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Cooperation between multiple Newsvendors with Warehouses[1]

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Abstract

This study considers a supply chain that consists of n retailers, each of them facing a newsvendor problem, m warehouses and a supplier. The retailers are supplied with a single product via some warehouses. In these warehouses, the ordered amounts of goods of these retailers become available after some lead time. At the time that the goods arrive at the warehouses, demand realizations are known by the retailers. The retailers can increase their expected joint profits by coordinating their orders and making allocations after demand realization. For this setting, we consider an associated cooperative game between the retailers. We show that this associated cooperative game has a nonempty core. Finally, we study a variant of this game, where the retailers are allowed to leave unsold goods at the warehouses.

KEYWORDS: SUPPLY CHAIN MANAGEMENT, NEWSVENDOR, WARE-HOUSE, GAME THEORY, BALANCEDNESS.

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1 Introduction

Recent developments in information and communication technology are changing the common way of doing business in distribution systems. Being able to access information of each other, firms may benefit from cooperative actions. For example, consider a retailer who is faced with a higher demand of a product than it has available on hand. If this retailer could find another retailer with excess stock close-by within an acceptable leadtime, it might be beneficial for both retailers to transship products to satisfy the excess demand. With the help of information technology, these retailers could benefit from the well-known concept of inventory pooling even if their stocks are not kept in the same physical location.

We consider a distribution system consisting of a group of retailers selling a single product in their local markets. Because of long manufacturing and transportation lead times (long supplier lead time), retailers have to decide on the order quantities before knowing the actual demand. For instance, shipments for fashion products such as running shoes from manufacturing facilities in Asia to markets in Europe and North-America have such long supplier lead times. How might the retailers benefit if they could be able to reallocate their orders after the demand realization?

To study this situation, we analyze a general distribution system where there are warehouses in addition to the retailers. The warehouses represent the physical areas (like harbors, regional distribution centers of the supplier, private warehouses where the retailers hire space to stock their goods because of physical constraints in the stores) where the ordered products are sent and available at the time that the demand is realized. The retailers might increase their joint profit by reallocating the available goods in the warehouses. The warehouses can also be considered as cross-docks, which do not hold any inventory. Eppen and Schrage (1981) consider a similar distribution system with cross-docks and make the following interpretation of the assumption that cross-docks do not hold inventory. The goods are physically shipped directly from the supplier to the cross-docks, where the orders are broken into smaller lots and shipped to the retailers. They investigate optimal order up to policies in such a system in a multi period framework and show how the system benefits from inventory pooling. We analyze the system as a single period model. Furthermore, we use cooperative game theory to answer the question whether it is beneficial for each retailer to cooperate.

Application of game theory in supply chains is not a new issue in the literature. The literature related with game theory can roughly be divided into cooperative and noncooperative. There are several tracks of interest in the literature using noncooperative game theory. One track of interest in this field is investigation of coordination mechanisms (like contracts) under horizontal and vertical competition. The main question is whether the proposed mechanism provides a solution that maximizes the total supply chain profit in a Stackelberg game setting or under Nash equilibrium. We refer to Cachon (1999, 2003) and Lariviere (1999) for reviews on analysis of contracts. The other track of interest is stock wars, where the retailers are fighting for limited supply of product. Cachon and Lariviere (1999) study a system where the supplier has a limited capacity and the retailer can affect the quantity that they will get by changing their order quantity. In this setting they investigate the performance of an allocation mechanism under different information scenarios like the retailers truthfully declaring their demand realization or keeping this private information for themselves. Another application of noncooperative game theory is to investigate the performance of some cooperation mechanisms where the players behave according to their individual preferences. Güllü et al. (2003) consider a decentralized supply chain under partial cooperation where the retailers can readjust their initial orders

(without changing the total order size) having more information of demand after some lead time so that both retailers can improve their expected costs. In our paper, we do not deal with the noncooperative behaviors of the players.

There are some differences between analysis using noncooperative game theory and ones using cooperative game theory. In the application of noncooperative game theory, it is assumed that each player acts individually according to its objective and preference, and usually the mechanisms (contracts) are investigated. One of the main focuses of concern is whether these mechanisms work well enough in providing coordination (maximum profit that the total system can obtain) under a competitive framework. In contrast, cooperative game theory assumes that binding agreements can be made between players on the advantage of the whole system. One of the main questions is whether the cooperation is stable, i.e. there is an allocation of the total benefit of the system among the players such that no group of players would like to leave the system. Cooperative game theory offers the concept of *core* as a direct answer to that question. Nonemptiness of the core means that there exists at least one allocation of the joint profits among the players such that no group of players has an incentive to leave.

We model our system in a newsvendor setting. The newsvendor models are well known single period models used especially for the products with high perishability or short life cycles. Initial applications appeared in modelling ordering decisions of newsvendors and in the fashion industry. The decreasing life cycles in the high tech industry, such as personal computers and mobile phones, extended the application areas of these models. For the method to compute the optimal solution of newsvendor problems see Silver et al. (1998) and see Khouja (1999) for a review.

In a newsvendor environment, retailers can increase their total profit by pooling their stock. The basic cooperation appears as follows: The retailers give a joint order and use this quantity to satisfy the total demand they are faced with. In this way, they can benefit from perfect allocation of the ordered quantity to the demands realized and coordination of the orders. Hartman et al. (2000) consider the game in the above mentioned setting, in which the value of the group of retailers is their optimal profit if they jointly determine an order size. In this study, they prove that the newsvendor game has a nonempty core for specific demand distributions of the retailers. Müller et al. (2002) and Slikker et al. (2001) independently come up with a more powerful result, namely that the core of the newsvendor games are nonempty regardless of the joint distribution of the random demands. Slikker et al. (2003) make several generalizations to this newsvendor setting. First, they consider non-anonymous wholesale and customer prices. In their model, orders are shipped directly to the retailers. Reallocation -after demand becomes known- is then subject to transshipment costs (for the lateral transshipment). Finally, they comment on the convexity of the game and prove that the cores of the associated games are nonempty.

We consider two cost components. The first one is transportation cost from the supplier to the warehouses. Besides transportation cost, this cost may include the purchasing cost of the products. The second cost component is the transportation cost from the warehouses to the retailers. In previous work, it has been assumed that the reallocation takes place at the retailer level. However, our setting covers a broad range of situations in which the reallocation of the ordered products could be made even if the ordered products are somewhere between the supplier and the retailer when the demands are realized at the retailers. Associated with this newsvendor situation, we formulate two coalitional games, which slightly differ from each other in terms of allocation problems considered. In one of the games, we force all the units to be sent to the retailers and in the other game we let the system leave the leftovers in the warehouses. We prove that these games have non-empty cores.

The remainder of the paper is organized as follows. In section 2, we introduce some preliminaries on cooperative game theory to make the reader familiar with the basic cooperative game theory concepts. Then, in section 3, we introduce the general model and we define the first associated cooperative game (forced allocation game). Additionally, we state our first main result. After discussing the second associated game (relaxed allocation game) and showing that this game has a nonempty core in section 4, we conclude our paper with some remarks in section 5.

2   Preliminaries on Cooperative Game Theory

Here, we give a brief introduction to cooperative game theory. Cooperative game theory covers the problem setting where different players act cooperatively to reach a common goal. Let N be a finite set of players, $N = \{1, ..., n\}$. A subset of N is called a *coalition* and denoted by S. A function v, assigning a value $v(S)$ for every coalition $S \subseteq N$ with $v(\emptyset) = 0$, is called a *characteristic function*. The value $v(S)$ is interpreted as the maximum total profit that coalition S can obtain through cooperation. Assuming that the benefit of a coalition S can be transferred between the players of S, a pair (N, v) is called a *cooperative game with transferable utility* (TU game). For a game (N, v), $S \subset N$ and $S \neq \emptyset$, the *subgame* $(S, v_{|S})$ is defined by $v_{|S}(T) = v(T)$ for each coalition $T \subseteq S$.

In reality, the players are not primarily interested in benefits of a coalition but in their individual benefits that they make out of that coalition. An allocation is a *payoff vector* $x = (x_i)_{i \in N} \in \mathbb{R}^N$, specifying for each player $i \in N$ the benefit x_i. An allocation x is called *efficient* if $\sum_{i \in N} x_i = v(N)$ and *individually rational* if $x_i \geq v(\{i\})$ for all $i \in N$. Individual rationality means that every player gets at least as much as what he could obtain by staying alone. The set of all individually rational and efficient allocations constitutes the *imputation set* $I(N, v) = \{x \in \mathbb{R}^N | \sum_{i \in N} x_i = v(N) \text{ and } x_i \geq v(\{i\}) \text{ for each } i \in N\}$. If this kind of rationality requirements are extended to all coalitions, we obtain the *core* $C(N, v) = \{x \in \mathbb{R}^N | \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \geq v(S) \text{ for each } S \subseteq N\}$. The interpretation of the core is that it consists of all imputations that are such that no group of retailers has an incentive to split off from the grand coalition N and form a smaller coalition because they collectively receive at least as much as what they can obtain for themselves.

Bondareva (1963) and Shapley (1967) independently made a general characterization of games with a non-empty core by the notion of balancedness. Let us define the vector e^S for all $S \subseteq N$ by $e_i^S = 1$ for all $i \in S$ and $e_i^S = 0$ for all $i \in N/S$. A map $\kappa : 2^N/\{\emptyset\} \to [0, 1]$ is called a *balanced map* if $\sum_{S \in 2^N/\{\emptyset\}} \kappa(S)e^S = e^N$. Further, a game (N, v) is called *balanced* if for every balanced map $\kappa : 2^N/\{\emptyset\} \to [0, 1]$ it holds that $\sum_{S \in 2^N/\{\emptyset\}} \kappa(S)v(S) \leq v(N)$. The following theorem is due to Bondareva (1963) and Shapley (1967).

Theorem 1 *Let (N, v) be a TU game. Then $C(N, v) \neq \emptyset$ if and only if (N, v) is balanced.*

A coalitional game (N, v) is called totally balanced if it is balanced and each of its subgames is balanced as well.

3   The Forced Allocation Game

In this section, we introduce our model and define the associated forced allocation game. Then, we show that the associated forced allocation game has a non-empty core.

Consider a distribution system that consists of a supplier, m warehouses and n retailers each experiencing a stochastic customer demand. Every retailer may decide on order quantities for one or more warehouses and transship goods from these warehouses. Assume that every retailer has to determine his order quantities, which maximizes his expected profit, before the stochastic demand is realized, as in the standard newsvendor setting. After some lead time, the ordered amounts will be sent to the warehouses and after the realization of demand becomes known they will be sent to the retailers in order to satisfy the demand. The following example illustrates the newsvendor problem in a 1 warehouse-1 retailer system.

Example 3.1 Consider a distribution system consisting of one supplier, one warehouse and one retailer. The retailer experiences a stochastic demand X of single item with probability mass function

$$p(x) = \begin{cases} 0 & \text{if } x \notin \{1,2\} \\ \frac{1}{2} & \text{if } x = 1 \\ \frac{1}{2} & \text{if } x = 2. \end{cases}$$

The retailer sells the goods at a price $p = 6$. The transportation cost $k = 1$ from the supplier to the warehouse and the transportation cost $f = 1$ from the warehouse to the retailer are charged to the retailer. While determining the optimal order quantity, the retailer should consider two costs defined in the standard newsvendor literature. The first one is the underage cost associated with each demand that is not satisfied. In this example, this is the opportunity cost of losing a customer demand, and defined by $c_u = p - (f + k) = 4$. The second one is the overage cost c_o, associated with each unit that is not sold. Here, this is simply the total cost of sending the units from the supplier to the retailer $c_o = k + f = 2$. It is well known that the optimal order quantity (q^*) of the retailer is the one that satisfies $(\frac{c_u}{c_u+c_o})$-quantile of demand distribution. This results in an optimal order quantity $q^* = 2$ and expected profit

$$\pi(2, X) = -2 * 1 + \frac{1}{2}(6 - 2) + \frac{1}{2}(12 - 2) = 5.$$

$\diamond$

In a situation with multiple warehouses and retailers, the retailers could increase their total expected profit just by reallocating the available quantities in the warehouses according to demand realizations. Furthermore, they could benefit more if they coordinate their orders. In order to analyze this situation, we define a *general newsvendor situation with warehouses* as a tuple $(N, W, (X_i)_{i \in N}, (k_w)_{w \in W}, (f_{wi})_{w \in W, i \in N}, (Z_i)_{i \in N}, (p_i)_{i \in N})$, where

N	:	Set of retailers, $N := \{1, ..., n\}$
W	:	Set of warehouses, $W := \{1, ..., m\}$
X_i	:	Stochastic demand at retailer i, with $E[X_i] < \infty$ for every $i \in N$
k_w	:	Transportation cost of the goods from the supplier to warehouse w including purchasing cost
f_{wi}	:	Transportation cost of goods from warehouse w to retailer i
$Z_i \subseteq W$	:	Non-empty set of warehouses related to retailer i
p_i	:	Selling price of the goods at retailer i

Throughout the study, we assume that p_i and f_{wi} are nonnegative, and $k_w > 0$ for all $i \in N$ and all $w \in W$. For notational convenience, we refer to a general newsvendor situation with warehouses as a general newsvendor situation.

The set of warehouses Z_i, via which retailer i is supplied, is a subset of all available warehouses W in the system. Therefore, our model covers two extreme situations. In the first one, every retailer has only one warehouse to provide goods from, ie., $|Z_i| = 1$ for all $i \in N$. In the second extreme, every retailer can use all warehouses, i.e., $Z_i = W$ for all $i \in N$. Consider a collection of retailers $S \subseteq N$. The coalition S is allowed to use any warehouse that could be used by at least one of its members, i.e., any warehouse in $Z_S := \bigcup_{i \in S} Z_i$.[2]

Let $(x_i)_{i \in S}$[3] be a realization of demand vector $X^S = (X_i)_{i \in S}$. For notational convenience, we will denote this realization as the vector $x^S \in \mathbb{R}^N$ where $x_i^S = 0$ for all $i \in N/S$ and $x_i^S = x_i$ for all $i \in S$.

Let Q^S be the collection of possible order vectors of coalition S defined by

$$Q^S := \{q \in \mathbb{R}^W | \ q_w = 0 \text{ for all } w \in W/Z_S \text{ and } q_w \geq 0 \text{ for all } w \in Z_S\}.$$

Suppose coalition S has order vector $q^S \in Q^S$. An allocation of q^S is a matrix $A^S \in \mathbb{R}_+^{W \times N}$ with

$$A_{wi}^S = 0 \text{ if } i \in N/S \text{ or } w \in W/Z_S \ ;$$
$$\sum_{i \in S} A_{wi}^S = q_w^S \text{ for all } w \in Z_S.$$

Here, A_{wi}^S denotes the amount of product that will be sent from warehouse w to retailer i. Note that we do not allow transshipment from the warehouses that are not in Z_S or to retailers that are not in coalition S. Moreover, we assume that at the end of the period all units should be transferred to the retailers. The set of all possible allocations of an order vector q^S is denoted by $M^S(q^S)$ for coalition S.

The profit of coalition S for order vector $q^S \in Q^S$, demand realization x^S of X^S, and allocation matrix $A^S \in M^S(q^S)$ can be expressed as

$$P^S(A^S, q^S, x^S) = - \sum_{w \in Z_S} k_w q_w^S + h^S(A^S, q^S, x^S),$$

where

$$h^S(A^S, q^S, x^S) = - \sum_{w \in Z_S} \sum_{i \in S} A_{wi}^S f_{wi} + \sum_{i \in S} p_i \min \left\{ \sum_{w \in Z_S} A_{wi}^S, x_i^S \right\}.$$

Here, h^S is the total revenue minus total transportation costs from warehouses to the retailers. Subtracting the transportation costs between supplier and the warehouses from h^S, we obtain the profit P^S.

In the following lemma, we show the existence of optimal allocations of order vectors for any demand realization.

Lemma 1 *Let $(N, W, (X_i)_{i \in N}, (k_w)_{w \in W}, (f_{wi})_{w \in W, i \in N}, (Z_i)_{i \in N}, (p_i)_{i \in N})$ be a general newsvendor situation with warehouses, let $S \subseteq N$, let $q^S \in Q^S$, and let x^S be a demand realization vector. Then there exists an allocation $A^{S,*}(q^S, x^S) \in M^S(q^S)$ that maximizes the profit $P^S(\cdot, q^S, x^S)$ of coalition S.*

---

[2] We remark that two retailers may prefer not to use a specific warehouse if they act alone. However, this warehouse may be used if they cooperate, for example because of a strategic position of this warehouse somewhere between the retailers.

[3] To cover some well known distributions (e.g., normal distribution), we assume that x_i can take negative values.

Proof : Because of the definition of P^S, it is sufficient to show that there exists an allocation that maximizes h^S. This allocation maximizes P^S as well. Since $h^S(A^S, q^S, x^S) = -\sum_{w \in Z_S} \sum_{i \in S} A^S_{wi} f_{wi} + \sum_{i \in S} p_i \min(\sum_{w \in Z_S} A^S_{wi}, x^S_i)$ is a continuous function of A^S for given q^S and x^S, and the domain of h^S is the compact set $M^S(q^S)$, we conclude that $h^S(\cdot, q^S, x^S)$ attains its maximum. $\square$

The following lemma provides us with an upper bound on the absolute difference of the values of h^S with different order quantities, which will be used to show that there exists an optimal order quantity for each coalition S in the following theorem. First, define $k_S = \max_{w \in Z_S} k_w$, $f_S = \max_{w \in Z_S, i \in S} f_{wi}$, and $p_S = \max_{i \in S} p_i$.

Lemma 2 *Let $(N, W, (X_i)_{i \in N}, (k_w)_{w \in W}, (f_{wi})_{w \in W, i \in N}, (Z_i)_{i \in N}, (p_i)_{i \in N})$ be a general newsvendor situation with warehouses, let $S \subseteq N$, let x^S be a demand realization vector, let q and q^{min} be two order vectors in Q^S such that $q_w \geq q^{min}_w$ for all $w \in Z_S$, and let $A^{S,*} \in M^S(q)$ and $B^{S,*} \in M^S(q^{min})$ be optimal allocation matrices for q^S and q^{min}, respectively. Then*

$$|h^S(A^{S,*}, q, x^S) - h^S(B^{S,*}, q^{min}, x^S)| \leq (p_S + f_S) \sum_{w \in Z_S} (q_w - q^{min}_w).$$

Proof : Let $B^S \in M^S(q^{min})$ be an allocation of q^{min} such that $A^{S,*}_{wi} \geq B^S_{wi}$ for all $w \in Z_S$ and $i \in S$. Then,

$$
\begin{aligned}
h^S(A^{S,*}, q, x^S) &= h^S(B^S, q^{min}, x^S) + \sum_{i \in S} p_i \left(\min\left\{ \sum_{w \in Z_S} A^{S,*}_{wi}, x_i \right\} - \min\left\{ \sum_{w \in Z_S} B^S_{wi}, x_i \right\} \right) \\
&\quad - \sum_{w \in Z_S} \sum_{i \in S} f_{wi}(A^{S,*}_{wi} - B^S_{wi}) \\
&\leq h^S(B^S, q^{min}, x^S) + p_S \sum_{w \in Z_S} (q_w - q^{min}_w).
\end{aligned}
$$

The equality follows from the definition of h^S. The inequality holds by $p_S = \max_{i \in S} p_i$, $min\{x, y\} - min\{z, y\} \leq |x - z|$ for all $x, y, z \in \mathbb{R}$, and $A^{S,*}_{wi} \geq B^S_{wi}$ for all $w \in Z_S$ and $i \in S$.

Since $B^{S,*}$ is an optimal allocation,

$$h^S(B^{S,*}, q^{min}, x^S) + p_S \sum_{w \in Z_S} (q_w - q^{min}_w) \geq h^S(A^{S,*}, q, x^S). \tag{1}$$

We repeat a similar argument for $B^{S,*}$. Let $A^S \in M^S(q)$ be an allocation of q such that $A^S_{wi} \geq B^{S,*}_{wi}$ for all $w \in Z_S$ and $i \in S$. Then,

$$
\begin{aligned}
h^S(B^{S,*}, q^{min}, x^S) &= h^S(A^S, q, x^S) - \sum_{i \in S} p_i \left(\min\left\{ \sum_{w \in Z_S} A^S_{wi}, x_i \right\} - \min\left\{ \sum_{w \in Z_S} B^{S,*}_{wi}, x_i \right\} \right) \\
&\quad + \sum_{w \in Z_S} \sum_{i \in S} f_{wi}(A^S_{wi} - B^{S,*}_{wi}) \\
&\leq h^S(A^S, q, x^S) + f_S \sum_{w \in Z_S} (q_w - q^{min}_w).
\end{aligned}
$$

The equality follows from the definition of h^S. The inequality holds since $f_S = \max_{w \in Z_S, i \in S} f_{wi}$ and $\min\{\sum_{w \in Z_S} A^S_{wi}, x_i\} \geq \min\{\sum_{w \in Z_S} B^{S,*}_{wi}, x_i\}$ for all $i \in S$. Since $A^{S,*}$ is an optimal allocation,

$$h^S(A^{S,*}, q, x^S) + f_S \sum_{w \in Z_S} (q_w - q^{min}_w) \geq h^S(B^{S,*}, q^{min}, x^S). \tag{2}$$

By (1) and (2), we have

$$p_S \sum_{w \in Z_S} (q_w - q_w^{min}) \geq h^S(A^{S,*}, q, x^S) - h^S(B^{S,*}, q^{min}, x^S) \geq -f_S \sum_{w \in Z_S} (q_w - q_w^{min}).$$

Consequently,

$$
\begin{aligned}
|h^S(A^{S,*}, q, x^S) - h^S(B^{S,*}, q^{min}, x^S)| \quad &\leq \quad \max\{p_S, f_S\} \sum_{w \in Z_S} (q_w - q_w^{min}) \\
&\leq \quad (p_S + f_S) \sum_{w \in Z_S} (q_w - q_w^{min}).
\end{aligned}
$$

$\square$

From now on, we refer to $h^S(A^{S,*}, q^S, x^S)$ as $r^S(q^S, x^S)$. The expected profit function of coalition S is defined by

$$\pi^S(q^S, X^S) \quad = \quad E_{X^S}[r^S(q^S, \cdot)] - \sum_{w \in Z_S} k_w q_w^S.$$

In the remainder of the paper, we refer to $E_{X^S}[r^S(q^S, \cdot)]$ as $R^S(q^S, X^S)$. In the following theorem, we show that each coalition has an optimal order vector.

Theorem 2 *Let $(N, W, (X_i)_{i \in N}, (k_w)_{w \in W}, (f_{wi})_{w \in W, i \in N}, (Z_i)_{i \in N}, (p_i)_{i \in N})$ be a general newsvendor situation and let $S \subseteq N$. Then, there exists an order vector $q^{S,*}$ that maximizes the expected profit function $\pi^S(\cdot, X^S)$ of coalition S.*

Proof : To prove this theorem, we will show that $\pi^S(\cdot, X^S)$ is a continuous function of the order vectors and any order vector outside a specific compact set results in lower expected profits than the order vector with all orders equal to zero.

Let x^S and $q \in Q^S$ be a demand realization vector and an order vector respectively, and let $\epsilon > 0$. Define $\delta = \frac{\epsilon}{|W|(k_S + f_S + p_S)}$. Let $q' \in Q^S$ be such that $|q - q'| < \delta$, where $|\cdot|$ denotes the Euclidean norm. Let $q^{min} \in Q^S$ be defined by

$$q_w^{min} = \min\{q_w, q_w'\}, \text{ for all } w \in Z_S.$$

Then,

$$
\begin{aligned}
|r^S(q, x^S) - r^S(q', x^S)| \quad &= \quad |r^S(q, x^S) - r^S(q^{min}, x^S) + r^S(q^{min}, x^S) - r^S(q', x^S)| \\
&\leq \quad |r^S(q, x^S) - r^S(q^{min}, x^S)| + |r^S(q^{min}, x^S) - r^S(q', x^S)| \\
&\leq \quad (p_S + f_S) \sum_{w \in Z_S} |q_w - q_w^{min}| + (p_S + f_S) \sum_{w \in Z_S} |q_w^{min} - q_w'| \\
&= \quad (p_S + f_S) \sum_{w \in Z_S} |q_w - q_w'|. \quad\quad (3)
\end{aligned}
$$

The first equality follows by adding and subtracting the term $r^S(q^{min}, x^S)$. The first inequality holds because of the triangle inequality. The second inequality holds by Lemma 2. The second equality holds because $|q_w - q_w^{min}| + |q_w^{min} - q_w'| = |q_w - q_w'|$ (which follows

by the definition of q_w^{min} and the fact that at least one of $|q_w - q_w^{min}|$ and $|q_w^{min} - q_w'|$ is zero for every $w \in Z_S$). Then

$$
\begin{aligned}
|\pi^S(q, X^S) - \pi^S(q', X^S)| &= \left| E_{X^S} \left[\sum_{w \in Z_S} k_w (q_w - q_w') + r^S(q, x^S) - r^S(q', x^S) \right] \right| \\
&\leq E_{X^S} \left[\left| \sum_{w \in Z_S} k_w (q_w - q_w') + r^S(q, x^S) - r^S(q', x^S) \right| \right] \\
&\leq E_{X^S} \left[\sum_{w \in Z_S} k_w |q_w - q_w'| + |r^S(q, x^S) - r^S(q', x^S)| \right] \\
&\leq E_{X^S} \left[k_S \sum_{w \in Z_S} |q_w - q_w'| + |r^S(q, x^S) - r^S(q', x^S)| \right] \\
&\leq E_{X^S} \left[k_S \sum_{w \in Z_S} |q_w - q_w'| + (p_S + f_S) \sum_{w \in Z_S} |q_w - q_w'| \right] \\
&= (k_S + p_S + f_S) \sum_{w \in Z_S} |q_w - q_w'| \\
&\leq (k_S + p_S + f_S) |q - q'| |W| \\
&< (k_S + p_S + f_S) * \delta * |W| \\
&= \epsilon.
\end{aligned}
$$

The first two inequalities follow by the triangle inequality. The third inequality follows by the definition of k_S. The fourth inequality follows by (3). The fifth inequality holds since $|q - q'| \geq |q_w - q_w'|$ for all $w \in Z_S$ and $|W| \geq |Z_S|$. We conclude that $\pi^S(\cdot, X^S)$ is a continuous function of the order vector.

Now we will show that any order vector outside a specific compact set results in lower expected profit than the expected profit of order vector $\mathbf{0}$, i.e., the order vector with all orders equal to zero. Let

$$
a^S = \frac{p_S}{\min_{i \in S, w \in Z_S} (k_w + f_{wi})} \left[\sum_{i \in S} E[X_i] + \sum_{i \in S} E[X_i^-] \right], \tag{4}
$$

where $X_i^- = \max\{-X_i, 0\}$[4]. Since for all $w \in W$ we have $k_w > 0$ by definition, a^S is well defined. Then, for all $q \in Q^S$ with $q_w > a^S$ for some $w \in Z_S$, say y, we have

$$
\begin{aligned}
\pi^S(q, X^S) &\leq -q_y * \min_{i \in S, w \in Z_S} (k_w + f_{wi}) + p_S * \sum_{i \in S} E[X_i] \\
&< -p_S * \sum_{i \in S} E[X_i^-] \\
&\leq \pi^S(\mathbf{0}, X^S).
\end{aligned}
$$

The first inequality follows by taking a lowerbound of transportation costs of ordered goods in the network and an upperbound of expected revenues into consideration. The second inequality follows by definition of a^S. The last inequality follows by $\pi^S(\mathbf{0}, X^S) = -\sum_{i \in S} p_i E[X_i^-]$. Hence, the optimal order vector exists and is an element of the compact set $\{q \in Q^S | 0 \leq q_w \leq a^S \text{ for all } w \in Z_S\}$. $\square$

---

[4]The introduction of X_i^- is superfluous if only nonnegative demands are possible.

Let Γ be a general newsvendor situation. The associated *forced allocation game* (N, v^Γ) is defined by

$$v^\Gamma(S) = \max_{q^S \in Q^S} \pi^S(q^S, X^S) \text{ for all } S \subseteq N.$$

This definition implies that $v^\Gamma(S)$ is the maximum expected profit that coalition S can obtain by coordinating orders and allocations. Since we are forcing the orders to be transferred to the retailers by restricting the feasible set of possible allocations $M^S(q^S)$ with constraint $\sum_{i \in S} A_{wi} = q_w^S$ for all $w \in Z_S$, we refer to this game as forced allocation game.

The following example illustrates the forced allocation game associated with a simple general newsvendor situation.

Example 3.2 Consider the 2-person newsvendor situation $\Gamma = (N, W, (X_i)_{i \in N}, (k_w)_{w \in W}, (f_{wi})_{w \in W, i \in N}, (Z_i)_{i \in N}, (p_i)_{i \in N})$, with $N = \{1, 2\}$, $W = \{w_1, w_2\}$, $k_{w_1} = k_{w_2} = 1$, $f_{w_1,1} = f_{w_1,2} = f_{w_2,1} = f_{w_2,2} = 1$, $Z_1 = \{w_1\}$, $Z_2 = \{w_2\}$, $p_1 = p_2 = 6$, and independent stochastic demands X_1 and X_2 defined by the same probability mass function

$$p_1(x) = p_2(x) = \begin{cases} 0 & \text{if } x \notin \{1, 2\} \\ \frac{1}{2} & \text{if } x = 1 \\ \frac{1}{2} & \text{if } x = 2. \end{cases}$$

The optimization problem for one person coalitions $S = \{1\}$ and $S = \{2\}$ is the same as in Example 3.1. So

$$v^\Gamma(\{1\}) = \pi^{\{1\}}(2, X_1) = 5.$$
$$v^\Gamma(\{2\}) = \pi^{\{2\}}(2, X_2) = 5.$$

The symmetry in costs and selling prices result in optimal order vectors $q \in \mathbb{R}^W$ with $q_{w_1} + q_{w_2} = 3$ for coalition $\{1, 2\}$. Let q^* be such an order vector. The expected optimal profit of coalition $\{1, 2\}$ can be calculated as

$$v^\Gamma(\{1, 2\}) = \pi^{\{1,2\}}(q^*, X^{\{1,2\}}) = -3 * 1 + \frac{1}{4}(12 - 3) + \frac{1}{2}(18 - 3) + \frac{1}{4}(18 - 3) = 10\frac{1}{2}.$$

Note that since we are considering a forced allocation game, for every demand realization we should send all ordered units to the retailer. Therefore, the transportation cost from warehouse to the retailer appears as 3 in the last tree terms of the expected profit function for coalition $\{1, 2\}$.

The associated forced allocation newsvendor game (N, v^Γ) is described by

$$v^\Gamma(S) = \begin{cases} 0 & \text{if } S = \emptyset; \\ 5 & \text{if } |S| = 1; \\ 10\frac{1}{2} & \text{if } S = N. \end{cases}$$

Note that this newsvendor game is balanced, i.e., it has a non-empty core, since any $y \in \mathbb{R}^2$ with $y_1 \geq 5$, $y_2 \geq 5$ and $y_1 + y_2 = 10\frac{1}{2}$ belongs to core, for example $y = (\frac{21}{4}, \frac{21}{4})$.

$\diamond$

In the following part of this section, we will prove that force allocation games have a nonempty core by showing that these games are balanced. The following lemma shows a relation between the expected profit functions of the grand coalition and of a balanced collection of subcoalitions, which will be used to prove that the forced allocation game is balanced.

Lemma 3 *Let* $(N, W, (X_i)_{i \in N}, (k_w)_{w \in W}, (f_{wi})_{w \in W, i \in N}, (Z_i)_{i \in N}, (p_i)_{i \in N})$ *be a general news-vendor situation. Let* κ *be an associated balanced map. Denote an optimal order vector of coalition* $S \subseteq N$ *by* $q^{S,*}$. *Then*

$$\pi^N \left(\sum_{S \subseteq N : S \neq \emptyset} \kappa(S) q^{S,*}, \sum_{S \subseteq N : S \neq \emptyset} \kappa(S) X^S \right) \geq \sum_{S \subseteq N : S \neq \emptyset} \pi^S \left(\kappa(S) q^{S,*}, \kappa(S) X^S \right).$$

Proof :   First, note that the total transportation costs of goods from the supplier to the warehouses are the same at both sides of the inequality. Hence, it suffices to show that

$$R^N \left(\sum_{S \subseteq N : S \neq \emptyset} \kappa(S) q^{S,*}, \sum_{S \subseteq N : S \neq \emptyset} \kappa(S) X^S \right) \geq \sum_{S \subseteq N : S \neq \emptyset} R^S \left(\kappa(S) q^{S,*}, \kappa(S) X^S \right). \tag{5}$$

Let x^N be a demand realization vector for the grand coalition and let x^S be the associated demand vector of any coalition $S \subset N$. As before, denote an allocation of $q^{S,*}$ that maximizes the profit of coalition S for demand realization x^S by $A^{S,*}$. Furthermore, let $B^{N,*}$ be the optimal allocation for the grand coalition for order vector $\sum_{S \subseteq N : S \neq \emptyset} \kappa(S) q^{S,*}$ and demand realization $\sum_{S \subseteq N : S \neq \emptyset} \kappa(S) x^S$. Then

$$r^N \left(\sum_{S \subseteq N : S \neq \emptyset} \kappa(S) q^{S,*} , \sum_{S \subseteq N : S \neq \emptyset} \kappa(S) x^S \right)$$

$$= h^N \left(B^{N,*}, \sum_{S \subseteq N : S \neq \emptyset} \kappa(S) q^{S,*}, \sum_{S \subseteq N : S \neq \emptyset} \kappa(S) x^S \right)$$

$$\geq h^N \left(\sum_{S \subseteq N : S \neq \emptyset} \kappa(S) A^{S,*}, \sum_{S \subseteq N : S \neq \emptyset} \kappa(S) q^{S,*}, \sum_{S \subseteq N : S \neq \emptyset} \kappa(S) x^S \right)$$

$$\geq \sum_{S \subseteq N : S \neq \emptyset} h^S \left(\kappa(S) A^{S,*}, \kappa(S) q^{S,*}, \kappa(S) x^S \right)$$

$$= \sum_{S \subseteq N : S \neq \emptyset} r^S \left(\kappa(S) q^{S,*}, \kappa(S) x^S \right)$$

The first inequality holds since $B^{N,*}$ is an optimal allocation for the grand coalition for order vector $\sum_{S \subseteq N : S \neq \emptyset} \kappa(S) q^{S,*}$ and demand realization vector $\sum_{S \subseteq N : S \neq \emptyset} \kappa(S) x^S$, while $\sum_{S \subseteq N : S \neq \emptyset} \kappa(S) A^{S,*}$ is a possible allocation in $M^N(\sum_{S \subseteq N : S \neq \emptyset} \kappa(S) q^{S,*})$. The second inequality holds since the transportation costs from the warehouses to the retailers are the same at both sides of the inequality, and

$$\min \left\{ \sum_{S \subseteq N : i \in S} \kappa(S) \sum_{w \in Z_S} A_{wi}^{S,*}, \sum_{S \subseteq N : i \in S} \kappa(S) x_i^S \right\} \geq \sum_{S \subseteq N : i \in S} \kappa(S) \min \left\{ \sum_{w \in Z_S} A_{wi}^{S,*}, x_i^S \right\}$$

for all $i \in N$, which means that every retailer in the grand coalition sells at least as much units as the balanced sum of sales that he could make in the balanced collection of coalitions. The last equality holds since $h^S \left(\lambda A^S, \lambda q^{S,*}, \lambda x^S \right) = \lambda h^S \left(A^S, q^{S,*}, x^S \right)$ for any A^S, which implies that $\lambda A^{S,*}$ is an optimal allocation for the order demand pairs $\lambda q^{S,*}, \lambda x^S$.

Since this inequality holds for any realization x^N of X^N, taking expectations proves that Equation (5) holds. This completes the proof. $\square$

Using the properties of the profit function shown in Lemma 3, we show that forced allocation games are balanced in the following theorem.

Theorem 3 *Let $\Gamma = (N, W, (X_i)_{i \in N}, (k_w)_{w \in W}, (f_{wi})_{w \in W, i \in N}, (Z_i)_{i \in N}, (p_i)_{i \in N})$ be a general news-vendor situation. The associated forced allocation game (N, v^Γ) is balanced.*

Proof : Let $\kappa : 2^N / \{\emptyset\} \longrightarrow [0, 1]$ be a balanced map. Recall that $X^N = (X_i)_{i \in N}$ and, for all $i \in N$, note that $\sum_{S \subseteq N : S \neq \emptyset} \kappa(S) X_i^S = X_i$ since κ is a balanced map.

Let $(q^{S,*})_{S \subseteq N : S \neq \emptyset}$ be optimal order vectors for the different coalitions. Then, z^N defined by $z^N = \sum_{S \subseteq N : S \neq \emptyset} \kappa(S) q^{S,*}$ denotes a possible order vector for coalition N, while $q^{N,*}$ is the optimal order vector. Hence, we know that $\pi^N(q^{N,*}, X^N) \geq \pi^N(z^N, X^N)$. Furthermore, for all $S \subseteq N$ with $S \neq \emptyset$ it holds that $\kappa(S) q^{S,*}$ maximizes profit $\pi^S(\cdot, \kappa(S) X^S)$ since $\pi^S(\lambda q, \lambda X) = \lambda \pi^S(q, X)$ for any stochastic demand vector X of coalition S and any order vector q of coalition S. Then,

$$
\begin{aligned}
v^\Gamma(N) &= \pi^N(q^{N,*}, X^N) \\
&\geq \pi^N(z^N, X^N) \\
&= \pi^N \left(\sum_{S \subseteq N : S \neq \emptyset} \kappa(S) q^{S,*}, \sum_{S \subseteq N : S \neq \emptyset} \kappa(S) X^S \right) \\
&\geq \sum_{S \subseteq N : S \neq \emptyset} \pi^S \left(\kappa(S) q^{S,*}, \kappa(S) X^S \right) \\
&= \sum_{S \subseteq N : S \neq \emptyset} \kappa(S) \pi^S \left(q^{S,*}, X^S \right) \\
&= \sum_{S \subseteq N : S \neq \emptyset} \kappa(S) v^\Gamma(S). \qquad (6)
\end{aligned}
$$

The first inequality follows from the fact that $q^{N,*}$ is the optimal order vector of the grand coalition. The second inequality holds because of Lemma 3. The third equality holds because π^S is homogeneous of degree one. We conclude that the associated game is balanced. $\square$

From Shapley (1967) and Bondareva (1963), the following corollary follows immediately.

Corollary 1 *Let $(N, W, (X_i)_{i \in N}, (k_w)_{w \in W}, (f_{wi})_{w \in W, i \in N}, (Z_i)_{i \in N}, (p_i)_{i \in N})$ be a general news-vendor situation. The associated forced allocation game has a non-empty core.*

Since every subgame of a forced allocation game associated with a general news-vendor situation is a forced allocation game itself, the following corollary follows immediately from Theorem 3.

Corollary 2 *Let $(N, W, (X_i)_{i \in N}, (k_w)_{w \in W}, (f_{wi})_{w \in W, i \in N}, (Z_i)_{i \in N}, (p_i)_{i \in N})$ be a general news-vendor situation with warehouses. The associated forced allocation game is totaly balanced.*

4 Relaxed Allocation Game

In this section, we drop the assumption of sending all units to the retailers. In other words, we allow the warehouses to have stock at the end of the selling period. By dropping this assumption, we actually change the allocation problem that cooperating retailers solve following demand realization. Consequently, we are considering a new game, which we call the relaxed allocation game.

In this section, we show that the core of the associated relaxed allocation game is non-empty as well. We will first introduce the relaxed allocation game formally. Then, we will define a new related newsvendor situation. After creating a relation between the relaxed allocation game and the forced allocation game associated with a general newsvendor situation and its related newsvendor situation, respectively, we can easily prove that the core of the relaxed allocation game is nonempty as well.

Consider a general newsvendor situation $\Gamma = (N, W, (X_i)_{i \in N}, (k_w)_{w \in W}, (f_{wi})_{w \in W, i \in N}, (Z_i)_{i \in N}, (p_i)_{i \in N})$. By dropping the assumption of the previous section to send all units of goods to the retailers at the end of the period, we actually enlarge the set of possible allocations $M^S(q^S)$. An allocation of q^S is a matrix $\hat{A}^S$ with

$$\hat{A}_{wi} = 0 \text{ if } i \in N/S \text{ or } w \in W/Z_S \ ;$$
$$\sum_{i \in S} \hat{A}_{wi} \leq q_w^S \text{ for all } w \in Z_S.$$

The set of all such possible allocations of order is denoted by $\hat{M}^S(q^S)$. In line with notation in the previous section, let $\hat{h}^S(\hat{A}^S, q^S, x^S)$ be the total transportation costs from the warehouses to the retailers if transportation takes place according to allocation $\hat{A}^S \in \hat{M}^S(q^S)$. Note that $\hat{h}^S$ coincides with h^S, but they are defined on different domains. Let $\hat{P}^S(\hat{A}^S, q^S, x^S)$ be the profit of the coalition S for an allocation $\hat{A}^S \in \hat{M}^S(q^S)$ under order vector q^S and demand realization vector x^S. Similarly, $\hat{P}$ is similar to P but we consider the allocations in $\hat{M}^S(q^S)$ instead of $M^S(q^S)$. Finally, let $\hat{\pi}^S(q^S, X^S)$ be the expected profit function of coalition S.

Then, the associated relaxed allocation game $(N, \hat{v}^\Gamma)$[5] is defined as follows:

$$\hat{v}^\Gamma(S) = \max_{q^S \in Q^S} \hat{\pi}^S(q^S, X^S) \text{ for all } S \subseteq N.$$

The following example illustrates the relaxed allocation game associated with the newsvendor situation in Example 3.2.

Example 4.1 Consider the 2-person newsvendor situation $\Gamma = (N, W, (X_i)_{i \in N}, (k_w)_{w \in W}, (f_{wi})_{w \in W, i \in N}, (Z_i)_{i \in N}, (p_i)_{i \in N})$, introduced in Example 3.2. Under the assumption that we do not need to send all ordered units to the retailers, the optimization problem for coalition $S = \{1\}$ is similar to a newsvendor problem with overage cost $c_o = 1$, underage cost $c_u = 4$ and stochastic demand X_1. Since in the relaxed allocation game it is allowed to keep the excess units at the warehouses, the overage cost of excess units appears as 1, which is only the transportation cost from the supplier to the warehouses. Then, the optimal order quantity for coalition $S = \{1\}$ can be found easily as 2, which results in expected profit

$$v^\Gamma(\{1\}) = \pi^{\{1\}}(2, X_1) = -2 * 1 + \frac{1}{2}(6 - 1) + \frac{1}{2}(12 - 2) = 5.5 \ .$$

Repeating the same argument, we come up with the optimal order quantity 2 and expected profit $\pi^{\{2\}}(2, X_2) = 5.5$ for coalition $S = \{2\}$. The symmetry in costs and selling prices results in the optimal order vectors $q \in \mathbb{R}^W$ with $q_{w_1} + q_{w_2} = 4$ for coalition $\{1, 2\}$. Let q^* be such an order vector. The expected optimal profit of coalition $\{1, 2\}$ can be calculated as

$$v^\Gamma(\{1, 2\}) = \pi^{\{1,2\}}(q^*, X_{1,2}) = -4 * 1 + \frac{1}{4}(12 - 2) + \frac{1}{2}(18 - 3) + \frac{1}{4}(24 - 4) = 11 \ .$$

---

[5]The relaxed allocation game $(N, \hat{v}^\Gamma)$ is well defined, since the existence of the optimal allocation and optimal order vectors can be shown similarly as in the previous chapter.

The associated *relaxed allocation game* (N, v^Γ) is described by

$$v^\Gamma(S) = \begin{cases} 0 & \text{if } S = \emptyset; \\ 5.5 & \text{if } |S| = 1; \\ 11 & \text{if } S = N. \end{cases}$$

Note that this newsvendor game is balanced, i.e., it has a non-empty core, since $y = (5.5, 5.5)$ belongs to the core.

$\diamond$

In the following part of the paper, we define a related newsvendor situation and consider the forced allocation game associated with this newsvendor situation. We will use this forced allocation game and Corollary 1 to prove that the cores of the relaxed allocation games associated with general newsvendor situation is nonempty .

Let Γ be a general newsvendor situation. The related situation of Γ is defined as $\Gamma' := (N \cup \bar{N}, W, (X_i)_{i \in N \cup \bar{N}}, (k_w)_{w \in W}, (f_{wi})_{w \in W, i \in N \cup \bar{N}}, (Z_i)_{i \in N \cup \bar{N}}, (p_i)_{i \in N \cup \bar{N}})$. Here $\bar{N} := \{\bar{1}, ..., \bar{n}\}$ represents a set of phantom retailers. In this situation, each retailer i has a phantom retailer $\bar{i}$ with the same set of warehouses that it works with, zero transportation cost ($f_{wi} = 0$, $\forall w \in W$, $i \in \bar{N}$), zero selling price ($p_i = 0$ $\forall i \in \bar{N}$) and zero demand ($X_i = 0$ $\forall i \in \bar{N}$). These phantom retailers work as trash cans with zero cost for the unsold product in the system. The other parameters are the same as in Γ. In the following part of the paper, $\bar{S}$ denotes the set of phantom retailers corresponding to a coalition S. Now consider the forced allocation game associated with Γ'. Recall that in this forced allocation game, an allocation matrix A^R of a coalition $R \subseteq N \cup \bar{N}$ is in the set

$$M^R(q^R) := \{A \in \mathbb{R}_+^{W*N \cup \bar{N}} \mid A_{wi} = 0 \text{ if } i \in N/R \text{ or } w \in W/Z_R ; \sum_{i \in R} A_{wi} = q_w^R \text{ for all } w \in Z_R\}.$$

Furthermore, the forced allocation game associated with Γ', $(N \cup \bar{N}, v^{\Gamma'})$, is defined as follows:

$$v^{\Gamma'}(R) = \max_{q^R \in Q^R} \pi^R(q^R, X^R) \text{ for all } R \subseteq N \cup \bar{N}.$$

The following lemma shows a relation between forced allocation games and relaxed allocation games.

Lemma 4 *Let Γ be a general newsvendor situation and let Γ' be the related newsvendor situation. Then*

$$\hat{v}^\Gamma(S) = v^{\Gamma'}(S \cup \bar{S}) \text{ for all } S \subseteq N$$

Proof : Consider a coalition $S \subseteq N$ and its corresponding coalition $S \cup \bar{S}$. Note that the set of possible orders is the same for coalition S in Γ and for coalition $S \cup \bar{S}$ in Γ', since $Z_S = Z_{\bar{S}} = Z_{S \cup \bar{S}}$. In other words, $Q^S = Q^{S \cup \bar{S}}$. Additionally, any possible demand realization vector x^S in Γ can be identified where $x^{S \cup \bar{S}} = (x^{S \cup \bar{S}})_{i \in S \cup \bar{S}}$ in Γ' with $x_i^{S \cup \bar{S}} = x_i^S$ if $i \in S$ and $x_i^{S \cup \bar{S}} = 0$ if $i \in \bar{S}$. Note that this defines a one-to-one correspondence between possible demand realization vectors in Γ and Γ' for coalitions S and $S \cup \bar{S}$, respectively, since the demands for all phantom retailers are defined as zero in Γ'.

Consider an order vector $q \in Q^S = Q^{S \cup \bar{S}}$, a demand realization $x^S \in X^S$ and its corresponding realization $x^{S \cup \bar{S}} \in X^{S \cup \bar{S}}$. Furthermore, consider an allocation $\hat{A}^S \in \hat{M}^S(q)$ for coalition S. Then, recall that

$$\hat{h}^S(\hat{A}^S, q, x^S) \;=\; -\sum_{w \in W} \sum_{i \in S} f_{wi} \hat{A}^S_{wi} + \sum_{i \in S} p_i \min \left\{ \sum_{w \in W} \hat{A}^S_{wi}, x^S_i \right\}$$

Similarly, consider an allocation $A^{S \cup \bar{S}} \in M^{S \cup \bar{S}}(q)$ for coalition $S \cup \bar{S}$. Then,

$$
\begin{aligned}
h^{S \cup \bar{S}}(A^{S \cup \bar{S}}, q, x^{S \cup \bar{S}}) = \;& -\sum_{w \in W} \sum_{i \in S} f_{wi} A^{S \cup \bar{S}}_{wi} - \sum_{w \in W} \sum_{i \in \bar{S}} 0 * A^{S \cup \bar{S}}_{wi} \\
& + \sum_{i \in S} p_i * \min \left\{ \sum_{w \in W} A^{S \cup \bar{S}}_{wi}, x^{S \cup \bar{S}}_i \right\} + \sum_{i \in \bar{S}} 0 * \min \left\{ \sum_{w \in W} A^{S \cup \bar{S}}_{wi}, x^{S \cup \bar{S}}_i \right\}
\end{aligned}
$$

It is easy to show that:

- For all $\hat{A}^S \in \hat{M}^S(q)$ there exists $A^{S \cup \bar{S}} \in M^{S \cup \bar{S}}(q)$ such that $\hat{h}^S(\hat{A}^S, q, x^S) = h^{S \cup \bar{S}}(A^{S \cup \bar{S}}, q, x^{S \cup \bar{S}})$ (*)

and

- For all $A^{S \cup \bar{S}} \in M^{S \cup \bar{S}}(q)$ there exists $\hat{A}^S \in \hat{M}^S(q)$ such that $h^{S \cup \bar{S}}(A^{S \cup \bar{S}}, q, x^{S \cup \bar{S}}) = \hat{h}^S(\hat{A}^S, q, x^S)$ (**)

Let $\hat{A}^{S,*}$ and $A^{S \cup \bar{S},*}$ be optimal allocation matrices for the forced allocation game and relaxed allocation game, respectively. Then,

$$
\begin{aligned}
\hat{h}^S(\hat{A}^{S,*}, q, x^S) \;&=\; \max_{\hat{A}^S \in \hat{M}^S(q)} \hat{h}^S(\hat{A}^S, q, x^S) \\
&=\; \max_{A^{S \cup \bar{S}} \in M^{S \cup \bar{S}}(q)} h^S(A^{S \cup \bar{S}}, q, x^{S \cup \bar{S}}) \\
&=\; h^{S \cup \bar{S},*}(A^{S \cup \bar{S},*}, q, x^{S \cup \bar{S}})
\end{aligned}
$$

the second equality holds by (*) and (**).

So

$$
\begin{aligned}
\hat{P}^S(\hat{A}^{S,*}, q, x^S) \;&=\; -\sum_{w \in Z_S} k_w q_w + \hat{h}^S(\hat{A}^{S,*}, q, x^S) \\
&=\; -\sum_{w \in Z_{S \cup \bar{S}}} k_w q_w + h^{S \cup \bar{S}}(A^{S \cup \bar{S},*}, q, x^{S \cup \bar{S}}) \\
&=\; P^{S \cup \bar{S}}(A^{S \cup \bar{S},*}, q, x^{S \cup \bar{S}}),
\end{aligned}
$$

and taking expectations,

$$
\begin{aligned}
\hat{\pi}^S(q, X^S) \;&=\; E_{X^S} \left[\hat{P}^S(\hat{A}^{S,*}, q, .) \right] \\
&=\; E_{X^{S \cup \bar{S}}} \left[P^{S \cup \bar{S}}(A^{S \cup \bar{S},*}, q, .) \right] \\
&=\; \pi^{S \cup \bar{S}}(q, X^{S \cup \bar{S}}),
\end{aligned}
$$

where the second equality holds since $X^S_i = X^{S \cup \bar{S}}_i$ for all $i \in N$ and $X^{\bar{S}}_i = 0$ for all $i \in \bar{N}$. Since $\hat{\pi}^S(q, X^S) = \pi^{S \cup \bar{S}}(q, X^{S \cup \bar{S}})$ holds for all $q \in Q^S = Q^{S \cup \bar{S}}$, we have

$$
\begin{aligned}
\hat{v}^{\Gamma}(S) &= \max_{q \in Q^S} \hat{\pi}^S(q, X^S) \\
&= \max_{q \in Q^{S \cup \bar{S}}} \pi^{S \cup \bar{S}}(q, X^{S \cup \bar{S}}) \\
&= v^{\Gamma'}(S \cup \bar{S}).
\end{aligned}
$$

$\square$

In the following theorem, we show that the relaxed allocation game has a non-empty core.

Theorem 4 *Let Γ be a general news-vendor situation. The associated relaxed allocation game has a non-empty core.*

Proof : Let Γ' be the related newsvendor situation and let $(N \cup \bar{N}, v^{\Gamma'})$ be the forced allocation game associated with Γ'. From Corollary 1, we know that the core of the forced allocation game is non-empty. Consider a payoff vector $z \in C(N \cup \bar{N}, v^{\Gamma'})$. Define payoff vector y for the relaxed allocation game $(N, \hat{v}^{\Gamma})$ by $y_i = z_i + z_{\bar{\imath}} \ \forall \ i \in N$, where $\bar{\imath} \in \bar{N}$ is the corresponding phantom retailer of retailer $i \in N$.

Then,

$$
\sum_{i \in N} y_i = \sum_{i \in N} z_i + \sum_{i \in \bar{N}} z_i = v^{\Gamma'}(N \cup \bar{N}) = \hat{v}^{\Gamma}(N).
$$

The second equality holds since $z \in C(N \cup \bar{N}, v^{\Gamma'})$, and the last equality follows from Lemma 4. So the payoff vector y is efficient.

Furthermore,

$$
\sum_{i \in S} y_i = \sum_{i \in S} z_i + \sum_{i \in \bar{S}} z_i \geq v^{\Gamma'}(S \cup \bar{S}) = \hat{v}^{\Gamma}(S), \text{ for all } S \subset N.
$$

The inequality holds since $z \in C(N, \cup \bar{N}, v^{\Gamma'})$, and the last equality holds by Lemma 4.

$\square$

Similar to the forced allocation games, the relaxed allocation games are totally balanced as well.

5 Remarks and Future Research

In this work, we conducted a game-theoretical analysis of a distribution structure with warehouses, in which retailers have an opportunity of reallocating their orders after demands are realized and they can coordinate their orders accordingly. Slikker et al.(2003) studied a situation without warehouses but with positive transshipment costs and showed that a stable division of expected profits exists. This study considered a general model related to a distribution structure with warehouses. We showed that a stable division of expected profits exists in either of two games defined in this setting.

The generality of the model arises from the cost network we utilized. The two step cost structure (k_w and $f_{w,i}$) and flexible number of available warehouses allow us to analyze different systems between two extreme cases, in which the allocation of the

ordered products takes place at the retailer level (only lateral transshipment) and at the supplier level.

In this study, we do not explicitly include the salvage value of the product in the model. However, it can be easily incorporated into the model just by resetting the selling prices and transportation costs properly for both of the games considered.

There are several potential directions of future research. One of them is to relax the assumption that the actual demand is known with certainty at the point of time the orders are reallocated. A second one is to consider risk seeking or risk averse natures of players. Stochastic game theory can provide us with the necessary framework to make an analysis where the players have different attitudes and preferences.

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