

Cooperative Phenomena in Coupled Oscillator Systems under External Fields

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Systems of many limit cycle oscillators are studied by using a phase description of the oscillation. Each oscillator interacts with all the other oscillators uniformly and is subject to external field. Two kinds of external fields are applied to the system: (1) periodic force and (2) random noises. Some effects of the external fields on the mutual entrainment are studied by analyses for steady macroscopic rotation and also by numerical simulations.

Large populations of coupled limit cycle oscillators are known to exhibit many interesting behaviors such as pattern formation and turbulent-like behavior.^{2),4)} Mutual synchronization is another type of important and peculiar behavior.^{1)~9)} A simple mathematical model for studying the synchronization is given by a set of differential equations.³⁾

$$(\text{Model 0}) \quad \dot{\phi}_i = \omega_i - \frac{K}{N} \sum_{j=1}^N \sin(\phi_i - \phi_j), \quad (1)$$

where ϕ_i represents the phase of the i -th oscillator, and N the total number of the oscillators. The natural frequencies ω_i are constant in time and they are distributed randomly. The normalized number density of the oscillators having natural frequency ω is denoted as $g(\omega)$. To simplify the analyses we treat the case that $g(\omega)$ is symmetric about the mean value ω_0 . For this model analytical expressions for various quantities can be obtained by a mean field theory.³⁾ An important quantity is the complex order parameter defined by

$$\sigma \exp(i\theta) = \frac{1}{N} \sum_{j=1}^N \exp(i\phi_j). \quad (2)$$

Equation (1) is rewritten by using (2) as

$$\dot{\phi}_i = \omega_i - K\sigma \sin(\phi_i - \theta) \quad (3)$$

which shows that each oscillator is subject to the mean field whose strength is $K\sigma$. If we assume that σ is time-independent, substitution of the solution of (3) into (2) yields a self-consistent equation for σ . It is known from the analysis of the self-consistent equation that there is a phase transition at a certain critical coupling strength K_c . The order parameter σ remains zero for $K < K_c$ and becomes nonzero for $K > K_c$ corresponding to the onset of macroscopic oscillation. Near and above the critical coupling strength σ is expressed as $\sigma \propto (K - K_c)^{1/2}$, and $\theta = \omega_0 t + \theta_0$. Then a macroscopic number of oscillators are entrained to the collective oscillation. We extend

Model 0 to study the influence of some external fields upon the phase transition and the collective mutual entrainment. Two kinds of external fields are applied to the oscillators. One is an external periodic force with frequency near the mean natural frequency ω_0 and the other is random noises.

A model in the presence of the external periodic force is given by^{4),5)}

$$(\text{Model 1}) \quad \dot{\phi}_i = \omega_i - \frac{K}{N} \sum_{j=1}^N \sin(\phi_i - \phi_j) - b \sin(\phi_i - \omega_f t), \quad (4)$$

where ω_f is the frequency of the periodic force and b is its strength. When K is zero, the oscillators are mutually independent and we have

$$\dot{\phi}_i = \omega_i - b \sin(\phi_i - \omega_f t)$$

or

$$\dot{\psi}_i = \omega_i - \omega_f - b \sin \psi_i, \quad (5)$$

where $\psi_i = \phi_i - \omega_f t$. If $|\omega_i - \omega_f| < b$, Eq. (5) has a stable stationary solution and therefore the i -th oscillator comes to have frequency ω_f , or it is entrained to the external force. When b is zero and K is larger than K_c , a macroscopic number of oscillators are synchronized to the collective oscillation whose frequency is the mean natural frequency ω_0 . When both b and K are not zero, behaviors of the model would be determined by competition of the forced entrainment and the mutual entrainment.

We first investigate a steadily rotating state of Model 1. The complex order parameter is assumed to rotate steadily at the frequency of the external periodic force.

$$\frac{1}{N} \sum_{j=1}^N \exp(i\phi_j) = \sigma \exp i(\omega_f t + \phi_0). \quad (6)$$

Equation (4) is rewritten with σ , ϕ_0 and $\psi_i = \phi_i - \omega_f t$ as

$$\dot{\psi}_i = \omega_i - \omega_f - b' \sin(\psi_i - \phi_0), \quad (7)$$

where $b' = \sqrt{b^2 + K^2 \sigma^2 + 2K\sigma b \cos \phi_0}$ and $\tan \phi_0 = K\sigma \sin \phi_0 / (b + K\sigma \cos \phi_0)$. Here we assume that σ and ϕ_0 are constant in time. Then we can derive a self-consistent equation for σ and ϕ_0 by using (6) and (7).⁹⁾ The solution of (7) takes the form,

$$\begin{aligned} \psi_i &= \phi_0 + \sin^{-1} \frac{\omega_i - \omega_f}{b'} & \text{for } |\omega_i - \omega_f|/b' \leq 1, \\ \bar{\omega}_i t + \phi_0 + h(\bar{\omega}_i t) & & \text{for } |\omega_i - \omega_f|/b' > 1, \end{aligned} \quad (8)$$

where $\bar{\omega}_i = \sqrt{(\omega_i - \omega_f)^2 - b'^2}$ and $h(x)$ is a certain 2π -periodic function of x . On the other hand, the order parameter σ is expressed in terms of ψ_i as

$$\begin{aligned} \sigma \exp(i\phi_0) &= \int_0^{2\pi} d\psi \, n(\psi) \exp(i\psi) \\ &= \int_{-\infty}^{\infty} d\omega \, g(\omega) \int_0^{2\pi} d\psi \, n(\psi; \omega) \exp(i\psi), \end{aligned} \quad (9)$$

where $n(\psi)$ is the normalized number density of the oscillators of phase ψ , and $n(\psi; \omega)$ its further decomposition into different natural frequencies. The phase distribution consists of the synchronized and desynchronized parts as $n(\psi) = n_s(\psi) + n_{ds}(\psi)$. These phase distributions are rewritten with the natural frequency distribution $g(\omega)$ through the solution (8).

$$n_s(\psi) = g(\omega_f + b' \sin(\psi - \phi_0)) b' \cos(\psi - \phi_0) \quad \text{where } |\psi - \phi_0| \leq \pi/2, \\ n_{ds}(\psi) = 1/2\pi \int_{|\omega - \omega_f| > b'} d\omega g(\omega) \sqrt{(\omega - \omega_f)^2 - b'^2} / (\omega - \omega_f - b' \sin(\psi - \phi_0)). \quad (10)$$

Substitution of (10) into (9) yields a self-consistent equation for σ and ϕ_0 in the form

$$\sigma \exp i(\phi_0 - \phi_0) = b' \left\{ \int_{-\pi/2}^{\pi/2} d\psi g(\omega_f + b' \sin \psi) \cos \psi \exp(i\psi) + iJ \right\}, \quad (11)$$

where

$$J = \int_0^{\pi/2} d\psi \frac{\cos \psi (1 - \cos \psi)}{\sin^3 \psi} \left\{ g\left(\omega_f + \frac{b'}{\sin \psi}\right) - g\left(\omega_f - \frac{b'}{\sin \psi}\right) \right\}. \quad (12)$$

Note that the solution in (11) is a particular solution of Model 1 because it has been obtained on the assumption that σ and ϕ_0 are constant in time, or a steadily rotating solution needs to satisfy Eq. (11).

When b is small enough, the periodic force may be treated as a perturbation on Model 0. The a.c. susceptibility defined by $\chi(\omega_f) = \lim_{b \rightarrow 0} \sigma/b$ is explicitly obtained when the coupling strength is near the critical one, i.e., $K \lesssim K_c = 2/\pi g(\omega_0)$. Then Eq. (11) is approximated by

$$\sigma \cos(\phi_0 - \phi_0) = \pi/2 g(\omega_f) b', \\ \sigma \sin(\phi_0 - \phi_0) = J_1(\omega_f) b', \quad (13)$$

where

$$J_1(\omega_f) = \int_0^\infty \frac{g(\omega_f + x) - g(\omega_f - x)}{2x} dx. \quad (14)$$

The a.c. susceptibility and the phase shift ϕ_0 are expressed by the solution of (13) when ω_f is close to ω_0 .

$$\chi = \frac{1}{\sqrt{\frac{\pi^2}{4} g^2(\omega_0) (K - K_c)^2 + \frac{16}{\pi^4} \frac{J_1'(\omega_0)^2}{g^4(\omega_0)} (\omega_f - \omega_0)^2}}, \quad (15)$$

$$\tan \phi_0 = \frac{8J_1'(\omega_0)(\omega_f - \omega_0)}{\pi^3 g^3(\omega_0)(K_c - K)}. \quad (16)$$

When ω_f is ω_0 , χ is inversely proportional to $K_c - K$. This is analogous to the Curie-Weiss law for ferromagnets. As far as the susceptibility is concerned, our coupled-oscillator system has the same property near the critical point as in the usual equilibrium phase transitions.

The above steadily rotating solution is not always realized. A computer simula-

tion on Model 1 with $N=1000$ was carried out with the Euler method to find a stable solution. The distribution $g(\omega)$ was assumed to be a Gaussian centered about 0 and with variance w^2 . In our simulation K was set to 2 and ω_f to 1 throughout, and b and w were changed continuously. The initial condition chosen was uniform, i.e., $\phi_i(0)=0$ for all i .

Figure 1 is a phase diagram determined by the simulation on a b - w plane, which shows that there are two different phases. When b or w is large, the particular solution of the self-consistent equation is realized and then the population of the oscillators is separated into two groups; one group is composed of oscillators synchronized to the external periodic force and the other group is composed of desynchronized oscillators. We call it the forced-entrainment phase. When b and w are small, the steadily rotating solution is not realized and then the order parameter σ comes to oscillate. We call it the mutual-entrainment phase. In the mutual-entrainment phase the population is separated into three groups. The third group is composed of oscillators which are mutually entrained. The mutual entrainment is realized at a series of different frequency ratios with nearly equal spacing.

The coupling-modified frequency $\tilde{\omega}_i$ is defined as $\tilde{\omega}_i = (\phi_i(T) - \phi_i(0))/T$ where T is 600 in our simulation. Figure 2 shows the relation of $\tilde{\omega}_i$ to the natural frequency ω_i . The occurrence of entrainment is reflected in this curve as a plateau where the coupling-modified frequency remains constant for some finite range of natural frequencies. When the parameters are set in the forced-entrainment phase, there is only one plateau at ω_f . There are two possible types of phase transitions from the forced-entrainment phase to the mutual-entrainment phase. When b is small and w is decreased continuously, it is seen that at $w=w_c$ a small-width plateau appears at ω_1 , apart from the one at ω_f . As w is decreased further, the plateau at ω_1 extends and other plateaus come out at $\omega_n \approx n\omega_1 - (n-1)\omega_f$. These plateaus correspond to higher-harmonic mutual entrainment. This transition is considered a modification of the phase transition in Model 0 by a weak periodic force. When w is small and b is decreased continuously, a qualitative change occurs at $b=b_c$ such that the plateau at ω_f starts to split into many plateaus to form a staircase. Their spacing increases from 0. The corresponding transition line on the b - w plane starts from the point $b = \omega_f$ and $w=0$ where each limit cycle oscillator starts to be entrained to the periodic force. We may say, therefore, that the macroscopic oscillation, which would exist if the system is not subject to the periodic force, is entrained to the periodic force at the transition point. The two lines corresponding to the two types of phase transitions seem to cross at $w \approx 0.7$ and $b \approx 1.05$. Details about the cross-over of these phase transitions are not known yet.

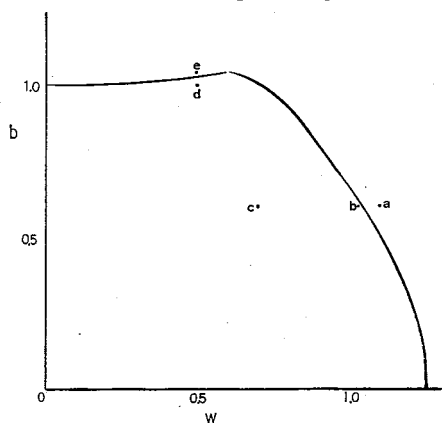


Fig. 1. Phase diagram in b - w plane with $K=2.0$ and $\omega_f=1.0$ and $\omega_0=0.0$ for Model 1.

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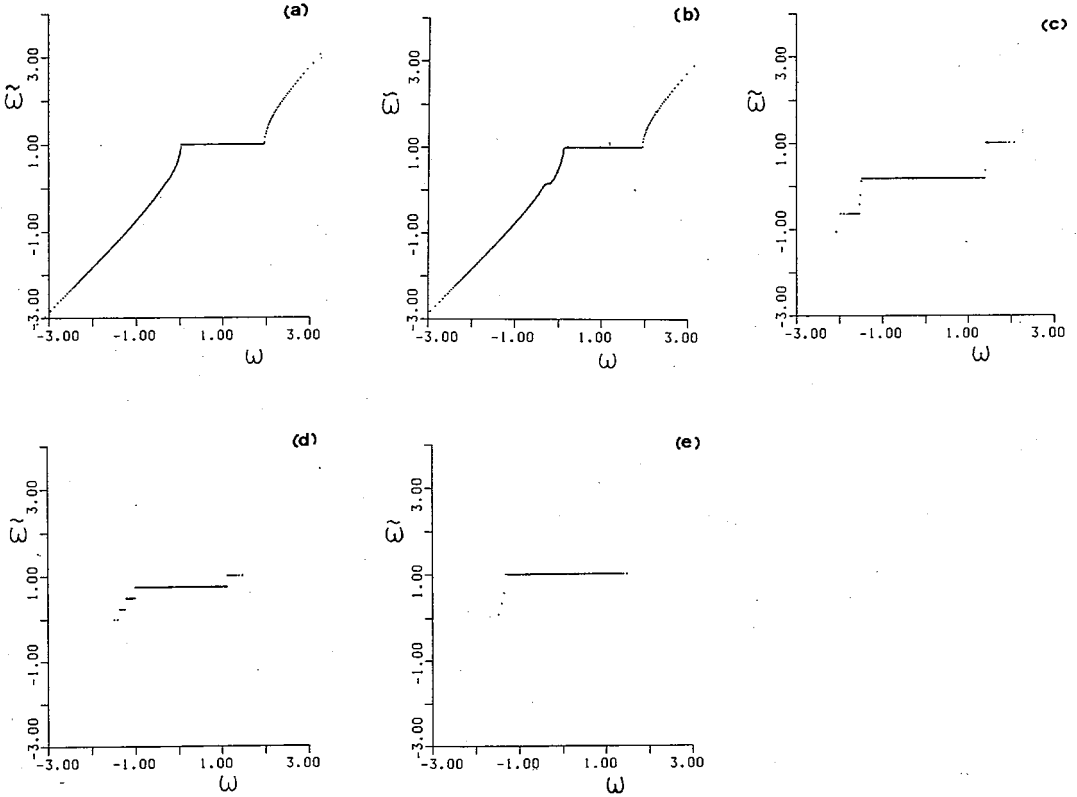


Fig. 2. Coupling-modified frequency $\tilde{\omega}$ as a function of natural frequency ω for the five set up parameter values indicated in Fig. 1 with (a)~(e).

(a) $b=0.6$, $w=1.1$. (b) $b=0.6$, $w=1.05$. (c) $b=0.6$, $w=0.7$. (d) $b=1.0$, $w=0.5$. (e) $b=1.04$, $w=0.5$.

tion. The particular solution of the self-consistent equation has been found to be the only stable state in the forced-entrainment phase. This solution may become unstable and mutual entrainment starts, which brings about oscillation of the order parameter. It is an important and future problem to seek for an equation of motion for σ which includes Eq. (11) as a stationary solution and explain the phase diagram theoretically.

Next we study another model in which some external random noises are present:

$$(\text{Model 2}) \quad \dot{\phi}_i = \omega_i - \frac{K}{N} \sum_{j=1}^N \sin(\phi_i - \phi_j) + f_i(t). \quad (17)$$

Here $f_i(t)$ are Gaussian white noises with properties

$$\overline{f_i(t)} = 0, \quad \overline{f_i(t)f_j(t')} = 2D \delta_{ij} \delta(t-t'). \quad (18)$$

When all oscillators are identical, i.e., $\omega_i = \omega_0$, Model 2 is equivalent to thermodynamic systems of classical XY spins and then the parameter D represents essentially the temperature of the spin systems. Random noises give rise to Brownian motion of phase difference between any two oscillators and therefore perfect entrainment is impossible. If the interactions are strong enough, however, a macroscopic oscillation may appear even in the presence of both random noises and random

natural frequencies.

Let us derive a self-consistent equation for Model 2. Since the complex order parameter is expected to rotate at the mean natural frequency, we put

$$\frac{1}{N} \sum_{j=1}^N \exp(i\phi_j) = \sigma \exp(i(\omega_0 t + \phi_0)). \quad (19)$$

Equation (17) is rewritten with the use of (19) as

$$\dot{\phi} = \omega - K\sigma \sin \phi + f_i(t), \quad (20)$$

where $\omega = \omega_i - \omega_0$ and $\phi = \phi_i - \omega_0 t - \phi_0$. If σ is constant in time, the Fokker-Planck equation for the probability distribution $n(\psi; \omega)$ is derived from the Langevin equation (20), and we get

$$\frac{\partial n(\psi; \omega)}{\partial t} = -\frac{\partial}{\partial \psi} \{(\omega - K\sigma \sin \psi) n(\psi; \omega)\} + D \frac{\partial^2}{\partial \psi^2} n(\psi; \omega). \quad (21)$$

The stationary solution of (21) satisfying the periodic boundary condition $n(\psi; \omega) = n(\psi + 2\pi; \omega)$ is given by¹⁰⁾

$$n(\psi; \omega) = \exp\left(\frac{-K\sigma + \omega\psi + K\sigma \cos \psi}{D}\right) n(0; \omega) \times \left\{ 1 + \frac{(e^{-2\pi\omega/D} - 1) \int_0^\psi e^{(-\omega\phi - K\sigma \cos \phi)/D} d\phi}{\int_0^{2\pi} e^{(-\omega\phi - K\sigma \cos \phi)/D} d\phi} \right\}, \quad (22)$$

where $n(0; \omega)$ is determined by the normalization condition

$$\int_0^{2\pi} n(\psi; \omega) d\psi = 1. \quad (23)$$

Substitution of (22) into (9) yields a self-consistent equation for σ .

$$\sigma = \int_{-\infty}^{\infty} d\omega g(\omega + \omega_0) \int_0^{2\pi} d\psi n(\psi; \omega) \exp(i\psi). \quad (24)$$

Let us find critical coupling strength K_c and a small amplitude solution near K_c . As $g(\omega + \omega_0)$ is symmetric about $\omega = 0$, the imaginary part on the right-hand side of (24) is always zero. The real part on the right-hand side may be expanded in powers of $K\sigma/D$ as

$$\sigma = K\sigma \left[\frac{1}{2} \int_{-\infty}^{\infty} g(D\omega + \omega_0) \frac{d\omega}{\omega^2 + 1} - \frac{K^2 \sigma^2}{4D^2} \int_{-\infty}^{\infty} g(D\omega + \omega_0) \left\{ \frac{1}{\omega^2 + 4} - \frac{\omega}{(\omega^2 + 1)^2} \right\} d\omega + O\left(\left(\frac{K\sigma}{D}\right)^4\right) \right]. \quad (25)$$

The critical coupling strength as a function of D is determined from (25), and we obtain

$$K_c(D) = 2 \left/ \int_{-\infty}^{\infty} g(D\omega + \omega_0) \frac{d\omega}{\omega^2 + 1} \right. . \quad (26)$$

As K is increased, a nontrivial solution branches off the trivial zero solution at $K = K_c$. Near and above the critical coupling strength σ is proportional to $(K - K_c)^{1/2}$. The coupling-modified frequency $\tilde{\omega}$ is similarly expanded in powers of $K\sigma/D$ as

$$\tilde{\omega} = \omega - \frac{1}{2} \frac{K\sigma(\omega - \omega_0)}{(\omega - \omega_0)^2 + D^2} + O\left(\frac{K^4\sigma^4}{D^4}\right). \quad (27)$$

Although the oscillator frequencies come nearer the frequency ω_0 of the macroscopic oscillation, there are no oscillators perfectly entrained to the macroscopic oscillation.

Figure 3 shows the critical curve (26) in the case that $g(\omega)$ is a Gaussian distribution centered about 0 and with variance w^2 . A computer simulation based on Model 2 with $N=1000$ was carried out with the Heun method and it was shown that the nontrivial solution of (25) is realized beyond the critical coupling strength. Figure 4 shows the distribution of the resultant frequencies $\tilde{\omega}$ in the ordered state. When D is zero, a delta peak and strong intensity decrease of the background spectrum is seen in the vicinity of the delta peak. When random noises are added, the delta peak broadens and the intensity depression disappears, that is, the mutual entrainment becomes less clear. Thus, the random noises turn out to make the critical coupling

strength larger without giving any qualitative change with respect to phase transition. Macroscopic oscillation emerges beyond the critical coupling strength, though there are no oscillators perfectly entrained to it.

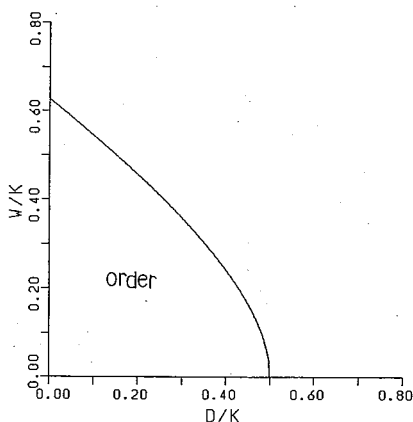


Fig. 3. Phase diagram for Model 2 in D/K - w/K plane.

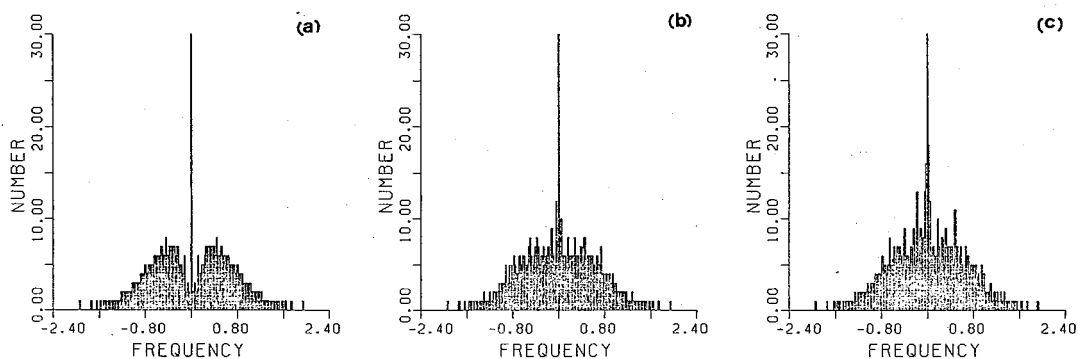


Fig. 4. Histogram of oscillator population as a function of the coupling-modified frequency $\tilde{\omega}$.

- (a) $K=1.0$, $w=0.6$, $D=0.0$.
- (b) $K=1.02$, $w=0.6$, $D=0.01$.
- (c) $K=1.04$, $w=0.6$, $D=0.02$.

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