COOPERATIVE n-PERSON STACKELBERG GAMES*

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Abstract. Stackelberg games and their resulting nonconvex programming problems can be used to model the behavior of independent decision-makers acting within a hierarchy. This paper examines the formation of coalitions within such organizations of optimizers for a large class of hierarchical problems. The mathematical characterizations of these games and the implications of their solutions are considered.

1 Introduction

Conflict and cooperation among groups of individuals are a natural part of the organizational process. An organization might consist of manufacturers competing within an economic system for the same market, or computers within a network sharing system resources. The types of communication and coordination of activities within the organizational structure can cause the system to flourish or decay. By understanding the behavior of such systems, we can improve their effectiveness and eliminate their inherent inefficiencies.

Chew [22] and Bialas and Chew [11] present a model of cooperation among decision-makers in a hierarchical organization. The model is based on Stackelberg games and related optimal control problems (see, for example, Simaan and Cruz [51], Başar and Olsder [6], and Tolwinski [53]). This early work was restricted to a linear objective function for each of the players and a requirement that all feasible decisions had to reside within a convex polytope. This paper extends these results to continuous objective functions over a bounded decision space.

2 Overview

This paper will consider models for the behavior of interacting decision-makers, each attempting to optimize individual objectives in view of decisions made by others. These problems can be found in many scientific disciplines, including operations research, control theory, economics, psychology, sociology and political science. Because of the pervasive nature of this topic, it has appeared in a variety of settings and adorned in different mathematical notation (see, for example, von Neumann and Morgenstern [55]).

With the development of the Dantzig-Wolfe Decomposition Principle [24] and its economic interpretation by Baumol and Fabian [7], mathematical programming has been used to describe the behavior of individuals interacting within organizations. Some of this work can be found in Anandalingam [1], Beckmann [8], Cassidy, *et al.* [21], Dirickx and Jennergren [25], Goreux [28], Haimes, *et al.* [29], Hax, *et al.* [30], Keeney and Raiffa [33], Koopmans [34], Marschak and Radner [40], and Wendell [57].

In addition, social scientists and psychologists have provided experimental studies of coalition formation. See, for example, Caplow [19, 20], Gamson [27], and Vinacke and Arkoff [54].

These citations represent only some of the research devoted to this subject. They do, however, suggest the broad interest in this topic.

3 The General Problem

We will briefly reintroduce the Stackelberg model presented by Bialas and Chew [11], and the related multilevel programming problem (see Bialas and Karwan [14, 15]). In this model, a decision-maker at one level of the hierarchy may have his objective function *and* decision space determined, in part, by decisions taken at other levels.

These Stackelberg games [52] have the following common characteristics:

1. The system has interacting players within a hierarchical structure.
2. The *leader* begins the game by announcing his decision, and the process continues for each player down through the hierarchy. Each subordinate player executes his policies after, and with the full knowledge of, his superior players.
3. The decision of a player can impact any other player's objective function, and a subsequent player's set of feasible choices.

Definition 1 *Let the vector $x \in \mathbb{R}^N$ be partitioned as $x = (x^a, x^b)$. Let $S \subset \mathbb{R}^N$ be compact, and let $f : \mathbb{R}^N \to \mathbb{R}$ be continuous on S. Then*

$$\Psi_f(S) \equiv \left\{ \hat{x} \in S \;\middle|\; f(\hat{x}) = \max_{x \in S \cap \{x^b = \hat{x}^b\}} \{f(x)\} \right\}$$

is the **set of rational reactions** *of f over S.*[1]

To formally define the n-player Stackelberg game, let the vector of decision variables for all players, denoted by $x \in \mathbb{R}^N$, be partitioned among n players with

$$x^k \equiv (x_1^k, x_2^k, \ldots, x_{N_k}^k) \in \mathbb{R}^{N_k} \quad (k = 1, 2, \ldots, n)$$

where $\sum_{k=1}^n N_k = N$. We will require that the n players choose values of $x \in S^1 \subset \mathbb{R}^N$, where S^1 is compact. The shape of S^1 will

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[1] Some authors call this set the *inducible region.*

determine the ability of one player to affect the set of feasible choices of the other players.

Let $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ be a set of continuous functions with $f_i(x) : S^i \to \mathbb{R}$ for all $i = 1, \ldots, n$.[2]

Definition 2 *Let the vector $x \in \mathbb{R}^N$ be partitioned as $x = (x^a, x^b)$ with $x^a = (x^1, \ldots, x^{k-1})$ and $x^b = (x^k, \ldots, x^n)$. The* **level-k feasible region**, *S^k, is recursively defined as $S^k = \Psi_{f_{k-1}}(S^{k-1})$ for $k = 2, 3, \ldots, n$.*

The set S^k represents the feasible outcomes resulting from the rational reactions of players at levels $1, 2, \ldots, k-1$. Hence S^k contains all of the information necessary for player i to assess the behavior of these players.

Given the preemptive decisions $(\hat{x}^{k+1}, \ldots, \hat{x}^n)$ of the first $n-k$ leading players, the optimization problem which must be solved by the player at level k is then

$$\begin{aligned} (L^k): \quad & \max\{f_k(x)\} \\ \text{st:} \quad & x \in S^k \\ & x^i = \hat{x}^i \quad (i = k+1, \ldots, n). \end{aligned}$$

This establishes a collection of nested mathematical programming problems $\{L^1, L^2, \ldots, L^n\}$ jointly representing the decision problems of n players in a hierarchical organization.[3]

Bard, *et al.* [3, 4, 5], Benson [9], Bialas and Karwan [13, 14, 15], Bialas and Wen [16], Candler and Townsley [18], Fortuny and McCarl [26], and others have characterized the properties of the nonconvex programming problem produced when the objective functions $f_i(x)$ are linear, and developed solution algorithms (of varying effectiveness) for $n = 2$ and $n = 3$. There has also been some work on solutions procedures for quadratic $f_i(x)$ (see Papavassilopoulos [44]).

4 An Illustration

Consider a game with three players, named 1, 2 and 3, each of whom controls an unlimited quantity of a commodity, with a different commodity for each player. Their task is to fill a container of unit capacity with amounts of their respective commodities, never exceeding the capacity of the container. This will be performed in a sequential fashion, with player 3 (the player at the "top" of the hierarchy) taking his turn first. A player cannot remove a commodity placed in the container by a previous player.

At the end of the sequence, a referee pays each player one dollar (or fraction, thereof) for each unit of his commodity which has been placed in the container. It is easy to see that, since player 3 has preemptive control over the container, he will fill it completely with his commodity, and collect one dollar.

Suppose, however, that the rules are slightly changed so that, in addition, player 3 could collect five dollars for each unit of *player one's* commodity which is placed in the container. Since player 2 does not receive any benefit from player one's commodity, player 2 would fill the container with his own commodity on his turn, if given the opportunity. This is the *rational reaction* of player 2. For this reason, player 3 has no choice but to fill the container with his commodity and collect only one dollar.

[2] This might also be written as $f_i(x) : S^1 \to \mathbb{R}$ for all $i = 1, \ldots, n$. The question of S^1 compact implying that $S^2 = \Psi_f(S^1)$ is compact is addressed for the linear case by Wen [56].

[3] Note that the leader is designated as player n, not player 1. Although not immediately intuitive, this convention results in many properties of L^k being invariant for fixed k and varying n. For example, L^2 will have the mathematical properties of a two-level problem in a system with any number of levels.

5 Coalition Formation

In the previous example, there are six dollars available to the three players. Divided equally, each of the players could receive two dollars. However, because of the sequential and independent nature of the decisions, such a solution cannot be attained.

The solution to the above problem is, thus, not Pareto-optimal (see Chew [22]). However, as suggested by the example, the formation of a coalition among subsets of the players could provide a means to achieve Pareto-optimality. The members of each coalition act for the benefit of the coalition as a whole. The question immediately raised are:

- which coalitions will tend to form,
- are the coalitions enforceable, and
- what will be the resulting distribution of wealth to each of the players?

The game in partition function form (see Lucas and Thrall [38] and Shenoy [49]) provides a framework for answering these questions in this Stackelberg setting.

Definition 3 *An* **abstract game** *is a pair (X, dom) where X is a set whose members are called* **outcomes** *and dom is a binary relation on X called* **domination**.

Let $G = \{1, 2, \ldots, n\}$ denote the set of n players. Let $\mathcal{P} = \{R_1, R_2, \ldots, R_M\}$ denote a **coalition structure** or partition of G into nonempty coalitions, where $R_i \cap R_j = \emptyset$ for all $i \neq j$ and $\cup_{i=1}^M R_i = G$.

Let $\mathcal{P}_0 \equiv \{\{1\}, \{2\}, \ldots, \{n\}\}$ denote the coalition structure where no coalitions have formed and let $\mathcal{P}_G \equiv \{G\}$ denote the **grand coalition**.

Consider $\mathcal{P} = \{R_1, R_2, \ldots, R_M\}$, an arbitrary coalition structure. Assume that utility is additive and transferable. As a result of the coalition formation, the objective function of each player in coalition R_j becomes

$$f'_{R_j}(x) = \sum_{i \in R_j} f_i(x).$$

Although the sequence of the players' decisions has not changed, their objective functions have. Let $R(i)$ denote the unique coalition $R_j \in \mathcal{P}$ such that player $i \in R_j$. Instead of maximizing $f_i(x)$, player i will now be maximizing $f'_{R(i)}(x)$. Let $\hat{x}(\mathcal{P})$ denote the solution to the resulting n-level optimization problem.

Definition 4 *Suppose that S^1 is compact and $\hat{x}(\mathcal{P})$ is unique. The* **value** *of coalition $R_j \in \mathcal{P}$, denoted by $v(R_j, \mathcal{P})$, is given by*

$$v(R_j, \mathcal{P}) \equiv \sum_{i \in R_j} f_i(\hat{x}(\mathcal{P})).$$

Bialas and Chew [12] have shown that $v(\cdot)$ need not be superadditive. Hence, one must be careful when applying some of the traditional game theory results which require superadditivity to this class of problems.

Definition 5 *A* **solution configuration** *is a pair $(r, \mathcal{P})$, where r is an n-dimensional vector (called an* **imputation***) whose elements r_i $(i = 1, \ldots, n)$ represent the payoff to each player i under coalition structure $\mathcal{P}$.*

Definition 6 *A solution configuration $(r, \mathcal{P})$ is a* **feasible solution configuration** *if and only if $\sum_{i \in R} r_i \leq v(R, \mathcal{P})$ for all $R \in \mathcal{P}$.*

Let Θ denote the set of all solution configurations which are feasible for the hierarchical decision-making problem under consideration. We can then define the binary relation dom, as follows:

Definition 7 *Let* $(r, \mathcal{P}_r)$, $(s, \mathcal{P}_s) \in \Theta$. *Then* $(r, \mathcal{P}_r)$ **dominates** $(s, \mathcal{P}_s)$, *denoted by* $(r, \mathcal{P}_r)\mathrm{dom}(s, \mathcal{P}_s)$, *if and only if there exists an nonempty* $R \in \mathcal{P}$, *such that*

$$r_i > s_i \quad \textit{for all} \quad i \in R \quad \textit{and} \tag{1}$$

$$\sum_{i \in R} r_i \leq v(R, \mathcal{P}_r). \tag{2}$$

Condition (1) implies that each decision maker in R prefers coalition structure $\mathcal{P}_r$ to coalition structure $\mathcal{P}_s$. Condition (2) ensures that R is a feasible coalition in $\mathcal{P}_r$. That is, R must not demand more for the imputation r than its value, $v(R, \mathcal{P}_r)$.

Definition 8 *The* **core**, $\mathcal{C}$, *of an abstract game is the set of undominated, feasible solution configurations.*

When the core is nonempty, each of its elements represents an enforceable solution configuration within the hierarchy.

6 Results

We have now defined a model of the formation of coalitions among players in a Stackelberg game. Perfect information is assumed among the players, and coalitions are allowed to form freely.

When coalitions form, the order of the players' actions remains unchanged. Each coalition earns the combined proceeds that each individual coalition member would have received in the original Stackelberg game. Utility is transferrable. A player now acts for the joint benefit of the members of his coalition, and utilmately himself. Therefore, a player's rational decision may change.

Using the above model, several results can be obtained regarding the formation of coalitions among the players. First, the distribution of wealth to any feasible coalition cannot exceed the value of the grand coalition. This is provided by the following lemma:

Lemma 1 *If solution configuration* $(z, \mathcal{P}) \in \Theta$ *then*

$$\sum_{i=1}^{n} z_i \leq \sum_{i=1}^{n} f_i(\hat{x}(\mathcal{P}_G)) = v(G, \mathcal{P}_G) \equiv V^*.$$

Proof. Let $\mathcal{P} = \{R_1, R_2, \ldots, R_m\}$. Since $\sum_{i \in R_k} z_i \leq v(R_k, \mathcal{P})$ for all $k = 1, 2, \ldots, m$.

$$\sum_{k=1}^{n} \sum_{j \in R_k} z_i \leq \sum_{k=1}^{m} v(R_k, \mathcal{P}) = \sum_{k=1}^{m} \sum_{j \in R_k} f_j(\hat{x}(\mathcal{P})).$$

Since $\mathcal{P}$ is a partition, we can rewrite the leftmost and rightmost terms to produce

$$\sum_{i=1}^{n} z_i \leq \sum_{i=1}^{n} f_i(\hat{x}(\mathcal{P})).$$

We also know that $\sum_{i=1}^{n} f_i(\hat{x}(\mathcal{P})) \leq V^*$ since $\hat{x}(\mathcal{P}) \in S^1$, and $\hat{x}(\mathcal{P}_G)$ is the solution to the mathematical programming problem

$$\max \sum_{i=1}^{n} f_i(x)$$
$$\text{st: } x \in S^1$$

Hence

$$\sum_{i=1}^{n} z_i \leq \sum_{i=1}^{n} f_i(\hat{x}(\mathcal{P}_G)) = V^*.$$

∎

Theorem 1 shows that coalition structures in the core have an even stricter requirement. Specifically, if the core of the abstract game is non-empty, the total value of any imputation must equal the value of the grand coalition.

Theorem 1 *If* $(z, \mathcal{P}) \in \mathcal{C} \neq \emptyset$ *then* $\sum_{i=1}^{n} z_i = V^*$.

Proof. From Lemma 1, we already have $\sum_{i=1}^{n} z_i \leq V^*$. We must now show that $\sum_{i=1}^{n} z_i \geq V^*$. We will do this by contradiction. Suppose that $(z, \mathcal{P}) \in \mathcal{C} \neq \emptyset$ and $\sum_{i=1}^{n} z_i < V^*$. Let

$$\Delta \equiv V^* - \sum_{i=1}^{n} z_i > 0.$$

Consider the solution configuration $(z', \mathcal{P}_G)$ with

$$z'_i = z_i + \frac{\Delta}{n} > z_i \quad \text{for all } i = 1, 2, \ldots, n.$$

Note that $(z', \mathcal{P}_G) \in \Theta$ and $(z', \mathcal{P}_G)\mathrm{dom}(z, \mathcal{P})$. Hence, $(z, \mathcal{P}) \notin \mathcal{C}$ which is a contradiction. ∎

We are now prepared to show that if the core is nonempty, then there always exists a solution configuration involving the grand coalition among the solution configurations in the core.

Theorem 2 *If* $\mathcal{C} \neq \emptyset$ *then there exists an imputation* $z = (z_1, \ldots, z_n)$ *such that* $(z, \mathcal{P}_G) \in \mathcal{C}$.

Proof. Let $(z, \mathcal{P}_z) \in \mathcal{C}$. From Theorem 1, we know that

$$\sum_{i=1}^{n} z_i = V^*. \tag{3}$$

From Equation (3) we see that $(z, \mathcal{P}_G)$ is feasible (i.e., $(z, \mathcal{P}_G) \in \Theta$).

To further show that $(z, \mathcal{P}_G) \in \mathcal{C}$, suppose that there is a solution configuration $(r, \mathcal{P}_r) \in \mathcal{C}$ and $(r, \mathcal{P}_r)\mathrm{dom}(z, \mathcal{P}_G)$. But then $(r, \mathcal{P}_r)\mathrm{dom}(z, \mathcal{P}_z)$ which yields a contradiction. Hence $(z, \mathcal{P}_G) \in \mathcal{C}$. ∎

It is also possible to construct a sufficient condition for the core to be empty. This is provided in Theorem 3.

Theorem 3 *The abstract game* (Θ, dom) *has* $\mathcal{C} = \emptyset$ *if there exists coalition structures* $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m$ *and coalitions* $R_j \in \mathcal{P}_j$ $(j = 1, \ldots, m)$ *with* $R_j \cap R_k = \emptyset$ *for all* $j \neq k$ *such that*

$$\sum_{j=1}^{m} v(R_j, \mathcal{P}_j) > V^*. \tag{4}$$

Proof. We will show that, given Condition (4), for any solution configuration $(z, \mathcal{P}_G) \in \Theta$, we can find a solution configuration $(y, \mathcal{P}_j) \in \Theta$ such that $(y, \mathcal{P}_j)\mathrm{dom}(z, \mathcal{P}_G)$.

We will prove the result by contradiction. Suppose that there does not exist a solution configuration $(y, \mathcal{P}_j) \in \Theta$ for any j which dominates $(z, \mathcal{P}_G)$. Then, for all $(y, \mathcal{P}_j) \in \Theta$ $(j = 1, \ldots, m)$, we have

$$\sum_{i \in R_j} y_i < \sum_{i \in R_j} z_i \quad \text{for all } R_j \in \mathcal{P}_j. \tag{5}$$

In particular, Relation (5) will be true for a solution configuration $(y, \mathcal{P}_j) \in \Theta$ $(j = 1, \ldots, m)$ with

$$\sum_{i \in R_j} y_i = v(R_j, \mathcal{P}_j) \quad \text{for all } R_j \in \mathcal{P}_j. \tag{6}$$

Summing both sides of Relations (5) and (6) over j yields

$$\sum_{j=1}^{m} v(R_j, \mathcal{P}_j) \leq \sum_{j=1}^{m} \sum_{i \in R_j} z_i \leq V^*$$

which is a contradiction to Condition (4).

Therefore, for any choice $(z, \mathcal{P}_G) \in \Theta$ we can find a solution configuration $(y, \mathcal{P}_j) \in \Theta$ such that $\sum_{i \in R_j} y_i > \sum_{i \in R_j} z_i$ for some $R_j \in \mathcal{P}_j$. Hence $(y, \mathcal{P}_j)\text{dom}(z, \mathcal{P}_G)$. ■

Finally, we can easily show that, in any 2-person game of this type, the core is always nonempty.

Theorem 4 *If* $n = 2$ *then* $\mathcal{C} \neq \emptyset$.

Proof. There are only two possible coalition structures, namely $\mathcal{P}_0 = \{\{1\}, \{2\}\}$ and $\mathcal{P}_G = \{\{1, 2\}\}$. Note that $\hat{x}(\mathcal{P}_0) \in S^1$ and $\hat{x}(\mathcal{P}_G)$ solves

$$\begin{aligned} &\max f_1(x) + f_2(x) \\ &\text{st:}\, x \in S^1. \end{aligned}$$

Therefore,

$$f_1(\hat{x}(\mathcal{P}_0)) + f_2(\hat{x}(\mathcal{P}_0)) \leq f_1(\hat{x}(\mathcal{P}_G)) + f_2(\hat{x}(\mathcal{P}_G)).$$

This can be rewritten as

$$v(\{1\}, \mathcal{P}_0) + v(\{2\}, \mathcal{P}_0) \leq v(\{1, 2\}, \mathcal{P}_G) \tag{7}$$

We will show that if $(r, \mathcal{P}_0) \in \Theta$ and $(s, \mathcal{P}_G) \in \Theta$ then $(r, \mathcal{P}_0)$ cannot dominate $(s, \mathcal{P}_G)$ for any r when $s = (s_1, s_2)$ with $s_1 + s_2 = f_1(\hat{x}(\mathcal{P}_G)) + f_2(\hat{x}(\mathcal{P}_G))$. We can assume, without loss of generality, that $r_1 = v(\{1\}, \mathcal{P}_0)$ and $r_2 = v(\{2\}, \mathcal{P}_0)$ since all other feasible solution configurations involving $\mathcal{P}_0$ are dominated by $((r_1, r_2), \mathcal{P}_0)$. We can also assume that $s_1 + s_2 = v(\{1, 2\}, \mathcal{P}_G)$ for a similar reason.

From Relation (7), we have $r_1 + r_2 \leq s_1 + s_2$, so we can choose

$$s_1 = \tfrac{1}{2}[v(\{1, 2\}, \mathcal{P}_G) + v(\{1\}, \mathcal{P}_0) - v(\{2\}, \mathcal{P}_0)] \geq v(\{1\}, \mathcal{P}_0)$$

and

$$s_2 = \tfrac{1}{2}[v(\{1, 2\}, \mathcal{P}_G) + v(\{2\}, \mathcal{P}_0) - v(\{1\}, \mathcal{P}_0)] \geq v(\{2\}, \mathcal{P}_0).$$

Therefore, there exists a feasible choice of $s = (s_1, s_2)$ such that $(r, \mathcal{P}_0)$ cannot dominate $(s, \mathcal{P}_G)$ for any feasible r. Hence $\mathcal{C} \neq \emptyset$. ■

7 An Example

We will expand on the illustration given in Section 4. Let c_{ij} represent the reward to player i if the commodity controlled by player j is placed in the container. Let C represent the matrix $[c_{ij}]$ and let x be an n-dimensional vector with x_j representing the amount of commodity j placed in the container. Note that $\sum_{j=1}^{n} x_j \leq 1$ and $x_j \geq 0$ for $j = 1, \ldots, n$. For the illustration provided in Section 4,

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}.$$

Note that Cx^{T} is a vector whose components represent the earnings to each player.

Chew [22] provides a simple procedure to solve this game. The algorithm requires $c_{11} > 0$.

Step 0: Initialize $i = 1$ and $j = 1$. Go to *Step 1*.

Step 1: If $i = n$, stop. The solution is $\hat{x}_j = 1$ and $\hat{x}_k = 0$ for $k \neq j$. If $i \neq n$, then go to *Step 2*.

Step 2: Set $i = i + 1$. If $c_{ii} > c_{ij}$, then set $j = i$. Go to *Step 1*.

If no ties occur in *Step 2* (i.e., $c_{ii} \neq c_{ij}$) then it can be shown that the above algorithm solves the problem (see Chew [22]).

Consider the three player game of this form with

$$C = C_{\mathcal{P}_0} = \begin{bmatrix} 4 & 1 & 4 \\ 1 & 0 & 3 \\ 2 & 5 & 1 \end{bmatrix}.$$

With coalition structure $\mathcal{P}_0 = \{\{1\}, \{2\}, \{3\}\}$, the solution is $(x_1, x_2, x_3) = (1, 0, 0)$ and the coalition values are $v(\{1\}, \mathcal{P}_0) = 4$, $v(\{2\}, \mathcal{P}_0) = 1$ and $v(\{3\}, \mathcal{P}_0) = 2$.

Under the formation of coalition structure $\mathcal{P}_0 = \{\{1\}, \{2, 3\}\}$, the resources of players 2 and 3 are combined. This yields a payoff matrix of

$$C_{\mathcal{P}} = \begin{bmatrix} 4 & 1 & 4 \\ 3 & 5 & 4 \\ 3 & 5 & 4 \end{bmatrix}$$

and a solution of $(0, 1, 0)$. The values of the coalitions in this case are $v(\{1\}, \mathcal{P}) = 1$ and $v(\{2, 3\}, \mathcal{P}) = 5$.

Finally, if all of the players join to form the grand coalition, $\mathcal{P}_G$, the payoff matrix becomes

$$C_{\mathcal{P}_G} = \begin{bmatrix} 7 & 6 & 8 \\ 7 & 6 & 8 \\ 7 & 6 & 8 \end{bmatrix}$$

with a solution of $(0, 0, 1)$ and $v(\{1, 2, 3\}, \mathcal{P}_G) = 8$. Note that

$$v(\{1\}, \mathcal{P}_0) + v(\{2, 3\}, \mathcal{P}) > v(\{1, 2, 3\}, \mathcal{P}_G).$$

From Theorem 3, we know that the core for this game is empty.

8 Conclusion

This paper has extended previous results for evaluating coalition formation for a class of games which can be called *cooperative Stackelberg games.* We have defined a Stackelberg game with an imbedded cooperative game which still retains the information framework of the original Stackelberg game.

The results shown here are among the first steps in using cooperative Stackelberg games to characterize the behavior of decision makers in an organized system. It would be surprising if some of the results provided here cannot be extended to even more general cases.

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