Coordinate Theorems for Affine Hjelmslev Planes (*).

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Summary. – *It is shown that an affine Hjelmslev plane* $\mathcal{H}$ *is a translation plane if and only if each of its coordinate biternary rings* $B = \langle k, T, T_0, 0, 1\rangle$ *are linear. Addition and multiplication in the ternary ring* $\langle k, T, 0, 1\rangle$ *are defined by* $a + b = T(a, 1, b)$ *and* $a\cdot b = T(a, b, 0)$, *respectively, and it is proved that every biternary ring of a translation plane has the additional properties that* $\langle k, +\rangle$ *is an abelian group,* $\langle k, +, \cdot\rangle$ *is right distributive, and* $T(a, 1, b) = T_0(a, 1, b)$. *Moreover, if a single linear biternary ring of* $\mathcal{H}$ *has these three properties, then* $\mathcal{H}$ *is a translation plane. It is shown that a translation plane is Desarguesian if and only if it has a linear biternary ring such that* $T = T_0$ *and* $\langle k, +, \cdot\rangle$ *is an affine Hjelmslev ring. Hessenberg's theorem for affine Hjelmslev planes is proved, and a special configurational condition which is equivalent to the commutativity of multiplication in each biternary ring is introduced.*

1. – Introduction.

Affine Hjelmslev planes, henceforth called A.H. planes, are generalizations of ordinary affine planes, where more than one line may pass through two distinct points.

The coordinatization of ordinary affine planes from the elements of an algebraic structure was discussed in [1] by E. Artin, and in [8] by M. Hall. In the former approach, which is valid for Desarguesian planes only, one constructs the coordinate ring first, and then introduces coordinates. In the latter case, the approach is reversed, and is valid in any affine plane. The coordinate ring in the Hall construction is called a Hall ternary ring.

Artin's ideas were generalized by Klingenberg in [10]. Lüneburg, in [14], and Lorimer and Lane, in [12], extended these notions. However, early attempts to coordinatize arbitrary A.H. planes by a generalization of a Hall ternary ring were unsuccessful. Kligenberg, in [10], introduced coordinates for points from an algebraic structure which was essentially a generalization of a double loop, but he had to assume that the plane was Pappian in order to coordinatize the lines; cf. [10], S 5.14. In [6], Drake coordinatized a subclass of A.H. planes, called radial H-planes, by H-modules, generalizing the concepts of Sperner in [15]. Finally, Cyganova, in [5], and the author, in his Ph.D. thesis, independently introduced coordinates into an arbitrary A.H. plane over a generalized ternary ring called an H-ternar. Moreover, in [5], it was shown that an A.H. plane can be constructed over every

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H-ternar. Recently, BACON, in [4], called an H-ternar a biternary ring, and showed that biternary rings and A. H. planes are categorically equivalent.

The purpose of this paper is to study the interaction between the geometric properties of an A.H. plane and the algebraic properties of its biternary ring, thus generalizing the results of HALL in [8].

In Section 2, we define an A.H. plane and introduce coordinates over a biternary ring $B \equiv \langle k, T, T_0, 0, 1 \rangle$, where k is the set of points incident with a line of the plane, T is a ternary operator, T_0 a partial ternary operator, and 0 and 1 are distinct points. Every partial ternary operator can be extended to a ternary operator on k. The operator T generates in the usual fashion an addition $(+)$ and a multiplication $(\cdot)$. We show that the neighbour relation restricted to k is a congruence on k; i.e., $a_i \sim b_i$; $i = 1, 2, 3$, implies $T(a_1, a_2, a_3) \sim T(b_1, b_2, b_3)$ and $T_0(a_1, a_2, a_3) \sim T_0(b_1, b_2, b_3)$. We use this result to obtain the algebraic properties of $\langle k, +, \cdot \rangle$. The structure of $\langle k, +, \cdot \rangle$ is essentially the same as that of the algebra introduced by KLINGENBERG in [10]. The operator T_0 also generates an addition, $\oplus$, and a multiplication, $\odot$. Later, we consider conditions under which T, $+$, and $\cdot$ coincide with T_0, $\oplus$, and $\odot$, respectively. For ordinary planes, T and T_0 always coincide.

In Section 3, we show that an A.H. plane is a translation plane if and only if each biternary ring is linear, i.e., $T(a, b, c) = a \cdot b + c$ and $T_0(a, b, c) = a \odot b \oplus c$. Moreover, we show that translation planes are exactly the ones which can be coordinatized by linear biternary rings with the properties (i) $\langle k, + \rangle$ is an abelian group, (ii) $\langle k, +, \cdot \rangle$ is right distributive, (iii) $T(1, a, b) = T_0(1, a, b)$. The translations, in this case, are the mappings $(x, y) \to (x + a, y + b)$.

In Section 4, a translation plane is called Desarguesian if for any collinear triple (P, Q, R), where P is not a neighbour of Q, there exists a dilatation fixing P and mapping Q into R. This definition of Desarguesian is weaker than of KLINGENBERG in [11]. An example of a Desarguesian plane which is not Desarguesian in the sense of Klingenberg was constructed in [12], 5. Given a translation plane, we show that if dilatations exist for collinear triples (PQR) such that P is a neighbour of neither Q nor R, then in each biternary ring, T and T_0 coincide, and $\langle k, +, \cdot \rangle$ is an associative ring. In fact, $\langle k, +, \cdot \rangle$ is an A.H. ring; cf. [12], 2.9 In view of [12], 3, the plane is Desarguesian if there exists a linear biternary ring $B \equiv \langle k, T, T_0, 0, 1 \rangle$ such that $T = T_0$ and $\langle k, +, \cdot \rangle$ is an A.H. ring. In [10], Klingenberg showed that his algebra is a commutative A.H. ring if the plane satisfies the minor Desarguesian and the Pappian configuration theorems for A.H. planes. He did not consider a Desargussian configuration theorem. This was introduced in [13] and was shown to be equivalent to the above mentioned weak definition of Desarguesian.

In Section 5, we use Klingenberg's results and those of our earlier sections to obtain an algebraic proof of a generalization of Hessenberg's theorem for A.H. planes. This was done for ordinary planes by KLINGENBERG in [9]. Finally, we introduce a special Pappus configurational condition and show that it is equivalent to the commutativity of multiplication in each biternary ring.

2. – Coordinatization of an A.H. plane.

In this section, we present, for the convenience of the reader, the essential ideas from [5] and the author's thesis.

An A.H. plane is an incidence structure with parallelism, $\{\boldsymbol{P}, \boldsymbol{L}, I, \|\}$. Here $\boldsymbol{P}$ and $\boldsymbol{L}$ are sets, $I \subset \boldsymbol{P} \times \boldsymbol{L}$, and $\| \subset \boldsymbol{L} \times \boldsymbol{L}$ is an equivalence relation. The elements of $\boldsymbol{P}[\boldsymbol{L}]$ are *points* [*lines*] and are denoted by $P, Q, \ldots$ $[l, m, \ldots]$. We write $l \| m$ for $(l, m) \in \|$ and PIl for $(P, l) \in I$. P, QIl shall mean PIl and QIl. We put $g \wedge h = \{P \in \boldsymbol{P} | PIg, h\}$, $g \vee h = \{P \in \boldsymbol{P} | PIg$ or $PIh\}$. $PIg \vee h$ shall mean PIg or PIh. If $A \subset \boldsymbol{P}$ and $l \in \boldsymbol{L}$, put $A \wedge l = \{P \in A | PIl\}$. $|A|$ is the cardinality of the set A.

Define $(P, Q) \in \sim_{\boldsymbol{P}}$ if there exist $l, m \in \boldsymbol{L}$, $l \neq m$, such that P, QIl, m. We usually write $P \sim_{\boldsymbol{P}} Q$ for $(P, Q) \in \sim_{\boldsymbol{P}}$. Define $(l, m) \in \sim_{\boldsymbol{L}}$ (or $l \sim_{\boldsymbol{L}} m$) if for every PIl there exists QIm such $P \sim_{\boldsymbol{P}} Q$ and for every QIm there exists PIl such that $Q \sim_{\boldsymbol{P}} P$. If there is no danger of ambiguity, we shall write $P \sim Q$ for $P \sim_{\boldsymbol{P}} Q$ and $l \sim m$ for $l \sim_{\boldsymbol{L}} m$. If $P \sim Q[l^{-} \sim m]$ we call P and $Q[l$ and $m)$ *neighbours*. If P and $Q[l$ and $m]$ are not neighbours, we write $P \nsim Q[l \nsim m]$.

An incidence structure with parallelism, $\mathcal{H} = \{\boldsymbol{P}, \boldsymbol{L}, I, \|\}$, is called an *affine Hjelmslev plane* (or an A.H. *plane*) if it satisfies the following system of axioms.

(A1) For any two points P and Q there exists $l \in \boldsymbol{L}$ such that P, QIl. We write $l = PQ$ if $P \nsim Q$.

(A2) There exist $P_1, P_2, P_3 \in \boldsymbol{P}$ such that $P_iP_j \nsim P_iP_k$; $i \neq j \neq k \neq i$; $i, j, k = 1, 2, 3$.

(A3) $\sim_{\boldsymbol{P}}$ is transitive on $\boldsymbol{P}$.

(A4) If PIv, h, then $v \nsim h$ iff $tv \wedge ht = 1$.

(A5) If $v \nsim h$; p, RIv; Q, RIh; and $P \sim Q$, then $R \sim P$, Q.

(A6) If $v \sim h$; $j \nsim v$; PIg, j; and QIh, j; then $P \sim Q$.

(A7) If $v \| h$; PIj, v; and $v \nsim j$; then $j \nsim h$ and there exists Q such that QIh, j.

(A8) For every $P \in \boldsymbol{P}$ and every $l \in \boldsymbol{L}$, there exists a unique line $L(P, l)$ such that $PIL(P, l)$ and $l \| L(P, l)$.

The set $\Pi_g = \{l \in \boldsymbol{L} | g \| l\}$ is a *pencil* of $\boldsymbol{L}$. We wirte $\Pi_1 \sim_{\boldsymbol{P}} \Pi_2$ (or $\Pi_1 \sim \Pi_2$) if there exist $l_1 \in \Pi_1$ and $l_2 \in \Pi_2$ such that $l_1 \sim l_2$. Any set of three points which satisfy the conditions of (A2) is a *triangle*.

Let $\overline{\boldsymbol{P}}$ and $\overline{\boldsymbol{L}}$ be the quotient spaces of $\sim_{\boldsymbol{P}}$ and $\sim_{\boldsymbol{L}}$ respectively; $\chi_{\boldsymbol{P}}$ and $\chi_{\boldsymbol{L}}$ will denote the quotient maps of $\sim_{\boldsymbol{P}}$ and $\sim_{\boldsymbol{L}}$ respectively. If $\overline{P} \in \overline{\boldsymbol{P}}$ and $\overline{l} \in \overline{\boldsymbol{L}}$, we define $\overline{P}I\overline{l}$ iff there exists $S \in \boldsymbol{P}$ such that SIl and $S \sim P$. If $\|$ is the parallelism relation for ordinary affine planes, then the incidence structure $\overline{\mathcal{H}} = \{\overline{\boldsymbol{P}}, \overline{\boldsymbol{L}}, I, \|\}$ associated with $\mathcal{H}$ in an ordinary affine plane. If $l \wedge m = \emptyset$, then $\overline{l} \| \overline{m}$; cf. [14], Sect. 2.6.

Select a triangle $\{O, X, Y\}$ of $\mathcal{H}$. Put $g = OX$, $h = OY$, $E = L(X, h)\wedge L(Y, g)$ and $k = OE$. The elements of k are denoted by $a, b, c, \ldots$: The lines g, h, and k are called the *essential lines* of the triangle $\{O, X, Y\}$. If P is any point, then

$$x = k\wedge L(P, h)\,, \qquad y = k\wedge L(P, g)$$

are the *coordinates* of P. The point with the coordinates x and y shall be denoted by $P(x, y)$, or simply by (x, y). Then $P(x, y)\ Ik$ if and only if $x = y$, and $x = P(x, x)$. We put $E = 1$ and $O = 0$. Call $\boldsymbol{L}_1 = \{l \in \boldsymbol{L} | \Pi_l \sim \Pi_k\}$ and $\boldsymbol{L}_2 = \{l \in \boldsymbol{L} | \Pi_l \nsim \Pi_h\}$, the sets of *lines of* the *first* and *second kind*, respectively. If $l \in \boldsymbol{L}_1$, then the elements u and v of k defined by

$$L(0, l)\wedge YE = P(u, 1)\,, \qquad l\wedge g = P(v, 0)$$

are the *coordinates* of l, and if $l \in \boldsymbol{L}_2$, the elements m, n defined by

$$L(0, l)\wedge XE = P(1, m)\,, \qquad l\wedge h = P(0, n)$$

are the *coordinates* of l. The line of the first (second) kind with the coordinates u, (m, n) shall be denoted by $l[u, v]_1$ $(l[m, n]_2)$, or simply by $[u, v]_1$ $([m, n]_2)$. $l[m, n]$ refers to a line whose kind is not specified.

We summarize the basic properties of coordinates; cf. [5], 2.

2.1. – *$l[m, n] \in \boldsymbol{L}_1$ implies $m \sim 0$.*

2.2. – $|l[m, n]_2 \wedge l[\mathrm{J}, v]_1| = 1$.

2.3. – *$P(a, b) \sim P(c, d) \Leftrightarrow a \sim c$ and $b \sim d$.*

2.4. – *$l[m, n] \sim l[u, v] \Leftrightarrow$ the lines are of the same kind, $m \sim u$, and $n \sim v$.*

2.5. – *$l[m, n] \| l[u, v] \Leftrightarrow$ the lines are of the same kind, and $m = u$.*

2.6. – *If $l[m, n] \wedge l[u, v] = \emptyset$, then the lines are of the same kind and $m \sim u$.*

Next, we introduce the notion of a biternary ring or an H-ternar; cf. [4], [5].

2.7. DEFINITIONS.

(*a*) An algebraic system $R \equiv \langle R, T, 0, 1\rangle$, where R is a set, T is a ternary operator, and $0, 1$ are distinct elements of R, is a *ternary ring* if the following axioms hold.

($\mathcal{T}_0$) $T(m, 0, n) = n = T(0, m, n)$, for all $m, n \in R$.

($\mathcal{T}_1$) $T(1, m, 0) = m = T(m, 1, 0)$, for all $m \in R$.

($\mathcal{T}_2$) $T(a, m, x) = b$ is uniquely solvable for x, for all a, m, $b \in R$.

The elements 0 and 1 are called the *zero* and *unit* of R, respectively; cf. [3], 2.5.

(*b*) If $R \equiv \langle R, T, 0, 1\rangle$ is a ternary ring, then $a + b = T(a, 1, b)$ and $a \cdot b = T(a, b, 0)$ are the *associated addition* and *multiplication* of R, respectively. The system $A(R) \equiv \langle R, +, \cdot, 0, 1\rangle$ is the *associated algebra* of R. R is *linear* if $T(a, b, c) = a \cdot b + c$.

(*c*) If R is a ternary ring, then $a \neq 0$ is a *right* (*left*) *divisor of zero* if there exists $b \neq 0$ such that $a \cdot b = 0 (b \cdot a = 0)$. $D_+ (D_-)$ is the set consisting of 0 and the right (left) divisors of zero. $D_0 = D_+ \cap D_-$ is the set of (two-sided) *divisors of zero*. If $a, b \in R$, we define $a \underset{R}{\sim} b$ if and only if every x which satisfies the equation $a = T(x, 1, b)$ is an element of D_+.

(*d*) T_0 is a *partial ternary operation* of R if and only if T_0 is a function from $R \times D_+ \times R$ into R with the properties

(i) $T_0(m, 0, n) = n = T_0(0, p, n)$ for $p \in D_+$ and $m, n \in R$.

(ii) $T_0(1, u, 0) = u$ for $u \in D_+$.

(iii) $T_0(a, m, x) = b$ is uniquely solvable for x, for all $(a, m, b) \in R \times D_+ \times R$.

(e) An algebraic system $B \equiv \langle R, T, T_0, 0, 1\rangle$ is a *biternary ring* iff the following axioms hold.

(β_1) $R \equiv \langle R, T, 0, 1\rangle$ is a ternary ring.

(β_2) $\underset{R}{\sim}$ is an equivalence relation on R.

(β_3) T_0 is a partial ternaryo peration on R. To $(b, u, r) \underset{R}{\sim} v$ if $(b, u, v) \in R \times D_+ \times R$.

(β_4) $T(x, m_1, n_1) = T(x, m_2, n_2)$ is uniquely solvable for x if and only if $m_1 \underset{R}{\not\sim} m_2$.

(β_5) The system $T(a_i, x, y) = b_i$; $i = 1, 2$, uniquely determines the pair x, y if $a_1 \underset{R}{\not\sim} a_2$. If $a_1 \underset{R}{\sim} a_2$ and $b_1 \underset{R}{\not\sim} b_2$, then the system cannot be solved. If $a_1 \underset{R}{\not\sim} a_2$ and $b_1 \underset{R}{\sim} b_2$, then $x \in D_+$.

(β_6) The system $y = T(x, m, n)$ and $x = T_0(y, u, v)$, where $u \in D_+$, determines uniquely the pair x, y.

(β_7) If $a_1 \underset{R}{\sim} a_2$ and $b_1 \underset{R}{\sim} b_2$ and $(a_1, b_1) \neq (a_2, b_2)$, then one and only one of the systems $T(a_i, x, y) = b_i$; $T_0(b_i, u, v) = a_i$; $i = 1, 2$, is solvable with respect to x, y; u, v. The solvable system has at least two solutions; and $x_1 \underset{R}{\sim} x_2$, $y_1 \underset{R}{\sim} y_2$; or $u_1 \underset{R}{\sim} u_2$, $v_1 \underset{R}{\sim} v_2$, according as the former or latter system is solvable.

(β_8) The system $T_0(b_i, x, y) = a_i$; $i = 1, 2$,

(i) determines uniquely $x \in D_+$ and $y \in R$ if $b_1 \underset{R}{\sim} b_2$ and $a_1 \underset{R}{\sim} a_2$,

(ii) has no solutions for x and y if $a_1 \underset{R}{\not\sim} a_2$.

(β_9) If $T_0(b, u_i, v_i) = a$; $i = 1, 2$, then $v_1 \underset{R}{\sim} v_2$ and there exists at least one pair a_1, b_1 such that $a_1 = T_0(b_i, u_i, v_i)$, $i = 1, 2$.

(β_{10}) The function T induces a function $\bar{T}$ in $R/\underset{R}{\sim}$ and $\langle M/\underset{R}{\sim}, \bar{T}, \bar{0}, \bar{1}\rangle$ is a ternary field with $\bar{0} = \{z|z \underset{R}{\sim} 0\}$ and $\bar{1} = \{z|z \underset{R}{\sim} 1\}$ in the sense of M. Hall; cf. [8].

The following two properties are immediate consequences of ($\mathcal{C}_1$), ($\mathcal{C}_2$), and (β_5).

(β_{11}) $T(x, m, n) = b$ is uniquely solvable for x if $m \underset{R}{\not\sim} 0$, for all $m, n, b \in R$.

(β_{12}) $T(a, x, n) = b$ is uniquely solvable for x, if $a \underset{R}{\not\sim} 0$, for all $a, n, b \in R$.

If $\underset{R}{\sim}$ is the identity relation, then $D_+ = \{0\}$, T_0 coincides with T, and $\langle R, T, 0, 1\rangle$ is a Hall ternary ring; cf. [8].

2.8. DEFINITION. – Let $\{O, X, Y\}$ be a triangle. Define

2.8.1. – $T(x, m, n) = k\wedge L\big(L(P(0, n), OP(1, m))\wedge L(x, h), g\big)$, for $(x, m, n \in k)$.

2.8.2. – $T_0(y, m, n) = k\wedge L\big(L(P(n, 0), OP(m, 1))\wedge L(y, g), h\big)$ for $(y, m, n) \in k\times D_+\times k$.

It was shown in [4] and in the author's thesis, that $B \equiv \langle k, T, T_0, 0, 1\rangle$ is a biternary ring,

$$P(x, y)\, Il[,u\, v]_1 \qquad \text{if and only if } x = T_0(y, u, v)\,,$$

and

$$P(x, y)\, Il[m, n]_2 \qquad \text{if and only if } y = T(x, m, n)\,.$$

B is the *biternary rinv of* $\mathcal{H}$ *with respect to* $\{O, X, Y\}$. $R \equiv \langle k, T, 0, 1\rangle$ is the *ternary ring of* $\mathcal{H}$ *with respect* $\{O, X, Y\}$, and $A(R)$ is the *associated algebra* of R. Notice that T_0 can be defined for all $(y, m, n) \in k$ and that $R_0 \equiv \langle k, T_0, 0, 1\rangle$ is the ternary ring of $\mathcal{H}$ with respect to $\{O, Y, X\}$.

The main result of [5] is given below; also cf. [4].

2.9. THEOREM. – *Let B be a biternary ring, and let $\mathcal{H}(B) = \langle \mathbf{P}, \mathbf{L}, \|, I\rangle$ be the incidence structure defined by*

$$\mathbf{P} = R\times R\,;$$

$$\mathbf{L} = \mathbf{L}_1 \cup \mathbf{L}_2\,,$$

where $\mathbf{L}_1$ consists of sets of the form

$$[u, v]_1 = \{(T_0(y, u, v), y) : y \in R\}\,, \qquad (u, v) \in D_+\times R,$$

and $\mathbf{L}_2$ consists of sets of the form

$$[m, n]_2 = \big\{(x, T(x, m, n)) : x \in R\big\}\,, \qquad (m, n) \in R\times R;$$

$[m, n] \| [u, v]$ if and only if the lines are of the same kind and $m = n$; and

incidence is given by set-theoretic containment.

Then $\mathcal{H}(B)$ is an A.H. *plane.*

In the rest of this paper, B, R and $A(R)$ will refer to the structures of 2.8 with respect to a fixed coordinate system $\{O, X, Y\}$ of a given A.H. plane $\mathcal{H}$.

2.10. REMARK. – *$\{\overline{O}, \overline{X}, \overline{Y}\}$ is a triangle of $\overline{\mathcal{H}}$ and $\overline{P(a, b)} = P(\bar{a}, \bar{b})$; $\overline{l[m, n]} = l[\overline{m}, \bar{n}]$.*

PROOF. – The first statement is clear. Now let $(\bar{x}, \bar{y})$ be the coordinates of $\overline{P(a, b)}$. Then

$$\bar{x} = \bar{k} \wedge L(\overline{P(a, b)}, \bar{h}) = \chi_L(k) \wedge \chi_L(L(P, h)) = \chi_P(k \wedge L(P, h)) = \chi_P(a) = \bar{n}\ .$$

Similarly, $\bar{y} = \bar{b}$ and $\overline{l[m, n]} = l[\overline{m}, \bar{u}]$.

2.11. LEMMA. – $D_+ = \bar{0}$ (cf. [5], Theorem 16).

PROOF. – Let $n \in D_+$. Then there exists $m \neq 0$ such that $n \cdot m = 0$. Hence $(0, n)$ as well as $(0, 0)$ lies on both $[0, 0]_1$ and $[m, 0]_1$. Thus $(0, 0) \sim (0, n)$ and so 2.3 implies $n \sim 0$.

Conversely, assume $n \sim 0$. Then $(0, n) \sim (0, 0)$ and so there exists $m \neq 0$ such that $[m, 0]_1$ passes through both points. Thus $0 = n \cdot m$ and so $n \in D_+$.

2.12. THEOREM. – *Let $\overline{B} = \langle OE, \overline{T}, \overline{T}_0, \bar{0}, \bar{1}\rangle$, where $\overline{T}_0 = \overline{T}_{\bar{k} \times \{\bar{0}\} \times \bar{k}}$, be the biternary ring of $\mathcal{H}$ associated with $\{O, X, Y\}$. Then the map $\chi_k \colon B \to \overline{B}(a \to \bar{a})$ is a biternary ring epimorphism. Hence $\sim_P \cap (k \times k)$ is the congruence associated with this homomorphism, i.e., $T(a_1, a_2, a_3) \sim T(b_1, b_2, b_3)$ if $a_i \sim b_i$; $i = 1, 2, 3$; and similarly for T_0. Moreover, $\sim_P \cap (k \times k) = \underset{R}{\sim}$ and $B/_{\underset{R}{\sim}} \cong \overline{B}$.*

PROOF. – Clearly χ_k is surjective. We must show that $\chi_k T = \overline{T} \chi_k^3$, and $\chi_k T_0 = \overline{T}_0 \chi_k^3$ restricted to $k \times D_+ \times k$. Since $\chi = (\chi_P, \chi_L)$ is a homomorphism and $\chi_k = \chi_P|_k$, 2.10 yields

$$\begin{aligned}\chi_k(T(x, m, n)) &= \chi_k\Big(k \wedge L\big(L(P(0, n), OP(1, m)) \wedge L(x, h), g\big)\Big)\\ &= \chi_L(k) \wedge \chi_L\Big(L\big(L(P(0, n), OP(1, m)) \wedge L(x, h), g\big)\Big)\\ &= OE \wedge L\big(L(P(\bar{0}, n), OP(\bar{1}, m)) \wedge L(x, h), g\big)\\ &= \overline{T}(x, m, n) = \overline{T} \chi_k^3(x, m, n)\,.\end{aligned}$$

It then follows that $T(a_1, a_2, a_3) \sim T(b_1, b_2, b_3)$ if $a_i \sim b_i$; $i = 1, 2, 3$. The analogous result for T_0 is verified in the same fashion. Hence $\sim_P \cap (k \times k)$ is a congruence on B. We next show that $\underset{R}{\sim} = \sim_P \cap (k \times k)$. By 2.11, it follows that $a \underset{R}{\sim} b$ implies $a \sim b$. Conversely, assume $a \sim b$. Put $M = L\big((a, b), k\big) \wedge h$ and $O = k \wedge L(M, g)$. Then from 2.8.1, we see that $a + c = b$. Now $(a, b) \sim (a, a)$ by 2.3, and so $L\big((a, b), k\big) \sim k$

Thus $0 \sim M$ and so $0 \sim c$. By 2.11, $a \underset{R}{\sim} b$. Finally, $B/_{\underset{R}{\sim}} \cong \bar{B}$ by [7], p. 57.

The fact that $\underset{R}{\sim}$ is a congruence yields the following corollary.

2.12.1. – COROLLARY. – *D_+ is a subalgebra of B, $T(k, D_+, D_+) \cup T(D_+, k, D_+) \subseteq D_+$, $\bar{a} = a + D_+$, and $B/D_+ \cong \bar{B}$.*

Using the fact that $\underset{R}{\sim}$ is a congruence, one can obtains horter algebraic proofs of many of the results in [5]. We give examples in 2.13 and 2.14.

2.13. LEMMA. – *If $P(a_1, b_1) \nsim P(a_2, b_2)$, then $P(a_1, b_1)P(a_2, b_2) \in \mathbf{L}_1$ if and only if $a_1 \sim a_2$.*

PROOF. – Suppose $l[m, n]_1$ is the given line. Then $m \sim 0$, by 2.1. Also $a_i = T_0(b_i, m, n)$; $i = 1, 2$. Since $\sim$ is a congruence, $T_0(b_i, m, n) \sim T_0(b_i, 0, n)$; $i = 1, 2$. Then 2.7 (*d*) (*i*) implies $a_1 \sim a_2$. Conversely, if $a_1 \sim a_2$ and $l[m, n]_2$ is the given line, then $T(a_1, m, n) \sim T(a_2, m, n)$ and so $b_1 \sim b_2$. Hence $P(a_1, b_1) \sim P(a_2, b_2)$; a contradiction.

2.14. THEOREM. – *$A(R)$ has the following properties.*

(1) *$\langle k, + \rangle$ is a loop with neutral element 0.*

(2) *$a \cdot 0 = 0 \cdot a = 0$ and $a \cdot 1 = 1 = 1 \cdot a = a$, for all $a \in k$.*

(3) *If $a \nsim 0$ and $b \in k$, then there exist unique x and y such that $xa = b$ and $ay = b$.*

(4) *If $x \nsim 0$ and $xy = xz$ or $yx = zx$, then $y = z$.*

(5) *If $x \nsim 0$ and $xy \sim xz$ or $yx \sim zx$, then $y \sim z$.*

(6) *$\bar{0} = D_+ = D_- = D_0$ and D_0 is an ideal of $A(R)$; cf.* [5], Theorems 16,17.

(7) *If $ab \in D_0$, then $a \in D_0$ or $b \in D_0$; cf.* [5], Theorem 15.

(8) *$\bar{0} = \eta_+ = \eta_-$, where $\eta_+(\eta_-)$ is the set of non-right (non-left) multiplicative inverses of $A(R)$.*

PROOF. – (1) The unique solutions of $x + a = b$ and $a + y = b$ are

$$x = k \wedge L(s, h), \qquad \text{where } S = L((0, a), k) \wedge L(b, g),$$

and

$$y = k \wedge L\big(L((a, b), k(\wedge h, g)\big).$$

Clearly, $a + 0 = 0 + a = a$, for all $a \in k$, by $(\mathcal{C}_0)$ and $(\mathcal{C}_1)$.

Assertion (2) follows from $(\mathcal{C}_0)$ and $(\mathcal{C}_1)$, and (3) and (4) are consequences of (β_{11}), (β_{12}), and 2.12. Assertion (5) follows from the application of (4) and 2.12 to the Hall ternary ring $\bar{B}$.

(6) From 2.11, we have $\bar{0} = D_+$. Now we show that $\bar{0} = D_-$. Let $a \sim 0$. Then $(1, 0) \sim (1, a)$ and so $0(1, a) \sim g$. Hence there exists $M \neq 0$, with $MI0(1, a)$, g. Thus $M = (c, 0)$ for some $c \neq 0$, and $c \cdot a = 0$. This proves that $\bar{0} \subseteq D_-$. Conversely, if $a \in D_-$, then there exists $b \neq 0$ such that $b \cdot a = 0$. If $a \notin \bar{0}$, then (2) and (4) imply $b = 0$; a contradiction. Finally, by 2.12.1, $\bar{0}$ is an ideal of $A(R)$.

(7) Let $ab \in D_0$. By (6), $D_0 = \bar{0}$. Since $a \cdot 0 = 0$, we have $a \cdot 0 \sim a \cdot b$. If $a \notin \bar{0}$, then (5) implies $b \in \bar{0}$.

(8) By 2.14 (3), $\eta_+ \subseteq \bar{0}$. Conversely, if $x \sim 0$ and $xy = 1$, then, from the conguence $\sim$, we obtain $xy \sim 0$; i.e., $1 \sim 0$; a contradiction.

2.14.1. – COROLLARY. – $A(R)/D_0 \cong A(R)$.

2.14.2. COROLLARY. – *$\mathcal{H}$ is a proper* A.H. *plane if and only if the disjointness relation is strictly finer than the parallel relation.*

PROOF. – Let $\mathcal{H}$ be proper. Then there exists $m \in R$ such that $m \sim 0$ and $m \neq 0$. By 2.5, $[0, 1]_2 \nparallel [m, 0]_2$. We shall show that $[m, 0]_2 \wedge [0, 1]_2 = \emptyset$. If (x, y) lies on both lines, then $xm = 1$ and so $0 \sim 1$; a contradiction. Conversely, suppose there exist f and l such that $f \nparallel l$ but $f \wedge l = \emptyset$. From 2.5 and 2.6, $f = [m, n]$ and $l = [u, v]$ are of the same kind, $m \sim u$, and $m \neq u$. Hence $\mathcal{H}$ is proper.

2.14.3. COROLLARY. – *If $a \nsim 0$, then the unique solution of $ax = 1$ is*

$$x = k \wedge L(XE \wedge 0(a, 1), g) .$$

2.15. COMMENT. – In [10], Klingenberg introduced coordinates into an associated A.H. plane of a projective plane ε_p in the following manner. A set $\{P_0, g, g'\}$ is a basis of $\mathcal{H}$ if $g \wedge g' = P_0$. Geometric notions of addition $(+)$ and multiplication $(\cdot)$ were then introduced into g, forming the algebraic structure $R \equiv \langle g, +, \cdot \rangle$. If $\mathcal{H}$ satisfies the minor Desarguesian property D_1, then $\langle g, + \rangle$ is an abelian group and $a \cdot (b + c) = a \cdot b + a \cdot c$. Symmetrically, he obtained a second structure $R' \equiv \langle g', \oplus, \odot \rangle$ on the points of g'. Using R, he introduced coordinates (x, y) for an arbitrary point P, and $[b, c]$ for a line l such that $\Pi_l \nsim \Pi_g$. If $\mathcal{H}$ has the property D_1, then (x, y) lies on $[b, c]$ provided that $by + x = c$. Symmetrically, we can use R' to coordinatize points and lines m where $\Pi_m \nsim \Pi_g$, such that $(x', y') I [b', c']$ when $b' \odot x' \oplus y' = c'$ where $x' = g' \wedge L(x, (1, 0)(0, 1))$. If D_1 holds, then the mapping $x \to x'$ from R to R' is an additive abelian group isomorphism. Using this fact, it is possible to prove that the translations of $\mathcal{H}$ are the maps $(x, y) \to (x + a, y + b)$. Hence we can show that $\mathcal{H}$ satisfies D_1 if and only if the translations form a transitive group. At this point in the construction, one is coordinatizing the plane by two algebraic structures R and R'. If the plane is also Pappian, then R and R' are commutative rings and the mapping $x \to x'$ is a ring isomorphism. One can then coordinatize the entire plane by the elements of R only.

In our approach, using biternary rings, we have coordinatized the plane with the single structure B. We shall now study the effects of the minor Desarguesian configurational conditions on B and prove that for Desarguesian planes, B and $A(R)$ coincide; i.e., $T(a, b, c) = T_0(a, b, c) = a\cdot b + c$. In a forthcoming paper, the author has verified this result for alternative projective Hjelmslev planes; i.e., projective H-planes which satisfy a projective (instead of an affine) minor Desarguesian configurational condition.

3. – Translation planes.

In this section, we shall use algebraic properties of biternary rings to derive two equivalent conditions for an A.H. plane to be a translation plane. We generalize the fact that an ordinary plane is a translation plane if and only if each ternary ring is linear and forms a quasifield under the associated addition and multiplication.

3.1. DEFINITION (cf. [10], D 12). – *A minor Desarguesian configuration* C_1 is a set of sic points, P_i, Q_i; $i = 1, 2, 3$, and eight lines, p_i, g_i; $i = 1, 2, 3$; q_1, q_2, satisfying the following conditions; g_1, g_2, and g_3 belong to a pencil p_i, $Q_i I g_i$; $i = 1$ $2, 3$; P_i, $P_j I p_k$, where (i, j, k) is a permutation of $(1, 2, 3)$; Q_1, $Q_3 I q_2$; Q_2, $Q_3 I q_1$; $p_1 \| q_1$, $p_2 \| q_2$; p_1, $p_2 \not\sim v_3$; and $p_1 \not\sim P_2$.

From [10], S 4.3, we have the following result.

3.2. LEMMA. – *Let* C_1 *be a minor Desarguesian configuration. Then*

(*a*) $p_1 \not\sim v_2$; $p_2 \not\sim v_1$; $q_1 \not\sim v_2$, v_3; $q_2 \not\sim v_1$, g_3.

(*b*) *If* $v_1 \not\sim v_2$, *then* $Q_1 \not\sim Q_2$. *If* $v_1 \sim v_2$, *then* $p_3 \sim v_1$, v_2; $p_3 \not\sim p_1$, p_2; $P_3 \not\sim P_1$, P_2; $v_3 \not\sim g_1$, g_2; $Q_3 \not\sim Q_1$, Q_2; $p_1 \not\sim p_2$; $q_1 \not\sim q_2$; *and* $Q_1 \not\sim Q_2$.

We observe that 3.2 (*b*) ensures that there exists a unique line q_3 through Q_1 and Q_2.

3.3. LEMMA. – *Let* C_1 *be a minor Desarguesian configuration. Then* $p_3 \not\sim g_3$.

PROOF. – First suppose that $g_1 \not\sim g_2$. If $p_3 \sim g_3$, then (A7) implies that $p_3 \sim g_1$, g_2, and so $g_1 \sim g_2$; a contradiction. Next, assume that $g_1 \sim g_2$. From 3.2, $p_3 \sim g_1$, g_2 and $g_3 \not\sim g_1$, g_2. Hence $p_3 \not\sim g_3$.

3.4. DEFINITION. – $\mathcal{H}$ has the property D_1 if for each minor Desarguesian configuration C_1, $p_3 \| q_3$.

3.5. THEOREM. – *The following are equivalent.*

(1) *Every ternary ring of* $\mathcal{H}$ *is linear.*

(2) $\mathcal{H}$ *has the property* D_1.

(3) $\mathcal{H}$ *is a translation plane.*

PROOF. – Assume that every ternary ring of $\mathcal{H}$ is linear. Let C_1 be a minor Desarguesian configuration. From 3.3, $g_3 \nsim p_3$, and so $g_3 \nsim L(P_3, p_3)$. Let $g = L(P_3, p_3)$. Choose k such that $P_3 I k$ and $k \nsim g_3$, g, using Satz 2.3 of [14]. Thus g_3, g, and k may be regarded as the essential lines of a coordinate system $\{O, X, Y\}$, where $P_3 = 0$, $g_3 = OY = h$, and $k = OE$. Let T be the associated ternary operator of $\{O, X, Y\}$. Then we see that $Q_3 = (0, n)$, $p_2 = [m_1, 0]_2$, $p_1 = [m_2, 0]_2$, for some n, m_1, $m_2 \in k$; and $q_2 = [m_3, n]_2$ and $q_1 = [m_2, n]_2$. It follows that $P_1 = (x, xm_1)$, $Q_1 = (x, T(x, m_1, n))$, $P_2 = (a, am_2)$, and $Q_2 = (a, T(a, m_2, n))$.

Now $p_3 \| g$ implies $xm_1 = am_2$. From the linearity of T, we obtain

$$T(x, m_1, n) = xm_1 + n = am_2 + n = T(a, m_2, n)\,.$$

Hence Q_1 and Q_2 have the same y-coordinates, and so $Q_1 Q_2 \| g$.

It follows from [10] and [11] that (2) implies (3); cf. 2.16. Finally, we show that (3) implies (1).

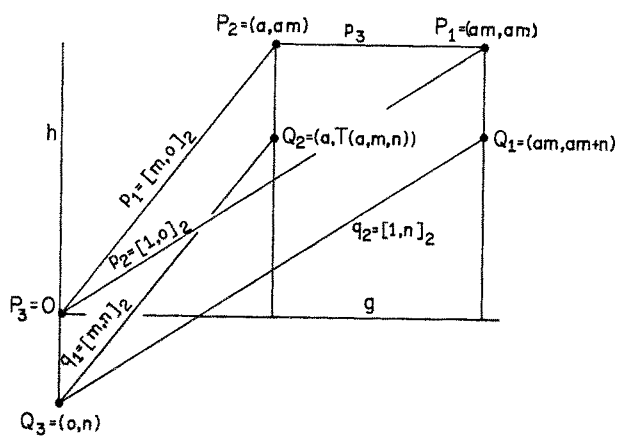

FIGURE 1.

Choose a ternary ring $\langle OE, T, 0, 1\rangle$ associated with $\{O, X, Y\}$. Put $P_3 = 0$, $Q_3 = (0, n)$, $P_2 = (a, am)$, $P_1 = (am, am)$, $Q_1 = (am, am + n)$, $Q_2 = (a, T(a, m, n))$. Consider $p_1 = [m, 0]_2$, $q_1 = [m, n]_2$, $p_2 = [1, 0]_2$, $q_2 = [1, n]_2$, and $p_3 = L(P_2, g)$. In view of $(\mathcal{C}_0)$ and $(\mathcal{C}_1)$, we may assume that $m \neq 0, 1$ and $n \neq 0$.

Let $\tau = \tau_{P_3 Q_3}$. From 2.2, $p_1, p_2 \nsim h$. Hence $P_2^\tau = Q_2$ and $P_1^\tau = Q_1$. Now

$$T(a, m, n) = am + n$$

if and only if $Q_1 I L(Q_2, g)$. But $P_1^\tau I L(P_2^\tau, P_3)$, and so our result follows.

Our objective now is to characterize translation planes in terms of their ternary rings.

Before we begin this task, we require the following generalization of an exercise found in [2] p. 74.

3.6. DEFINITION. – A configuration C_3 consisting of eight lines l, m, p_{12}, p_{23}, p_{21}, p_{32}, p_{11}, p_{33}, and six points P_i, Q_i; $i=1,2,3$, is called a *parallel Pappus configuration* if the following conditions are satisfied:

(i) $l \| m$ and $l \nsim m$.

(ii) $P_i I l$ and $Q_i I m$; $i=1,2,3$.

(iii) $P_i, Q_j I p_{ij}$; $i, j = 1, 2, 3$.

(iv) $p_{12} \| p_{23}$ and $p_{21} \| p_{32}$:

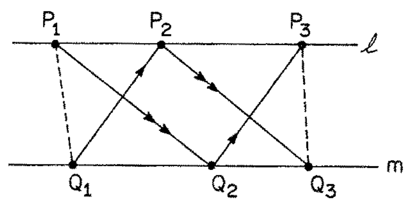

FIGURE 2.

3.7. THEOREM. – *If* $\mathcal{H}$ *is a translation plane, then for any parallel Pappus configuration* C_3, *we have* $p_{11} \| p_{33}$:

PROOF. – Let $\tau_1 = \tau_{P_1Q_2}$ and $\tau_2 = \tau_{Q_2P_3}$. By (i) of 3.6 and Satz 2.5 of [14], we obtain $P \nsim Q$ for each PIl and QIm. Hence $P_i \nsim Q_j$, $p_{1\cdot} \nsim l$ and $p_{33} \nsim m$. Hence $p_2^{\tau_1} = Q_3$ and $Q_1^{\tau_1} = P_2$. Thus $Q_1^{\tau_1\tau_2} = Q_3$. Since the translations form an abelian group, we have $P_2^{\tau_1} = Q_3 = Q_1^{\tau_2\tau_1}$ and $P_1^{\tau_2\tau_1} = P_1^{\tau_1\tau_2} = P_3$; cf. [14], Satz 3.7. Thus $P_1^{\tau_2\tau_1} \cdot IL(Q_1^{\tau_2\tau_1}, p_{11})$, or $p_{33} \| p_{11}$.

3.8. THEOREM. – *If* $\mathcal{H}$ *is a translation plane, then every biternary ring* $B \equiv \equiv \langle k, T, T_0, 0, 1\rangle$ *of* $\mathcal{H}$ *has the following properties,*

(*a*) $\langle k, T, 0, 1\rangle$ *and* $\langle k, T_0, 0, 1\rangle$ *are linear.*

(*b*) $\langle k, +\rangle$ *is an abelian group.*

(*c*) $(a+b)\cdot c = a\cdot c + b\cdot c$ *and* $(a \oplus b) \odot c = a \odot c \oplus b \oplus c$ *for all* $a, b, c \in k$.

(*d*) $T(1, a, b) = T_0(1, a, b)$.

Moreover, if a single biternary ring of an A.H. *plane* $\mathcal{H}$ *has the properties* (*a*), (*b*), (*c*), (*d*), *then* $\mathcal{H}$ *is a translation plane, and the translation group* $\mathfrak{Z}$ *consists of the mappings* $(P(x, y))^\tau = P(x+a, y+b)$, *for* $a, b \in k$. *Also,* $\langle k, +\rangle$ *is isomorphic to* $\mathfrak{Z}_{\Pi_k}$, where $\mathfrak{Z}_{\Pi_k}$ is group of translations with the direction Π_k; cf. [14], p. 273.

We prove the first part of our theorem in the following sequence of Lemmas in which $\mathcal{H}$ is assumed to be a translation plane. τ_{PQ} or $\tau(P, Q)$ will denote the translation mapping P to Q; in particular, for a given coordinate system of $\mathcal{H}$, τ_{0a} is the translation mapping 0 to a.

3.9. LEMMA. – $\tau_{0a}\tau_{0b} = \tau_{0,a+b}$.

PROOF. – Since $\Pi_k \nsim \Pi_g, \Pi_h$, we readily verify that $\tau_{0a} = \tau_{(0b),(a,a+b)} = \tau_{b,a+b}$. Then $\tau_{0,a+b} = \tau_{b,a+b}\tau_{0b} = \tau_{0a}\tau_{0b}$.

3.10 LEMMA. – *If* $b \in \nsim_0$, *then* $b+1,\ b-1 \notin D_0$.

PROOF. – Suppose $b, b+1 \in D_0$. Then τ_{0b}^{-1} nad $\tau_{0,b+1}$ are neighbour translations by 2.14. Also, $\tau_{0b}\tau_{01} = \tau_{0,b+1}$ by 3.9. Thus $\tau_{01} = \tau_{0b}^{-1}\tau_{0,b+1}$ is a neighbour translation, and so $0 \sim 1$; a contradiction. Similarly, $b-1 \notin D_0$.

3.11. LEMMA. – $\langle k, +\rangle$ *is an abelian group and* $\langle k, +\rangle \cong \mathfrak{Z}_{\Pi_k}$.

PROOF. – Clearly, $\mathfrak{Z}_{\Pi_k} = \{\tau_{0a} | a \in k\}$. Now consider the injection $f\colon \langle k, +\rangle \to \mathfrak{Z}_{\Pi_k}$ defined by $f(a) = \tau_{0a}$. Then 3.9 implies that $f(a+b) = f(a)f(b)$ and $f(0) = 1$. Since $\mathfrak{Z}_{\Pi_k}$ is an abelian group, it follows that $\langle k, +\rangle$ also is an abelian group and f is an isomorphism.

3.12. (LEMMA. – *For each ternary ring R, $A(R)$ is right distributive.*

PROOF. – Suppose at first that $a \nsim 0$. Then by 2.3, $(0, b) \nsim (a, a+b)$. By 22.1 and 3.11, $b \nsim a+b$. Hence $(0, bc) \nsim (a, ac+bc)$ and $(b, bc) \nsim (a+b, ac+bc)$. Thus

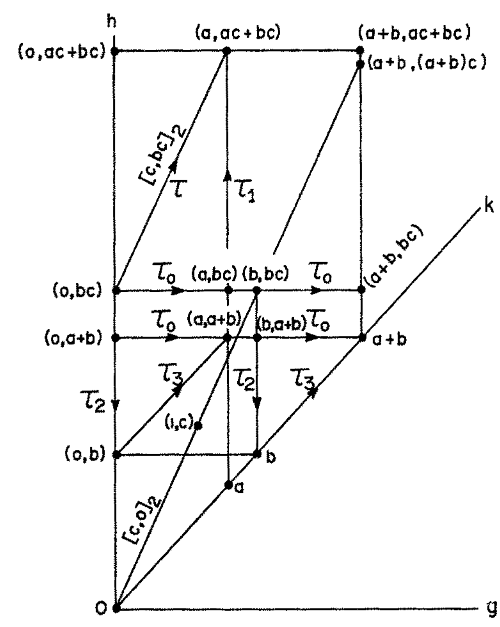

FIGURE 3.

to prove our claim, we need only to show that $(0, bc)(a, ac+bc)\|(b, bc)(a+b, ac+bc)$ in Figure 3. Then, since $(0, bc)(a, ac+bc)$ is the line $[c, bc]_2$, we obtain $(0, 0)(b, bc) = (b, bc)(a+b, ac+bc)$ and our result follows.

Put $\tau_0 = \tau((0, bc), (a, bc))$; $\tau_1 = \tau((a, bc), (a, ac+bc))$; $\tau_2 = \tau((0, a+b), (0, b))$; $\tau_3 = \tau((0, b), (a, a+b))$. Then it follows that $(0, a+b)^{\tau_0} = (a, a+b)$; $(b, a+b)^{\tau_2} = b$; and $b^{\tau_3} = a+b$. Clearly, $\tau_0 = \tau_2\tau_3$, and thus $(b, a+b)^{\tau_0} = a+b$. Consequently, $(b, bc)^{\tau_0} = (a+b, bc)$. Also, $(a+b, bc)^{\tau_1} = (a+b, ac+bc)$. Let $\tau = \tau_1\tau_0$. Then $(0, bc)^{\tau} = (a, ac+bc)$; and $(b, bc)^{\tau} = (a+b, ac+bc)$. Since $(0, bc) \nsim (a, ac+bc)$, τ is not a neighbour translation. Hence by [11], S 12,

$$(0, bc)(a, ac+bc)\|(b, bc)(a+b, ac+bc).$$

Next, suppose that $a \in D_0$. By 3.10, $a+1 \notin D_0$. Thus

$$(a+b)c = \big((1+a)+(-1+b)\big)c = (1+a)c+(-1+b)c = c+ac-c+bc = ac+bc\,.$$

3.13. LEMMA. – *$T(1, a, b) = T_0(1, a, b)$, for all $a, b \in k$.*

PROOF. – By 3.5, each ternary ring of $\mathcal{H}$ is linear. Hence $\langle k, T_0, 0, 1\rangle$ is linear and $T_0(1, a, b) = 1 \odot a \oplus b = a \oplus b$. Thus it suffices to show that $a+b = a\oplus b$. From 2.8.2, we obtain $a \oplus b = k \wedge L(S, h)$, where $S = L((b, 0), k)\wedge L(a, g)$. From 3.11, we deduce that $L((b, 0), k)\wedge h = (0, -b)$ and $L((b, 0), k) = [1, -b]_2$. Now $S = (x, a)$ for some $x \in k$. But $SIL((b, 0), k)$ and hence $a = x-b$ and $x = a+b$. Thus

$$a \oplus b = k\wedge L((a+b, a), h) = a+b\,.$$

PROOF OF THEOREM 3.8. – By 3.5 and the preceding sequence of lemmas, every biternary ring of a translation plane has the properties (*a*), (*b*), (*c*) and (*d*).

Conversely, assume that an A.H. plane $\mathcal{H}$ satisfies (*a*), (*b*), (*c*), and (*d*). We shall show that the translations are exactly the maps of the form

$$(x, y)^{\tau} = (x+a, y+b)\,, \qquad \text{for } a, b \in k.$$

First we prove such a map is a translation. Let $(x, y), (u, v) I[m, n]_2$. Then

$$L((u+a, v+b), [m, n]_2) = [m, v+b-(u+a)m]_2\,.$$

It is easy to see that $(x, y)^{\tau} IL((u, v)^{\tau}, [m, n]_2)$, by using (*c*). Next, suppose that $(x, y), (u, v) I[m, n)_1$. Since T_0 is linear, (*c*) and (*d*) yield

$$L((u+a, v+b), [m, n]_1) = [m, u+a-(v+b)\odot m]_1\,.$$

From (c) and (d), $(a+b)\odot c = a\odot c + b\odot c$. It follows that $(x, y)^\tau IL((u, v)^\tau, [m, n]_1)$. It is clear that $\tau = 1$ or τ has no fixed points. Again, using (c) and (d), we can show that every line parallel to a trace of τ is also a trace of τ and so τ is a translation. It is easily verified that each translation has the above form, and so the set of translations forms a transitive group.

4. – Desarguesian A.H. planes.

In this section, we generalize the result that an ordinary affine plane is Desarguesian if and only if $A(R)$ is a division ring. In [10], Klingenberg showed that if an A.H. translation plane is Pappian, then the coordinate ring which he introduced is a commutative A.H. ring. In that paper, no notion of a Desarguesian plane was given. This was introduced in [11]. However, Lorimer and Lane have shown in [12] that the definition in [11] of a Desarguesian plane is too strong and they have introduced a weaker condition. Several equivalent definitions are presented there, and one of these, Axiom (A10)(P) is the characterization of Desarguesian A.H. planes which we shall use in the ensuing discussion.

The following axioms refer to a given point P.

(A10) (P) If P, Q, and R are collinear, and $P \not\sim Q$, then there exists a dilatation $\sigma = \sigma(PQP)$ which maps P into P and Q into R.

(A10) $(P, \not\sim)$ If P, Q, and R are collinear, $P \not\sim Q$ and $P \not\sim R$, then there exists a dilatation $\sigma = \sigma(PQR)$ which maps P into P and Q into R.

4.1. REMARK. – *Let* $\sigma = \sigma(PQR)$. *If* S *is any point such that* S, PIf; S, QIj, *and* $f \not\sim j$, *then* $S^\sigma = f \wedge L(R, j)$.

4.2. LEMMA. – *If* $\mathcal{H}$ *is a translation plane satisfying axiom* (A10)$(P, \not\sim)$, *then for each ternary ring* R, $A(R)$ *is left distributive.*

PROOF. – First we show that $a(b+c(= ab + ac$ when $a \notin D_0$. Consider Figure 4. Since $a \not\sim 0$, (A10) $(0, \not\sim)$ implies that there exists $\sigma = \sigma(0, (1, c), (a, ac))$. Since $XE \not\sim [b+c, 0]_2$ and $h \not\sim [0, c]_2$, we obtain $(1, b+c)^\sigma = (a, a(b+c))$ and $(0, c)^\sigma = (0, ac)$. Since $L((0, ac), [b, c]_2) = [b, ac]_2$, we have $(a, a(b+c))I[b, ac]_2$. Thus $a(b+c) = ab + ac$.

Next, suppose $a \in D_0$. Thus $a - 1 \notin D_0$ by 3.10, and so by 3.8 (c),

$$\begin{aligned} a(b+c) &= (a-1+1)(b+c) = (a-1)(b+c) + b + c \\ &= (a-1)b + (a-1)c + b + c = ab - b + ac - c + b - c \\ &= ab + ac. \end{aligned}$$

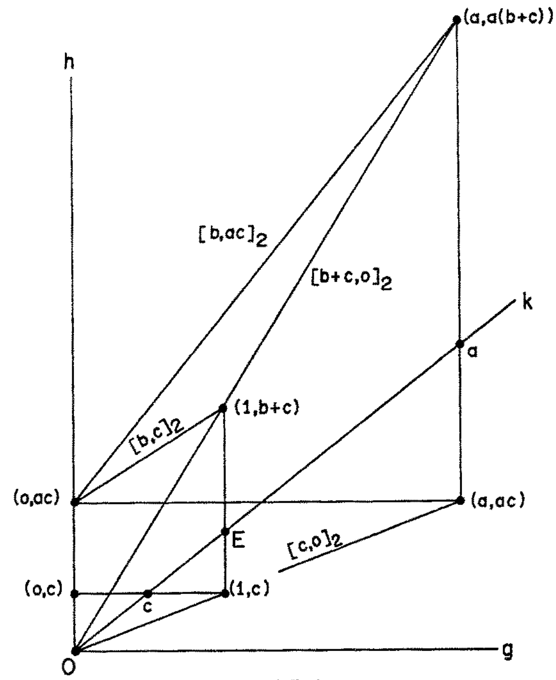

FIGURE 4.

4.3. LEMMA. – *If* $\mathcal{H}$ *is a* $\mathfrak{Z}$*-plane with* (A10) $(P, \sim)$, *then in each* $A(R)$ *multiplication is associative.*

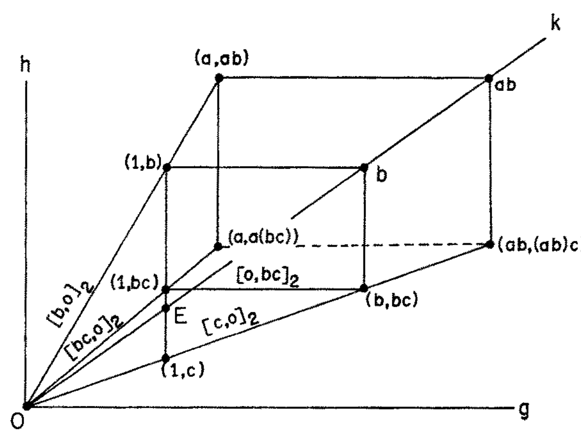

FIGURE 5.

PROOF. – It is enough to show that $a(bc) = a(bc)c$ when $b \notin D_0$. For if $b \in D_0$, then $d = b - 1 \notin D_0$ and 3.8 and 4.2 imply

$$(ab)c = \big(a(d+1)\big)c = (ad + a)c = (ad)c + ac$$
$$= a(dc) + ac = a(dc + c) = a\big((d+1)c\big) = a(bc)\,.$$

Assume $b \notin D_0$.

Case 1: $a \notin D_0$. Consider Figure 5. Now $\sigma = \sigma(0, b, ab)$ exists by 2.14 (7), since $a, b \notin D_0$. Because $L(b, h) \sim [c, 0]_2$ and $[0, b]_2 \sim [b, 0]_2$, 4.1 yields $(b, bc)^\sigma = (ab, (ab)c)$ and $(1, b)^\sigma = (a, ab)$. Since $\sigma = \sigma(0, (1, b), (a, ab))$ and $XE \sim [bc, 0]_2$, we obtain $(1, bc)^\sigma = (a, a(bc))$. Hence $(a, a(bc))\, IL((ab, (ab)c), [0, bc]_2$. Thus $(a, a(bc))$ and $(ab, (ab)c)$ have the same y-coordinate and so $a(bc) = (ab)c$.

Case 2: $a \in D_0$. Then $a - 1 \notin D_0$. Hence

$$\begin{aligned}(ab)c &= ((a-1+1)b)c = ((a-1)b + b)c\\ &= ((a-1)b)c + bc = (a-1)(bc) + bc\\ &= ((a-1)+1)bc = a(bc)\,.\end{aligned}$$

4.4. LEMMA. – *Let $\mathcal{K}$ be a translation plane and let B be a biternary ring of $\mathcal{K}$. If*

(i) *$A(R)$ and $A(R_0)$ are left distributive*

and

(ii) *$A(R)$ satisfies the right inversive property* (i.e., $ab = 1$ *implies* $(ca)b = c$ *for all* $c \in R$), *then* $T(a, b, c) = T_0(a, b, c)$.

PROOF. – Since $\mathcal{K}$ is a translation plane, T and T_0 are linear. In view of (d) of 3.8. it is sufficient to show that $T(a, b, 0) = T_0(a, b, 0)$. From 2.8.1 and 2.8.2, we have $T_0(a, b, 0) = k \wedge L(S, h)$, where $S = 0(b, 1) \wedge L(a, g)$.

First we assume that $b \notin D_0$. Then by 2.13, $0(b, 1)$ is a line of the second kind. Hence $0(b, 1) = [m, 0]_2$, where $b \cdot m = 1$. Thus $S = (x, a)$, where $a = x \cdot m$. By (ii), we have $(ab)m = a = xm$. Since $m \notin D_0$, 2.14 (4) implies that $ab = x$. Hence

$$(T_0(a, b, 0), a) = S = (ab, a) = (T(a, b, 0), a)\,.$$

Next, suppose $b \in D_0$. Let $c = b - 1$. Thus $c \notin D_0$. Then (i) yields

$$\begin{aligned}T_0(a, b, 0) &= a \odot b = a \odot (c + 1) = a \odot c + a\\ &= a \cdot c + a = a \cdot (c + 1) = a \cdot b\,.\end{aligned}$$

We may now state the main result of this section.

4.5. THEOREM. – The following are equivalent.

(a) $\mathcal{K}$ is Desarguesian.

(b) At least one biternary ring $B = \langle R, T, T_0, 0, 1\rangle$ has the properties: (i) T and T_0 are linear, (ii) $A(R)$ is an A.H. ring, (iii) $T = T_0$.

(c) $\mathcal{K}$ is isomorphic to $\mathcal{K}(H)$ for some A.H. ring H; cf. 3.10 of [12].

PROOF. – 2.14 and Lemmas 4.1 to 4.4 show (a) implies (b). (b) clearly implies (c), and the last implication follows from 3.4 and 3.11 of [12].

4.5.1. COROLLARY. – *For any ternary ring R of $\mathcal{H}$, $A(R) \cong H$, where H is the ring of trace-preserving endomorphisms of $\mathcal{H}$. Hence any two ternary rings of $\mathcal{H}$ are isomorphic.*

PROOF. – $\mathcal{H}$ may be regarded as the analytic model of an A.H. plane over $A(R)$. The result then follows from 3.10 of [12].

4.5.2. COROLLARY. – *If $\mathcal{H}$ is a translation plane, then* (A10) $(P, \sim)$ *is equivalent to* (A10) (P).

PROOF. – Let B be any biternary ring of $\mathcal{H}$. Then R is linear and by Lemmas 4.2 and 4.3, $A(R)$ is an associative local ring. Lemma 4.4 then implies that $T(a, b, c) = T_0(a, b, c)$. By 2.7 (e), $(\beta 7)$ and $(\beta 13)$, $A(R)$ is an A.H. ring. Hence $\mathcal{H}$ is Desarguesian.

5. – Pappian planes.

In [10], a Pappian configurational property was introduced into an associated A.H. plane of a projective Hjelmslev plane. The A.H. plane was called Pappian if it satisfied the minor Desarguesian configurational property and the Pappian configurational property. We shall say that $\mathcal{H}$ is *Pappian* if $\mathcal{H}$ is a translation plane and satisfies the Pappian configurational property. It was shown in [10] that the coordinate ring introduced there is a commutative A.H. ring if and only if the plane is Pappian. In view of 4.5, we therefore have the following result.

5.1. THEOREM. – *Let $\mathcal{H}$ be an A.H. plane.*

(1) *If $\mathcal{H}$ is Pappian, then it is Desarguesian.*

(2) *If $\mathcal{H}$ is Desarguesian, then $\mathcal{H}$ is Pappian if and only if at least one ternary ring of $\mathcal{H}$ is commutative.*

5.2. – The special Pappus configuration.

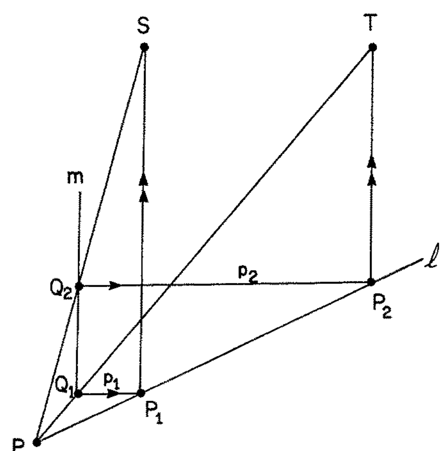

FIGURE 6.

A *special Pappus configuration* is a set of four lines l, m, p_1, p_2, and five points P, P_1, P_2, Q_1, Q_2 satisfying the conditions:

(i) $\Pi_l \nsim \Pi_m$.

(ii) $p_1 \nsim m$; $P \nsim X$ for all XIm.

(iii) $P, P_1, P_2 Il$; $Q_1, Q_2 Im$.

(iv) $p_1 \| p_2$.

Because of (ii) and (A7), we can define the additional points $S = PQ_2 \wedge L(P_1, m)$ and $T = PQ_1 \wedge L(P_2, m)$.

We say that $\mathcal{H}$ has the *special Pappus property* P^* if for each special Pappus configuration, $TIL(S, p_1)$.

5.3. THEOREM. – *$\mathcal{H}$ has the special Pappus property if and only if multiplication is commutative in each ternary ring R of $\mathcal{H}$.*

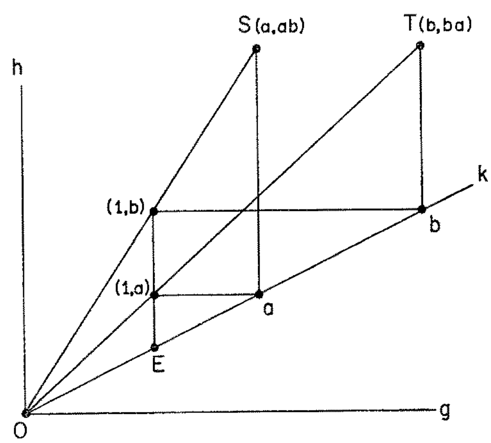

FIGURE 7.

PROOF. – Suppose that $\mathcal{H}$ has property P^*. Let R be the ternary ring of $\{O, X, Y\}$. We shall show that $ab = ba$. Now the lines $L(E, h)$, k, $L(a, g)$, $L(b, g)$, and the points 0, a, b, $(1, a)$ and $(1, b)$ form a special Pappus configuration, with $S = (a, ab)$ and $T = (b, ba)$. Then property P^* implies that $(a, ab) IL((b, ba), g)$ and so $ab = ba$.

Conversely, let l, m, p_1, p_2 and P, P_1, P_2, Q_1, Q_2 form a special Pappus configuration and assume that multiplication in each ternary ring is commutative. Then conditions (i) and (ii) imply that l, $L(P, m)$, and $L(P, p_1)$ are three mutually non-neighbouring lines through P, and hence determine a triangle in which $g = L(P, p_1)$, $h = L(P, m)$, $k = l$, $P = 0$ and $E = l \wedge m$. Let R be the associated ternary ring. Then $S = (a, ab)$ and $T = (b, ba)$. Since $ab = ba$, we conclude that $TIL(S, g)$.

5.3.1. COROLLARY. – *If $\mathcal{H}$ is Desarguesian, then the special Pappus property is equivalent to the Pappus property.*

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