance.

Results

Memory-One Strategies and Akin's Lemma. ZD strategies are memory-one strategies (23, 36); they only condition their behavior on the outcome of the previous round. Memory-one strategies can be written as a vector $\mathbf{p} = (p_{C,n-1}, \ldots, p_{C,0}, p_{D,n-1}, \ldots, p_{D,0})$. The entries $p_{S,j}$ denote the probability to cooperate in the next round, given that the player previously played $S \in \{C, D\}$ and that j of the coplayers cooperated (in the *SI Text*, we present an extension in which players additionally take into account who of the coplayers cooperated). A simple example of a memory-one strategy is the strategy Repeat, $\mathbf{p}^{Rep}$, which simply reiterates the own move of the previous round, $p^{Rep}_{C,j} = 1$ and $p^{Rep}_{D,j} = 0$. In addition, memory-one strategies need to specify a cooperation probability p_0 for the first round. However, our results will often be independent of the initial play, and in that case we will drop p_0.

Let us consider a repeated game in which a focal player with memory-one strategy $\mathbf{p}$ interacts with $n-1$ arbitrary coplayers (who are not restricted to any particular strategy). Let $v_{S,j}(t)$ denote the probability that the outcome of round t is (S,j). Let $\mathbf{v}(t) = [v_{C,n-1}(t), \ldots, v_{D,0}(t)]$ be the vector of these probabilities. A limit distribution $\mathbf{v}$ is a limit point for $t \to \infty$ of the sequence $[\mathbf{v}(1) + \ldots + \mathbf{v}(t)]/t$. The entries $v_{S,j}$ of such a limit distribution correspond to the fraction of rounds in which the focal player finds herself in state (S,j) over the course of the game.

There is a surprisingly powerful relationship between a focal player's memory-one strategy and the resulting limit distribution of the iterated game. To show this relationship, let $q_C(t)$ be the probability that the focal player cooperates in round t. By definition of $\mathbf{p}^{Rep}$ we can write $q_C(t) = \mathbf{p}^{Rep} \cdot \mathbf{v}(t) = [v_{C,n-1}(t) + \ldots + v_{C,0}(t)]$. Similarly, we can express the probability that the focal player cooperates in the next round as $q_C(t+1) = \mathbf{p} \cdot \mathbf{v}(t)$. It follows that $q_C(t+1) - q_C(t) = (\mathbf{p} - \mathbf{p}^{Rep}) \cdot \mathbf{v}(t)$. Summing up over all rounds from 1 to t, and dividing by t, yields $(\mathbf{p} - \mathbf{p}^{Rep}) \cdot [\mathbf{v}(1) + \ldots \mathbf{v}(t)]/t = [q_C(t+1) - q_C(1)]/t$, which has absolute value at most $1/t$. By taking the limit $t \to \infty$ we can conclude that

$$\left(\mathbf{p} - \mathbf{p}^{Rep}\right) \cdot \mathbf{v} = 0. \qquad [1]$$

This relation between a player's memory-one strategy and the resulting limit distribution will prove to be extremely useful. Because the importance of Eq. **1** has been first highlighted by Akin (27) in the context of the pairwise prisoner's dilemma, we will refer to it as Akin's lemma. We note that Akin's lemma is remarkably general, because it neither makes any assumptions on the specific game being played, nor does it make any restrictions on the strategies applied by the remaining $n-1$ group members.

Zero-Determinant Strategies in Multiplayer Social Dilemmas. As an application of Akin's lemma, we will show in the following that single players can gain an unexpected amount of control over the resulting payoffs in a multiplayer social dilemma. To this end, we first need to introduce some further notation. For a focal player i, let us write the possible payoffs in a given round as a vector $\mathbf{g}^i = (g^i_{S,j})$, with $g^i_{C,j} = a_j$ and $g^i_{D,j} = b_j$. Similarly, let us write the average payoffs of i's coplayers as $\mathbf{g}^{-i} = (g^{-i}_{S,j})$, where the entries are given by $g^{-i}_{C,j} = [ja_j + (n-j-1)b_{j+1}]/(n-1)$ and $g^{-i}_{D,j} = [ja_{j-1} + (n-j-1)b_j]/(n-1)$. Finally, let $\mathbf{1}$ denote the $2n$-dimensional vector with all entries being one. Using this notation, we can write player i's payoff in the repeated game as $\pi^i = \mathbf{g}^i \cdot \mathbf{v}$, and the average payoff of i's coplayers as $\pi^{-i} = \mathbf{g}^{-i} \cdot \mathbf{v}$. Moreover, by definition of $\mathbf{v}$ as a limit distribution, it follows that $\mathbf{1} \cdot \mathbf{v} = 1$. After these preparations, let us assume player i applies the memory-one strategy

$$\mathbf{p} = \mathbf{p}^{Rep} + \alpha \mathbf{g}^i + \beta \mathbf{g}^{-i} + \gamma \mathbf{1}, \qquad [2]$$

with α, β, and γ being parameters that can be chosen by player i (with the only restriction that $\beta \neq 0$). Due to Akin's lemma, we can conclude that such a player enforces the relationship

$$0 = \left(\mathbf{p} - \mathbf{p}^{Rep}\right) \cdot \mathbf{v} = \left(\alpha \mathbf{g}^i + \beta \mathbf{g}^{-i} + \gamma \mathbf{1}\right)\mathbf{v} = \alpha \pi^i + \beta \pi^{-i} + \gamma. \quad [3]$$

Player i's strategy thus guarantees that the resulting payoffs of the repeated game obey a linear relationship, irrespective of how the other group members play. Moreover, by appropriately choosing the parameters α, β, and γ, the player has direct control on the form of this payoff relation. As in Press and Dyson (23), who were first to discover such strategies for the prisoner's dilemma, we refer to the memory-one strategies in Eq. **2** as zero-determinant strategies or ZD strategies.

For our purpose, it will be convenient to proceed with a slightly different representation of ZD strategies. Using the parameter transformation $l = -\gamma/(\alpha+\beta)$, $s = -\alpha/\beta$, and $\phi = -\beta$, ZD strategies take the form

$$\mathbf{p} = \mathbf{p}^{Rep} + \phi\left[(1-s)\left(l\mathbf{1} - \mathbf{g}^i\right) + \mathbf{g}^i - \mathbf{g}^{-i}\right], \quad [4]$$

and the enforced payoff relationship according to Eq. **3** becomes

$$\pi^{-i} = s\pi^i + (1-s)l. \quad [5]$$

We refer to l as the baseline payoff of the ZD strategy and to s as the strategy's slope. Both parameters allow an intuitive interpretation: when all players adopt the same ZD strategy $\mathbf{p}$ such that $\pi^i = \pi^{-i}$, it follows from Eq. **5** that each player yields the payoff l. The value of s determines how the mean payoff of the other group members π^{-i} varies with π^i. The parameter ϕ does not have a direct effect on Eq. **5**; however, the magnitude of ϕ determines how fast payoffs converge to this linear payoff relationship as the repeated game proceeds (37).

The parameters l, s, and ϕ of a ZD strategy cannot be chosen arbitrarily, because the entries $p_{S,j}$ are probabilities that need to satisfy $0 \le p_{S,j} \le 1$. In general, the admissible parameters depend on the specific social dilemma being played. In *SI Text*, we show that exactly those relations **5** can be enforced for which either $s = 1$ (in which case the parameter l in the definition of ZD strategies becomes irrelevant) or for which l and $s < 1$ satisfy

$$\max_{0 \le j \le n-1}\left\{b_j - \frac{j}{n-1}\frac{b_j - a_{j-1}}{1-s}\right\} \le l \le \min_{0 \le j \le n-1}\left\{a_j + \frac{n-j-1}{n-1}\frac{b_{j+1} - a_j}{1-s}\right\}. \quad [6]$$

It follows that feasible baseline payoffs are bounded by the payoffs for mutual cooperation and mutual defection, $b_0 \le l \le a_{n-1}$, and that the slope needs to satisfy $-1/(n-1) \le s \le 1$. With s sufficiently close to 1, any baseline payoff between b_0 and a_{n-1} can be achieved. Moreover, because the conditions in Eq. **6** become increasingly restrictive as the group size n increases, larger groups make it more difficult for players to enforce specific payoff relationships.

Important Examples of ZD Strategies. In the following, we discuss some examples of ZD strategies. At first, let us consider a player who sets the slope to $s = 1$. By Eq. **5**, such a player enforces the payoff relation $\pi^i = \pi^{-i}$, such that i's payoff matches the average payoff of the other group members. We call such ZD strategies fair. As shown in Fig. 2*A*, fair strategies do not ensure that all group members get the same payoff; due to our definition of social dilemmas, unconditional defectors always outperform unconditional cooperators, no matter whether the group also contains fair players. Instead, fair players can only ensure that they do not take any unilateral advantage of their peers. Our characterization **6** implies that all social dilemmas permit a player to be fair, irrespective of the group size. As an example, consider the strategy proportional Tit-for-Tat (*pTFT*), for which the probability to cooperate is simply given by the fraction of cooperators among the coplayers in the previous round

$$pTFT_{S,j} = \frac{j}{n-1}. \quad [7]$$

For pairwise games, this definition of *pTFT* simplifies to Tit-for-Tat, which is a fair ZD strategy (23). However, also for the public goods game and for the volunteer's dilemma, *pTFT* is a ZD strategy, because it can be obtained from Eq. **4** by setting $s = 1$ and $\phi = 1/c$, with c being the cost of cooperation.

As another interesting subclass of ZD strategies, let us consider strategies that choose the mutual defection payoff as baseline payoff, $l = b_0$, and that enforce a positive slope $0 < s < 1$. The enforced payoff relation **5** becomes $\pi^{-i} = s\pi^i + (1-s)b_0$, implying that on average the other group members only get a fraction s of any surplus over the mutual defection payoff. Moreover, as the slope s is positive, the payoffs π^i and π^{-i} are positively related. As a consequence, the collective best reply for the remaining group members is to maximize i's payoffs by cooperating in every round. In analogy to Press and Dyson (23), we call such ZD strategies extortionate, and we call the quantity $\chi = 1/s$ the extortion factor. For games in which $l = b_0 = 0$, Eq. **5** shows that the extortion factor can be written as $\chi = \pi^i/\pi^{-i}$. Large extortion factors thus signal a substantial inequality in favor of player i. Extortionate strategies are particularly powerful in social dilemmas in which mutual defection leads to the lowest group payoff (as in the public goods game and in the volunteer's dilemma). In that case, they enforce the relation $\pi^i \ge \pi^{-i}$; on average, player i performs at least as well as the other group members (as also depicted in Fig. 2*B*). As an example, let us consider a public goods game and a ZD strategy $\mathbf{p}^{Ex}$ with $l = 0$, $\phi = n/[(n-r)sc + rc]$, for which Eq. **4** implies

$$p_{S,j}^{Ex} = \frac{j}{n-1}\left[1 - (1-s)\frac{n(r-1)}{r + (n-r)s}\right], \quad [8]$$

independent of the player's own move $S \in \{C, D\}$. In the limit $s \to 1$, $\mathbf{p}^{Ex}$ approaches the fair strategy *pTFT*. As s decreases from 1, the cooperation probabilities of $\mathbf{p}^{Ex}$ are increasingly biased to the own advantage. Extortionate strategies exist for all social dilemmas (this follows from condition **[6]** by setting $l = b_0$ and choosing an s close to 1). However, larger groups make extortion more difficult. For example, in public goods games with $n > r/(r-1)$, players cannot be arbitrarily extortionate any longer as **[6]** implies that there is an upper bound on χ (*SI Text*).

As the benevolent counterpart to extortioners, Stewart and Plotkin described a set of generous strategies for the iterated prisoner's dilemma (24, 28). Generous players set the baseline payoff to the mutual cooperation payoff $l = a_{n-1}$ while still enforcing a positive slope $0 < s < 1$. These parameter choices result in the payoff relation $\pi^{-i} = s\pi^i + (1-s)a_{n-1}$. In particular, for games in which mutual cooperation is the optimal outcome for the group (as in the public goods game and in the prisoner's dilemma but not in the volunteer's dilemma), the payoff of a generous player satisfies $\pi^i \le \pi^{-i}$ (Fig. 2*C*). For the example of a public goods game, we obtain a generous ZD strategy $\mathbf{p}^{Ge}$ by setting $l = rc - c$ and $\phi = n/[(n-r)sc + rc]$, such that

$$p_{S,j}^{Ge} = \frac{j}{n-1} + (1-s)\frac{n-j-1}{n-1}\frac{n(r-1)}{r + (n-r)s}. \quad [9]$$

For $s \to 1$, $\mathbf{p}^{Ge}$ approaches the fair strategy *pTFT*, whereas lower values of s make $\mathbf{p}^{Ge}$ more cooperative. Again, such generous strategies exist for all social dilemmas, but the extent to which players can be generous depends on the particular social dilemma and on the size of the group.

As a last interesting class of ZD strategies, let us consider players who choose $s = 0$. By Eq. **5**, such players enforce the payoff relation $\pi^{-i} = l$, meaning that they have unilateral control over the mean payoff of the other group members (for the prisoner's dilemma, such equalizer strategies were first discovered in ref. 38). However,

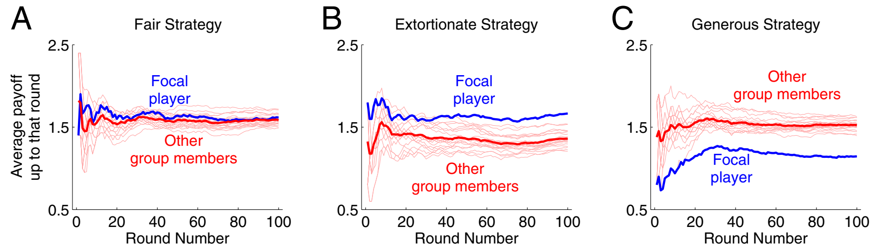

Fig. 2. Characteristic dynamics of payoffs over the course of the game for three different ZD strategies. Each panel depicts the payoff of the focal player π^i (blue) and the average payoff of the other group members π^{-i} (red) by thick lines. Additionally, the individual payoffs of the other group members are shown as thin red lines. (*A*) A fair player ensures that the own payoff matches the mean payoff of the other group members. However, fair strategies cannot ensure that all group members yield the same payoff. (*B*) For games in which mutual defection leads to the lowest group payoff, extortionate players ensure that their payoffs are above average. (*C*) In games in which mutual cooperation is the social optimum, generous players let their coplayers gain higher payoffs. The three graphs depict the case of a public goods game with $r=4$, $c=1$, and group size $n=20$. For the strategies of the other group members, we used random memory-one strategies, where the cooperation probabilities were independently drawn from a uniform distribution. For the strategies of the focal player, we used (*A*) *pTFT*, (*B*) $\mathbf{p}^{Ex}$ with $s=0.8$, and (*C*) $\mathbf{p}^{Ge}$ with $s=0.8$.

unlike extortionate and generous strategies, equalizer strategies typically cease to exist once the group size exceeds a critical threshold. For the example of a public goods game this threshold is given by $n=2r/(r-1)$. For larger groups, single players cannot determine the mean payoff of their peers any longer.

Stable Cooperation in Multiplayer Social Dilemmas. Let us next explore which ZD strategies give rise to a Nash equilibrium with stable cooperation. In *SI Text*, we prove that such ZD strategies need to have two properties: they need to be generous (by setting $l=a_{n-1}$ and $s>0$), but they must not be too generous [the slope needs to satisfy $s\geq(n-2)/(n-1)$]. In particular, whereas in the repeated prisoner's dilemma any generous strategy with $s>0$ is a Nash equilibrium (27, 28), larger group sizes make it increasingly difficult to uphold cooperation. In the limit of infinitely large groups, it follows that s needs to approach 1, suggesting that ZD strategies need to become fair. For the public goods game, this implies that stable cooperation can always be achieved when players cooperate in the first round and adopt proportional Tit-for-Tat thereafter. Interestingly, this strategy has received little attention in the previous literature. Instead, researchers have focused on other generalized versions of Tit-for-Tat, which cooperate if at least k coplayers cooperated in the previous round (4, 39, 40). Such memory-one strategies take the form $p_{S,j}=0$ if $j<k$ and $p_{S,j}=1$ if $j\geq k$. Unlike pTFT, these threshold strategies neither enforce a linear relation between payoffs, nor do they induce fair outcomes, suggesting that *pTFT* may be the more natural generalization of Tit-for-Tat in large-scale social dilemmas.

In addition to the stable ZD strategies, Akin's lemma also allows us to characterize all pure memory-one strategies that sustain mutual cooperation. In *SI Text*, we show that any such strategy $\mathbf{p}$ needs to satisfy the following four conditions

$$p_{C,n-1}=1,\quad p_{C,n-2}=0,\quad p_{D,1}\leq\frac{a_{n-1}-a_0}{b_{n-1}-a_{n-1}},\quad \text{and}\quad p_{D,0}\leq\frac{a_{n-1}-b_0}{b_{n-1}-a_{n-1}}, \tag{10}$$

with no restrictions being imposed on the other entries $p_{S,j}$. The first condition $p_{C,n-1}=1$ ensures that individuals continue to play C after mutual cooperation; the second condition $p_{C,n-2}=0$ guarantees that any unilateral deviation is punished; and the last two conditions describe whether players are allowed to revert to cooperation after rounds with almost uniform defection. Surprisingly, only these last two conditions depend on the specific payoffs of the social dilemma. As an application, condition **10** imply that the threshold variants of Tit-for-Tat discussed above are only a Nash equilibrium if they use the most stringent threshold: $k=n-1$. Such unforgiving strategies, however, have the disadvantage that they are often susceptible to errors: already a small probability that players fail to cooperate may cause a complete breakdown of cooperation (41). Instead, the stochastic simulations by Hauert and Schuster (5) showed that successful strategies tend to cooperate after mutual cooperation and after mutual defection [i.e., $p_{C,n-1}=p_{D,0}=1$ and $p_{S,j}=0$ for all other states (S,j)]. We refer to such a behavior as *WSLS*, because for pairwise dilemmas it corresponds to the Win-Stay, Lose-Shift strategy described by ref. 36. Because of condition **[10]**, *WSLS* is a Nash equilibrium if and only if the social dilemma satisfies $(b_{n-1}+b_0)/2\leq a_{n-1}$. For the example of a public goods game, this condition simplifies to $r\geq 2n/(n+1)$, which is always fulfilled for $r\geq 2$. For social dilemmas that meet this condition, *WSLS* provides a stable route to cooperation that is robust to errors.

Zero-Determinant Alliances. In agreement with most of the theoretical literature on repeated social dilemmas, our previous analysis is based on the assumption that individuals act independently. As a result, we observed that a player's strategic options typically diminish with group size. As a countermeasure, subjects may try to gain strategic power by coordinating their strategies with others. In the following, we thus extend our theory of ZD strategies for single individuals to subgroups of players. We refer to these subgroups as ZD alliances. Because the strategic power of ZD alliances is likely to depend on the exact mode of coordination between the allies, we consider two different models: when subjects form a strategy alliance, they only agree on the set of alliance members and on a common ZD strategy that each ally independently applies. During the actual game, there is no further communication between the allies. Strategy alliances can thus be seen as a boundary case of coordinated play, which requires a minimum amount of coordination. Alternatively, we also analyze synchronized alliances, in which all allies synchronize their actions in each round (i.e., the allies cooperate collectively, or they defect collectively). In effect, such a synchronized alliance thus behaves like a new entity that has a higher leverage than each player individually. Synchronized alliances thus may be considered as a boundary case of coordinated play that requires substantial coordination.

To model strategy alliances, let us consider a group of $n^{\mathcal{A}}$ allies, with $1\leq n^{\mathcal{A}}<n$. We assume that all allies make a binding agreement that they will play according to the same ZD strategy $\mathbf{p}$ during the repeated game. Because the ZD strategy needs to allow allies to differentiate between the actions of the other allies and the outsiders, we need to consider a more general state space than before. The state space now takes the form $(S,j^{\mathcal{A}},j^{-\mathcal{A}})$. The first entry S corresponds to the focal player's own play in the previous round, $j^{\mathcal{A}}$ gives the number of cooperators among the other allies, and $j^{-\mathcal{A}}$ is the number of cooperators among the outsiders. A memory-one strategy $\mathbf{p}$ again needs to specify a cooperation probability $p_{S,j^{\mathcal{A}},j^{-\mathcal{A}}}$ for each of the possible states. Using this state space, we can define ZD strategies for a player i in a strategy alliance as

$$\mathbf{p}=\mathbf{p}^{Rep}+\phi\left[(1-s)(l\mathbf{1}-\mathbf{g}^i)+\mathbf{g}^i-\left(n^{\mathcal{A}}-1\right)w^{\mathcal{A}}\mathbf{g}^{\mathcal{A}}-\left(n-n^{\mathcal{A}}\right)w^{-\mathcal{A}}\mathbf{g}^{-\mathcal{A}}\right]. \tag{11}$$

The vector $\mathbf{g}^{\mathcal{A}}$ contains the average payoff of the other allies for each possible state, and $\mathbf{g}^{-\mathcal{A}}$ is the corresponding vector for the outsiders. The weights $w^{-\mathcal{A}} \geq 0$ and $w^{\mathcal{A}} \geq 0$ are additional parameters that determine the relative importance of outsiders and other allies, being subject to the constraint $(n^{\mathcal{A}}-1)w^{\mathcal{A}} + (n-n^{\mathcal{A}})w^{-\mathcal{A}} = 1$. In the special case of a single player forming an alliance, $n^{\mathcal{A}} = 1$, this guarantees that the two definitions of ZD strategies **4** and **11** are equivalent.

Similarly to the case of single individuals, we can apply Akin's lemma to show that strategy alliances enforce a linear relationship between their own mean payoff $\pi^{\mathcal{A}}$ and the mean payoff of the outsiders $\pi^{-\mathcal{A}}$ (for details, see *SI Text*)

$$\pi^{-\mathcal{A}} = s^{\mathcal{A}}\pi^{\mathcal{A}} + \left(1-s^{\mathcal{A}}\right)l, \qquad [12]$$

where the slope of the alliance is given by $s^{\mathcal{A}} = \left[s-(n^{\mathcal{A}}-1)w^{\mathcal{A}}\right] / \left[1-(n^{\mathcal{A}}-1)w^{\mathcal{A}}\right]$. A strategy alliance can enforce exactly those payoff relationships **12** for which either $s^{\mathcal{A}} = 1$ or for which l and $s^{\mathcal{A}} < 1$ satisfy the conditions

$$\max_{0 \leq j \leq n-n^{\mathcal{A}}} \left\{ b_j - \frac{j}{n-n^{\mathcal{A}}} \frac{b_j - a_{j-1}}{1-s^{\mathcal{A}}} \right\} \leq l \leq \min_{n^{\mathcal{A}}-1 \leq j \leq n-1} \left\{ a_j + \frac{n-j-1}{n-n^{\mathcal{A}}} \frac{b_{j+1}-a_j}{1-s^{\mathcal{A}}} \right\}. \qquad [13]$$

Interestingly, to reach this strategic power, an alliance needs to put a higher weight on the within-alliance payoffs (i.e., $w^{\mathcal{A}}$ needs to exceed $w^{-\mathcal{A}}$; *SI Text*), such that the allies are stronger affected by what the other allies do, as opposed to the actions of the outsiders. For single player alliances, $n^{\mathcal{A}} = 1$, condition **13** again simplifies to the previous condition **6**. However, as the alliance size $n^{\mathcal{A}}$ increases, condition **13** becomes easier to satisfy. Larger alliances can therefore enforce more extreme payoff relationships. For the example of a public goods game, we noted that single players cannot be arbitrarily extortionate when $n > r/(r-1)$. Alliances, on the other hand, only need to be sufficiently large, $n^{\mathcal{A}}/n \geq (r-1)/r$. Once an alliance has this critical mass, there are no bounds to extortion.

In a similar way, we can also analyze the strategic possibilities of a synchronized alliance. Because synchronized alliances act as a single entity, they transform the symmetric social dilemma between n independent players to an asymmetric game between $n-n_{\mathcal{A}}+1$ independent players. From the perspective of the alliance, the state space now takes the form (S,j), where $S \in \{C,D\}$ is the common action of all allies and where $0 \leq j \leq n-n_{\mathcal{A}}$ is the number of cooperators among the outsiders. ZD strategies for the synchronized alliance can be defined analogously to ZD strategies for single players

$$\mathbf{p} = \mathbf{p}^{Rep} + \phi\left[\left(1-s^{\mathcal{A}}\right)\left(l\mathbf{1}-\mathbf{g}^{\mathcal{A}}\right) + \mathbf{g}^{\mathcal{A}} - \mathbf{g}^{-\mathcal{A}}\right], \qquad [14]$$

with $\mathbf{g}^{\mathcal{A}}$ being the payoff vector for the allies and $\mathbf{g}^{-\mathcal{A}}$ being the payoff vector of the outsiders. For a single player alliance, $n^{\mathcal{A}} = 1$, this again reproduces the definition of ZD strategies in **4**. By applying Akin's lemma to Eq. **14**, we conclude that synchronized alliances enforce $\pi^{-\mathcal{A}} = s^{\mathcal{A}}\pi^{\mathcal{A}} + (1-s^{\mathcal{A}})l$, which is the same as relationship **12** for strategy alliances. Surprisingly, we even find that for reasonable alliance sizes, $n^{\mathcal{A}} \leq n/2$, strategy alliances and synchronized alliances have the same set of enforceable parameters l and $s^{\mathcal{A}}$, as given by Eq. **13** (see *SI Text* for details). Thus, for the two models of ZD alliances considered here, the exact mode of coordination is irrelevant for the alliance's strategic power unless the alliance has reached a substantial size.

Table 1 gives an overview of our findings on ZD strategies and ZD alliances in multiplayer social dilemmas. It shows that, although generally, ZD strategies exist for all group sizes, the power of single players to enforce particular outcomes typically diminishes or disappears in large groups. Forming ZD alliances then allows players to increase their strategic scope. The impact of a given ZD alliance, however, depends on the specific social dilemma: although ZD alliances can become arbitrarily powerful in public goods games, their strategic options remain limited in the volunteer's dilemma.

Discussion

When Press and Dyson (23) discovered the new class of ZD strategies for the repeated prisoner's dilemma, this came as a big surprise (24, 25): after more than five decades of research, it seemed unlikely that any major property of the prisoner's dilemma has been overlooked. For repeated multiplayer dilemmas the situation is different. Although various Folk theorems guarantee that cooperation is also feasible in large groups (42, 43), there has been considerably less theoretical research on the evolution of cooperation in repeated multiplayer dilemmas (4, 5, 39, 40). This lack of research may be due to the higher complexity: the mathematics of repeated n-player dilemmas seems to be more intricate, and numerical investigations are impeded because the time to compute payoffs increases exponentially in the number of players (5). Nevertheless, we showed here that many of the results for the repeated prisoner's dilemma can be directly transferred to general social dilemmas, with an arbitrary number of involved subjects. The foundation for this progress is a new framework, provided by Akin's lemma and the theory of Press and Dyson.

Using this framework, we extended the theory of repeated multiplayer dilemmas into three directions. The first and most immediate direction is our finding that ZD strategies exist in all social dilemmas. These strategies allow players to unilaterally dictate linear payoff relations, irrespective of the specific social dilemma being played, irrespective of the group size, and irrespective of the counter measures taken by the other group members. In particular, we showed that any social dilemma allows players to be fair, extortionate, or generous. Each of these strategy classes has its own particular strengths: extortionate strategies give a player a relative advantage compared with the other group members; fair strategies help to avoid further inequality within a group; and generous strategies allow players to revert to mutual cooperation when a coplayer defected by accident. At the same time, ZD strategies are remarkably simple. For example, to be fair in a public goods game, players only need to apply a rule called proportional Tit-for-Tat: if j of the $n-1$ other group members cooperated in the previous round, then cooperate with probability $j/(n-1)$ in the following round. Extortionate and generous strategies can be obtained in a similar way, by slightly modifying *pTFT* to the own advantage or to the advantage of the others.

As the second direction, we explored which ZD strategies and which pure memory-one strategies can be used to sustain cooperation in multiplayer dilemmas. Among ZD strategies, such strategies need to be generous (such that players never try to outperform their peers) (27, 28), but at the same time they must not be too generous. The right degree of generosity depends on the size of the group but not on the specific social dilemma being played. As a rule of thumb, we obtain that in larger groups, subjects are required to show less generosity.

As the last direction, we extended the concept of zero-determinant strategies from single players to subgroups of players, to which we refer to as ZD alliances. Depending on the degree of coordination, we explored two forms of ZD alliances: members of a strategy alliance only agree on using a common ZD strategy during the game, but they do not coordinate each of their decisions; members of a synchronized alliance, on the other hand, act as a single entity—they either all cooperate or they all defect in a given round. The effect of such ZD alliances depends on the size of the alliance, the applied strategy, and the properties of the underlying social dilemma. In general, we find that by coordinating their play with others, subjects can increase their strategic options considerably. The exact mode of coordination, however, only turns out to play a minor role: As long as the size of the ZD alliance is below half the group size, strategy alliances and synchronized alliances have the same strategic power. In addition to their static properties, ZD strategies for the prisoner's dilemma also have a remarkable dynamic component (23, 44): when a player commits himself to an extortionate ZD strategy, then adapting coplayers learn to cooperate over time. Numerical simulations in the SI show an analogous result for multiplayer dilemmas: when ZD alliances apply strategies with a positive

Table 1. Strategic power of different ZD strategies for three different social dilemmas

Strategy class	Typical property	Prisoner's dilemma	Public goods game	Volunteer's dilemma
Fair strategies	$\pi^{-A} = \pi^{A}$	Always exist	Always exist	Always exist
Extortionate strategies	$\pi^{-A} \leq \pi^{A}$	Always exist	In large groups, single players cannot be arbitrarily extortionate, but sufficiently large ZD alliances can be arbitrarily extortionate	Even large ZD alliances cannot be arbitrarily extortionate
Generous strategies	$\pi^{-A} \geq \pi^{A}$	Always exist	In large groups, single players cannot be arbitrarily generous, but sufficiently large ZD alliances can be arbitrarily generous	Do not ensure that own payoff is below average
Equalizers	$\pi^{-A} = l$	Always exist	May not be feasible for single players, but is always feasible for sufficiently large ZD alliances	Only feasible if size of ZD alliance is $n^{A} = n-1$, can only enforce $l = b - c$

Analogously to the case of individual players, ZD alliances are fair when they set $s^{A} = 1$; they are extortionate when $l = b_0$ and $0 < s^{A} < 1$; they are generous for $l = a_{n-1}$ and $0 < s^{A} < 1$; and they are equalizers when $s^{A} = 0$. For each of the three considered social dilemmas, we explore whether a given ZD strategy is feasible by examining the respective conditions in Eq. **13**. In the repeated prisoner's dilemma, single players can exert all strategic behaviors (23, 28, 29). Other social dilemmas either require players to form alliances to gain sufficient control (as in the public goods game), or they only allow for limited forms of control (as in the volunteer's dilemma). These results hold both for strategy alliances and for synchronized alliances.

slope, they can trigger a positive group dynamics among the outsiders. The magnitude of this dynamic effect again depends on the size of the alliance, and on the applied strategy of the allies.

Here, we focused on ZD strategies; but the toolbox that we apply (in particular Akin's lemma) is more general. As an example, we identified all pure memory-one strategies that allow for stable cooperation, including the champion of the repeated prisoner's dilemma, Win-Stay Lose-Shift (36, 45). We expect that there will be further applications of Akin's lemma to come. Such applications may include, for instance, a characterization of all Nash equilibria among the stochastic memory-one strategies or an analysis of how alliances are formed and whether evolutionary forces favor particular alliances over others (46, 47).

Overall, our results reveal how single players in multiplayer games can increase their control by choosing the right strategies and how they can increase their strategic options by joining forces with others.

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Supporting Information

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SI Text

In the following, we show how the mathematical framework introduced by Press and Dyson (1) and Akin (2) can be extended to explore cooperation and control in multiplayer social dilemmas. We begin by defining the setup of repeated social dilemmas, and then we discuss the existence and the properties of zero-determinant strategies. In particular, we study ZD strategies that allow a player to differentiate between the actions of different coplayers. We also identify strategies that give rise to stable cooperation. To this end, we focus on two strategy classes: ZD strategies and pure memory-one strategies. Then, we investigate how individuals can extend their strategic options by coordinating their behaviors with others, and we apply our results to two examples of multiplayer dilemmas: the public goods game and the volunteer's dilemma. The appendix contains the proofs for our propositions.

Setup of the Model: Repeated Multiplayer Dilemmas. We consider repeated social dilemmas between n players (as illustrated in Fig. 1). In each round, players may either cooperate (C) or defect (D), and the players' payoffs for each round depend on their own action and on the number of cooperators among the other group members. Specifically, in a group with j other cooperators, a cooperator receives the payoff a_j, whereas a defector obtains b_j. To qualify as a social dilemma, we assume that one-shot payoffs satisfy the following three conditions (3):

i) Independent of the own action, players prefer their coplayers to be cooperative

$$a_{j+1} \geq a_j \quad \text{and} \quad b_{j+1} \geq b_j \text{ for all } j \text{ with } 0 \leq j < n-1. \qquad \textbf{[S1]}$$

ii) Within each mixed group, defectors strictly outperform cooperators

$$b_{j+1} > a_j \text{ for all } j \text{ with } 0 \leq j < n-1. \qquad \textbf{[S2]}$$

iii) Mutual cooperation is preferred over mutal defection

$$a_{n-1} > b_0. \qquad \textbf{[S3]}$$

As particular examples of such social dilemmas, we discuss the linear public goods game (4) and the volunteer's dilemma (5) later.

We will assume here that the social dilemma is repeated infinitely often. This assumption is merely made for simplicity of the argument; similar results can be obtained for finitely many rounds (6). In repeated games, a player's strategy needs to specify how to act in given round, depending on the outcomes of the previous rounds. Given the strategies of all group members, let us denote player i's expected payoff in round t as $\pi_i(t)$. The payoffs for the repeated game are defined as the average payoff per round

$$\pi^i = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \pi^i(t). \qquad \textbf{[S4]}$$

In the following, we will assume that these limits exist. This assumption holds, for example, when players only base their decisions on a finite (but arbitrarily large) number of past rounds.

Zero-Determinant Strategies for Multiplayer Dilemmas.

Memory-one strategies and Akin's lemma. Although in general, strategies for repeated games can be arbitrarily complicated, we showed that players can achieve a remarkable control over the possible payoff relations by referring to the outcome of the last round only. In particular, we focused on players who only consider their own move in the previous round and the number of cooperators in the previous round (this is a consequence of our assumption that the game is symmetric, such that payoffs do not depend on who of the coplayers cooperated, but only on how many). Such strategies are particularly relevant when players can only observe the outcome of the game, but not the coplayers' individual actions. In the context of alliances, however, it is useful to consider a slightly more general strategy set, which allows players to distinguish between different coplayers. In this section, we will therefore develop a more general theory of ZD strategies.

To this end, let us denote player i's action in a given round as $S_i \in \{C, D\}$, and let $\sigma = (S_1, \ldots, S_n) \in \{C, D\}^n$ denote the overall outcome of that round. A memory-one strategy is a rule that tells a player what to do in the next round, given the outcome of the previous round. Formally, such memory-one strategies correspond to a map that takes the outcome of the previous round σ as input and that returns the cooperation probability p_σ for the next round, $\mathbf{p} = (p_\sigma)_{\sigma\in\{C,D\}^n}$. For example, player i's strategy Repeat, which simply reiterates the own move from the previous round, takes the form $\mathbf{p}^{\text{Rep}}$ with

$$\mathbf{p}^{\text{Rep}}_{(S_1,\ldots,S_n)} = \begin{cases} 1 & \text{if } S_i = C \\ 0 & \text{if } S_i = D \end{cases}. \qquad \textbf{[S5]}$$

Additionally a memory-one strategy also needs to specify a cooperation probability for the first round of the game. Because the outcome of infinitely repeated games is often independent of the first round, the initial cooperation probability is typically neglected (1, 2, 7–12). In the following, we will therefore only specify a player's initial cooperation probability when necessary.

If all members of the group use memory strategies, the calculation of payoffs according to Eq. **S4** becomes particularly simple. In that case, the repeated game can be described as a Markov chain, as the outcome in the next round only depends on the outcome of the previous round (13–15). Although the assumption of memory-one strategies often simplifies calculations, we will show below that the properties of ZD strategies hold irrespective of the strategies of the other group members (in particular, ZD strategists do not require their coplayers to apply memory-one strategies).

For a game between n players with arbitrary but fixed strategies, let $v_\sigma(t)$ be the probability that the outcome of the tth round is σ and let $\mathbf{v}(t)=[v_\sigma(t)]_{\sigma\in\{C,D\}^n}$ be the vector of these probabilities. For example, for pairwise games $\mathbf{v}(t)$ becomes $[v_{CC}(t), v_{CD}(t), v_{DC}(t), v_{DD}(t)]$. A limit distribution $\mathbf{v}$ is a limit point for $t\to\infty$ of the sequence $[\mathbf{v}(1)+\ldots\mathbf{v}(t)]/t$. The entries of v_σ of such a limit distribution correspond to the fraction of rounds in which the group members find themselves in state $\sigma\in\{C,D\}^n$ over the course of the game. If one of the players applies a memory-one strategy $\mathbf{p}$, Akin's lemma again guarantees that there is a powerful relationship between $\mathbf{p}$ and $\mathbf{v}$ (which can be shown with literally the same proof as in the main text).

Lemma (Akin's Lemma). *Suppose the focal player applies an arbitrary memory-one strategy* $\mathbf{p}$. *Then, for any limiting distribution* $\mathbf{v}$ *(irrespective of the outcome of the initial round), we have*

$$\left(\mathbf{p}-\mathbf{p}^{Rep}\right)\cdot\mathbf{v}=0, \qquad \textbf{[S6]}$$

where the product refers to the usual scalar product, $\mathbf{p}\cdot\mathbf{v}=\sum_{\sigma\in\{C,D\}^n}p_\sigma v_\sigma$.

We note that Akin's lemma makes no assumptions on the payoff structure of the game (in particular it also applies to games that do not have the form of a social dilemma). Moreover, there are no restrictions on the strategies applied by the remaining $n-1$ group members.

Zero-determinant strategies. To define zero-determinant strategies, let us first introduce some further notation. For a given round outcome $\sigma\in\{C,D\}^n$, let $|\sigma|$ denote the total number of cooperators (i.e., $|\sigma|$ equals to the number of Cs in σ). Then we can write player i's payoff g^i_σ for that round as

$$g^i_{(S_1,\ldots,S_n)} = \begin{cases} a_{|\sigma|-1} & \text{if } S_i=C \\ b_{|\sigma|} & \text{if } S_i=D. \end{cases} \qquad \textbf{[S7]}$$

Let $\mathbf{g}^i=(g^i_\sigma)_{\sigma\in\{C,D\}^n}$ be the corresponding payoff vector. Using this notation, we can write player i's payoff in round t as $\pi^i(t)=\mathbf{g}^i\cdot\mathbf{v}(t)$, and i's expected payoff for the repeated social dilemma according to Eq. **S4** becomes $\pi^i=\mathbf{g}^i\cdot\mathbf{v}$. Finally, let $\mathbf{1}=(1)_{\sigma\in\{C,D\}^n}$ denote the vector with all entries being equal to one. By definition of $\mathbf{v}$, it follows that $\mathbf{1}\cdot\mathbf{v}=1$. We can now introduce ZD strategies as follows.

Theorem (Press and Dyson). *Let* α, β_j, *and* γ *be parameters such that* $\sum_{j\neq i}\beta_j\neq 0$. *If player i applies a memory-one strategy of the form*

$$\mathbf{p}=\mathbf{p}^{Rep}+\alpha\mathbf{g}^i+\sum_{j\neq i}^{n}\beta_j\mathbf{g}^j+\gamma 1, \qquad \textbf{[S8]}$$

then, irrespective of the strategies of the remaining $n-1$ *group members, payoffs obey the equation*

$$\alpha\pi^i+\sum_{j\neq i}\beta_j\pi^j+\gamma=0. \qquad \textbf{[S9]}$$

*We refer to strategies of the form **S8** as zero-determinant strategies or ZD strategies.*

Proof: Follows immediately from Akin's lemma, because

$$0=\left(\mathbf{p}-\mathbf{p}^{Rep}\right)\cdot\mathbf{v}=\left(\alpha\mathbf{g}^i+\sum_{j\neq i}^{n}\beta_j\mathbf{g}^j+\gamma\mathbf{1}\right)\cdot\mathbf{v}=\alpha\pi^i+\sum_{j\neq i}\beta_j\pi^j+\gamma. \qquad \textbf{[S10]}$$

By using ZD strategies, a player can thus enforce a linear payoff relation between his own payoff and the payoffs of the coplayers. Moreover, by appropriately choosing the parameters α, β_j, and γ, the player has direct control on the form of this payoff relation.

For our purpose, it will be convenient to use a slightly different representation of ZD strategies. For a player i who applies a ZD strategy, let us consider the following parameter transformation:

$$l=\frac{-\gamma}{\alpha+\sum_{k\neq i}\beta_k},\quad s=\frac{-\alpha}{\sum_{k\neq i}\beta_k},\quad w_{j\neq i}=\frac{\beta_j}{\sum_{k\neq i}\beta_k},\quad w_i=0,\quad \phi=-\sum_{k\neq i}\beta_k. \qquad \textbf{[S11]}$$

Using these new parameters, ZD strategies take the form

$$\mathbf{p}=\mathbf{p}^{Rep}+\phi\left[s\mathbf{g}^i-\sum_{j\neq i}w_j\mathbf{g}^j+(1-s)l\mathbf{1}\right], \qquad \textbf{[S12]}$$

subject to the constraints $\phi \neq 0$, $w_i = 0$, and $\sum_{j=1}^{n} w_j = 1$, which directly arise from the definitions in Eq. **S11**. When player i applies such a ZD strategy, the enforced payoff relation according to Eq. **S9** becomes

$$\pi^{-i} = s\pi^i + (1-s)l, \qquad \textbf{[S13]}$$

where $\pi^{-i} = \sum_{j=1}^{n} w_j \pi^j$ is the weighted average payoff of the coplayers. We refer to l as the baseline payoff of the ZD strategy, to s as the slope, and to $\mathbf{w} = (w_j)$ as the strategy's weights. The parameter ϕ does not have a direct effect on Eq. **S13**; however, the magnitude of ϕ determines how fast payoffs converge to the enforced payoff relation as the game proceeds (6).

Examples (the impact of different weights):

i) Equal weight on all coplayers. Suppose player i applies a ZD strategy with weights $w_j = 1/(n-1)$ for all $j \neq i$. According to Eq. **S12**, the entries of such a ZD strategy have the form

$$p_\sigma = \begin{cases} 1 + \phi\left[(1-s)\left(l - a_{|\sigma|-1}\right) - \dfrac{n-|\sigma|}{n-1}\left(b_{|\sigma|} - a_{|\sigma|-1}\right)\right] & \text{if } S_i = C \\[2ex] \phi\left[(1-s)\left(l - b_{|\sigma|}\right) + \dfrac{|\sigma|}{n-1}\left(b_{|\sigma|} - a_{|\sigma|-1}\right)\right] & \text{if } S_i = D. \end{cases} \qquad \textbf{[S14]}$$

The cooperation probabilities of player i thus only depend on the player's own action S_i in the previous round and on the number of cooperators $|\sigma|$. That is, the ZD strategies that we discussed in the main text are exactly those ZD strategies that use the same weight for each of their coplayers. According to Eq. **S13**, such strategies enforce $\pi^{-i} = s\pi^i + (1-s)l$, with π^{-i} being the arithmetic mean of all coplayers' payoffs $\pi^{-i} = \sum_{j \neq i} \pi^j/(n-1)$. In Fig. S1, we illustrate this relationship for two different social dilemmas (the public goods game and the volunteer's dilemma) and for two different ZD strategies (a generous and an extortionate ZD strategy).

ii) Full weight on one coplayer. Let us now suppose instead that player i chooses $\mathbf{w}$ such that all entries are zero except for $w_j = 1$ for some $j \neq i$. It follows that

$$p_\sigma = \begin{cases} 1 + \phi(1-s)\left(l - a_{|\sigma|-1}\right) & \text{if } S_i = S_j = C \\ 1 + \phi\left[s a_{|\sigma|-1} - b_{|\sigma|} + (1-s)l\right] & \text{if } S_i = C, S_j = D \\ \phi\left[s b_{|\sigma|} - a_{|\sigma|-1} + (1-s)l\right] & \text{if } S_i = D, S_j = C \\ \phi(1-s)\left(l - b_{|\sigma|}\right) & \text{if } S_i = S_j = D. \end{cases} \qquad \textbf{[S15]}$$

That is, player i's reaction only depends on the number of cooperators, the own move, and on player j's move. The enforced payoff relation **S13** becomes $\pi^j = s\pi^i + (1-s)l$.

A player cannot enforce arbitrary payoff relations **S13** because the parameters l, s, $\mathbf{w}$, and ϕ need to be set such that the resulting cooperation probabilities according to Eq. **S12** are in the unit interval. We thus say that a payoff relation $(l, s, \mathbf{w})$ is enforceable if there is a $\phi \neq 0$ such that the resulting ZD strategy $\mathbf{p}$ satisfies $p_\sigma \in [0,1]$ for all possible outcomes $\sigma \in \{C, D\}^n$. The following gives some necessary conditions for enforceable payoff relations.

Proposition 1 (Necessary Conditions for Enforceable Payoff Relations). *Any enforceable payoff relation* $(l, s, \mathbf{w})$ *satisfies* $-\frac{1}{n-1} \leq s \leq 1$, *and if* $s < 1$ *then* $b_0 \leq l \leq a_{n-1}$. *Moreover, the parameter* $\phi \neq 0$ *needs to be chosen such that* $\phi > 0$.

In addition to these necessary conditions, one can also give a characterization of all possible payoff relations.

Proposition 2 (Enforceable Payoff Relations). *Consider a payoff relation* $(l, s, \mathbf{w})$ *for player i such that* $w_j \geq 0$ *for all j. Let* $\hat{w}_j$ *denote the sum of the j smallest entries of* $\mathbf{w}$ *(excluding the entry* w_i*), and let* $\hat{w}_0 = 0$. *Then* $(l, s, \mathbf{w})$ *is enforceable for a given social dilemma if and only if either* $s = 1$ *or*

$$\max_{0 \leq j \leq n-1} \left\{ b_j - \frac{\hat{w}_j\left(b_j - a_{j-1}\right)}{1-s} \right\} \leq l \leq \min_{0 \leq j \leq n-1} \left\{ a_j + \frac{\hat{w}_{n-j-1}\left(b_{j+1} - a_j\right)}{1-s} \right\}. \qquad \textbf{[S16]}$$

Remark (some observations on enforceable payoff relations):

i) A direct consequence of *Proposition 2* is that ZD strategies exist for all social dilemmas and for all weights $\mathbf{w}$ with $w_j \geq 0$ (one only needs to set $s = 1$). Moreover, if all weights are positive, i.e., $w_j > 0$ for all $j \neq i$, it also follows that any baseline payoff between $b_0 \leq l \leq a_{n-1}$ is enforceable for some $s < 1$ (because $b_j > a_{j-1}$ for all j; one only needs to choose an s that is sufficiently close to one).

ii) For a given slope and given weights, we also note that if the baseline payoffs l_1 and l_2 are enforceable, then so is any baseline payoff l between l_1 and l_2.

iii) In the special case of equal weights on all coplayers, $w_j = 1/(n-1)$ for all j, the sum of the j smallest entries of $\mathbf{w}$ is simply given by $\hat{w}_j = j/(n-1)$. In that case, a payoff relation is enforceable if and only if either $s = 1$ or

$$\max_{0 \leq j \leq n-1} \left\{ b_j - \frac{j}{n-1} \frac{b_j - a_{j-1}}{1-s} \right\} \leq l \leq \min_{0 \leq j \leq n-1} \left\{ a_j + \frac{n-j-1}{n-1} \frac{b_{j+1} - a_j}{1-s} \right\}. \qquad \textbf{[S17]}$$

Fig. S2 gives an illustration of the enforceable payoff relations, again for the two examples of a public goods game and a volunteer's dilemma. In particular, in both examples the space of enforceable payoff relations shrinks with group size. More generally, if the payoff advantage of a defector $b_{j+1} - a_j$ does not increase with group size, then *Proposition 2* implies that larger group sizes n make it more difficult to enforce specific payoff relations.

iv) In the special case of full weight on one coplayer, i.e., $w_k=1$ for some $k \neq i$ and all other entries of $\mathbf{w}$ are zero, the sum of the j smallest entries is either $\hat{w}_j=0$ (if $j<n-1$) or $\hat{w}_j=1$ (if $j=n-1$). Because $b_{j+1} \geq b_j$ and $a_{j+1} \geq a_j$, a payoff relation is enforceable if either $s=1$ or

$$\max\left\{b_{n-2}, \frac{a_{n-2}-sb_{n-1}}{1-s}\right\} \leq l \leq \min\left\{\frac{b_1-sa_0}{1-s}, a_1\right\}. \quad \textbf{[S18]}$$

For games in which $b_{n-2}>a_1$, condition **S18** cannot be satisfied. In particular, the condition cannot be satisfied for social dilemmas with group size $n>3$ (because for such social dilemmas $b_{n-2}>a_1$ follows from **S1** and **S2**). Thus, in large groups, the only feasible ZD strategies are those with $s=1$, i.e., the only payoff relationship that player i can enforce between his own payoff and player k's payoff is the fair payoff relationship, $\pi^k=\pi^i$.

v) A comparison of Eqs. **S17** and **S18** thus shows that the set of enforceable payoff relations is larger if player i uses the same weight for all coplayers. More generally, if a payoff relation $(l,s,\mathbf{w})$ is enforceable for some weight vector $\mathbf{w}$, then it is also enforceable for the weight vector that puts equal weight on all coplayers (this follows from *Proposition 2* because $\hat{w}_j$ becomes largest when all coplayers have the same weight).

Nash Equilibria of Repeated Multiplayer Dilemmas. Various Folk theorems have shown how repetition can be used to sustain cooperation in social dilemmas (16, 17). To prove these Folk theorems, one typically constructs specific strategies that give rise to given payoff combinations $(\pi^1,\ldots,\pi^n)$ and shows then that these strategies are indeed an equilibrium of the repeated game. Herein, we ask a somewhat different question: within a certain strategy class, what are all Nash equilibria? We respond to this question for two different strategy classes, the class of ZD strategies, and the class of pure memory-one strategies. The equilibria among these two strategy classes will prove to be stable against any mutant strategy (i.e., we do not need to assume that mutants are restricted to ZD strategies or memory-one strategies).

To simplify the analysis, we will focus on symmetric memory-one strategies for which the cooperation probability only depends on the own move in the previous round and on the number of cooperators in the previous round (but not on who of the coplayers cooperated). In that case, each possible outcome σ can be identified with a pair (S,j), where $S\in\{C,D\}$ is the player's own move and j is the number of cooperators among the other $n-1$ group members. Then we can represent memory-one strategies as vectors of the form $\mathbf{p}=(p_{S,j})$. Using an analogous notation as in the previous section, $\mathbf{v}=(v_{S,j})$ corresponds to the frequency of observing each of the states (S,j) over the course of the game, $\mathbf{g}^i=(g^i_{S,j})$ corresponds to player i's payoffs, and $\mathbf{g}^{-i}=(g^{-i}_{S,j})$ corresponds to the average payoffs of i's coplayers (using the arithmetic mean $\mathbf{g}^{-i}=\frac{1}{n-1}\sum_{j\neq i}\mathbf{g}^j$). Because symmetric memory-one strategies are a subset of all memory-one strategies, the previous results (in particular Akin's lemma and the Press and Dyson theorem) naturally carry over.

Nash equilibria among ZD strategies. Consider a group in which all players apply the ZD strategy $\mathbf{p}$ with parameters l, s, ϕ, and $w_j=1/(n-1)$ for all $j\neq i$, and let us suppose the nth player considers deviating. We will refer to the first $n-1$ players as the residents and to the nth player as the mutant, and we denote a resident's payoff by π and the mutant's payoff by $\hat{\pi}$. Because residents apply a ZD strategy, Eq. **S13** implies that each of them enforces the relationship

$$\frac{n-2}{n-1}\pi+\frac{1}{n-1}\hat{\pi}=s\pi+(1-s)l, \quad \textbf{[S19]}$$

which can be rewritten as

$$\hat{\pi}=s^{\mathcal{R}}\pi+\left(1-s^{\mathcal{R}}\right)l, \quad \textbf{[S20]}$$

with

$$s^{\mathcal{R}}=s(n-1)-(n-2). \quad \textbf{[S21]}$$

That is, the $n-1$ residents collectively enforce a linear relationship between their own payoff π and the payoff of the mutant $\hat{\pi}$, with the same baseline payoff l and with slope $s^{\mathcal{R}}$. By using the same strategy as the residents, the mutant yields the payoff $\hat{\pi}=\pi$. For $s^{\mathcal{R}}<1$, Eq. **S20** then implies that $\hat{\pi}=\pi=l$. For $s^{\mathcal{R}}=s=1$, the value of l is a free parameter that does not have an effect on the entries of the ZD strategy; to be consistent with the case $s<1$, we define l to be the payoff that the strategy yields if applied by all group members (this payoff depends on the strategy's cooperation probability in the initial round). Thus, by using the residents' strategy the mutant yields the payoff l.

For $\mathbf{p}$ to be a Nash equilibrium, a minimum requirement is that the mutant must not have an incentive to switch to a different ZD strategy $\hat{\mathbf{p}}$, with parameters $\hat{l}$ and $-\frac{1}{n-1}<\hat{s}<1$. Such a mutant would enforce the payoff relation

$$\pi=\hat{s}\hat{\pi}+(1-\hat{s})\hat{l}. \quad \textbf{[S22]}$$

Solving the two linear equations **S20** and **S22** for the mutant's payoff yields

$$\hat{\pi}=\frac{l\left(1-s^{\mathcal{R}}\right)+\hat{l}s^{\mathcal{R}}(1-\hat{s})}{1-\hat{s}s^{\mathcal{R}}}. \quad \textbf{[S23]}$$

The mutant has no incentive to deviate when $\hat{\pi}\leq l$, that is when

$$\hat{\pi}-l=\frac{\left(\hat{l}-l\right)s^{\mathcal{R}}(1-\hat{s})}{1-\hat{s}s^{\mathcal{R}}}\leq 0. \quad \textbf{[S24]}$$

Because $-\frac{1}{n-1} \le s \le 1$ (by *Proposition 1*) and because $-\frac{1}{n-1} < \hat{s} < 1$ (by assumption), the denominator of Eq. **S24** is positive. We can therefore distinguish three cases.

i) $s^{\mathcal{R}} = 0$. In that case $\hat{\pi} - l = 0$, and player n cannot improve his payoff by deviating.

ii) $s^{\mathcal{R}} > 0$. In that case $\hat{\pi} - l \le 0$ if and only if $\hat{l} \le l$. To prevent the mutant from deviating, the residents thus need to apply a strategy with maximum possible baseline payoff, $l = a_{n-1}$.

iii) $s^{\mathcal{R}} < 0$. Then $\hat{\pi} - l \le 0$ if and only if $\hat{l} - l \ge 0$. To prevent the mutant from deviating, the residents' ZD strategy needs to set l to the minimum value $l = b_0$.

This result also holds if mutants are not restricted to ZD strategies.

Proposition 3 (Nash Equilibria Among ZD Strategies). *Consider a social dilemma in which mutual cooperation is the best outcome for the group, whereas mutual defection is the worst possible outcome*

$$b_0 \le \min_{0 \le j \le n} \frac{j a_{j-1} + (n-j) b_j}{n} \le \max_{0 \le j \le n} \frac{j a_{j-1} + (n-j) b_j}{n} \le a_{n-1}. \quad \textbf{[S25]}$$

Let **p** *be a ZD strategy with parameters l, s, and* ϕ *and let* $s^{\mathcal{R}} = (n-1)s - (n-2)$. *Then* **p** *is a Nash equilibrium if and only if one of the following three cases holds:*

$$s^{\mathcal{R}} = 0 \text{ and } b_0 \le l \le a_{n-1}; \quad s^{\mathcal{R}} > 0 \text{ and } l = a_{n-1}; \quad \text{and} \quad s^{\mathcal{R}} < 0 \text{ and } l = b_0.$$

Remark (some observations for stable ZD strategies):

i) The three conditions in *Proposition 3* do not depend on ϕ. Whether a ZD strategy is stable only depends on the payoff relation that it enforces, but not on the exact strategy that gives rise to this payoff relation.

ii) The three conditions do not directly depend on the sign of s, but on the sign of $s^{\mathcal{R}} = (n-1)s - (n-2)$. Whether a given ZD strategy is stable thus depends on the group size.

iii) The second condition is of particular interest, because it states that stable mutual cooperation can be achieved by ZD strategies with $l = a_{n-1}$ and $s > (n-2)/(n-1)$. For pairwise games these are exactly the generous ZD strategies with $l = a_{n-1}$ and $s > 0$ (2, 10). As the group size increases, generous strategies need to approach the fair strategies (with $s = 1$) to allow for stable cooperation.

Corollary 1 (Convexity of the Set of Nash Equilibria). *Consider a social dilemma in which mutual cooperation is the best outcome for the group. Suppose* $\mathbf{p}'$ *and* $\mathbf{p}''$ *are two ZD strategies that both give rise to stable cooperation [i.e.,* $l' = l'' = a_{n-1}$ *and* $s' \ge (n-2)/(n-1)$, $s'' \ge (n-2)/(n-1)$]. *Then any linear combination* $\mathbf{p} = \lambda \mathbf{p}' + (1-\lambda)\mathbf{p}''$ *with* $0 \le \lambda \le 1$ *is also a ZD strategy that gives rise to stable cooperation.*

Proof: A direct computation shows that **p** can be written as a ZD strategy with parameters $l = a_{n-1}$, $\phi = \lambda\phi' + (1-\lambda)\phi'' > 0$, and $s = [\lambda s'\phi' + (1-\lambda)s''\phi'']/[\lambda\phi' + (1-\lambda)\phi''] \ge (n-2)/(n-1)$.

A similar result can also be shown for ZD strategies that lead to mutual defection.

Nash equilibria among pure memory-one strategies. As another application of Akin's lemma, we will show in the following which pure memory-one strategies allow for stable cooperation in multiplayer social dilemmas. To this end, let us again consider a group in which the first $n-1$ players apply some pure memory-one strategy $\mathbf{p} = (p_{S,j})$ and in which the nth player considers deviating. Let **v** denote a limit distribution from the perspective of the $n-1$ resident players and let $\hat{\mathbf{v}}$ be the corresponding limit distribution from the perspective of the mutant player. The following observation will be useful:

Lemma 1 (Relationship Between Limit Distributions). *If the residents apply a pure memory-one strategy,* $p_{S,j} \in \{0,1\}$ *for all* (S,j), *the entries of* **v** *satisfy*

$$\begin{aligned} v_{C,j} &= 0 \quad \text{for all } j < n-2 \\ v_{D,j} &= 0 \quad \text{for all } j > 1. \end{aligned} \quad \textbf{[S26]}$$

Moreover, the limit distributions **v** *and* $\hat{\mathbf{v}}$ *are related by*

$$\begin{aligned} \hat{v}_{C,n-1} &= v_{C,n-1}, \quad \hat{v}_{C,0} = v_{D,1} \\ \hat{v}_{D,n-1} &= v_{C,n-2}, \quad \hat{v}_{D,0} = v_{D,0}, \end{aligned} \quad \textbf{[S27]}$$

and $\hat{v}_{S,j} = 0$ *for all other states* (S,j).

Proof: As all residents use the same pure strategy, they play the same action in any given round. Thus, if one of the residents cooperates, then there are at least $n-2$ other cooperators, and therefore the probability to end up in a state with less than $n-2$ other cooperators is zero, $v_{C,j} = 0$ for $j < n-2$. The same argument shows that $v_{D,j} = 0$ for $j > 1$. Finally, for Eq. **S27**, we note that the mutant is in the state $(C, n-1)$ if and only if each of the residents is in the state $(C, n-1)$. Similarly, the mutant is in the states $(C,0)$, $(D, n-1)$, and $(D,0)$ if and only if the residents are in the state $(D,1)$, $(C, n-2)$, and $(D,0)$, respectively.

Proposition 4 (Pure Memory-One Strategies That Give Rise to Stable Cooperation). *Consider a social dilemma in which there is an incentive to deviate from mutual cooperation,* $b_{n-1} > a_{n-1}$. *Let* **p** *be a pure memory-one strategy that cooperates in the first round and that sticks to cooperation as long as all other players do so (i.e.,* $p_{C,n-1} = 1$). *Then* **p** *is a Nash equilibrium if and only if the entries of* **p** *satisfy the three conditions*

$$p_{C,n-2} = 0 \quad \text{[S28a]}$$

$$p_{D,1} \leq \frac{a_{n-1} - a_0}{b_{n-1} - a_{n-1}} \quad \text{[S28b]}$$

$$p_{D,0} \leq \frac{a_{n-1} - b_0}{b_{n-1} - a_{n-1}}. \quad \text{[S28c]}$$

Remark (some remarks on Proposition 4):

i) The full proof of *Proposition 4* is given in the appendix; the step (⇒) follows from a straightforward computation of payoffs for two possible mutant strategies. The step (⇐) requires a more sophisticated argument; it is exactly in this second step where Akin's lemma comes into play.

ii) According to *Proposition 4*, the stability of a cooperative and pure memory-one strategy **p** is solely determined by the four entries $p_{C,n-1} = 1, p_{C,n-2} = 0, p_{D,1}$, and $p_{D,0}$. This observation is a consequence of *Lemma 1*, which allowed us to neglect all other entries of **p.** For pairwise games, *Lemma 1* is not required, and thus pairwise games allow more general versions of *Proposition 4*, which are then also valid for mixed memory-one strategies (2).

Examples (for stable memory-one strategies):

i) The proofs of the various Folk theorems are often based on trigger strategies, which relentlessly play *D* after any deviation from the equilibrium path (16). An example of such a strategy is *Grim*, for which $p_{C,n-1} = 1$ and $p_{S,j} = 0$ for all other states (S,j). Because *Grim* satisfies the three conditions in Eq. **S28**, *Grim* is indeed a Nash equilibrium for all multiplayer dilemmas.

ii) In their analysis of multiplayer dilemmas, refs. 18 and 19 consider the performance of generalized versions of Tit-for-Tat. These TFT_k strategies cooperate if at least *k* other group members have cooperated in the previous round, i.e., $p_{S,j} = 1$ if and only if $j \geq k$. As the conditions **S28** are only satisfied for $k = n-1$, it follows that TFT_{n-1} is the only Nash equilibrium among these generalized versions of Tit-for-Tat.

iii) Unfortunately, neither *Grim* nor TFT_{n-1} is robust under the realistic assumption that players sometimes commit errors (15). For stochastic simulations of variants of the public goods game, ref. 14 found that evolution may promote strategies that only cooperate after mutual cooperation or after mutual defection, i.e., $p_{C,n-1} = p_{D,0} = 1$, and $p_{S,j} = 0$ for all other states (S,j). We refer to such strategies as *WSLS.* For the prisoner's dilemma, this strategy corresponds to the Win-Stay Lose-Shift behavior described by ref. 13. According to Eq. **S28**, *WSLS* is a Nash equilibrium if and only if the social dilemma satisfies $(b_{n-1} + b_0)/2 \leq a_{n-1}$. In Fig. S3, we illustrate the stability of these strategies when the social dilemma is a public goods game.

With a similar approach, we can also characterize the pure memory-one strategies that result in defection.

Proposition 5 (Pure Memory-One Strategies That Give Rise to Stable Defection). *Consider a social dilemma with $b_0 \geq a_0$ and $b_{n-1} \geq a_{n-1}$. Let* **p** *be a pure memory-one strategy that defects in the first round and that sticks to defection as long as all other players do so (i.e., $p_{D,0} = 0$). Then* **p** *is a Nash equilibrium if and only if at least one of the following two conditions is satisfied*

$$p_{D,1} = 0; \quad \text{and} \quad p_{C,n-1} = p_{C,n-2} = 0 \text{ and } \frac{b_{n-1} + a_0}{2} \leq b_0. \quad \text{[S29]}$$

Remark (some remarks on Proposition 5):

i) As a special case of the above proposition, we conclude that *AllD* is an equilibrium for any social dilemma with $b_{n-1} \geq a_{n-1}$ and $b_0 \geq a_0$.

ii) Conversely, mutual defection is never stable in social dilemmas with $b_0 < a_0$ (such as the volunteer's dilemma). If $b_0 < a_0$, it follows from condition **S1** that $b_0 < a_j$ for all *j.* As a consequence, an equilibrium in which everyone defects can always be invaded by a player who switches to *AllC.*

Coordinated Play and Zero-Determinant Alliances. In the previous sections, we analyzed the amount of control that single individuals can exert over their coplayers in repeated multiplayer dilemmas. Now we are interested in the question whether individuals can gain a higher amount of control by coordinating their play with other group members. To this end, let us consider a set of $n^A < n$ players, who agree on a joint strategy. We will refer to these players as the allies and to the remaining group members as the outsiders (as depicted in Fig. 1). Depending on the degree of coordination, one can think of various forms of coordinated play. In the following we are going to explore two modes of coordination:

i) Strategy alliances. Here, players only agree on the set of alliance members and on a common memory-one strategy that each ally applies. During the actual game, there is no further communication taking place. Strategy alliances thus require a minimum amount of coordination. This alliance type seems particularly relevant when allies do not have the possibility to communicate during the game or when it is too costly to coordinate the allies' actions in each round.

ii) Synchronized alliances. Alternatively, players may synchronize their decisions in each round such that all allies play the same action. In effect, a synchronized alliance thus behaves like a new entity that has a higher leverage than each single player.

We will use the symbols π^A to refer to the payoff of an ally and π^{-A} to refer to the average payoff of the outsiders. In the following, we investigate which linear payoff relationships ZD alliances can enforce and how their strategic strength depends on the mode of coordination.

Strategy alliances. To model strategy alliances, let us consider a group of $n^{\mathcal{A}}$ allies, with $1 \le n^{\mathcal{A}} < n$. We assume that the allies agree on using the same ZD strategy $\mathbf{p}$ during the game. The parameters of this ZD strategy are given by l, s, and $\phi > 0$. To allow allies to differentiate between the actions of other allies and outsiders, the weights $\mathbf{w} = (w_j)$ of the ZD strategy may depend on a player's membership in the alliance. That is, if player i is a member of the alliance, then

$$w_{j \ne i} = \begin{cases} w^{\mathcal{A}} & \text{if } j \text{ is a member of the alliance} \\ w^{-\mathcal{A}} & \text{if } j \text{ is an outsider,} \end{cases} \qquad \text{[S30]}$$

with $w^{\mathcal{A}} \ge 0$ and $w^{-\mathcal{A}} \ge 0$ such that the weights sum up to one, $(n^{\mathcal{A}}-1)w^{\mathcal{A}} + (n-n^{\mathcal{A}})w^{-\mathcal{A}} = 1$. Because all allies use the same strategy, it follows that they all get the same payoff. Moreover, by Eq. **S13**, each of the allies enforces the payoff relationship

$$(n^{\mathcal{A}}-1)w^{\mathcal{A}}\pi^{\mathcal{A}} + (n-n^{\mathcal{A}})w^{-\mathcal{A}}\pi^{-\mathcal{A}} = s\pi^{\mathcal{A}} + (1-s)l. \qquad \text{[S31]}$$

This payoff relationship can be rewritten as

$$\pi^{-\mathcal{A}} = s^{\mathcal{A}}\pi^{\mathcal{A}} + (1-s^{\mathcal{A}})l, \qquad \text{[S32]}$$

such that

$$s^{\mathcal{A}} = \frac{s-(n^{\mathcal{A}}-1)w^{\mathcal{A}}}{1-(n^{\mathcal{A}}-1)w^{\mathcal{A}}}, \qquad \text{[S33]}$$

is the effective slope of the strategy alliance. For $w^{\mathcal{A}} = 1/(n-1)$, we recover the case of equal weights on all group members. For general weights $w^{\mathcal{A}}$, we say that the payoff relationship **S32** with parameters $(l, s^{\mathcal{A}})$ is enforceable if we can find $\phi > 0$ and $0 \le w^{\mathcal{A}} < 1/(n^{\mathcal{A}}-1)$ such that all entries of the resulting ZD strategy according to Eq. **S12** are in the unit interval. The following gives the corresponding characterization.

Proposition 6 (Enforceable Payoff Relations for Strategy Alliances). *A strategy alliance can enforce the payoff relation* $(l, s^{\mathcal{A}})$ *if and only if either* $s^{\mathcal{A}} = 1$ *or*

$$s^{\mathcal{A}} < 1 \;\; and \;\; \max_{0 \le j \le n-n^{\mathcal{A}}} \left\{ b_j - \frac{j}{n-n^{\mathcal{A}}} \frac{b_j - a_{j-1}}{1-s^{\mathcal{A}}} \right\} \le l \le \min_{n^{\mathcal{A}}-1 \le j \le n-1} \left\{ a_j + \frac{n-j-1}{n-n^{\mathcal{A}}} \frac{b_{j+1}-a_j}{1-s^{\mathcal{A}}} \right\}. \qquad \text{[S34]}$$

Moreover, if $n^{\mathcal{A}} \le n/2$, *then* $-1 \le -n^{\mathcal{A}}/(n-n^{\mathcal{A}}) \le s^{\mathcal{A}} \le 1$.

Remark (on strategy alliances):

i) Earlier we saw that individuals typically lose their strategic power when groups become large. The above proposition shows that players can regain control by forming alliances. In particular, the space of enforceable payoff relations $(l, s^{\mathcal{A}})$ increases with the size of the alliance $n^{\mathcal{A}}$. Larger alliances can therefore enforce more extreme payoff relationships, as illustrated in Fig. S4.

ii) Somewhat surprisingly, it follows from the proof of *Proposition 6* that the set of enforceable payoff relationships becomes maximal when $w^{\mathcal{A}}$ approaches $1/(n^{\mathcal{A}}-1)$ (and therefore $w^{-\mathcal{A}} \to 0$). The most powerful alliances are those in which the outsiders' actions only have an infinitesimal influence.

iii) In contrast to the case of ZD strategies for individual players, we note that the effective slope $s^{\mathcal{A}}$ for alliances does not need to be bounded from below. As an example, let us assume the alliance has reached a size $n^{\mathcal{A}}$ such that $b_{n-n^{\mathcal{A}}} \le a_{n^{\mathcal{A}}-1}$ (which can only happen when $n^{\mathcal{A}} > n/2$). Because $b_{n-n^{\mathcal{A}}}$ is an upper bound for the left side of Eq. **S34** and $a_{n^{\mathcal{A}}-1}$ is a lower bound for the right side of Eq. **S34**, it follows that any l with $b_{n-n^{\mathcal{A}}} \le l \le a_{n^{\mathcal{A}}-1}$ can be enforced, irrespective of the value of $s^{\mathcal{A}} < 1$.

iv) However, for alliances that have reached a size $n^{\mathcal{A}}$ such that $b_{n-n^{\mathcal{A}}} < a_{n^{\mathcal{A}}-1}$, the theory of ZD strategies becomes somewhat less relevant: such alliances are better off by cooperating in each round (if all allies cooperate, their payoff is at least $a_{n^{\mathcal{A}}-1}$, whereas if they all defect, their payoff is at most $b_{n-n^{\mathcal{A}}}$). In other words, if $n^{\mathcal{A}}$ individuals are able to make a binding agreement that they will all play the same strategy, in a social dilemma with $b_{n-n^{\mathcal{A}}} < a_{n^{\mathcal{A}}-1}$, then unconditional cooperation is a dominant strategy for the alliance.

Synchronized alliances. In the previous scenario, we assumed that each of the allies decides independently whether to cooperate in a given round. Let us now turn to a scenario in which allies meet after each round to decide which action they collectively play in the next round. As a result, the alliance members act as a single entity, in a game with $n - n^{\mathcal{A}}$ coplayers. To investigate such a scenario, let us first adapt our notation correspondingly. For a given set of allies, let (S, j) refer to the outcome in which all allies choose $S \in \{C, D\}$, and in which j of the outsiders cooperate. As in previous sections, the limit distribution $\mathbf{v} = (v_{S,j})$ corresponds to the fraction of rounds the alliance finds herself in state (S, j) over the course of the game. A memory-one strategy for a synchronized alliance is a vector $\mathbf{p} = (p_{S,j})$—given the outcome (S, j) of the previous round, the cooperation probability $p_{S,j}$ is used to determine whether all allies cooperate or all allies defect in the next round. The synchronized alliance uses the strategy Repeat if cooperation probabilities are given by

$$p_{S,j}^{\text{Rep}} = \begin{cases} 1 & \text{if } S = C \\ 0 & \text{if } S = D. \end{cases} \qquad \text{[S35]}$$

With literally the same proof as in the main text, one can verify Akin's lemma for synchronized alliances: if the alliance applies a memory-one strategy $\mathbf{p}$ then any corresponding limit distribution $\mathbf{v}$ satisfies $(\mathbf{p}-\mathbf{p}^{\mathrm{Rep}})\cdot\mathbf{v}=0$. Let us next write down the possible payoffs in a given round. The payoff vector $\mathbf{g}^{\mathcal{A}}$ for the synchronized alliance has the entries

$$g^{\mathcal{A}}_{S,j}=\begin{cases} a_{n_A+j-1} & \text{if } S=C \\ b_j & \text{if } S=D, \end{cases} \quad \textbf{[S36]}$$

and the corresponding vector $\mathbf{g}^{-\mathcal{A}}$ that contains the average payoffs of the outsiders (using the arithmetic mean) takes the form

$$g^{-\mathcal{A}}_{S,j}=\begin{cases} \dfrac{ja_{n^{\mathcal{A}}+j-1}+\left(n-n^{\mathcal{A}}-j\right)b_{n^{\mathcal{A}}+j}}{n-n^{\mathcal{A}}} & \text{if } S=C \\[2ex] \dfrac{ja_{j-1}+\left(n-n^{\mathcal{A}}-j\right)b_j}{n-n^{\mathcal{A}}} & \text{if } S=D. \end{cases} \quad \textbf{[S37]}$$

Using these payoff vectors, the payoff of each ally is given by $\pi^{\mathcal{A}}=\mathbf{g}^{\mathcal{A}}\cdot\mathbf{v}$, and the mean payoff of the outsiders is $\pi^{-\mathcal{A}}=\mathbf{g}^{-\mathcal{A}}\cdot\mathbf{v}$. Analogously to the case of individual players, we can define ZD strategies for synchronized alliances as strategies of the form

$$\mathbf{p}=\mathbf{p}^{Rep}+\alpha\mathbf{g}^{\mathcal{A}}+\beta\mathbf{g}^{-\mathcal{A}}+\gamma\mathbb{1}, \quad \textbf{[S38]}$$

with $\mathbb{1}$ being the memory-one strategy with all entries being one and with α, β, and γ being parameters of the ZD strategy (with $\beta\neq 0$). Akin's lemma then implies that such alliances enforce the relationship

$$\alpha\pi^{\mathcal{A}}+\beta\pi^{-\mathcal{A}}+\gamma=0. \quad \textbf{[S39]}$$

Again, we stress the fact that this relationship holds irrespective of the strategies of the outsiders: even if outsiders notice that they are facing a synchronized alliance, there is nothing they can do to prevent the above payoff relationship. As above, we use the parameter transformation $l=-\gamma/(\alpha+\beta)$, $s^{\mathcal{A}}=-\alpha/\beta$, and $\phi=-\beta$ to write ZD strategies as follows:

$$\mathbf{p}=\mathbf{p}^{Rep}+\phi\left[s^{\mathcal{A}}\mathbf{g}^{\mathcal{A}}-\mathbf{g}^{-\mathcal{A}}+\left(1-s^{\mathcal{A}}\right)l\mathbb{1}\right]. \quad \textbf{[S40]}$$

With these new parameters, the enforced payoff relationship according to Eq. **S39** takes the usual form

$$\pi^{-\mathcal{A}} = s^{\mathcal{A}}\pi^{\mathcal{A}}+\left(1-s^{\mathcal{A}}\right)l. \quad \textbf{[S41]}$$

Proposition 7 (Enforceable Payoff Relations for Synchronized Alliances). *A synchronized alliance can enforce the payoff relation* $(l,s^{\mathcal{A}})$ *if and only if either* $s^{\mathcal{A}}=1$ *or*

$$s^{\mathcal{A}}\neq 1 \ \textit{and} \ \max_{0\le j\le n-n^{\mathcal{A}}}\left\{b_j-\frac{j}{n-n^{\mathcal{A}}}\,\frac{b_j-a_{j-1}}{1-s^{\mathcal{A}}}\right\}\le l\le \min_{n^{\mathcal{A}}-1\le j\le n-1}\left\{a_j+\frac{n-j-1}{n-n^{\mathcal{A}}}\,\frac{b_{j+1}-a_j}{1-s^{\mathcal{A}}}\right\}. \quad \textbf{[S42]}$$

Moreover, if $n^{\mathcal{A}}\le n/2$, *then* $-1\le -n^{\mathcal{A}}/(n-n^{\mathcal{A}})\le s^{\mathcal{A}}\le 1$.

Remark (on synchronized alliances):

i) For not too large alliances with $n^{\mathcal{A}}\le n/2$, *Propositions 6* and *7* imply exactly the same conditions on enforceable payoff relationships. Thus, although strategy alliances require considerably less coordination between the allies, they have the same strategic power as synchronized alliances.

ii) When $n^{\mathcal{A}}>n/2$, synchronized alliances may be able to enforce a strictly larger set of payoff relationships, because they are not restricted to payoff relationships with $s^{\mathcal{A}}\le 1$. Whether relationships with $s^{\mathcal{A}}>1$ are enforceable depends on the social dilemma. When the social dilemma satisfies $b_{n-n^{\mathcal{A}}}<a_{n^{\mathcal{A}}-1}$ then condition **S42** can be satisfied for any $b_{n-n^{\mathcal{A}}}<l<a_{n^{\mathcal{A}}-1}$ by choosing $s^{\mathcal{A}}>1$ sufficiently large. Conversely, when $b_{n-n^{\mathcal{A}}}\ge a_{n^{\mathcal{A}}-1}$ then only slopes $s^{\mathcal{A}}\le 1$ are feasible (because for $s^{\mathcal{A}}>1$ the left side of Eq. **S42** is strictly larger than $b_{n-n^{\mathcal{A}}}$, whereas the right side is strictly lower than $a_{n^{\mathcal{A}}-1}$).

iii) Overall, we conclude that synchronized alliances are more powerful than strategy alliances if and only if the alliance has reached a size $n^{\mathcal{A}}$ such that $b_{n-n^{\mathcal{A}}}<a_{n^{\mathcal{A}}-1}$. However, as noted for strategy alliances, the condition $b_{n-n^{\mathcal{A}}}<a_{n^{\mathcal{A}}-1}$ transforms the social dilemma into a game in which mutual cooperation is the best strategy for the alliance, such that the notion of ZD alliances becomes less important.

Here we explored the strategic power of alliances, assuming that the allies agree on a joint ZD strategy. Given these results, one may ask which ZD strategy the allies should agree on and which combinations of alliance strategies and outsider strategies form an equilibrium of the game between allies and outsiders. This question is different from the questions explored in *Nash Equilibria of Repeated Multiplayer Dilemmas*. There, we considered a homogeneous group of players, and we explored which strategies are stable if applied by all group members. To explore equilibria for games with ZD alliances, one needs to distinguish between the strategies of the allies and the strategies of the outsiders. This inherent asymmetry makes the equilibrium analysis more intricate, and we thus leave this question for future work.

Applications. In this section, we apply our theory to two particular examples of multiplayer social dilemmas: the public goods game and the volunteer's dilemma. For simplicity, we will focus here on symmetric strategies that only depend on the number of cooperators but not on the cooperators' identities (for ZD strategies this implies that we consider the case of equal weights on all coplayers).

Public goods games. In a public goods game, each player of a group can cooperate by contributing an amount $c>0$ to a public pool. Total contributions are multiplied by a factor r with $1<r<n$ and evenly shared among all group members. Thus, payoffs are given by

$$a_j = \frac{(j+1)rc}{n} - c, \quad \text{and} \quad b_j = \frac{jrc}{n}. \qquad \textbf{[S43]}$$

Some of the properties of ZD strategies for public goods games have been recently described by (12), using an independent approach. Here we complement and extend these results.

ZD strategies for public goods games. Plugging the payoff values **S43** into representation **S14** shows that ZD strategies have the form

$$p_{S,j} = \begin{cases} 1+\phi\left[(1-s)\left(l-\frac{(j+1)rc}{n}+c\right)-\frac{n-j-1}{n-1}c\right] & \text{if } S_i = C \\ \phi\left[(1-s)\left(l-\frac{jrc}{n}\right)+\frac{j}{n-1}c\right] & \text{if } S_i = D. \end{cases} \qquad \textbf{[S44]}$$

To explore which payoff relationships (l,s) a single player can enforce, we use the characterization given in Eq. **S17**. Because the payoffs of the public goods game are linear in the number of coplayers j, the corresponding conditions become particularly simple (as only the boundary cases $j=0$ and $j=n-1$ need to be considered). We conclude that a single player can enforce a linear payoff relation with parameters (l,s) if either $s=1$ or

$$\begin{gathered} 0 \le l \le rc - c \\ \frac{(n-1)rc}{n} - \frac{c}{1-s} \le l \le \frac{(r-n)c}{n} + \frac{c}{1-s}. \end{gathered} \qquad \textbf{[S45]}$$

Fig. S2 shows the set of all pairs (l,s) that satisfy the above constraints for various group sizes n. We get the following conclusions for the existence of extortionate strategies, generous strategies, and equalizers, depending on the size n of the group:

i) Extortionate strategies ($l=b_0=0$). Let us ask which slopes s an extortionate player can enforce. The inequalities **S45** then imply that slopes $s \ge (r-1)/r$ can always be enforced, irrespective of the group size n. However, slopes $s<(r-1)/r$ are only enforceable if the group size is sufficiently small

$$n \le \frac{r(1-s)}{r(1-s)-1}. \qquad \textbf{[S46]}$$

We conclude that in large groups, $n\to\infty$, only extortionate strategies with $s<(r-1)/r$ are feasible.

i) Generous strategies ($l=a_{n-1}=rc-c$). Using the inequalities **S45**, one can show that generous players can enforce exactly the same slopes as extortionate players.

ii) Equalizers ($s=0$). For equalizers, the inequalities **S45** imply there are three regimes: (*a*) if $n \le r/(r-1)$, all baseline payoffs $0\le l\le rc-c$ can be enforced; (*b*) if $r/(r-1)<n\le 2r/(r-1)$, only a limited subset of baseline payoffs $0<l<rc-c$ can be enforced; and (*c*) if $n>2r/(r-1)$, there are no equalizers.

In particular, we conclude that for a given multiplication factor $r>1$ the set of equalizer strategies disappears as groups become large.

Strategy alliances in the public goods game. By *Proposition 6*, strategy alliances with $n^{\mathcal{A}}$ members can enforce a linear relation with parameters $(l,s^{\mathcal{A}})$ if and only if either $s^{\mathcal{A}}=1$ or if the two following inequalities hold:

$$\begin{gathered} 0 \le l \le rc - c \\ \frac{(n-n^{\mathcal{A}})rc}{n} - \frac{c}{1-s^{\mathcal{A}}} \le l \le \frac{(rn^{\mathcal{A}}-n)c}{n} + \frac{c}{1-s^{\mathcal{A}}}. \end{gathered} \qquad \textbf{[S47]}$$

For the special cases of extortionate strategies, generous strategies, and equalizers, these inequalities allow us to derive the following conclusions:

i) Extortionate strategies ($l=b_0=0$). We can rewrite the inequalities **S47** to obtain a critical threshold on the fraction of alliance members that is needed to enforce a certain slope $s^{\mathcal{A}}$

$$\frac{n^{\mathcal{A}}}{n} \ge \frac{r(1-s^{\mathcal{A}})-1}{r(1-s^{\mathcal{A}})}. \qquad \textbf{[S48]}$$

In particular, if an alliance wants to enforce arbitrarily high extortion factors $\chi\to\infty$, then $s^{\mathcal{A}}=1/\chi\to 0$ and the critical threshold becomes $n^{\mathcal{A}}/n \ge (r-1)/r$. This condition is always satisfied if $n^{\mathcal{A}}=n-1$, implying that an alliance with $n-1$ members can always be arbitrarily extortionate toward the remaining group member.

ii) Generous strategies ($l=a_{n-1}=rc-c$). The inequalities **S47** lead to the same threshold for $n^{\mathcal{A}}/n$ as in the case of extortionate strategies, as given in Eq. **S48**.

iii) Equalizers ($s=0$). For equalizers, the inequalities **S47** lead to two critical thresholds; to be able to set the payoffs of the outsiders to any value between $0\le l\le rc-c$, the fraction of the allies needs to satisfy

$$\frac{n^{\mathcal{A}}}{n} \geq \frac{r-1}{r}. \qquad \textbf{[S49]}$$

However, to be able to set the payoffs of the outsiders to some value between $0 \leq l \leq rc - c$, the number of allies only needs to exceed

$$\frac{n^{\mathcal{A}}}{n} \geq \frac{(n-2)(r-1)}{n+(n-2)r}. \qquad \textbf{[S50]}$$

Nash equilibria for the public goods game. In the following, let us describe a few strategies that allow for stable cooperation in the public goods game. According to *Proposition 3*, this can be achieved by using a ZD strategy with parameters $l = a_{n-1}$, $\phi > 0$, and $(n-2)/(n-1) \leq s \leq 1$. When we choose the boundary case $s = 1$ and $\phi = 1/c$ (which is the maximum value of ϕ, given the constraint $0 \leq p_{S,j} \leq 1$), the resulting ZD strategy according to Eq. **S44** is proportional Tit-for-Tat with entries

$$pTFT_{S,j} = \frac{j}{n-1}, \qquad \textbf{[S51]}$$

which is independent of the player's own move. This rule says that the player's cooperation probability is given by the fraction of cooperators among the coplayers in the previous round (additionally, we need to specify the cooperation probability for the first round, which needs to be set to one).

Another boundary case is given by the choice $s = (n-2)/(n-1)$ and $\phi = [n(n-1)]/\{c[n(n-2)+r]\}$ (which is again the maximum value of ϕ). We refer to the resulting ZD strategy as generous Tit-for-Tat, which has entries

$$gTFT_{S,j} = \frac{j}{n-1} + \frac{n-j-1}{n-1}\,\frac{n(r-1)}{(n-2)n+r}. \qquad \textbf{[S52]}$$

Also $gTFT$ is independent of the player's own move, and it is generally more cooperative than $pTFT$, because $gTFT_{S,j} > pTFT_{S,j}$ for all $j < n-1$.

A last boundary case is given by $\phi \to 0$, in which case the resulting ZD strategy approaches the strategy Repeat, independent of the choice of $(n-2)/(n-1) \leq s \leq 1$. Due to *Corollary 1*, we can conclude that any linear combination of these three strategies of the form

$$\mathbf{p} = \lambda_1 \cdot pTFT + \lambda_2 \cdot gTFT + \lambda_3 \cdot \text{Repeat}, \qquad \textbf{[S53]}$$

(with $0 \leq \lambda_k \leq 1$, $\lambda_3 < 1$, and $\lambda_1 + \lambda_2 + \lambda_3 = 1$) is also a stable ZD strategy.

Among the pure memory-one strategies, *Proposition 4* allows us to conclude that *Grim* and TFT_{n-1} are always Nash equilibria. Moreover, the strategy *WSLS* is a Nash equilibrium if $r \geq 2n/(n+1)$, as illustrated in Fig. S3.

Volunteer's dilemma. In the volunteer's dilemma, at least one of the players needs to cooperate and pay a cost $c > 0$ in order for all group members to derive a benefit $b > c > 0$. Thus, the payoffs are given by

$$a_j = b - c \;\text{ for all } j, \text{and } \; b_j = b \;\text{ if }\; j \geq 1 \;\text{ and }\; b_0 = 0. \qquad \textbf{[S54]}$$

ZD strategies for the volunteer's dilemma. According to Eq. **S14**, the ZD strategies with equal weight on all coplayers have the form

$$p_{S,j} = \begin{cases} 1 + \phi\left[(1-s)(l-b+c) - \dfrac{n-j-1}{n-1}c\right] & \text{if } S_i = C \\ \phi\left[(1-s)(l-b) + \dfrac{j}{n-1}c\right] & \text{if } S_i = D, j \geq 1 \\ \phi\ (1-s)l & \text{if } S_i = D, j = 0. \end{cases} \qquad \textbf{[S55]}$$

By condition **S17**, exactly those parameters l and s can be enforced for which either $s = 1$ or

$$\max\left\{0, b - \frac{1}{n-1}\,\frac{c}{1-s}\right\} \leq l \leq b - c. \qquad \textbf{[S56]}$$

This set of enforceable payoff relations is illustrated in Fig. S2*B*. In the special case of extortionate strategies ($l = 0$), condition **S56** implies that a given slope s can only be enforced for sufficiently small groups

$$n \leq 1 + \frac{c}{b(1-s)}. \qquad \textbf{[S57]}$$

For equalizers ($s = 0$), the inequalities in Eq. **S56** imply that a player can only determine the average payoff of the coplayers if $n = 2$, and then the only enforceable baseline-payoff is $l = b - c$.

Strategy alliances in the volunteer's dilemma. For strategy alliances, the enforceable payoff relationships according to Eq. **S34** become

$$\max\left\{0, b-\frac{1}{n-n^{\mathcal{A}}}\frac{c}{1-s^{\mathcal{A}}}\right\} \leq l \leq b-c. \quad \textbf{[S58]}$$

In particular, alliances that aim to enforce an extortionate relationship (with $l=0$ and some $s^{\mathcal{A}} \geq 0$) need to have a critical size

$$\frac{n^{\mathcal{A}}}{n} \geq 1-\frac{c}{nb(1-s^{\mathcal{A}})}. \quad \textbf{[S59]}$$

It follows that alliances cannot be arbitrarily extortionate (setting $s^{\mathcal{A}}=0$ on the right side implies that such alliances would need to satisfy $n^{\mathcal{A}}>n-1$). Instead, even a large alliance of size $n^{\mathcal{A}}=n-1$ can only enforce slopes with $s^{\mathcal{A}} \geq (b-c)/b$.

The performance of ZD alliances against adapting outsiders. In addition to the static properties of ZD strategies, Press and Dyson (1) also highlighted a remarkable dynamic property of ZD strategies: when a player with a fixed extortionate ZD strategy is matched with an adapting opponent who is free to change his strategy over time, then the adapting opponent will move toward more cooperation. In the following we present simulations suggesting that ZD alliances can have a similar effect.

To explore the performance of ZD alliances against adapting outsiders, we consider a group of n subjects. Let us assume that the players $\{1,\ldots,n^{\mathcal{A}}\}$ form a synchronized alliance with $n^{\mathcal{A}}<n$ and that the allies commit themselves to act according to a ZD strategy with parameters l, $s^{\mathcal{A}}$, and ϕ (similar results could also be obtained under the assumption that the allies form a strategy alliance instead of a synchronized alliance). Moreover, let us assume that each of the outsiders applies some arbitrary memory-one strategy. To parametrize memory-one strategies, we note that the possible outcomes of a single round of the game can be written as $\sigma=(S^{\mathcal{A}}, S_{n^{\mathcal{A}}+1},\ldots,S_n)$, where $S^{\mathcal{A}} \in \{C,D\}$ is the joint action of the allies and where $S_j \in \{C,D\}$ is the action of each of the outsiders. As a result, there are $2^{n-n^{\mathcal{A}}+1}$ possible outcomes σ. The memory-one strategies for the outsiders are thus modeled as vectors $\mathbf{p}=(p_\sigma)$ with $2^{n-n^{\mathcal{A}}+1}$ entries $p_\sigma \in [0,1]$.

We assume that the ZD alliance and the outsiders interact in a series of repeated games. The strategy of the ZD alliance is assumed to be fixed, but outsiders are allowed to adapt their strategy from one repeated game to the next. Specifically, we assume that in each time step, the group interacts in a repeated public goods game, resulting in the payoff $\pi^{\mathcal{A}}$ for each of the allies and the payoffs π_j for each outsider j. Because all players use memory-one strategies, these payoffs can be calculated using a Markov-chain approach (14). In the next time step, one of the outsiders is randomly picked to change his strategy from $\mathbf{p}$ to $\mathbf{p}'$. The entries of the new strategy $\mathbf{p}'$ are independently drawn from a normal distribution around the old strategy (using an SD of 0.01). If the outsider's payoff using the new strategy is π_j', then we assume that the outsider keeps the new strategy with probability

$$\rho=\frac{1}{1+\exp\left[-\omega\left(\pi_j'-\pi_j\right)\right]}. \quad \textbf{[S60]}$$

Otherwise, the outsider rejects the new strategy and continues to use the old strategy. The parameter $\omega>0$ corresponds to the strength of selection. In the limit of weak selection $\omega \to 0$, this yields $\rho=1/2$, such that the choice between the new and the old strategy is fully random. For the simulations, we consider the case of strong selection (we used $\omega=100$); in that case, the new strategy is likely to be adopted if $\pi'>\pi$, and it is likely to be rejected when $\pi'<\pi$. This elementary process, in which outsiders are allowed to experiment with new strategies, is repeated for τ time steps. For the initial population of outsiders, we assume that all outsiders start with unconditional defection.

Fig. S5 reports the outcome of this evolutionary scenario for three different ZD strategies: a fair strategy, an extortionate strategy, and a generous strategy. In all three scenarios, the outsiders become more cooperative over time; as a consequence, also the allies cooperate more often, because all three ZD strategies are conditionally cooperative (Fig. S5, *Upper*). However, there are clear differences in the final cooperation rates between the three scenarios. Extortionate alliances seem to be least successful to incentivize cooperation, whereas fair alliances tend to achieve full cooperation in the long run. This success of fair strategies can be attributed to their higher slope values: because fair strategies use $s=1$, they perfectly recoup outsiders for increasing their cooperation rates. Both other strategy classes use slopes with $s<1$, which makes it less attractive for the outsiders to become more cooperative. As depicted in the lower panels of Fig. S5, fair ZD alliances therefore also yield the highest payoffs by the end of the simulation.

Of course, the numerical simulations presented here only provide a snapshot of the full dynamical properties of ZD strategies (which deserve a careful analysis on their own). The simulations serve as a proof of principle: as previously shown for the iterated prisoner's dilemma (1, 20), ZD strategies can be used to generate a positive group dynamics in multiplayer dilemmas.

Appendix: Proofs.

Proof of Proposition 1: By the definition of ZD strategies, the cooperation probabilities after mutual cooperation and mutual defection are given by

$$\begin{aligned} p_{(C,\ldots,C)} &= 1+\phi(1-s)(l-a_{n-1}) \\ p_{(D,\ldots,D)} &= \phi(1-s)(l-b_0). \end{aligned} \quad \textbf{[S61]}$$

As these two entries need to be in the unit interval, it follows that

$$\begin{aligned} \phi(1-s)(l-a_{n-1}) &\leq 0 \\ 0 &\leq \phi(1-s)(l-b_0). \end{aligned} \quad \textbf{[S62]}$$

Adding up these two inequalities implies $\phi(1-s)(b_0-a_{n-1})\le 0$, and because of Eq. **S3**

$$\phi(1-s)\ge 0. \tag{S63}$$

Analogously, let us consider outcomes σ in which all players but one cooperate (i.e., σ is a permutation of (C,. . .,C, D), in which case

$$p_\sigma=\begin{cases}1+\phi\left[sa_{n-2}-(1-w_j)a_{n-2}-w_jb_{n-1}+(1-s)l\right] & \text{if the defector is a coplayer } j\neq i\\ \phi[sb_{n-1}-a_{n-2}+(1-s)l] & \text{if the defector is player } i\end{cases} \tag{S64}$$

Because all these entries p_σ need to be in the unit interval

$$\begin{gathered}\phi\left[sa_{n-2}-(1-w_j)a_{n-2}-w_jb_{n-1}+(1-s)l\right]\le 0\\ 0\le\phi[sb_{n-1}-a_{n-2}+(1-s)l].\end{gathered} \tag{S65}$$

Adding up these inequalities yields $\phi(s+w_j)(b_{n-1}-a_{n-2})\ge 0$ for all $j\neq i$, and because of Eq. **S2**

$$\phi(s+w_j)\ge 0 \quad \text{for all } j\neq i. \tag{S66}$$

Combining the inequalities **S63** and **S66** then yields

$$\phi(1+w_j)\ge 0 \quad \text{for all } j\neq i, \tag{S67}$$

and because at least one of the w_j is larger than zero (because all w_j sum up to one), it follows that $\phi\ge 0$. The restriction $\phi\neq 0$ then implies $\phi>0$. Due to the inequalities **S63** and **S66**, we may also conclude that $-\min_{j\neq i}w_j\le s\le 1$. Because $\min_{j\neq i}w_i\le 1/(n-1)$ (again because the w_j sum up to one), it follows that $-1/(n-1)\le s\le 1$.

For $s\neq 1$, the inequalities **S62** and **S63** imply $b_0\le l\le a_{n-1}$.

Proof of Proposition 2: For a given $\sigma=(S_1,\ldots,S_n)$, the entries p_σ of a ZD strategy according to Eq. **S12** can be written as

$$p_\sigma=p_\sigma^{Rep}+\phi\left[(1-s)\left(l-g_\sigma^i\right)+\sum_{j\neq i}w_j\left(g_\sigma^i-g_\sigma^j\right)\right], \tag{S68}$$

with p_σ^{Rep} given by Eq. **S5** and with g_σ^i and g_σ^j given by Eq. **S7**. Let σ^C denote the set of i's coplayers who cooperate in state σ, nd let σ^D denote the corresponding set of coplayers who defect. Using this notation, the entry p_σ is given by

$$p_\sigma=\begin{cases}1+\phi\left[(1-s)\left(l-a_{|\sigma|-1}\right)-\sum_{j\in\sigma^D}w_j\left(b_{|\sigma|}-a_{|\sigma|-1}\right)\right] & \text{if } S_i=C\\ \phi\left[(1-s)\left(l-b_{|\sigma|}\right)+\sum_{j\in\sigma^C}w_j\left(b_{|\sigma|}-a_{|\sigma|-1}\right)\right] & \text{if } S_i=D.\end{cases} \tag{S69}$$

Because $\phi>0$ can be chosen arbitrarily small, the condition $p_\sigma\in[0,1]$ is thus satisfied for all σ if and only if the following inequalities hold

$$\begin{aligned}(1-s)\left(l-a_{|\sigma|-1}\right)-\sum_{j\in\sigma^D}w_j\left(b_{|\sigma|}-a_{|\sigma|-1}\right)\le 0 &\quad \text{for all } \sigma \text{ with } S_i=C\\ (1-s)\left(l-b_{|\sigma|}\right)+\sum_{j\in\sigma^C}w_j\left(b_{|\sigma|}-a_{|\sigma|-1}\right)\ge 0 &\quad \text{for all } \sigma \text{ with } S_i=D.\end{aligned} \tag{S70}$$

If $s=1$, these inequalities are independent of the parameter l, and they are satisfied for any social dilemma (because $w_j\ge 0$ for all j by assumption and because $b_{|\sigma|}>a_{|\sigma|-1}$ by Eq. **S2**). For $s<1$, we may divide the above inequalities by $(1-s)>0$, implying that Eq. **S70** is equivalent to

$$\begin{aligned}a_{|\sigma|-1}+\frac{\sum_{j\in\sigma^D}w_j\left(b_{|\sigma|}-a_{|\sigma|-1}\right)}{1-s}\ge l &\quad \text{for all } \sigma \text{ with } S_i=C\\ b_{|\sigma|}-\frac{\sum_{j\in\sigma^C}w_j\left(b_{|\sigma|}-a_{|\sigma|-1}\right)}{1-s}\le l &\quad \text{for all } \sigma \text{ with } S_i=D.\end{aligned} \tag{S71}$$

The inequalities **S71** in turn are satisfied if and only if

$$\max_{\sigma|_{S_i=D}}\left\{b_{|\sigma|}-\frac{\sum_{j\in\sigma^C}w_j\left(b_{|\sigma|}-a_{|\sigma|-1}\right)}{1-s}\right\}\le l\le\min_{\sigma|_{S_i=C}}\left\{a_{|\sigma|-1}+\frac{\sum_{j\in\sigma^D}w_j\left(b_{|\sigma|}-a_{|\sigma|-1}\right)}{1-s}\right\}. \tag{S72}$$

Because the terms $(b_{|\sigma|}-a_{|\sigma|-1})/(1-s)$ are positive, the respective maxima and minima are attained by choosing the weights w_j as small as possible. That is, for a given number of total cooperators $|\sigma|$, the extrema are attained for those states σ for which $\sum_{j\in\sigma^C}w_j$ and $\sum_{j\in\sigma^D}w_j$ are minimal. This observation implies that condition **S72** is equivalent to

$$\max_{0\le j\le n-1}\left\{b_j-\frac{\hat{w}_j\left(b_j-a_{j-1}\right)}{1-s}\right\}\le l\le \min_{0\le j\le n-1}\left\{a_j+\frac{\hat{w}_{n-j-1}\left(b_{j+1}-a_j\right)}{1-s}\right\}, \qquad \textbf{[S73]}$$

with $\hat{w}_j$ being the sum of the j smallest entries in $(w_j)_{j\neq i}$.

Proof of Proposition 3: We already know that for a ZD strategy to be a Nash equilibrium, one of the three conditions need to be fulfilled (otherwise there would be a different ZD strategy that yields a higher payoff). Conversely, let us assume that one of the three conditions of the proposition is fulfilled.

i) If $s^{\mathcal{R}}=0$, then by Eq. **S20**, the mutant's payoff is $\hat{\pi}=l$, irrespective of the mutant's strategy. In particular, there is no incentive to deviate.

ii) Suppose $s^{\mathcal{R}}>0, l=a_{n-1}$, and let us assume to the contrary that the zero-determinant strategy is not a Nash equilibrium. Then there is a mutant strategy such that $\hat{\pi}>a_{n-1}$. Because the residents collectively enforce the relation $\hat{\pi}=s^{\mathcal{R}}\pi+(1-s^{\mathcal{R}})a_{n-1}$ and because $s^{\mathcal{R}}>0$, we can conclude $\pi>a_{n-1}$. However, then the average payoff of all group members exceeds a_{n-1}, contradicting the assumption that a_{n-1} is the maximum average payoff per round.

iii) Under the assumption that b_0 is the minimum average payoff per round, the case $s^{\mathcal{R}}<0$ and $l=b_0$ can be treated analogously to the previous case.

Proof of Proposition 4: ($\Rightarrow$) Because $\mathbf{p}$ is a Nash equilibrium, the payoff of any mutant strategy $\hat{\mathbf{p}}$ satisfies $\hat{\pi}\le a_{n-1}$. Let us first consider a mutant who applies the strategy $\hat{\mathbf{p}}=AllD$, i.e., $\hat{p}_{S,j}=0$ for all (S,j). Because the mutant never cooperates, $\hat{v}_{C,j}=0$ for all j, and by *Lemma 1* also $\hat{v}_{D,j}=0$ for all j except $j\in\{0,n-1\}$. The values of $\hat{v}_{D,n-1}$ and $\hat{v}_{D,0}$ can be obtained by calculating the left eigenvectors of the transition matrix

	$(D,n-1)$	$(D,0)$
$(D,n-1)$	$p_{C,n-2}$	$1-p_{C,n-2}$
$(D,0)$	$p_{D,0}$	$1-p_{D,0}$

[S74]

If we had $p_{C,n-2}=1$, then the assumption that $\mathbf{p}$ players start with cooperation would imply $\hat{v}_{D,n-1}=1$, such that the payoff of *AllD* was $\hat{\pi}=b_{n-1}>a_{n-1}$. Because this contradicts the assumption that $\mathbf{p}$ is a Nash equilibrium, we conclude that $p_{C,n-2}=0$.

A calculation of the left eigenvector of Eq. **S74** with respect to the eigenvalue 1 then yields

$$\begin{pmatrix}\hat{v}_{D,n-1}\\ \hat{v}_{D,0}\end{pmatrix}=\begin{bmatrix}p_{D,0}\Big/\left(1+p_{D,0}\right)\\ 1\Big/\left(1+p_{D,0}\right)\end{bmatrix}. \qquad \textbf{[S75]}$$

As a result, the payoff of *AllD* is

$$\hat{\pi}=b_{n-1}\hat{v}_{D,n-1}+b_0\hat{v}_{D,0}=\frac{b_{n-1}\cdot p_{D,0}+b_0}{1+p_{D,0}}. \qquad \textbf{[S76]}$$

The requirement $\hat{\pi}\le a_{n-1}$ then implies the condition **S28c**.
As another special case of a possible mutant strategy, let us consider a mutant $\hat{\mathbf{p}}$ such that $\hat{p}_{C,n-1}=0$, and $\hat{p}_{S,j}=1$ for all other states (S,j). With a similar calculation as in the previous case, one can determine this mutant's payoff as

$$\hat{\pi}=a_{n-1}\hat{v}_{C,n-1}+b_{n-1}\hat{v}_{D,n-1}+a_0\hat{v}_{C,0}=\frac{(a_{n-1}+b_{n-1})\cdot p_{D,1}+a_0}{1+2p_{D,1}}. \qquad \textbf{[S77]}$$

The requirement $\hat{\pi}\le a_{n-1}$ implies condition **S28b**.

($\Leftarrow$) Suppose the $n-1$ residents apply the pure memory-one strategy $\mathbf{p}$. Due to Akin's lemma

$$\sum_{j=0}^{n-1}\left(p_{C,j}-1\right)v_{C,j}+\sum_{j=0}^{n-1}p_{D,j}v_{D,j}=0, \qquad \textbf{[S78]}$$

which because of *Lemma 1* and $p_{C,n-1}=1$, $p_{C,n-2}=0$ simplifies to

$$v_{C,n-2}=p_{D,1}v_{D,1}+p_{D,0}v_{D,0}. \qquad \textbf{[S79]}$$

Then, irrespective of the mutant's strategy, the payoff $\hat{\pi}$ satisfies

$$\begin{aligned}\hat{\pi}-a_{n-1}&=\sum\nolimits_{j=0}^{n-1}(a_j-a_{n-1})\hat{v}_{C,j}+(b_j-a_{n-1})\hat{v}_{D,j}\overset{\text{Lemma 1}}{=}(a_0-a_{n-1})\hat{v}_{C,0}+(b_{n-1}-a_{n-1})\hat{v}_{D,n-1}+(b_0-a_{n-1})\hat{v}_{D,0}\\&\overset{\text{Lemma 1}}{=}(a_0-a_{n-1})v_{D,1}+(b_{n-1}-a_{n-1})v_{C,n-2}+(b_0-a_{n-1})v_{D,0}\\&\overset{\text{Eq.[S79]}}{=}(a_0-a_{n-1})v_{D,1}+(b_{n-1}-a_{n-1})\left(p_{D,1}v_{D,1}+p_{D,0}v_{D,0}\right)+(b_0-a_{n-1})v_{D,0}\\&=\left[(b_{n-1}-a_{n-1})p_{D,1}-(a_{n-1}-a_0)\right]\cdot v_{D,1}+\left[(b_{n-1}-a_{n-1})p_{D,0}-(a_{n-1}-b_0)\right]\cdot v_{D,0}\overset{[S28b,c]}{\le}0,\end{aligned}$$

that is, $\mathbf{p}$ is a Nash equilibrium.

Proof of Proposition 5: The proof follows along the same lines as the proof of *Proposition 4*.

($\Rightarrow$) Suppose $\mathbf{p}$ is a Nash equilibrium and assume that $p_{D,1} \neq 0$ (because $\mathbf{p}$ is a pure strategy it follows that $p_{D,1} = 1$). We have to show that these assumptions imply $p_{C,n-1} = p_{C,n-2} = 0$ and $(b_{n-1} + a_0)/2 \le b_0$. To this end, let us first consider a mutant with strategy *AllC*, such that $\hat{p}_{S,j} = 1$ for all (S, j). Because of *Lemma 1*, and because the mutant always cooperates, the only possible outcomes in a given round (from the perspective of the mutant) are $(C, n-1)$ and $(C, 0)$. The transition matrix is given by

$$\begin{array}{c|cc} & (C,n-1) & (C,0) \\ \hline (C,n-1) & p_{C,n-1} & 1-p_{C,n-1} \\ (C,0) & 1 & 0 \end{array}. \quad \text{[S80]}$$

The limit distribution of this transition matrix is

$$\begin{pmatrix} \hat{v}_{C,n-1} \\ \hat{v}_{C,0} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{2-p_{C,n-1}} \\ \dfrac{1-p_{C,n-1}}{2-p_{C,n-1}} \end{pmatrix}, \quad \text{[S81]}$$

such that the payoff of *AllC* becomes

$$\hat{\pi} = a_{n-1}\hat{v}_{C,n-1} + a_0\hat{v}_{C,0} = \frac{a_{n-1} + \left(1-p_{C,n-1}\right)a_0}{2-p_{C,n-1}}. \quad \text{[S82]}$$

If we had $p_{C,n-1} = 1$, this payoff would equal to $\hat{\pi} = a_{n-1} > b_0 = \pi$, contradicting our assumption that $\mathbf{p}$ is a Nash equilibrium. Thus, $p_{C,n-1} = 0$.

Let us now consider another mutant strategy $\hat{\mathbf{p}}$ with $\hat{p}_{D,0} = 1$ and $\hat{p}_{S,j} = 0$ for all other states. Again by constructing the transition matrix [with possible states $(C, n-1)$, $(C, 0)$, $(D, n-1)$, and $(D, 0)$], one can compute the payoff of this mutant as

$$\hat{\pi} = \frac{b_{n-1} + (1-p_{C,n-2})(b_0 + a_0)}{3 - 2p_{C,n-2}}. \quad \text{[S83]}$$

If $\hat{p}_{C,n-2} = 1$, then $\hat{\pi} = b_{n-1} > b_0 = \pi$, again contradicting the assumption that $\mathbf{p}$ is a Nash equilibrium. Therefore, $\hat{p}_{C,n-2} = 0$, and the mutant's payoff becomes $\hat{\pi} = (b_{n-1} + b_0 + a_0)/3$. For $\mathbf{p}$ to be an equilibrium, this payoff needs to satisfy $\hat{\pi} \le b_0$, which yields $(b_{n-1} + a_0)/2 \le b_0$.

($\Leftarrow$) Let us first consider the case $p_{D,1} = 0$. Because the memory-one strategy also prescribes to defect in the first round and because $p_{D,0} = 0$, it follows that all residents play defect throughout the game, irrespective of the strategy of the mutant. Thus, any mutant's payoff can be written as $\hat{\pi} = \hat{v}_{D,0}b_0 + \hat{v}_{C,0}a_0 \le b_0 = \pi$, showing that $\mathbf{p}$ is a Nash equilibrium.

Let us now consider the second case, $p_{C,n-1} = p_{C,n-2} = 0$. Without loss of generality, we can also set $p_{D,1} = 1$, and by assumption, $p_{D,0} = 0$. Under these conditions, Akin's lemma becomes

$$v_{D,1} = v_{C,n-1} + v_{C,n-2}. \quad \text{[S84]}$$

Then, irrespective of the strategy of the mutant, the mutant's payoff satisfies

$$\begin{aligned} \hat{\pi} - b_0 &= \sum_{j=0}^{n-1} (a_j - b_0)\hat{v}_{C,j} + (b_j - b_0)\hat{v}_{D,j} \\ &\overset{\text{Lemma 1}}{=} (a_{n-1} - b_0)\hat{v}_{C,n-1} + (a_0 - b_0)\hat{v}_{C,0} + (b_{n-1} - b_0)\hat{v}_{D,n-1} \\ &\overset{\text{Lemma 1}}{=} (a_{n-1} - b_0)v_{C,n-1} + (a_0 - b_0)v_{D,1} + (b_{n-1} - b_0)v_{C,n-2} \\ &\overset{\text{Eq. [S84]}}{=} (a_{n-1} - b_0)v_{C,n-1} + (a_0 - b_0)\left(v_{C,n-1} + v_{C,n-2}\right) + (b_{n-1} - b_0)v_{C,n-2} \\ &= (a_{n-1} + a_0 - 2b_0)\cdot v_{C,n-1} + (b_{n-1} + a_0 - 2b_0)\cdot v_{C,n-2} \le 0, \end{aligned}$$

with the last inequality being due to the fact that $(b_{n-1} + a_0)/2 \le b_0$ and $a_{n-1} \le b_{n-1}$. Again, we conclude that $\mathbf{p}$ is a Nash equilibrium.

Proof of Proposition 6: Due to our construction, strategy alliances require each ally to apply a ZD strategy $\mathbf{p}$ with parameter l, s, and ϕ and weights $\mathbf{w}$. To enforce an effective slope $s^{\mathcal{A}}$, Eq. **S33** implies that the parameter s needs to be chosen such that

$$s = s^{\mathcal{A}} + \left(n^{\mathcal{A}} - 1\right)w^{\mathcal{A}}\left(1 - s^{\mathcal{A}}\right). \quad \text{[S85]}$$

For $s^{\mathcal{A}} = 1$, we get $s = 1$, and *Proposition 2* guarantees that the payoff relationship is enforceable (independent of l and the weights $w^{\mathcal{A}}$).

Let us now assume that $s < 1$ (and therefore $s^{\mathcal{A}} < 1$). Because of *Proposition 2*, the alliance can enforce the payoff relationship $(l, s^{\mathcal{A}})$ if and only if we can find an appropriate weight vector $\mathbf{w} = (w_j)$ such that

$$\max_{0\le j\le n-1}\left\{b_j-\frac{\hat{w}_j}{1-(n^{\mathcal{A}}-1)w^{\mathcal{A}}}\frac{b_j-a_{j-1}}{1-s^{\mathcal{A}}}\right\}\le l\le\min_{0\le j\le n-1}\left\{a_j+\frac{\hat{w}_{n-j-1}}{1-(n^{\mathcal{A}}-1)w^{\mathcal{A}}}\frac{b_{j+1}-a_j}{1-s^{\mathcal{A}}}\right\}. \quad \textbf{[S86]}$$

As in *Proposition 2*, $\hat{w}_j$ refers to the sum of the j smallest entries in $\mathbf{w}$ (excluding the entry corresponding to the focal player). Because $\mathbf{w}$ only has the two possible entries $w^{\mathcal{A}}$ and $w^{-\mathcal{A}}$, we can write $\hat{w}_j$ as

$$\hat{w}_j=\begin{cases} jw^{\mathcal{A}} & \text{if } w^{\mathcal{A}}\le w^{-\mathcal{A}},\ j\le n^{\mathcal{A}}-1\\ (n^{\mathcal{A}}-1)w^{\mathcal{A}}+(j-n^{\mathcal{A}}+1)w^{-\mathcal{A}} & \text{if } w^{\mathcal{A}}\le w^{-\mathcal{A}},\ j>n^{\mathcal{A}}-1\\ jw^{-\mathcal{A}} & \text{if } w^{\mathcal{A}}>w^{-\mathcal{A}},\ j\le n-n^{\mathcal{A}}\\ (j+n^{\mathcal{A}}-n)w^{\mathcal{A}}+(n-n^{\mathcal{A}})w^{-\mathcal{A}} & \text{if } w^{\mathcal{A}}>w^{-\mathcal{A}},\ j>n-n^{\mathcal{A}} \end{cases} \quad \textbf{[S87]}$$

The constraint $(n^{\mathcal{A}}-1)w^{\mathcal{A}}+(n-n^{\mathcal{A}})w^{-\mathcal{A}}=1$ implies $w^{-\mathcal{A}}=[1-(n^{\mathcal{A}}-1)w^{\mathcal{A}}]/(n-n^{\mathcal{A}})$. By plugging this into Eq. **S87**, we can calculate the expression

$$\frac{\hat{w}_j}{1-(n^{\mathcal{A}}-1)w^{\mathcal{A}}}=\begin{cases} \dfrac{jw^{\mathcal{A}}}{1-(n^{\mathcal{A}}-1)w^{\mathcal{A}}} & \text{if } w^{\mathcal{A}}\le\dfrac{1}{n-1},\ j\le n^{\mathcal{A}}-1\\ \dfrac{1}{1-(n^{\mathcal{A}}-1)w^{\mathcal{A}}}-\dfrac{n-j-1}{n-n^{\mathcal{A}}} & \text{if } w^{\mathcal{A}}\le\dfrac{1}{n-1},\ j>n^{\mathcal{A}}-1\\ \dfrac{j}{n-n^{\mathcal{A}}} & \text{if } w^{\mathcal{A}}>\dfrac{1}{n-1},\ j\le n-n^{\mathcal{A}}\\ \dfrac{1-(n-j-1)w^{\mathcal{A}}}{1-(n^{\mathcal{A}}-1)w^{\mathcal{A}}} & \text{if } w^{\mathcal{A}}>\dfrac{1}{n-1},\ j>n-n^{\mathcal{A}}. \end{cases} \quad \textbf{[S88]}$$

Due to condition **S86**, the space of enforceable payoff relations becomes maximal when we choose the weight $w^{\mathcal{A}}$ such that $\hat{w}_j/[1-(n^{\mathcal{A}}-1)w^{\mathcal{A}}]$ becomes maximal. Eq. **S88** suggests that $\hat{w}_j/[1-(n^{\mathcal{A}}-1)w^{\mathcal{A}}]$ is monotonically increasing in $w^{\mathcal{A}}$. Thus, considering the restriction $0\le w^{\mathcal{A}}<1/(n^{\mathcal{A}}-1)$, the maximum is attained for $w^{\mathcal{A}}\to 1/(n^{\mathcal{A}}-1)$, which also implies $w^{\mathcal{A}}>1/(n-1)$. From Eq. **S88** we obtain

$$\lim_{w^{\mathcal{A}}\to 1/(n^{\mathcal{A}}-1)}\frac{\hat{w}_j}{1-(n^{\mathcal{A}}-1)w^{\mathcal{A}}}=\begin{cases}\dfrac{j}{n-n^{\mathcal{A}}} & \text{if } j\le n-n^{\mathcal{A}}\\ \infty & \text{if } j>n-n^{\mathcal{A}}.\end{cases} \quad \textbf{[S89]}$$

Thus, for $w^{\mathcal{A}}$ sufficiently close to $1/(n^{\mathcal{A}}-1)$, condition **S86** is satisfied if and only if

$$\max_{0\le j\le n-n^{\mathcal{A}}}\left\{b_j-\frac{j}{n-n^{\mathcal{A}}}\frac{b_j-a_{j-1}}{1-s^{\mathcal{A}}}\right\}\le l\le\min_{n^{\mathcal{A}}-1\le j\le n-1}\left\{a_j+\frac{n-j-1}{n-n^{\mathcal{A}}}\frac{b_{j+1}-a_j}{1-s^{\mathcal{A}}}\right\}. \quad \textbf{[S90]}$$

S90 coincides with condition **S34**. Moreover, if $n^{\mathcal{A}}\le n/2$, we can choose a j with $n^{\mathcal{A}}-1<j\le n-n^{\mathcal{A}}$ such that Eq. **S90** suggests

$$b_j-\frac{j}{n-n^{\mathcal{A}}}\frac{b_j-a_{j-1}}{1-s^{\mathcal{A}}}\le l. \quad \textbf{[S91]}$$

Similarly, using $j-1$ in the right side of Eq. **S90** leads to

$$l\le a_{j-1}+\frac{n-j}{n-n^{\mathcal{A}}}\frac{b_j-a_{j-1}}{1-s^{\mathcal{A}}}. \quad \textbf{[S92]}$$

Summing up these two inequalities shows that $s^{\mathcal{A}}\ge-[n^{\mathcal{A}}/(n-n^{\mathcal{A}})]$.

Proof of Proposition 7: The proof follows the lines of *Propositions 1* and *2*. By its definition (Eq. **S40**), a ZD strategy for a synchronized alliance has the form

$$p_{S,j}=\begin{cases}1+\phi\left[\left(1-s^{\mathcal{A}}\right)\left(l-a_{n^{\mathcal{A}}+j-1}\right)-\dfrac{n-n^{\mathcal{A}}-j}{n-n^{\mathcal{A}}}\left(b_{n^{\mathcal{A}}+j}-a_{n^{\mathcal{A}}+j-1}\right)\right] & \text{if } S=C\\ \phi\left[\left(1-s^{\mathcal{A}}\right)\left(l-b_j\right)+\dfrac{j}{n-n^{\mathcal{A}}}\left(b_j-a_{j-1}\right)\right] & \text{if } S=D,\end{cases} \quad \textbf{[S93]}$$

for $0\le j\le n-n^{\mathcal{A}}$. The conditions $p_{C,j}\le 1$ and $p_{D,j}\ge 0$ imply the following sign constraints

$$0\ge\phi\left[\left(1-s^{\mathcal{A}}\right)\left(l-a_{n^{\mathcal{A}}+j-1}\right)-\frac{n-n^{\mathcal{A}}-j}{n-n^{\mathcal{A}}}\left(b_{n^{\mathcal{A}}+j}-a_{n^{\mathcal{A}}+j-1}\right)\right] \quad \textbf{[S94a]}$$

$$0 \le \phi\left[\left(1-s^{\mathcal{A}}\right)\left(l-b_j\right)+\frac{j}{n-n^{\mathcal{A}}}\left(b_j-a_{j-1}\right)\right], \quad \textbf{[S94b]}$$

for all $0 \le j \le n-n^{\mathcal{A}}$. Setting $j=n-n^{\mathcal{A}}$ in Eq. **S94a** yields

$$\phi\left(1-s^{\mathcal{A}}\right)\left(l-a_{n-1}\right) \le 0, \quad \textbf{[S95]}$$

and setting $j=0$ in Eq. **S94b** yields

$$0 \le \phi\left(1-s^{\mathcal{A}}\right)\left(l-b_0\right). \quad \textbf{[S96]}$$

Adding up Eqs. **S95** and **S96** shows that

$$\phi\left(1-s^{\mathcal{A}}\right) \ge 0. \quad \textbf{[S97]}$$

For $s^{\mathcal{A}}=1$, we can ensure that $0 \le p_{S,j} \le 1$ for all entries in Eq. **S93** by choosing a $\phi>0$ that is sufficiently small. Let us therefore assume $s^{\mathcal{A}} \ne 1$. Because we also have $\phi \ne 0$ by definition, condition **S97** becomes $\phi(1-s^{\mathcal{A}})>0$. Dividing the sign constraints in Eq. **S94** by $\phi(1-s^{\mathcal{A}})$ then implies

$$\max_{0 \le j \le n-n^{\mathcal{A}}}\left\{b_j-\frac{j}{n-n^{\mathcal{A}}}\frac{b_j-a_{j-1}}{1-s^{\mathcal{A}}}\right\} \le l \le \min_{0 \le j \le n-n^{\mathcal{A}}}\left\{a_{n^{\mathcal{A}}+j-1}+\frac{n-n^{\mathcal{A}}-j}{n-n^{\mathcal{A}}}\cdot\frac{b_{n^{\mathcal{A}}+j}-a_{n^{\mathcal{A}}+j-1}}{1-s^{\mathcal{A}}}\right\}. \quad \textbf{[S98]}$$

This condition in turn is equivalent to condition **S42**.

Conversely, suppose Eq. **S42** is satisfied for some $(l,s^{\mathcal{A}})$ with $s^{\mathcal{A}} \ne 1$. It follows that the sign constraints **S94** are satisfied for every choice of ϕ subject to the condition $\phi(1-s^{\mathcal{A}})>0$, which implies that the entries $p_{S,j}$ in Eq. **S93** satisfy $p_{C,j} \le 1$ and $p_{D,j} \ge 0$ for all j. By choosing ϕ sufficiently close to zero, we can also ensure that $p_{C,j} \ge 0$ and $p_{D,j} \le 1$. Therefore, the payoff relationship $(l,s^{\mathcal{A}})$ is enforceable.

Finally, suppose $n^{\mathcal{A}} \le n/2$. Setting $j=0$ in Eq. **S94a** yields

$$\phi\left[\left(1-s^{\mathcal{A}}\right)\left(l-a_{n^{\mathcal{A}}-1}\right)-\left(b_{n^{\mathcal{A}}}-a_{n^{\mathcal{A}}-1}\right)\right] \le 0, \quad \textbf{[S99]}$$

and setting $j=n^{\mathcal{A}}$ in Eq. **S94b** results in

$$0 \le \phi\left[\left(1-s^{\mathcal{A}}\right)\left(l-b_{n^{\mathcal{A}}}\right)+\frac{n^{\mathcal{A}}}{n-n^{\mathcal{A}}}\left(b_{n^{\mathcal{A}}}-a_{n^{\mathcal{A}}-1}\right)\right]. \quad \textbf{[S100]}$$

Adding up these two inequalities shows that

$$\phi\left(s^{\mathcal{A}}+\frac{n^{\mathcal{A}}}{n-n^{\mathcal{A}}}\right) \ge 0. \quad \textbf{[S101]}$$

Combining Eqs. **S97** and **S101** gives $\phi>0$, which in turn implies $-[n^{\mathcal{A}}/(n-n^{\mathcal{A}})] \le s^{\mathcal{A}} \le 1$.

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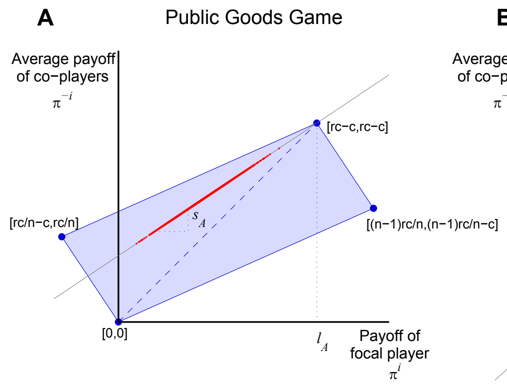

Fig. S1. Illustration of ZD strategies in the case of equal weights, $w_j = 1/(n-1)$, for all $j \neq i$, and for (*A*) the linear public goods game and (*B*) the volunteer's dilemma. The blue-shaded area represents all feasible payoffs, with the *x* axis representing the payoff of player *i* and the *y* axis representing the mean payoff of *i*'s coplayers. The dashed diagonal gives the payoff combinations for which $\pi^i = \pi^{-i}$. In both graphs, the strategy of player *i* is fixed to some ZD strategy, whereas for the coplayers we sampled 10^4 random memory-one strategies. Red dots represent the resulting payoff combinations, and the gray line gives the prediction according to Eq. **S13**. For both graphs, we considered an infinitely repeated game in a group of size $n = 4$. Parameters: (*A*) public goods game [$a_j = (j+1)rc/n - c$ and $b_j = jrc/n$] with $r = 2.4$ and $c = 1$. For the strategy of player *i*, we used a generous ZD strategy with parameters $l = rc - c$, $s = 2/3$, $\phi = 1/2$. (*B*) Volunteer's dilemma ($a_j = b - c$, $b_{j>0} = b$, and $b_0 = 0$) with $b = 1.5$, $c = 1$; player *i* applies an extortionate strategy with parameters $l = 0$, $s = 9/10$, $\phi = 1/2$.

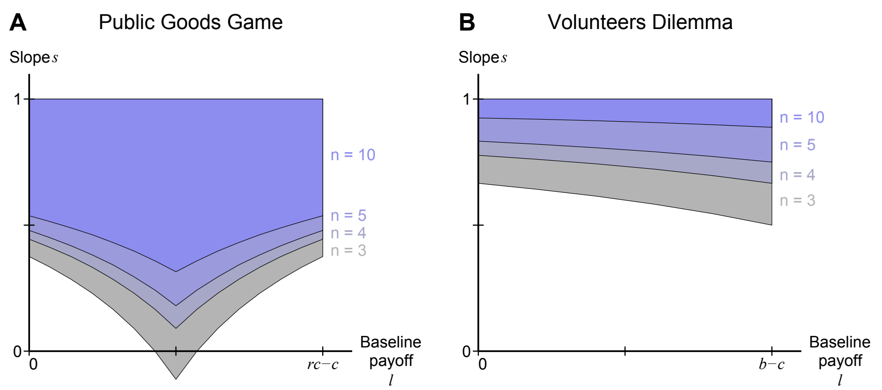

Fig. S2. Enforceable payoff relations in the case of equal weights on all coplayers for (*A*) the linear public goods game and (*B*) the volunteer's dilemma. A pair (l,s) is enforceable for a given group size *n* if the point is within the respectively shaded area. The set of enforceable pairs for large *n* is a subset of the respective set for smaller *n*, i.e., the set of enforceable pairs shrinks with increasing group size. Parameters: (*A*) linear public goods game with $r = 2.4$, $c = 1$; (*B*) volunteer's dilemma with $b = 1.5$, $c = 1$.

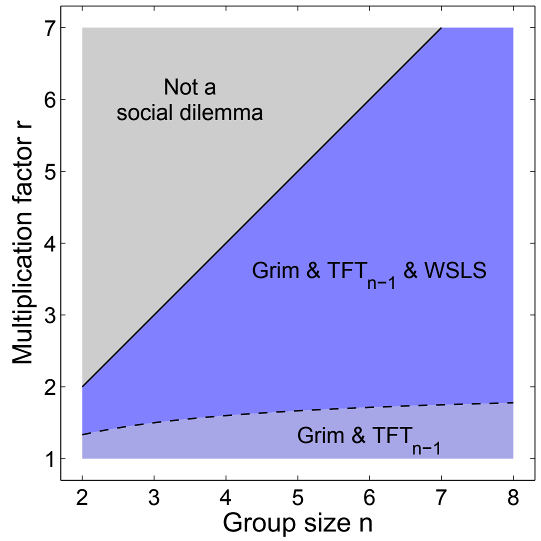

Fig. S3. Stable memory-one strategies for the linear public goods game. The figure illustrates for which parameter regions the strategies *Grim*, *TFT$_k$*, and *WSLS* are Nash equilibria, provided that the public goods game constitutes a social dilemma (i.e., $1<r<n$). *Grim* and *TFT$_{n-1}$* are always Nash equilibria. *TFT$_k$* for $k<n-1$ is never a Nash equilibrium. *WSLS* is a Nash equilibrium when $(b_{n-1}+b_0)/2 \leq a_{n-1}$, which yields $r \geq 2n/(n+1)$. In particular, *WSLS* is always a Nash equilibrium when $r \geq 2$, irrespective of group size.

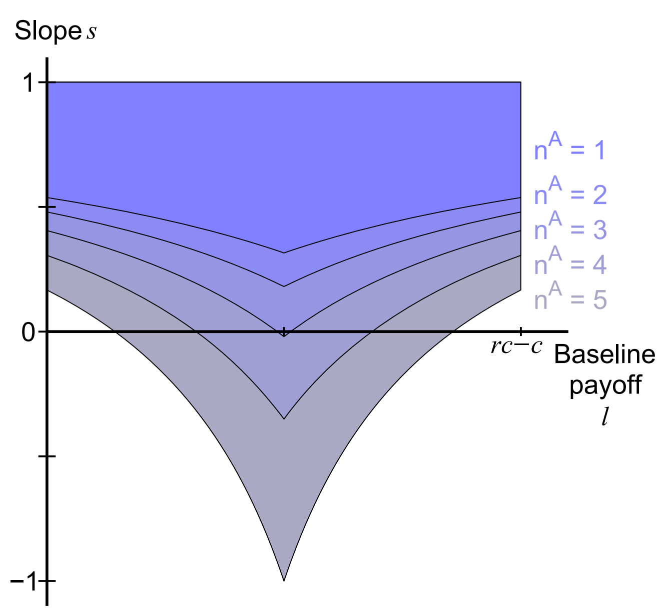

Fig. S4. Strategic power of strategy alliances in the public goods game. Each colored area illustrates the set of enforceable payoff relations according to *Proposition 6* for different alliance sizes n^A. The set of enforceable payoff relations for small n^A is a subset of the respective set with larger n^A. Consequently, the larger the alliance, the more extreme payoff relations it can enforce. Parameters: linear public goods game with $r=2.4$, $c=1$, and group size $n=10$.

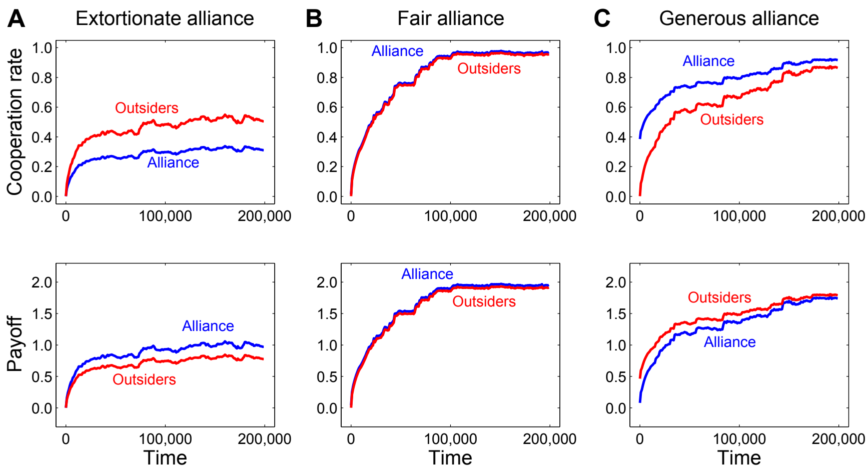

Fig. S5. Performance of three different ZD alliances against adaptive outsiders. For each simulation, the strategy of the ZD alliance was fixed, whereas outsiders were allowed to adapt their strategy as described in the text. (*Upper*) Average cooperation rate during each repeated game. (*Lower*) Resulting payoffs for allies and outsiders. Each panel depicts the average of 20 simulations. All simulations were run for a public goods game with $r=3$ and $c=1$, in a group of size $n=5$ with $n^{\mathcal{A}}=2$ allies. For the strategies of the ZD alliances we used (*A*) an extortionate strategy with $l=0$, $s=0.8$, and $\phi=1/2$; (*B*) proportional Tit-for-Tat with $s=1$ and $\phi=1$; and (*C*) a generous strategy with $l=rc-c=2$, $s=0.8$, and $\phi=1/2$.