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COORDINATED PAIR SYSTEMS; PART II: SPARSE STRUCTURE OF DYCK WORDS AND OGDEN'S LEMMA (*)

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Abstract. – *In this paper we continue the investigation of the structure of computations in cp systems which was initiated in Part I of this paper. Here again our main combinatorial tool is the structure of Dyck words (and the Exchange Theorem). However in this paper we investigate the "sparse structure" of Dyck words (i. e., the structure of sparse subworks of Dyck words) and use our results about this sparse structure to derive Ogden's pumping lemma for context-free languages.*

Résumé. – *Dans cet article, nous poursuivons l'étude de la structure des calculs dans les systèmes cp, étude commencée dans la première partie de cet article. A nouveau, notre outil combinatoire principal est la structure des mots de Dyck (et le théorème d'échange). Ici, nous étudions la « structure dispersée » des mots de Dyck (i.e. la structure de sous-mots fractionnés de mots de Dyck) et nous employons nos résultats sur cette structure pour obtenir le lemme d'itération d'Ogden pour les langages algébriques.*

INTRODUCTION

The aim of this paper is to present the results of an investigation which continues the line of research initiated in [EHR1], [EHR2] and [EHR3]. We continue the investigation of the structure of computations in cp systems and again (as in Part I of this paper) our main combinatorial tools are results on the combinatorial structure of *Dyck words*. Now however we are interested in the structure of *sparse subwords* of Dyck words. We obtain a number of results concerning this "sparse structure" of Dyck words (Section 1) and then

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combine these results with the Exchange Theorem (given in [EHR2]) to prove Ogden's pumping lemma for context-free (i. e., cp) languages (*see*, e. g., [O], [H]).

0. PRELIMINARIES

We assume the reader to be familiar with Part I of this paper [EHR3]; we use without recalling terminology, notation and results from there. If we refer to a result (or a definition) from Part I, then we precede its reference number by I-hence, e. g., Lemma 1.2.1 refers to Lemma 2.1 from Part I.

In considerations of this paper we will often embed a given word as a sparse subword in another word. Hence we need the following technical notion.

DEFINITION 0.1: Let $U=(i_1, i_2, \ldots, i_n)$ be a support in a word w. The *U-embedding*, denoted φ_U, is the bijection from $\{1, 2, \ldots, n\}$ onto U defined by $\varphi_U(t)=i_t$ for $1 \leqq t \leqq n$. ■

If $U=(i_1, i_2, \ldots, i_n)$ is a support in a word w and $V=(j_1, j_2, \ldots, j_m) \subseteqq \{1, 2, \ldots, n\}$, then following the usual convention we use $\varphi_U(V)$ to denote $(\varphi_U(j_1), \ldots, \varphi_U(j_m))$. Moreover, if $\kappa=V_1, \ldots, V_m$ is a sequence of subsets of $\{1, \ldots, n\}$, then we use $\varphi_U(\kappa)$ to denote the sequence $\varphi_U(V_1), \ldots, \varphi_U(V_m)$.

1. SPARSE SUBWORDS IN DYCK WORDS

In this section we investigate the structure of sparse subwords in Dyck words. We start by introducing a number of basic notions that formalize such a structure.

DEFINITION 1.1. Let U be a support in $w \in D_\Sigma$.

(1) U is *w-complete* if, for every w-nested pair (i, j), $i \in U$ if and only if $j \in U$.

(2) The *w-completion of* U, denoted by $cpl_w(U)$, is the set

$$cpl_w(U)=\bigcup \{p \mid p \text{ is a } w\text{-nested pair with } p \cap U \neq \varnothing\}. \quad ■$$

It should be obvious that $cpl_w(U)$ is w-complete for any support U in w.

The following is an easy observation concerning D_Σ.

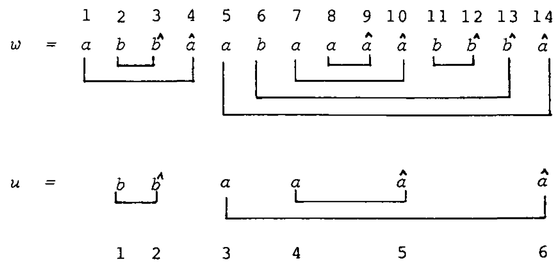
LEMMA 1.1: *Let $w \in D_\Sigma$ and let $u = w(U)$ for some w-complete support U in w. Then $u \in D_\Sigma$. Moreover, if p is a u-nested pair, then $\varphi_U(p)$ is w-nested. If κ is a w-chain, then $\varphi_U(\kappa)$ is a u-chain.* ■

The following example shows that even balanced pairs are not preserved under φ_U (where U is a w-complete support) and consequently that φ_U does not preserve cochains.

Example 1.1: Let $w = ab\hat{b}\hat{a}abaa\hat{a}\hat{a}b\hat{b}\hat{b}\hat{a}$ and let $U = \{2, 3, 5, 7, 10, 14\}$.

Then U is a w-complete support.

w and $u = w(U)$ have the following nested structures.



$\kappa_1 = (3,6), (4,5)$ is a u-chain and $\varphi_U(\kappa_1) = (5,14), (7,10)$ is a w-chain.

On the other hand, $p = (1,6)$ is u-balanced, but the corresponding pair $\varphi_U(p) = (2,14)$ is not w-balanced.

Consequently $\kappa_2 = (1,2), (3,6)$ is a u-cochain, while $\varphi_U(\kappa_2) = (2,3), (5,14)$ is not a w-cochain. ■

This example motivates the following notion.

DEFINITION 1.2: Let U be a w-complete support in $w \in D_\Sigma$ and let $u = w(U)$. U is *w-proper* if $\varphi_U(\kappa)$ is a w-cochain for every u-cochain κ. ■

The following result is obvious.

LEMMA 1.2: *Let $w \in D_\Sigma$ and let U be a w-complete segment in w. Then U is w-proper.* ■

Example 1.1 (continued):

$U = \{2, 3, 5, 7, 10, 14\}$ is not w-proper.

Let $V = \{1, 2, 3, 4, 5, 7, 10, 14\}$. Then V is w-proper. ■

The next result shows that, for a w-proper support U, φ_U preserves balanced pairs.

LEMMA 1.3: *Let U be a w-proper support in $w \in D_\Sigma$ and let $u = w(U)$. If p is a u-balanced pair, then $\varphi_U(p)$ is w-balanced.*

Proof: Let $p = (i, j)$ be a u-balanced pair.

If p is u-nested, then $\varphi_U(p)$ is w-nested and so our lemma clearly holds.

If p is not u-nested, then there exist occurrences j_1 and i_1 such that $\kappa = (i, j_1), (i_1, j)$ is a u-cochain. Since U is w-proper, $\varphi_U(\kappa)$ is a w-cochain. Consequently $(\varphi_U(i), \varphi_U(j)) = \varphi_U(p)$ is a w-balanced pair. ■

In order to extend arbitrary supports to proper ones we need the following notions.

DEFINITION 1.3: Let U be a support in $w \in D_\Sigma$ and let $p = (i, j)$, $p_1 = (i_1, j_1)$ be two w-nested pairs.

(1) p is *U-relevant* if $\{i, i+1, \ldots, j\} \cap U \neq \varnothing$.

(2) p and p_1 are *U-equivalent*, denoted by $p \overset{U}{\equiv} p_1$, if

$$\{i, i+1, \ldots, j\} \cap U = \{i_1, i_1+1, \ldots, j_1\} \cap U. \quad ■$$

We use $[p]_U$ to denote the equivalence class of p with respect to $\overset{U}{\equiv}$; that is the set of all w-nested pairs U-equivalent with p.

If p is U-relevant, then clearly $[p]_U$ consists of U-relevants pairs; we say then that $[p]_U$ is *U-relevant.* The set of w-nested pairs that are not U-relevant forms an equivalence class of $\overset{U}{\equiv}$.

Example 1.1 (continued):

(6, 13) is U-relevant because $\{6, 7, \ldots, 13\} \cap U = \{7, 10\}$.

(7, 10) and (6, 13) are U-equivalent w-nested pairs because also $\{7, 8, \ldots, 10\} \cap U = \{7, 10\}$.

(8, 9) and (11, 12) are not U-relevant and consequently $(8, 9) \overset{U}{\equiv} (11, 12)$.

$[(7, 10)]_U = \{(6, 13), (7, 10)\}$ and

$[(8, 9)]_U = \{(8, 9), (11, 12)\}$. ■

LEMMA 1.4: *Let U be a support in $w \in D_\Sigma$. The elements of a U-relevant equivalence class of $\overset{U}{\equiv}$ form a w-chain.*

Proof: Let (i_1, j_1) and (i_2, j_2) be two different U-equivalent w-nested pairs. We may assume that $i_1 < i_2$.

Then either $i_1 < i_2 < j_2 < j_1$ or $i_1 < j_1 < i_2 < j_2$.

Assume that the latter (i. e., the "or" case) holds.

Then $\{i_1, i_1+1, \ldots, j_1\} \cap \{i_2, i_2+1, \ldots, j_2\} = \varnothing$. Consequently, since

$$(i_1, j_1) \overset{U}{\equiv} (i_2, j_2), \{i_1, i_1+1, \ldots, j_1\} \cap U = \{i_2, i_2+1, \ldots, j_2\} \cap U = \varnothing.$$

This implies that (i_1, j_1) and (i_2, j_2) are not U-relevant.

Thus, if we have two U-relevant w-nested pairs that are U-equivalent, then the former (i. e., the "either" case) holds: one pair lies within the other. It is now easy to see that a set of U-relevant U-equivalent w-nested pairs forms a w-chain. ■

Let $\kappa = p_1, \ldots, p_m$ be a w-chain. Then we write out $(\kappa) = p_1$; hence $out(\kappa)$ denotes the outer pair of κ. Moreover, somewhat informally, we will use the notation $out([p]_U)$ to denote $out(\kappa)$ where κ is the chain consisting of the elements of $[p]_U$ (*see* the above lemma).

DEFINITION 1.4: Let $w \in D_\Sigma$ and let U be a support in w. The *extension of* U (*in* w), denoted $ext_w(U)$ is a support in w defined by

$ext_w(U) = cpl_w(U) \cup \{ out([p]_U) \mid p$ is a U-relevant w-nested pair$\}$. ■

Note that the extension of a support U in $w \in D_\Sigma$ is w-complete. Furthermore, w-nested pairs contained in $ext_w(U)$ are U-relevant.

Example 1.1 (continued):

For $U = \{2, 3, 5, 7, 10, 14\}$ we have the following U-relevant classes: $\{(1, 4), (2, 3)\}$, $\{(5, 14)\}$ and $\{(6, 13), (7, 10)\}$.

Hence $ext_w(U) = U \cup (1,4) \cup (5, 14) \cup (6, 13) = \{1, 2, 3, 4, 5, 6, 7, 10, 13, 14\}$. ■

LEMMA 1.5: *Let U be a support in $w \in D_\Sigma$ and let $V = ext_w(U)$. Then*

(1) $U \subseteq V$,

(2) *if $p = (i, j)$ is a w-nested pair such that $p \subseteq V - U$, then $\{i+1, i+2, \ldots, j-1\} \cap U \neq \varnothing$, and*

(3) *if $\kappa = (i_1, j_1), (i_2, j_2)$ is a w-chain such that $(i_1, j_1) \cup (i_2, j_2) \subseteq V - U$, then either $\{i_1+1, i_1+2, \ldots, i_2-1\} \cap U \neq \varnothing$,*

or $\{j_2+1, j_2+2, \ldots, j_1-1\} \cap U \neq \varnothing$.

Proof: (1) Obvious.

(2) $V = ext_w(U)$ contains only U-relevant pairs. Hence $\{i, i+1, \ldots, j\} \cap U \neq \varnothing$, but $(i, j) \cap U = \varnothing$ and consequently $\{i+1, \ldots, j-1\} \cap U \neq \varnothing$.

(3) $(i_1, j_1) \cap U = \varnothing$ and $(i_2, j_2) \cap U = \varnothing$.

If moreover both

$$\{i_1+1, i_1+2, \ldots, i_2-1\} \cap U = \varnothing \quad \text{and} \quad \{j_2+1, \ldots, j_1-1\} \cap U = \varnothing,$$

then (i_1, j_1) and (i_2, j_2) are U-equivalent.

Since $(i_1, j_1) \cap U = \varnothing$, (i_1, j_1) was added in the extension so we have $(i_1, j_1) = out\ ([(i_1, j_1)]_U)$.

Analogously we conclude that $(i_2, j_2) = out\ ([(i_2, j_2)]_U)$.

But $[(i_1, j_1)]_U = [(i_2, j_2)]_U$ and consequently $(i_1, j_1) = (i_2, j_2)$; a contradiction. ■

We are now able to prove our main result concerning the extension of a support in a word.

LEMMA 1.6: *Let U be a support in $w \in D_\Sigma$. Then $V = ext_w(U)$ is w-proper.*

Proof. Let $v = w\ (V)$ and let κ be a v-cochain. In order to prove the lemma we have to show that $\varphi_V(\kappa)$ is a w-cochain.

Assume to the contrary that $\varphi_V(\kappa)$ is not a w-cochain.

This implies that there are two w-nested pairs (i_1, j_1) and (i_2, j_2) in $\varphi_V(\kappa)$ with $j_1 < i_2$ such that (i_1, j_2) is not w-balanced.

Hence there has to exist a w-nested pair (i_0, j_0) such that either

$$i_1 < j_1 < i_0 < i_2 < j_2 < j_0 \quad \text{or} \quad i_0 < i_1 < j_1 < j_0 < i_2 < j_2.$$

Since these two cases are symmetric we discuss only the former one (leaving the latter one to the reader).

Thus assume $i_1 < j_1 < i_0 < i_2 < j_2 < j_0$ for some w-nested pair (i_0, j_0).

The pair (i_0, j_0) is U-relevant because (i_2, j_2) is U-relevant. So $(i, j) = out\ ([(i_0, j_0)]_U)$ is a well-defined w-nested pair. $(i, j) \subseteq ext_w(U) = V$. Since (i_1, j_1) is U-relevant, $\{i_1, i_1+1, \ldots, j_1\} \cap U \neq \varnothing$. This implies that $i \geqq j_1$, because by definition (i, j) is U-equivalent with (i_0, j_0).

Hence we have found $(i_1, j_1, i, i_2, j_2, j) \subseteq V$ for some w-nested pairs (i_1, j_1), (i_2, j_2) and (i, j). This contradicts our assumption that $(\varphi_V^{-1}(i_1), \varphi_V^{-1}(j_1))$ and $(\varphi_V^{-1}(i_2), \varphi_V^{-1}(j_2))$ are nested pairs in a v-cochain κ.

Consequently $\varphi_V(\kappa)$ is a w-cochain for every v-cochain κ. ■

2. A *SPARSE* PROOF OF OGDENS LEMMA

The Exchange Theorem given in [EHR2] has turned out to be very useful in Part I of this paper; it will also play a crucial role in the present part.

Our basic techniques are the same as before. We use Lemma I.4.2 to relate balanced pairs in the weak description of a computation to equivalent pieces in its trail. We can "pump" these pieces using the Exchange Theorem. Lemma 1.4.1 is used to establish a relationship between occurrences in the result of a computation and occurrences of right letters in its weak description.

Since Odgen's Lemma deals with "special" occurrences in a word of a context-free language, now we are not interested in all occurrences but rather in "special" subsets of these letters. In dealing with these subsets the results on the sparse structure of Dyck words presented in the previous section become important. They enable us to embed properly "special" balanced pairs in the weak description of a computation.

THEOREM 2.1 (Ogden's Lemma): *Let K be a context-free language over an alphabet Θ. Then there exists a constant $d \in \mathbf{N}^+$ such that, for every $w \in K$ and every $\Delta \subseteqq fs(w)$ with $\# \Delta \geqq d$, there exist segments $U_1 < U_2 < U_3 < U_4 < U_5$ satisfying*

(i) $fs(w) = \bigcup_{i=1}^{5} U_i$,

(ii) *either U_1, U_2 and U_3 contain elements from Δ or U_3, U_4 and U_5 contain elements from Δ,*

(iii) *$U_2 \cup U_3 \cup U_4$ contains at most d elements from Δ, and*

(iv) *$w_1 w_2^n w_3 w_4^n w_5 \in K$ for every $n \in \mathbf{N}$, where $w_i = w(U_i)$ for all $1 \leqq i \leqq 5$.*

Proof: Let $G = (G_1, G_2, R)$ be a real-time cp system computing $K = L(G)$, where $G_1 = (\Sigma_1, P_1, S_1, \Theta)$ and $G_2 = (\Sigma_2, P_2, S_2)$.

Let $d = 4(2r^2)^{(5r^2)}$, where $r = \# \Gamma(G)$. We will show that the theorem holds for this choice of d.

Thus consider a word $w \in K$ and a set $\Delta \subseteqq fs(w)$ of occurrences in w such that $\# \Delta \geqq d$. Then let ρ be a successful computation in G such that $res(\rho) = w$ and let $\alpha = trl(\rho)$, $\xi = wdes(\alpha)$. As usual, we consider α to be a $\Gamma(G)$-coloring of ξ; α maps every occurrences in $fs(\alpha) = fs(\xi)$ to an element of $\Gamma(G)$. Then of course, $ind(\alpha) = r$.

G is a real-time cp system, thus every occurrence k of a right letter in ξ corresponds in a natural way to an occurrence of a letter-$ctb(\alpha(k))$-in w (*see* Lemma 1.4.1). This correspondence is described by the V-embedding φ_V, where $V = \{ k \in fs(\xi) \mid \xi(k) \in \hat{\Sigma}_2 \}$ is the set of occurrences of right letters in ξ. Thus, for a support W in ξ, we have $ctb(\alpha(W)) = w(\varphi_V^{-1}(W))$.

Let $\Delta_\xi = \varphi_V(\Delta)$, hence Δ_ξ consists of those occurrences of letters in ξ that contribute to occurrences of "distinguished" letters in w (that is occurrences in Δ).

Then let $\Xi = ext_\xi(\Delta_\xi)$ be the extension of Δ_ξ in ξ. Δ_ξ consists of occurrences of right letters in ξ only, but Ξ contains at least these *and* the matching occurrences of left letters in ξ. Consequently, for any ξ-complete support W, $\#(W \cap \Xi) \geqq 2(W \cap \Delta_\xi)$, and in particular we have $\# \Xi \geqq 2 \# \Delta_\xi \geqq 2d$.

CLAIM 1: *There exists a balanced segment W in ξ such that:*

(a) $\#(W \cap \Delta_\xi) \leqq d$, and

(b) W contains

either an α-uniform ξ-chain κ with $|\kappa| = 6$,

or an α-uniform ξ-cochain κ with $|\kappa| = 3$,

each pair of which is contained in Ξ.

Proof of Claim 1: Ξ is a ξ-complete support, hence $\bar{\xi} = \xi(\Xi)$ is an element of D_{Σ_2}. It has length $|\bar{\xi}| = \# \Xi \geqq 2d$.

According to Theorem I.3.1. there exists a balanced segment $\bar{W} = (\bar{i}, \bar{i}+1, \ldots, \bar{j})$ in $\bar{\xi}$ such that $d < \# \bar{W} \leqq 2d$. If we consider $\bar{\alpha} = \alpha(\Xi)$ as a $\Gamma(G)$-coloring of $\bar{\xi}$, then $ind(\bar{\alpha}) = r$.

Theorem I.3.5. implies that either $\bar{W}$ contains an $\bar{\alpha}$-uniform $\bar{\xi}$-chain $\bar{\kappa}$ with $|\bar{\kappa}| = 6$, or $\bar{W}$ contains an $\bar{\alpha}$-uniform $\bar{\xi}$-cochain $\bar{\kappa}$ with $|\bar{\kappa}| = 3$. So let $\bar{\kappa}$ be either a $\bar{\xi}$-chain or a $\bar{\xi}$-cochain as above. We consider both cases at the same time.

By Lemma 1.6. Ξ is a ξ-proper support. Hence $\kappa = \varphi_\Xi(\bar{\kappa})$ is a ξ-(co)chain. The pair $(\bar{i}, \bar{j})$ is ξ-balanced, thus $(i, j) = (\varphi_\Xi(\bar{i}), \varphi_\Xi(\bar{j}))$ is also ξ-balanced, because Ξ is ξ-proper (see Lemma 1.3).

Let $W = (i, i+1, \ldots, j)$. Then W is a ξ-balanced segment that satisfies our claim. This is seen as follows.

$$(a) \qquad \#(W \cap \Delta_\xi) \leqq (1/2) \# (W \cap \Xi) = (1/2) \bar{W} \leqq d.$$

(Here we have used the fact that φ_Ξ is a bijection between $\bar{W}$ and $W \cap \Xi$.)

(*b*) Obviously κ is contained in W. Furthermore, κ is α-uniform. This follows from the $\bar{\alpha}$-uniformness of $\bar{\kappa}$ and the fact that $\bar{\alpha}(k) = \bar{\alpha}(k_1)$ implies that $\alpha(\varphi_\Xi(k)) = \alpha(\varphi_\Xi(k_1))$.

Hence our claim holds. ■

It seems helpful to illustrate some of the notions used in this claim with an example. Since the constants used in the proof become rather large even in simple (but nontrivial) cases we give a "scaled" example: the longest uniform chain in the trail of the computation we present has length 2 and it contains no non-trivial (longer than 1) uniform conchains.

Example 2.1: Let $G=(G_1, G_2, R)$ be a cp system which has the following rewrites:

$$\psi_1=(X \to aX,\ A \to BA),$$
$$\psi_2=(X \to bY,\ A \to \Lambda),$$
$$\psi_3=(Y \to bX,\ B \to A),$$
$$\psi_4=(Y \to cY,\ B \to \Lambda)$$

and

$$\psi_0=(Y \to c,\ B \to \Lambda).$$

Furthermore, let $G_1=(\{X, Y, a, b, c\}, P_1, X, \{a, b, c\})$ and $G_2=(\{A, B\}, P_2, A)$, where P_1 and P_2 are chosen in such a way that they "fit" the set of rewrites.

Consider $w=aaaabbbcbbc \in L(G)$ together with the set $\Delta=\{2, 6, 7\}$ of "distinguished" positions in w.

A possible computation ρ for w in G is determined by the control sequence

$$cont(\rho)=\psi_1, \psi_1, \psi_1, \psi_1, \psi_2, \psi_3, \psi_2, \psi_4, \psi_3, \psi_2, \psi_0.$$

This computation has the trail

$$\alpha=trl(\rho)=[X; A]\,[\psi_1, 0]\,[\psi_1, 1]\,[\psi_1, 2]\,[\psi_1, 0] \ldots [\psi_2, 0]\,[\psi_0, 0].$$

The weak description ξ of ρ is given by

$$\xi=A\hat{A}BA\hat{A}BA\hat{A}BA\hat{A}BA\hat{A}\hat{B}A\hat{A}\hat{B}\hat{B}A\hat{A}\hat{B}$$

and obviously $fs(\xi)=fs(\alpha)=\{1, 2, \ldots, 22\}$.

The occurrences of right letters in ξ, that is occurrences in α "contributing" to symbols in w, form the set

$$V=\{2, 5, 8, 11, 14, 15, 17, 18, 19, 21, 22\}.$$

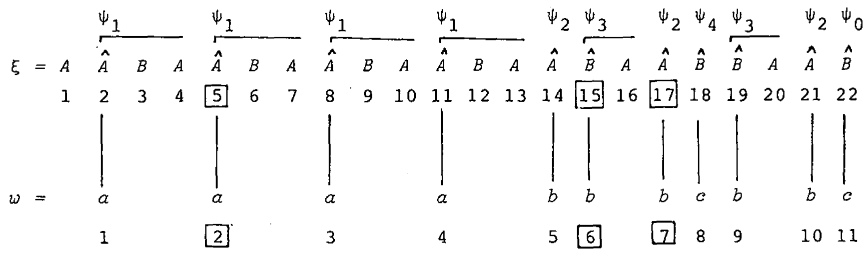
In V we distinguish the set $\Delta_\xi=\{5, 15, 17\}$.
$cpl_\xi(\Delta_\xi)=\{4, 5, 12, 15, 16, 17\}$.

ξ contains the following Δ_ξ-relevant pairs:

$$(3, 22), (4, 5), (6, 19), (9, 18), (12, 15) \quad \text{and} \quad (16, 17).$$

Of these only (6, 19) and (9, 18) are Δ_ξ-equivalent. So

$$out([(9, 18)]_{\Delta_\xi})=out((6, 19), (9, 18))=(6, 19).$$



Thus

$$\Xi = ext_\xi(\Delta_\xi) = \{3, 4, 5, 6, 12, 15, 16, 17, 19, 22\}.$$

Then

$$\bar{\xi} = \xi(\Xi) = BA\hat{A}BB\hat{B}A\hat{A}\hat{B}\hat{B}$$

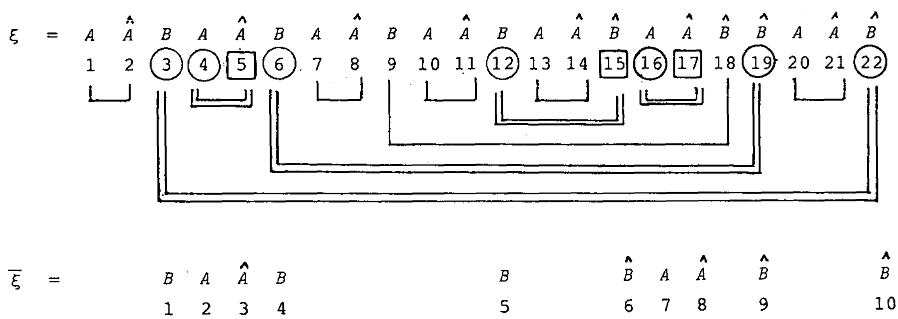
and

$\bar{\alpha} =$

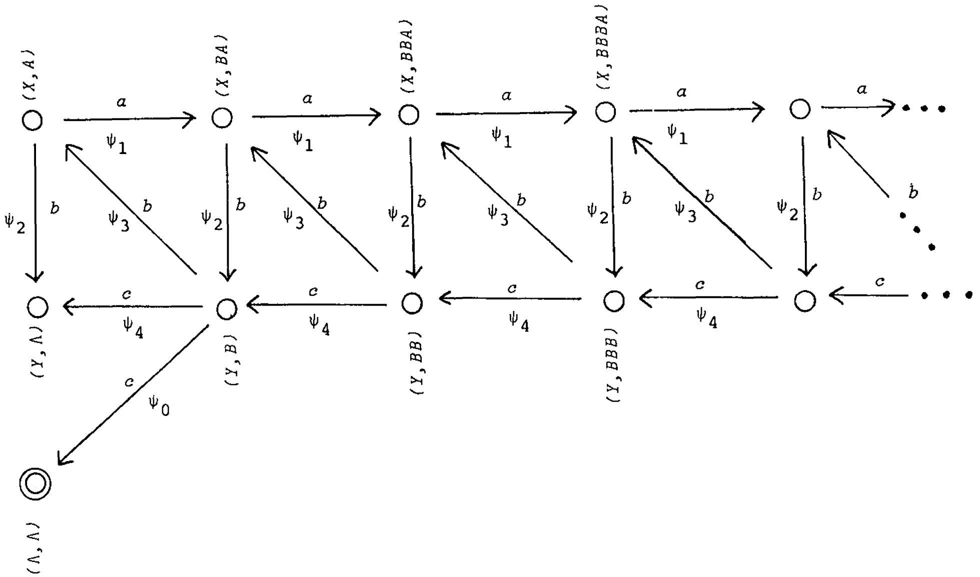
$$\alpha(\Xi) = [\psi_1, 1][\psi_1, 2][\psi_1, 0][\psi_1, 1][\psi_1, 1][\psi_3, 0][\psi_3, 1][\psi_2, 0][\psi_3, 0][\psi_0, 0].$$

$\bar{\xi}$ has an $\bar{\alpha}$-uniform chain $\bar{\kappa} = (4, 9), (5, 6)$ which is mapped by φ_Ξ to the α-uniform ξ-chain $\kappa = (6, 19), (12, 15)$.

The above may be depicted as follows.



All the computations in our cp system are given by the following diagram.



(End of Example 2.1.). ■

Proof of Theorem 2.1 (continued):

The above claim enables us to find a splitting of α suitable for the application of the Exchange Theorem.

Let W be as in the statement of Claim 1. We consider separately the case when W contains a chain and the case when W contains a cochain.

(1) Let $\kappa=(i_1, j_1), \ldots, (i_6, j_6)$ be a α-uniform ξ-chain contained in W. Then let $W_0, W_1, \ldots, W_{12}$ be the κ-splitting of ξ.

The following two claims are helpul in proving the second condition from the statement of Theorem 2.1.

CLAIM 2: $W_6 \cap \Delta_\xi \neq \varnothing$.

Proof of Claim 2: κ contains only Δ_ξ-relevant pairs, especially $(i_6, j_6) \subseteq \Xi = ext_\xi(\Delta_\xi)$. Hence $W_6 \cap \Delta_\xi \neq \varnothing$. ■

CLAIM 3: *There exist two pairs* $P_s=(i_s, j_s)$ *and* $P_t=(i_t, j_t)$ *of* κ, *where* $1 \leqq s < t \leqq 5$, *such that*

either $\quad W_s \cap \Delta_\xi \neq \varnothing \quad$ *and* $\quad W_t \cap \Delta_\xi \neq \varnothing$

or $W_{12-t} \cap \Delta_\xi \neq \varnothing$ *and* $W_{12-s} \cap \Delta_\xi \neq \varnothing$.

Proof of Claim 3: Observe that by definition Δ_ξ contains only occurrences of right letters in ξ. Hence the occurrences $i_1, \ldots, i_5$ all belong to $\Xi - \Delta_\xi$.

We consider separately three cases, depending on how many of the occurrences $j_1, \ldots, j_5$ are contained in Δ_ξ.

(*a*) There exist s, t with $1 \leqq s < t \leqq 5$ such that $j_s, j_t \in \Delta_\xi$.

Then clearly "or" holds.

(*b*) There exists exactly one r, $1 \leqq r \leqq 5$, such that $j_r \in \Delta_\xi$.

Since obviously $W_{12-r} \cap \Delta_\xi \neq \varnothing$, "or" holds whenever for some $p \neq r$ $W_{12-p} \cap \Delta_\xi \neq \varnothing$. So assume that this is not the case; for every $p \neq r$ we have $W_{12-p} \cap \Delta_\xi = \varnothing$.
Then let

$$s = \begin{cases} 3, & \text{if } r \in \{1, 2\}, \\ 1, & \text{if } r \in \{3, 4, 5\}, \end{cases}$$

and

$$t = \begin{cases} 4, & \text{if } r \in \{1, 2, 3\}, \\ 2, & \text{if } r \in \{4, 5\}. \end{cases}$$

Note that r is different from all elements of $\{s, s+1, t, t+1\}$.

Since $(i_s, j_s) \cup (i_{s+1}, j_{s+1}) \subseteqq \Xi - \Delta_\xi$, Lemma 1.5.(3) implies that either $W_s \cap \Delta_\xi \neq \varnothing$ or $W_{12-s} \cap \Delta_\xi \neq \varnothing$.

But we have assumed that $W_{12-s} \cap \Delta_\xi = \varnothing$, consequently $W_s \cap \Delta_\xi \neq \varnothing$.

In the same way we deduce that $W_t \cap \Delta_\xi \neq \varnothing$.

Hence we are left with the "either" case of the claim.

(*c*) For all $1 \leqq r \leqq 5$ we have $j_r \in \Xi - \Delta_\xi$.

Then applying Lemma 1.5.(3) to (i_r, j_r), (i_{r+1}, j_{r+1}) we find that, for each $1 \leqq r \leqq 4$, either $W_r \cap \Delta_\xi \neq \varnothing$ or $W_{12-r} \cap \Delta_\xi \neq \varnothing$. A simple counting argument yields that at least two of the sets W_1, W_2, W_3, W_4, W_8, W_9, W_{10} and W_{11} that have a nonempty intersection with Δ_ξ must lie at "the same side" of W_6.

This implies that our claim holds. ■

Let s, t be as in the above claim. We write

$$\bar{W}_1 = \bigcup_{k=0}^{t-1} W_k,$$

$$\bar{W}_2 = \bigcup_{k=t}^{5} W_k,$$

$$\bar{W}_3 = W_6,$$

$$\bar{W}_4 = \bigcup_{k=7}^{12-t} W_k$$

and

$$\bar{W}_5 = \bigcup_{k=12-t+1}^{12} W_k.$$

Now we show that, for $i=1, \ldots, 5$, $U_i = \varphi_V^{-1}(\bar{W}_i)$ satisfies the statement of the theorem. This is seen as follows.

(0) Obviously $\bar{W}_1 < \bar{W}_2 < \ldots < \bar{W}_5$. Hence $U_1 < U_2 < \ldots < U_5$. because φ_V^{-1} is an increasing function.

(i) $\bigcup_{i=1}^{5} \bar{W}_i = \bigcup_{k=0}^{12} W_k = fs(\xi) \supseteq V$.

Hence

$$\bigcup_{i=1}^{5} U_i = \varphi_V^{-1}\left(\bigcup_{i=1}^{5} \bar{W}_i\right) \supseteq \varphi_V^{-1}(V) = fs(w).$$

The reverse inclusion is obvious.

Consequently $\bigcup_{i=1}^{5} U_i = fs(w)$.

(ii) Clearly, if $\bar{W}_i \cap \Delta_\xi \neq \varnothing$, then $U_i \cap \Delta \neq \varnothing$ for $i=1, 2, \ldots, 5$.

Thus, by Claim 2, U_3 contains an element from Δ. Moreover Claim 3 implies that either U_1 and U_2 or U_4 and U_5 contain elements from Δ.

(iii) The chain κ is contained in W, so $\bigcup_{k=1}^{11} W_k \subseteqq W$.

On the other hand, according to Claim 1, $\#(W \cap \Delta_\xi) \leqq d$.

Hence, because $\bigcup_{i=2}^{4} \bar{W}_i \subseteqq \bigcup_{k=1}^{11} W_k$ we have $\#\left(\bigcup_{i=2}^{4} \bar{W}_i \cap \Delta_\xi\right) \leqq d$. From this it follows that $\bigcup_{i=2}^{4} U_i = \varphi_V^{-1}\left(\bigcup_{i=2}^{4} \bar{W}_i\right)$ contains at most d elements from Δ.

(iv) Let $\alpha_i = \alpha(\bar{W}_i)$ for $i=1, \ldots, 5$.

Then $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5 = trl(\rho)$.

(i_t, j_t) and (i_6, i_6) are α-equivalent ξ-nested pairs. Hence by Lemma 1.4.2. $\alpha_2\alpha_3\alpha_4 \sim \alpha_3$.

Exchanging these pieces in the trails of two copies of the computation ρ leads to (unique) successful computations ρ_0 and ρ_2 in G such that

$$trl(\rho_0) = \alpha_1\alpha_3\alpha_5 \quad \text{and} \quad trl(\rho_2) = \alpha_1\alpha_2\alpha_2\alpha_3\alpha_4\alpha_4\alpha_5 = \alpha_1\alpha_2^2\alpha_3\alpha_4^2\alpha_5.$$

We apply the Exchange Theorem once again, this time to the equivalent pieces α_3 (in ρ_2) and $\alpha_2\alpha_3\alpha_4$ (in ρ) to obtain a successful computation ρ_3 in G such that

$$trl(\rho_3) = \alpha_1\alpha_2^2\alpha_2\alpha_3\alpha_4\alpha_4^2\alpha_5 = \alpha_1\alpha_2^3\alpha_3\alpha_4^3\alpha_5.$$

Continuing in this way we get an infinite sequence of successful computations $\rho_0, \rho_1 = \rho, \rho_2, \rho_3, \ldots$ in G such that, for every $n \in \mathbf{N}$, $trl(\rho_n) = \alpha_1\alpha_2^n\alpha_3\alpha_4^n\alpha_5$.

This implies that $res(\rho_n) \in L(G) = K$ for every $n \in \mathbf{N}$.

Hence, for all $n \in \mathbf{N}$, $w_1 w_2^n w_3 w_4^n w_5 \in L(G) = K$, where

$$w_i = ctb(\alpha_i) = ctb(\alpha(\bar{W}_i)) = w(\varphi_V^{-1}(\bar{W}_i)) = w(U_i).$$

This proves the theorem in the "chain-case".

Example 2.1 (continued): Let $\bar{W}_1, \bar{W}_2, \ldots, \bar{W}_5$ be the κ-splitting of ξ and let for $i = 1, 2, \ldots, 5$, $U_i = \varphi_V^{-1}(\bar{W}_i)$.

Then

$U_1 = \{1, 2\}$, $U_2 = \{3, 4\}$, $U_3 = \{5, 6\}$, $U_4 = \{7, 8, 9\}$ and $U_5 = \{10, 11\}$.

Applying the Exchange Theorem to the equivalent subwords of ξ

$$BA\hat{A}BA\hat{A}BA\hat{A}\hat{B}A\hat{A}\hat{B}\hat{B} = \xi(\bar{W}_2 \cup \bar{W}_3 \cup \bar{W}_4) \quad \text{and} \quad BA\hat{A}\hat{B} = \xi(\bar{W}_3)$$

it is possible to find it computations in G for the words

$$w(U_1)\,w(U_2)^n\,w(U_3)\,w(U_4)^n\,w(U_5) = aa\,(aa)^n\,bb\,(bcb)^n\,bc, \quad \text{for all } n \in \mathbf{N}.$$

Note that $U_2 \cap \Delta = \varnothing$ and $U_5 \cap \Delta = \varnothing$, thus this partition of $fs(w)$ does not satisfy the second condition of Theorem 2.1 (End of Example 2.1.). ■

(2) Let $\kappa = (i_1, j_1), (i_2, j_2), (i_3, j_3)$ be a α-uniform ξ-cochain contained in W.

Let $W_0, W_1, \ldots, W_6$ be the κ-splitting of ξ. The following result can be proved in the same way as Claim 2.

CLAIM 4: *$W_1 \cap \Delta_\xi \neq \varnothing$, $W_3 \cap \Delta_\xi \neq \varnothing$ and $W_5 \cap \Delta_\xi \neq \varnothing$.* ■

We choose now $U_i = \varphi_V^{-1}(\bar{W}_i)$, where $\bar{W}_1 = W_0 \cup W_1 \cup W_2$, $\bar{W}_2 = W_3 \cup W_4$, $\bar{W}_3 = W_5$, $\bar{W}_4 = \varnothing$ and $\bar{W}_5 = W_6$.

This choice satisfies conditions (i) through (iv) from the statement of Theorem 2.1. The proof of this fact is omitted, because it can be done analogously to the proof given for the "chain"-case. As a matter of fact, now the proof is quite simpler: in the "cochain" case our construction implies that the "either" part of condition (ii) from the statement of the theorem holds-hence now Claim 4 can replace Claims 2 and 3.

We would also like to remark the following concerning the proof of (iv) in the "cochain" case: now $\alpha_2 \alpha_3 \alpha_4 \sim \alpha_3$, where $\alpha_i = \alpha(\bar{W}_i)$ for $i = 1, \ldots, 5$, follows from the fact that (i_2, j_3) and (i_3, j_3) are equivalent ξ-balanced pairs.

Hence the theorem holds also in the "cochain" case.

This concludes the proof of Theorem 2.1. ■

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