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COPRIME FACTORIZATION OVER A CLASS OF NONLINEAR SYSTEMS*

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SUMMARY

In this paper, we consider the problem of generalizing elements of linear coprime factorization theory to a nonlinear context. The idea is to work with a suitably wide class of nonlinear systems to cover many practical situations, yet not cope with so broad a class as to disallow useful generalizations to the linear results. In particular, we work with nonlinear systems characterized in terms of (possibly time-varying) state-dependent matrices $A(x), B(x), C(x), D(x)$ and an initial state x_0. (This class clearly does contain the class of finite-dimensional linear (time-varying) systems.) We achieve first right coprime factorizations for idealized situations. To achieve stable left factorizations we specialize to the case where the matrices are output-dependent. Alternatively, we work with systems, perhaps augmented by a direct feedthrough term, where the input is reconstructible from the output. For nonlinear feedback control systems, with plant and controller having stable left factorizations, then under appropriate regularity-conditions earlier results have allowed the generation of the class of stabilizing controllers for a system in terms of an arbitrary stable system (parameter). Plant uncertainties, including unknown initial conditions are modelled by means of a Yula–Kucera-type parametrization approach developed for nonlinear systems. Certain robust stabilization results are also shown, and simulations demonstrate the regulation of nonlinear plants using the techniques developed. All the results are presented in such a way that specialization for the case of linear systems is immediate.

KEY WORDS Nonlinear systems Coprime factorizations

1. INTRODUCTION

Coprime factorization results for linear systems have proved powerful tools for characterizing the class of all stabilizing controllers for linear systems. Such characterizations have led to robust stabilization results and has set the stage for (robust) optimal controller design for linear systems.[1,2] The challenge is to develop coprime factorization tools to cope with nonlinear systems.

The class of all stabilizing controllers for linear, continuous-time, time-invariant systems have been characterized in terms of polynomial matrix function descriptions[3] and for discrete time using stable transfer function matrix fraction descriptions.[4] State-space form matrix fraction (transfer function) descriptions were first developed in,[5] so opening the way for

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conditions, we achieve matrix fraction descriptions in terms of an arbitrary stable system (parameter). Then certain robust stabilization results from Reference 9 are shown to be applicable to this case. All the results are presented in such a way that specialization for the case of linear systems is immediate. We generalize certain key linear results pertaining to cascading and inverting linear plants to this class and then use these results to create sets of right coprime and stable left factorizations for this subclass which are pertinent in idealized nominal plant, stabilizing controller arrangements. For some of this work we need certain augmentation techniques. When used in conjunction with existing nonlinear theory, the resulting factorizations allow us to generate the class of all stabilizing (augmented) controllers for a given (augmented) nonlinear plant. Of course, it is trivial to dualize to the class of all (augmented) plants stabilized by a given (augmented) controller. We relate these back to our original unaugmented plant/controller systems, and explore some bounds of possible nonlinear stabilization/factorization theories of this type.

Section 2 generalizes the linear cascade and inverse operations, and also introduces right coprime factorizations for systems (2). It also sets up a general theorem proof methodology used in the rest of the paper. Section 3 specializes to the case where the state-dependence is reconstructible from the output of the plant alone, giving right coprime and stable left factorizations as well as certain Bezout identities, at least for idealized nominal plant/controller arrangements. Section 4 includes the augmentation method to obtain further results for the stable left factorizations, and justifies this approach by proving that stability results for the augmented plant carry over to certain arrangements including the nominal plant. Then results from Reference 9 and a companion paper Reference 14 are reviewed, and coupled with these factorizations lead to the controller class K_Q which stabilizes a given nominal plant. Also, stabilization results are quoted for an Yula–Kucera type parametrization of nonlinear plants, and this is used to extend the theory to the case of unequal initial conditions between the plant and controller. Section 5 presents simulation studies for the control of certain nonlinear plants. Conclusions are drawn in Section 6.

2. NONLINEAR FACTORIZATIONS

Nonlinear system class

The nominal plants, and controllers and derivative systems studied in this paper, belong to a class of nonlinear systems (operators)

$$G(\gamma, x_0) : \begin{array}{l} \dot{x} = A(\gamma)x + B(\gamma)u, \\ y = C(\gamma)x + D(\gamma)u \end{array} \qquad x(0) = x_0 : \left[\begin{array}{c|c} A(\gamma) & B(\gamma) \\ \hline C(\gamma) & D(\gamma) \end{array}\right]_{x(0) = x_0} \tag{7}$$

where, either $\gamma = \text{constant}$, $\gamma = t$, $\gamma = x(t)$, or indeed $\gamma = (x(t), t)$, although a number of our results exclude this latter time-varying case. Variations such as $\gamma = u(t)$ or $y(t)$, or more generally $\gamma = (u(t), x(t))$, or indeed causally filtered $x(t)$ or strictly causally filtered $y(t)$, denoted $x_w(t), y_w(t)$ can be handled in our technical approach, although for simplicity of presentation we work primarily with the cases $\gamma = x(t)$ and $\gamma = x_w(t)$. The partitioned matrix notation with an initial state subscript is a mild generalization of the common notation for the $\gamma = \text{constant}$ case. The following assumption is crucial to certain results to follow:

Assumption: The matrices $A(\gamma), B(\gamma)$, etc. are assumed to exist, and are bounded, for all finite γ, and are such that $x(.), y(.)$ of (7) exist for all $x(0)$, $t \geqslant 0$, and are unique. (8)

working with time varying stable linear operators instead of transfer functions, see Reference 6 and its references.

For nonlinear systems, a number of generalizations are available, building on the work of Reference 7. See also References 8 to 10.

The less restrictive the assumptions on the nonlinearities, the less closely one can echo the linear results. Thus, at the this stage there is incentive to work with restricted classes of nonlinear systems which commonly arise in practise, and yet allow a factorization theory to develop which goes some of the way to match in elegance and power the well-established linear results.

It is desirable that nonlinear factorization and stabilization results are developed which transparently specialize to the familiar results associated with state-space descriptions for a linear system G as follows. Let us denote such linear systems

$$G : \begin{matrix} \dot{x} = Ax + Bu \\ y = Cx + Du \end{matrix} : \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{1}$$

where $x(t)$ is the state vector, $u(t)$ the input vector, and $y(t)$ the output vector. A useful class of such nonlinear generalizations are denoted

$$G(x_0) : \begin{matrix} \dot{x} = A(x)x + B(x)u \\ y = C(x)x + D(x)u \end{matrix} : \left[\begin{array}{c|c} A(x) & B(x) \\ \hline C(x) & D(x) \end{array}\right]_{x(0)} \tag{2}$$

initialized by $x(0) = x_0$.

Such systems can arise, for example, from linearization of more general nonlinear systems

$$\dot{x} = f(x, u); \qquad y = h(x, u) \tag{3}$$

in the vicinity of a known trajectory x^*. Thus with $\delta x = x - x^*$

$$\delta\dot{x} = \left.\frac{\partial f}{\partial x}\right|_x \delta x + \left.\frac{\partial f}{\partial u}\right|_x \delta u + \cdots \tag{4}$$

$$\delta y = \left.\frac{\partial h}{\partial x}\right|_x \delta x + \left.\frac{\partial h}{\partial u}\right|_x \delta u + \cdots \tag{5}$$

Neglecting higher-order terms, and setting $x = \delta x + x^*$, gives a nonlinear system of the form (2) with state δx,

$$A(\delta x) = \left.\frac{\partial f}{\partial x}\right|_{x^* + \delta x},$$

etc.

Work has been done to generate coprime factorizations of a class of systems which includes (2), the class

$$\dot{x} = f(x) + G(x)u \tag{6}$$

The existence of coprime right factorizations for systems in (6) is shown in Reference 12 for the case when the smooth feedback stabilization problem is solvable for the system, and its follows that feedback linearizable systems amidst such factorizations. Under the assumptions of stabilizability and detectability, Reference 11 gives right coprime factorizations, and under the assumption of existence of controller and observer forms, Reference 13 gives both right and left coprime factorizations.

In the paper, working with nonlinear systems of the form of (2) under appropriate regularity derived from the cascade form (10) as,

$$G(x_0)R(x_0) = \left[\begin{array}{cc|c} A(x_R) - B(x_R)D^{-1}(x_R)C(x_R) & 0 & B(x_R)D^{-1}(x_R) \\ -B(x)D^{-1}(x_R)C(x_R) & A(x) & B(x)D^{-1}(x_R) \\ \hline -D(x)D^{-1}(x_R)C(x_R) & C(x) & D(x)D^{-1}(x_R) \end{array}\right]_{\substack{x_R(0)=x_0 \\ x(0)=x_0}} \tag{14}$$

Let us denote the input to the system as u, the output as y, and also define the output from $R(x_0)$ as y_R. Now, from (11) $y_R = D^{-1}(x_R)[u - C(x_R)x_R]$, then from (14),

$$\begin{aligned} \dot{x} &= A(x)x + B(x)y_R, \quad x_0 \\ \dot{x}_R &= A(x_R)x_R + B(x_R)y_R, \quad x_0 \\ y &= D(x_R)D^{-1}(x_R)[u - C(x_R)x_R] + C(x)x \end{aligned} \tag{15}$$

Thus in (15), x and x_R obey the same differential equation. Now under the solution uniqueness assumption (8) on the class of systems (2) of this section, and with $x(0) = x_R(0)$, then $x(t) = x_R(t)$ for all $t \geqslant 0$, and consequently, $A(x_R) \equiv A(x)$, etc. From (15) we then have $y = u$, giving $G(x_0)R(x_0) = I$ as required.

Another proof of this latter result is instructive. From (14) consider a co-ordinate basis change from the state $[x_R' \ x']'$ to state $[x_R' \ (x' - x_R')]'$, achieved by elementary row and column operators on the partitioned matrix (column two is added to column one, then the first row is subtracted from the second).

$$G(x_0)R(x_0) = \left[\begin{array}{cc|c} A(x) - B(x)D^{-1}(x)C(x) & 0 & B(x)D^{-1}(x) \\ 0 & A(x) & 0 \\ \hline 0 & C(x) & I \end{array}\right]_{\substack{x_R(0) = x_0 \\ [x(0) - x_R(0)] = 0}} = I \tag{16}$$

The second equality follows from deletion of the unobservable mode x_R and the uncontrollable mode with zero initial condition $[x_R(t) - x(t)]$.

To demonstrate the left inverse case, first note from application of (10)

$$R(x_R(0))G(x_0)$$

$$= \left[\begin{array}{cc|c} A(x) & 0 & B(x) \\ B(x_R)D^{-1}(x_R)C(x) & A(x_R) - B(x_R)D^{-1}(x_R)C(x_R) & B(x_R)D^{-1}(x_R)D(x) \\ \hline D^{-1}(x_R)C(x) & -D^{-1}(x_R)C(x_R) & D^{-1}(x_R)D(x) \end{array}\right]_{\substack{x(0) = x_0 \\ x_R(0) = x_0}}$$

Also, defining $y_R \equiv C(x)x + D(x)u$, gives

$$\begin{aligned} \dot{x} &= [A(x) - B(x)D^{-1}(x)C(x)]\,x + [B(x)D^{-1}(x)]\,y_R, \quad x_0 \\ \dot{x}_R &= [A(x_R) - B(x_R)D^{-1}(x_R)C(x_R)]\,x_R + [B(x_R)D^{-1}(x_R)]\,y_R, \quad x_0 \\ y &= D^{-1}(x_R)[C(x)x - C(x_R)x_R] + D^{-1}(x_R)D(x) \end{aligned} \tag{17}$$

Thus $x_R(t)$ and $x(t)$ obey the same differential equation and so by the uniqueness assumption (8), when $x_R(0) = x(0)$, then $x_R(t) = x(t)$ and $y = u$ for $t \geqslant 0$. Note also

$$R(x_0)G(x_0) = \left[\begin{array}{cc|c} A(x) & 0 & B(x) \\ 0 & A(x) & 0 \\ \hline 0 & -D^{-1}(x)C(x) & I \end{array}\right]_{\substack{x(0) \\ [x(0) - x_R(0)] = 0}} = I \tag{18}$$

□

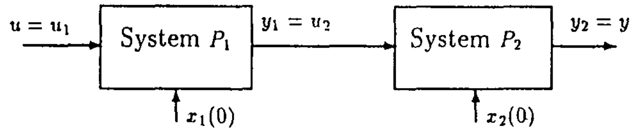


Figure 1. Cascade of system P_2 and system P_1

There exists a complete factorization stability theory for the cases $\gamma = \text{constant}$ and $\gamma = t$ leading to a description of the class of all stabilizing controllers for the plant. Here we show that the nonlinear (time-varying) case when $\gamma = x(t)$, so that $G(\gamma, x_0)$ is G of (2), likewise, yields a 'partial' theory along similar lines. We proceed by first considering in turn, the cascade of nonlinear systems as in (2) and the inverse, for the case when $D^{-1}(x)$ exists.

Cascade. First consider the cascade of systems P_1, P_2 as in Figure 1 where each is of the form of (2). The state equations of the cascade P_2P_1 with input $u = u_1$ and output $y = y_2$ and state $x' = [x_1' \; x_2']$ are

$$\begin{aligned} \dot{x}_1 &= A_1(x_1)x_1 + B_1(x_1)u_1, \qquad x_1(0) \\ \dot{x}_2 &= A_2(x_2)x_2 + B_2(x_2)y_1 \qquad ,x_2(0) \\ u_2 = y_1 &= C_1(x_1)x_1 + D_1(x_1)u_1 \\ y_2 &= C_2(x_2)x_2 + D_2(x_2)y_1 \end{aligned} \tag{9}$$

That is, in the partitioned matrix operator notation of (2), the following cascade relations is established

$$\begin{aligned} P_2P_1 &= \left[\begin{array}{c|c} A_2(x_2) & B_2(x_2) \\ \hline C_2(x_2) & D_2(x_2) \end{array}\right]_{x_2(0)} \left[\begin{array}{c|c} A_1(x_1) & B_1(x_1) \\ \hline C_1(x_1) & D_1(x_1) \end{array}\right]_{x_1(0)} \\ &= \left[\begin{array}{cc|c} A_1(x_1) & 0 & B_1(x_1) \\ B_2(x_2)C_1(x_1) & A_2(x_2) & B_2(x_2)D_1(x_1) \\ \hline D_2(x_2)C_1(x_1) & C_2(x_2) & D_2(x_2)D_1(x_1) \end{array}\right]_{\substack{x_1(0) \\ x_2(0)}} \end{aligned} \tag{10}$$

Inverse. Let us consider the following system $R(x_0)$ defined in terms of the system matrices (2) where $D^{-1}(x)$ exists for all x.

$$R(x_0) = \left[\begin{array}{c|c} A(x_R) - B(x_R)D^{-1}(x_R)C(x_R) & B(x_R)D^{-1}(x_R) \\ \hline -D^{-1}(x_R)C(x_R) & D^{-1}(x_R) \end{array}\right]_{x_R(0) = x_0} \tag{11}$$

Lemma 1

Consider the system $G(x_0)$ of (2), where $D^{-1}(x)$ exists for all x and the associated system $R(x_0)$ of (11). Then for the cascade $G(x_0)R(x_0)$ and $R(x_0)G(x_0)$,

$$x_R(t) = x(t) \text{ for all } t > 0 \tag{12}$$

where x_R denotes the state of the system $R(x_0)$ in each of the cascades. Moreover $R(x_0)$ is the inverse operator $G^{-1}(x_0)$ satisfying $G^{-1}(x_0)G(x_0) = G(x_0)G^{-1}(x_0) = I$, that is

$$R(x_0) = G^{-1}(x_0) \tag{13}$$

Proof. In the right inverse case, the state equations of the cascaded system $G(x_0)R(x_0)$ are familiar linear system results, let us restrict attention to the time-invariant (nonlinear) system case and assume the following

Assumption: The state estimate feedback gain $F(\xi)$, is constructed such that

$$\dot{\xi} = [A(\xi) + B(\xi)F(\xi)]\,\xi, \qquad \xi(0) = \xi_0 \tag{20}$$

is exponentially stable for arbitrary initial conditions ξ_0.

Of course, it is necessary that the pair $[A(.), B(.)]$ be appropriately controllable. Also, it should be noted that in the time-invariant case exponential stability is equivalent to bounded-input, bounded-output (BIBO) stability. Under Assumption (20), it is clear that the feedback pair $\{G(x_0)|_{x_0=\hat{x}_0}, K(\hat{x}_0)\}$ has certain exponential stability properties by virtue of the following lemma.

Lemma 2

Referring to (2), (19), consider the plant $G(x_0)|_{x_0=\hat{x}_0}$ with states $x(t)$, and a feedback controller $K(\hat{x}_0)$ with states $\hat{x}(t)$ as in Figure 2(b). Then

$$x(t) = \hat{x}(t) \text{ for all } t \geqslant 0 \tag{21}$$

Moreover, the states $x(t)$, of both plant and controller satisfy

$$\dot{x} = [A(x) + B(x)F(x)]\,x, \quad x(0) = x_0 \tag{22}$$

which is exponentially stable under (20).

Proof. Defining $u^* = -H(\hat{x})C(x)x - H(\hat{x})D(x)F(\hat{x})\hat{x}$, the relevant equations can be organized as

$$\begin{aligned} \dot{x} &= [A(x) + H(\hat{x})C(x)]\,x + B(x)u + u^* + H(\hat{x})D(x)F(\hat{x})\hat{x}, \quad x(0) = x_0 \\ \dot{\hat{x}} &= [A(\hat{x}) + H(\hat{x})C(\hat{x})]\,\hat{x} + B(\hat{x})u + u^* + H(\hat{x})D(\hat{x})F(\hat{x})\hat{x}, \quad \hat{x}(0) = x_0 \end{aligned} \tag{23}$$

Apply Assumption (8), then (21) holds as required. Also, given that $x(t) = \hat{x}(t)$, and since $u = F(\hat{x})\hat{x}$ then (23) becomes equivalent to (22). □

Remark. The equations for $\delta x = x - \hat{x}$ appear instructive only in special cases, such as the linear case, when $\delta\dot{x} = (A + HC)\delta x$. Yet it is the stability of the δx equations, fed from the x state equation, along with the stability of (22), that determines the internal stability of the feedback system $\{G(x_0), K(\hat{x}_0)\}$, when $\hat{x}_0 \neq x_0$.

Right coprime factorization

Lemma 3

Consider the nominal plant/controller arrangement of Figure 2(b) with the definitions (2), (19), and stability Assumption (20). Define also the system with state $\hat{x}(t)$

$$\begin{bmatrix} M(x_1(0)) & U(x_2(0)) \\ N(x_1(0)) & V(x_2(0)) \end{bmatrix} \equiv \left[\begin{array}{cc|cc} A(x_1) + B(x_1)F(x_1) & 0 & B(x_1) & 0 \\ 0 & A(x_2) + B(x_2)F(x_2) & 0 & -H(x_2) \\ \hline F(x_1) & F(x_2) & I & 0 \\ C(x_1) + D(x_1F(x_1) & C(x_2) + D(x_2)F(x_2) & D(x_1) & I \end{array}\right]_{\substack{x_1(0) \\ x_2(0)}} \tag{24}$$

Remarks

(1) The results of this lemma are critically dependent on the initial condition constraints $x_R(0) = x(0)$. It does not appear straightforward to give robustness conditions which would also achieve the limit $\lim_{t\to\infty}[x_R(t) - x(t)] = 0$ for unequal initial conditions, as in the well-understood linear system case when $G(x)$, $R(x_R)$ are both linear and asymptotically stable. Our approach will be to deal with initial condition mismatching along with unmodelled dynamics and external inputs/disturbances in subsequent sections.

(2) In manipulations it is important to notice that for cascading P_1 and $(P_2 + P_3)$ then $P_1(P_2 + P_3) \neq P_1P_2 + P_1P_3$, in general, whereas of course for matrix multiplication $A(x)[B(x) + C(x)] = A(x)B(x) + A(x)C(x)$.

Nominal plant and stabilizing controller for plants $G(x_0)$

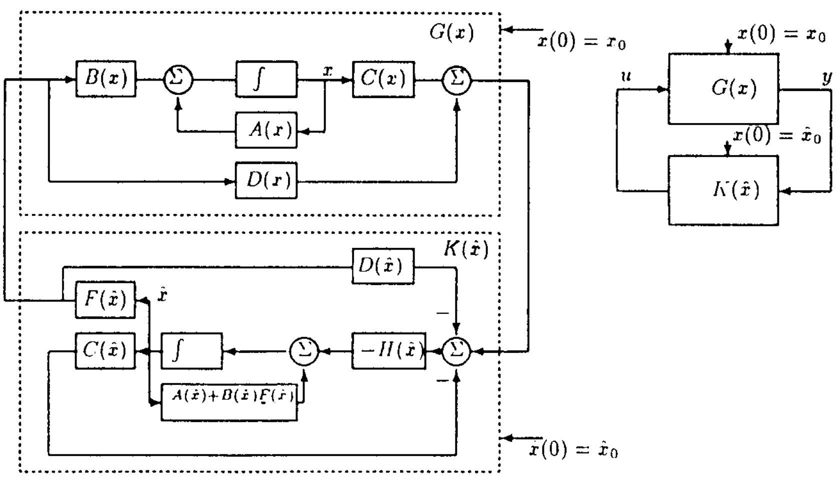
Let us consider first a familiar state estimate feedback controller arrangement $K(\hat{x}_0)$ as, see also Figure 2(a),

$$K(\hat{x}_0) = \left[\begin{array}{c|c} A(\hat{x}) + B(\hat{x})F(\hat{x}) + H(\hat{x})[C(\hat{x}) + D(\hat{x})F(\hat{x})] & -H(\hat{x}) \\ \hline F(\hat{x}) & 0 \end{array}\right]_{\hat{x}(0) = \hat{x}_0} \tag{19}$$

where $F(\hat{x})$ is the nonlinear state feedback gain, and $-H(\hat{x})$ is the nonlinear output injection in the estimator.

Of course, in the linear case, when $A(.)$, $B(.)$, $F(.)$ etc. are not state-dependent, then $K(\hat{x}_0)$ stabilizes $G(\hat{x}_0)$ for arbitrary $\hat{x}_0$, x_0 when $\dot{\xi} = (A + BF)\xi$ and $\dot{\zeta} = (A + HC)\zeta$ are asymptotically stable. Moreover, the effects of initial conditions $\hat{x}_0 \neq x_0$ decay exponentially. Stabilizing F, H are readily found given the conditions $[A, B]$ completely controllable and $[A, C]$ completely observable.

In the nonlinear case studied here, let us first consider the nominal plant/controller pair $\{G(x_0)|_{x_0 = \hat{x}_0}, K(\hat{x}_0)\}$. Also, in order to proceed with a theory that transparently specializes to



(a) Usual state estimate feedback arrangement. (b) Nominal plant $G(x_0)$ with controller $K(\hat{x}_0)$

Figure 2. Equivalent loops for the pair $\{G(\hat{x}), K(\hat{x})\}$

cannot, in general, be factored as

$$\begin{bmatrix} \tilde{V}(x_v(0)) & \tilde{U}(x_u(0)) \\ \tilde{N}(x_n(0)) & \tilde{M}(x_m(0)) \end{bmatrix}$$

where

$$G = \tilde{M}^{-1}(x_m(0))\tilde{N}(x_n(0)), \qquad K = \tilde{V}^{-1}(x_v(0))\tilde{U}(x_u(0)) \tag{31}$$

as in the linear case since superposition does not hold for nonlinear systems. To see the difficulties, note that, omitting the initial conditions, then for all u_1, u_2

$$\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & U \\ N & V \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \tilde{V}(Mu_1 + Uu_2) - \tilde{U}(Nu_1 + Vu_2) \\ -\tilde{N}(Mu_1 + Uu_2) + \tilde{M}(Nu_1 + Vu_2) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\neq \begin{bmatrix} (\tilde{V}M - \tilde{U}N)u_1 + (\tilde{V}U - \tilde{U}V)u_2 \\ (\tilde{M}N - \tilde{N}M)u_1 + (\tilde{M}V - \tilde{N}U)u_2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} \tilde{V}M - \tilde{U}N = I; & \tilde{V}U - \tilde{U}V = 0 \\ \tilde{M}N - \tilde{N}M = 0; & \tilde{M}V - \tilde{N}U = I \end{bmatrix}$$

Consequently, since $\tilde{M}N = \tilde{N}M$, $\tilde{V}U = \tilde{U}V$ by assumption, then both $\tilde{V}M - \tilde{U}N = I$ and $\tilde{M}V - \tilde{N}U = I$ cannot be simultaneously satisfied in general.

(2) To demonstrate why we cannot achieve the left factorizations (31) for our class of systems, in general, consider the cascade $K = \tilde{V}^{-1}\tilde{U}$, omitting the initial conditions. Now, in general, the state space matrices of $\tilde{U}$ are a function of the input to $\tilde{U}$. When this input can not be recovered (without differentiation) from the output of $\tilde{U}$, the generic case, then in the cascade $\tilde{V}^{-1}\tilde{U}$, the state-space matrices of $\tilde{V}^{-1}$ do not have access to this input, and thus cannot, in general, equal those of the state space formulation of $\tilde{U}$. Consequently, there is not the possibility of the state space matrices of $\tilde{V}^{-1}$ tracking those of $\tilde{U}$. This situation is avoided in the next sections by guaranteeing via restrictions and or augmentations that the information needed to reconstruct the state-dependence is always available to both members of a cascade.

Robustness properties

Thus far, the work in this section has dealt with the special case of equal initial conditions in the nominal plant and controller and no external disturbances. Such disturbances are dealt with in a later section by introducing certain differential boundedness constraints. Let us now recall a lemma from Reference 14 which we specialize and mildly extend to the class of systems (2), obeying assumptions (8), (20).

Theorem 1[14]

Consider a well-posed and stable system $\{G(x_0), K(x_0)\}$, where $G(x_0), K(x_0)$ fall within the class (2), and the functions obey assumptions (8), (20). Then

$$\begin{bmatrix} M(x_0) & -U(x_0) \\ -N(x_0) & V(x_0) \end{bmatrix}^{-1} \text{ exists and is internally stable} \tag{32}$$

Consider also an arbitrary map, $S(x_0, \hat{x}_0)$ within the class of systems (2). Then $S(x_0, \hat{x}_0)$ within the class of systems (2). Then $S(x_0, \hat{x}_0)$ has a right factorization

$$S(x_0, \hat{x}_0) = P_G(x_0, \hat{x}_0) D_G(x_0, \hat{x}_0)^{-1} \tag{33}$$

The stable right factorizations of the nominal plant $G(x_0)$ and controller $K(\hat{x}_0)$ are given from

$$G(x_0) = N(x_0)M^{-1}(x_0), \qquad K(\hat{x}_0) = U(\hat{x}_0)V^{-1}(\hat{x}_0) \tag{25}$$

Moreover, internal stability of $\{G(x_0), K(\hat{x}_0)\}$ is equivalent to the BIBO stability condition,

$$\begin{bmatrix} I & -K(\hat{x}_0) \\ -G(x_0) & I \end{bmatrix}^{-1} \text{BIBO stable} \Leftarrow \begin{bmatrix} M(x_0) & -U(\hat{x}_0) \\ -N(x_0) & V(\hat{x}_0) \end{bmatrix}^{-1} \text{BIBO stable} \tag{26}$$

$$\begin{bmatrix} M(x_0) & -U(\hat{x}_0) \\ -N(x_0) & V(\hat{x}_0) \end{bmatrix}^{-1} \text{BIBO stable} \Rightarrow \text{the factorizations in (25) are right coprime} \tag{27}$$

Notation and definitions. The definition (24) should be interpreted as

$$M(x_0) = \left[\begin{array}{c|c} A(x) + B(x)F(x) & B(x) \\ \hline F(x) & I \end{array}\right]_{x(0)}, \quad \text{and } U(\hat{x}_0) = \left[\begin{array}{c|c} A(\hat{x}) + B(\hat{x})F\hat{x} & -H(\hat{x}) \\ \hline F(\hat{x}) & 0 \end{array}\right]_{\hat{x}(0)}, \quad \text{etc.}$$

Given M, N, a right $G = NM^{-1}$, then M, N is a *right coprime* factorization of G iff for all unbounded inputs u, Mu or Nu is unbounded. (In the linear case this is the standard definition that N, M have no common zero in the right half plane). The pair $\{G, K\}$ here denotes the feedback system consisting of plant G and controller K as shown in Figure 2(b). *Internal stability* of a feedback pair is defined as being BIBO stability for all possible additional inputs to the loop with outputs being the outputs of the systems in the feedback loop.

Proof. Defining x_M and x as the states of M and G respectively, then cascade $G(x_0)$ with $M(x_0)$,

$$G(x_0)M(x_0) = \left[\begin{array}{cc|c} A(x_M) + B(x_M)F(x_M) & 0 & B(x_M) \\ B(x)F(x_M) & A(x) & B(x) \\ \hline D(x)F(x_M) & C(x) & D(x) \end{array}\right]_{\substack{x_M(0) = x_0 \\ x(0)}} \tag{28}$$

From (28), and defining the output of the block M driven by u as $y_M \equiv u + F(x_M)x_M$, then

$$\begin{aligned} \dot{x}_M &= A(x_M)x_M + B(x_M)y_M; \qquad & x_M(0) &= x_0 \\ \dot{x} &= A(x)x + B(x)y_M; & x(0) &= x_0 \end{aligned} \tag{29}$$

Now from (29) and under the uniqueness Assumption (8), we have $x_M(t) = x(t)$, $t \geqslant 0$, so that

$$G(x_0)M(x_0) = \left[\begin{array}{cc|c} A(x) + B(x)F(x) & 0 & B(x) \\ 0 & A(x) + B(x)F(x) & 0 \\ \hline C(x) + D(x)F(x) & C(x) & D(x) \end{array}\right]_{\substack{x(0) \\ x(0) - x(0) = 0}} = N(x_0) \tag{30}$$

where the last equation follows from a co-ordinate basis change, and removal of uncontrollable and unobservable modes. Then right multiplication by $M^{-1}(x_0)$ gives $G(x_0) = N(x_0)M^{-1}(x_0)$ as required. Likewise the dual case for the controller factorization is established, and stability is given by the assumption (20). The coprimeness and stability conditions are Theorem 2.1 and Lemma 2.2 of Reference 14.

Remarks

(1) The second inverse in (26) can be written down from (24) via Lemma (1), but appears instructive only in special cases, such as the linear case when its stability is guaranteed by a H selection such that $\dot{\zeta} = (A + HC)\zeta$ is asymptotically stable. However, this inverse

The equation (40) can be inverted by Lemma 1, since the 'D' function is the identity, which is trivially invertible.

From Figure 3 we have

$$y_1 = M(x_0)^{-1}(U(x_0)S(x_0, \hat{x}_0)y_1 + u_1) \tag{41}$$

$$e_1 = (N(x_0) - V(x_0)S(x_0, \hat{x}_0))y_1 \tag{42}$$

Then since it has been established that $M(x_0) - U(x_0)S(x_0, \hat{x}_0)$ is invertible, (41) can be reformulated in the form

$$y_1 = (M(x_0) - U(x_0)S(x_0, \hat{x}_0))^{-1}u_1 \tag{43}$$

Then combining (43) with (42) gives (36) as required. □

Corollary 1

With the conditions of Theorem 1 holding then

$$S(x_0, \hat{x}_0) \text{ BIBO stable} \Leftrightarrow N_S(x_0, \hat{x}_0), M_S(x_0, \hat{x}_0) \text{ coprime}$$

Proof. We can express $N_S(x_0, \hat{x}_0)$, $M_S(x_0, \hat{x}_0)$ in the form

$$\begin{bmatrix} I \\ S(x_0, \hat{x}_0) \end{bmatrix} = \begin{bmatrix} M(x_0) & -U(x_0) \\ -N(x_0) & V(x_0) \end{bmatrix}^{-1} \begin{bmatrix} M_S(x_0, \hat{x}_0) \\ -N_S(x_0, \hat{x}_0) \end{bmatrix} \tag{45}$$

($\Rightarrow$)

The stability property (32), and the above equation (45) give $N_S(x_0, \hat{x}_0)$, $M_S(x_0, \hat{x}_0)$ BIBO stable. From above, and pre-multiplying by [1 0] we have the Bezout

$$I = [I \;\; 0] \begin{bmatrix} M(x_0) & -U(x_0) \\ -N(x_0) & V(x_0) \end{bmatrix}^{-1} \begin{bmatrix} M_S(x_0) \\ N_S(x_0) \end{bmatrix} \tag{46}$$

The stability property (32) also guarantees stability of the matrix

$$[I \;\; 0] \begin{bmatrix} M(x_0) & -U(x_0) \\ -N(x_0) & V(x_0) \end{bmatrix}^{-1}$$

and consequently we have $M_S(x_0, \hat{x}_0)$, $N_S(x_0, \hat{x}_0)$ coprime by Lemma 2.1 of Reference 14.

($\Leftarrow$)

If $N_S(x_0, \hat{x}_0)$, $M_S(x_0, \hat{x}_0)$ are coprime then they are stable, and (45) gives $S(x_0, \hat{x}_0)$ BIBO stable. □

Remarks

(1) We have reached a major objective of this section, namely to achieve a right factorization for the feedback pair $\{G_S(\hat{x}_0), K(x_0)\}$ in terms of a factorization of the pairs $\{G(x_0), K(x_0)\}$. An interesting special case is when $G_S(\hat{x}_0) = G(\hat{x}_0)$. This case represents a nominal plant, but with initial conditions not necessarily equal to those of the controller.

(2) It is not possible to generate a complete robustness theory based only on the material in this section. To facilitate the robustness theory, in the following sections we restrict the class of plant and controller or work with augmentation forms, then achieve stabilization results for these situations. For the case of augmentations, results are generated which relate back to the standard plants and controllers.

where the inverse is guaranteed to exist, and

$$\begin{bmatrix} D_G(x_0,\hat{x}_0) \\ P_G(x_0,\hat{x}_0) \end{bmatrix} = \begin{bmatrix} I \\ S(x_0,\hat{x}_0) \end{bmatrix} (M(x_0) - U(x_0)S(x_0,\hat{x}_0))^{-1} \tag{34}$$

$$M(x_0)D_G(x_0,\hat{x}_0) - U(x_0)P_G(x_0,\hat{x}_0) = I \tag{35}$$

Further there exists a plant $G_S(\hat{x}_0)$ as depicted in Figure 3 such that

$$\begin{aligned} G_S(\hat{x}_0) &= N(x_0)D_G(x_0,\hat{x}_0) - V(x_0)P_G(x_0,\hat{x}_0) \\ &= N_S(x_0,\hat{x}_0)M_S(x_0,\hat{x}_0)^{-1} \end{aligned} \tag{36}$$

where

$$N_S(x_0,\hat{x}_0) = N(x_0) - V(x_0)S(x_0,\hat{x}_0); \qquad M_S(x_0,\hat{x}_0) = M(x_0) - U(x_0)S(x_0,\hat{x}_0) \tag{37}$$

Also, given an arbitrary plant $G_S(\hat{x}_0)$ in the class (2), then it can be parametrized in terms of $S(.)$ given by (33) where

$$\begin{bmatrix} D_G(x_0,\hat{x}_0) \\ P_G(x_0,\hat{x}_0) \end{bmatrix} = \begin{bmatrix} M(x_0) & -U(x_0) \\ -N(x_0) & V(x_0) \end{bmatrix}^{-1} \begin{bmatrix} I \\ -G_S(\hat{x}_0) \end{bmatrix} \tag{38}$$

$G_S(\hat{x}_0)$ has a right factorization (36), and again (33), (34), (35) hold.

Proof. Most of the proof follows as in Reference 14. It remains only to observe that with the definitions of the theorem holding, then the factor $(M(x_0) - U(x_0)S(x_0,\hat{x}_0))^{-1}$ will exist for any $S(x_0, \hat{x}_0)$, and to show that Figure 3 represents $G_S(x_0)$.

From the definitions in (24) and denoting by $*$ functions not relevant to the argument,

$$M(x_0) = \left[\begin{array}{c|c} * & * \\ \hline * & I \end{array}\right]; \qquad U(x_0) = \left[\begin{array}{c|c} * & * \\ \hline * & 0 \end{array}\right]; \qquad S(x_0,\hat{x}_0) = \left[\begin{array}{c|c} * & * \\ \hline * & * \end{array}\right]$$

Then by (10) we have

$$U(\hat{x}_0)S(x_0,\hat{x}_0) = \left[\begin{array}{c|c} * & * \\ \hline * & 0 \end{array}\right] \tag{39}$$

thus we can express

$$M(x_0) - U(x_0)S(x_0,\hat{x}_0) = \left[\begin{array}{c|c} * & * \\ \hline * & I \end{array}\right] \tag{40}$$

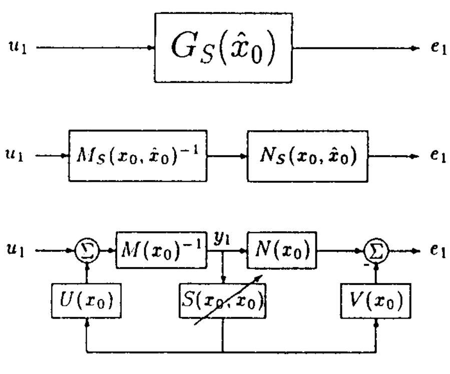


Figure 3. The system $G_S(x_0)$

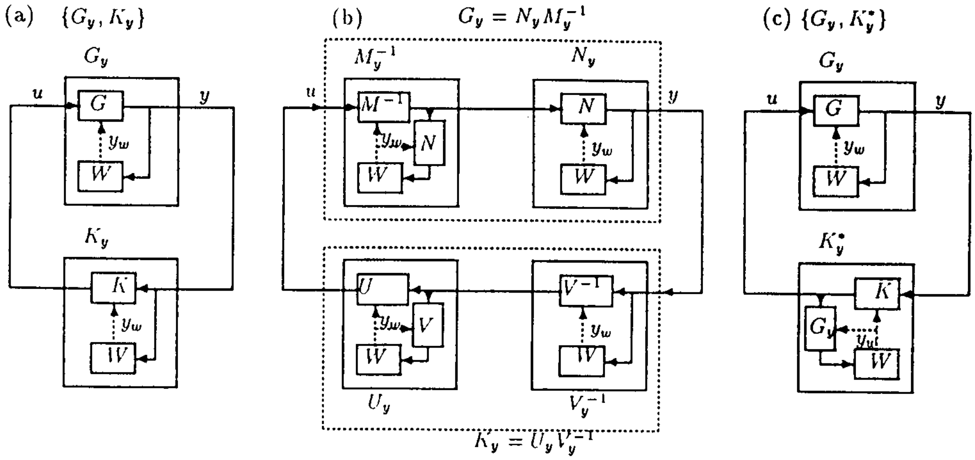


Figure 4. The feedback systems $\{G_y, K_y\}$, $\{G_y, K_y^*\}$

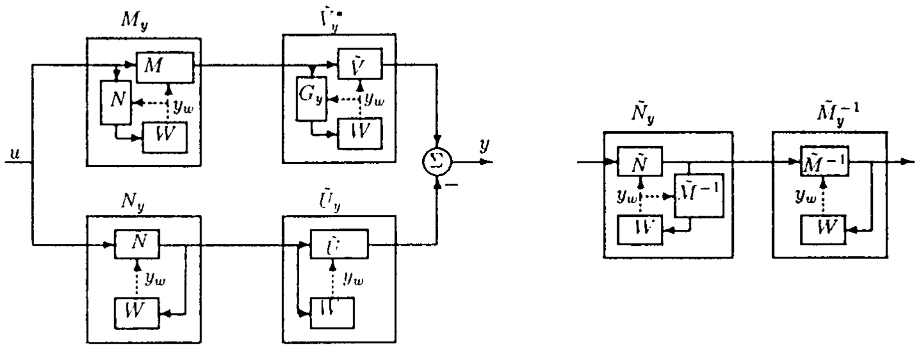


Figure 5. (a) The Bezout $\tilde{V}_y^* M_y - \tilde{U}_y N_y = I$; (b) $G_y = \tilde{M}_y^{-1}\tilde{N}$

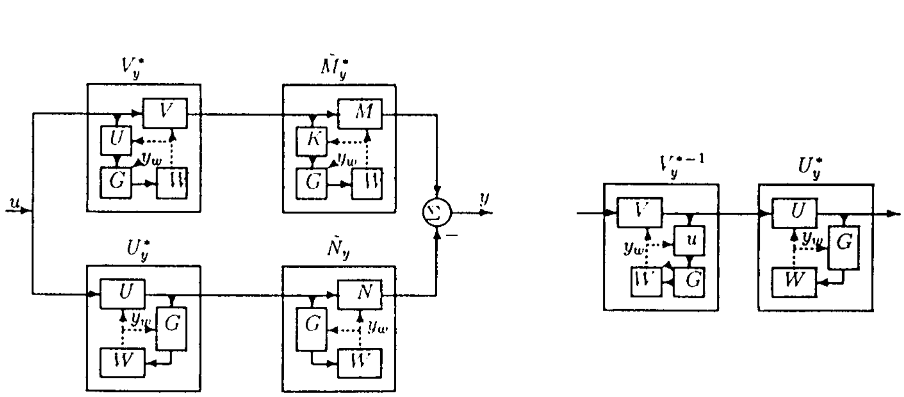


Figure 6. (a) The Bezout $\tilde{M}_y^* V_y^* - \tilde{N}_y U_y^* = I$; (b) $K_y^* = U_y^* V_y^{*-1}$

3. SYSTEMS WITH OUTPUT-DEPENDENT NONLINEARITIES

In the previous section, the systems considered had *state*-dependent nonlinearities. Of course, a mild generalization would have permitted filtered state-dependent nonlinearities. Here we specialize to *output* dependent nonlinearities to achieve a more complete factorization theory, including stable left factorizations, and Bezout identities. To avoid any algebraic loop that might arise in an implementation of $y = C(y)x + D(y)u$, and to widen the class of systems, we introduce a strictly causal filter on y giving y_w, so that $y = C(y_w)x + D(y_w)u$. We foreshadow that to achieve our objectives of stable left factorization, not only do we need the restrictions introduced so far on initial conditions and output nonlinearities, but also we need to work with plant/controller nominal models that only make sense in a feedback arrangement.

We proceed by considering the plant/controller arrangement as depicted in Figure 4(a). More precisely, let us define a plant as follows.

$$G_y(x_0) : \begin{array}{l} \dot{x} = A(y_w)x + B(y_w)u; \quad x(0) = x_0 \\ \dot{x}_w = A_w x_w + B_w y; \quad x_w(0) \\ y = C(y_w)x + D(y_w)u \\ y_w = C_w x_w \end{array} \quad : \left[\begin{array}{c|c} A(y_w) & B(y_w) \\ \hline C(y_w) & D(y_w) \end{array}\right]^{W(y)}_{x(0)} \tag{47}$$

Here y_w is the output of a filter W driven by y where

$$W : \left[\begin{array}{c|c} A_w & B_w \\ \hline C_w & 0 \end{array}\right]_{x_w(0)} \tag{48}$$

Of course, any member of (47) with states $[x' \;\; x_w']'$ is a specialization of the more general class of nonlinear systems (7).

Note also that when the inverse of the nonlinear system exists, as when $D^{-1}(x)$ exists, then W can be taken to have the state tracking properties of this inverse (requiring generalization of A_w to $A_w(y_w)$ and B_w to $B_w(y_w)$). Now setting $C_w = I$ makes y_w equivalent to the state of the inverse system which is x itself.

The feedback controller K_y: $u \to y$ is likewise more precisely defined as

$$K_y(x_0) : \left[\begin{array}{c|c} A(y_w) + B(y_w)F(y_w) + H(y_w)[C(y_w) + D(y_w)F(y_w)] & -H(y_w) \\ \hline F(y_w) & 0 \end{array}\right]^{W(u)}_{\hat{x}(0) = \hat{x}_0} \tag{49}$$

We claim below right factorizations of G_y, K_y, as depicted in Figures 4(b), 5 and 6 where the operator notation can be interpreted in state-space terms in the following example for N_y.

$$N_y : \left[\begin{array}{cc|c} A(y_w) + B(y_w)F(y_w) & 0 & B(y_w) \\ B_w(y_w)[C(y_w) + D(y_w)F(y_w)] & A_w(y_w) & B_w(y_w)D(y_w) \\ \hline C(y_w) + D(y_w)F(y_w) & 0 & D(y_w) \end{array}\right]_{\substack{x(0) = x_0 \\ x_w(0)}}$$

$$y_w = \text{causal filtered version of } C_w(y_w)x_w \tag{50}$$

Likewise, the definitions allow state-space definitions for other operators V_y, U_y, M_y etc. depicted in Figures 4, 5 and 6 can be formulated, and the left fractional descriptions for $\tilde{M}_y^*, \tilde{M}_y, \tilde{N}_y, \tilde{V}_y^*, \tilde{U}_y$ can be generated from the linear versions as in Reference 6 with the appropriate state dependence added as in Figures 4, 5 and 6.

Note that the pair $\{G_y, K_y\}$ here denotes the feedback system consisting of plant G_y and controller K_y as shown in Figure 4(a).

directly establish coprimeness of the factorization $G_y = \tilde{M}_y^{-1}\tilde{N}_y$, but this does not give rise to a Bezout identity, in general. Also in the linear case, the BIBO stability of factories is guaranteed with $\dot{\zeta} = (A + BF)\zeta$ asymptotically stable.

(2) To ensure BIBO stability of the factors in the nonlinear case, it makes sense to examine the following assumption:

Assumption: The state estimate feedback gain $F(y_w)$, and state output injection $H(y_w)$ are constructed such that

$$\dot{\xi} = [A(y_w) + B(y_w)F(y_w)]\,\xi, \quad \xi(0)$$
$$\dot{\zeta} = [A(y_w) + H(y_w)C(y_w)]\,\zeta, \quad \zeta(0)$$

are exponentially stable for arbitrary initial conditions $\xi(0), \zeta(0)$, for any admissible trajectory y_w. (55)

We don't claim here that such an assumption can be satisfied, except possibly for a limited set of trajectories y_w, or even that a complete theory can be based on this assumption.

(3) Factors such as $\tilde{V}_y^*$ are introduced in the lemma since it is not possible, in general, to find a $\tilde{V}_y$ such that $K_y = \tilde{V}_y^{-1}\tilde{U}_y$ and also $\tilde{V}_y M_y - \tilde{U}_y N_y = I$, at least with the factors being obvious generalizations of the linear ones where the matrices $A(.), B(.)$, etc. are all functions of the one variable, viz. y_w.

(4) A further limitation of the nonlinear theory is evident from Figure 7, as now explained. Let us express $K = \tilde{V}_X^{-1}\tilde{U}_Y$ as in Figure 7(b), where X and Y are filters generating the variables x_i for $i = 1, 2, 3, 4$ which feed into the relevant state space matrices $A(.), B(.)$, etc. In order for $\tilde{V}_X^{-1}\tilde{U}_Y$ of Figure 7(b) to generalize the linear results using the methodology of this paper, we require $x_3 = x_4$. But, from Figure 7(b),

$$Y = X\tilde{V}_X^{-1}\tilde{U}_Y = XK \Leftrightarrow \{x_3 = x_4\} \tag{56}$$

Similarly, for the Bezout $\tilde{V}_X M_y - \tilde{U}_Y N_y = I$ to hold as required in Figure 7(a), using our methodology we require $x_1 = x_2$. But from Figure 7(a)

$$XM_y = YN_y \Leftrightarrow X = YG \Leftrightarrow \{x_1 = x_4\} \tag{57}$$

Combining (56) and (57) we have

$$\{x_1 = x_2\} \quad \text{and} \quad \{x_3 = x_4\} \Rightarrow X = XKG \tag{58}$$

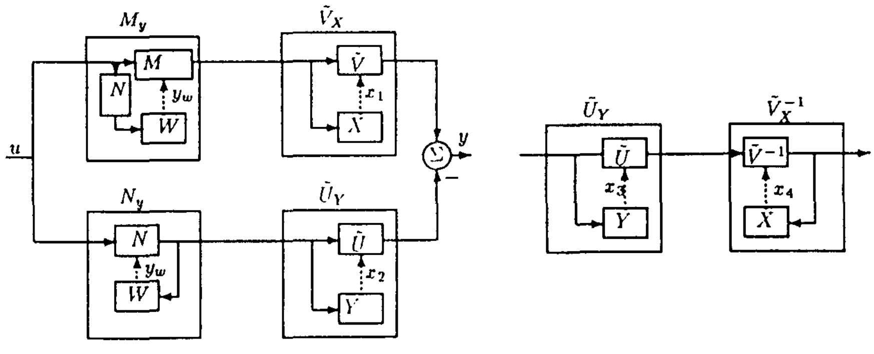


Figure 7. (a) The Bezout $\tilde{V}_X M_y - \tilde{U}_Y N_y = I$; (b) The system $\tilde{V}_X^{-1}\tilde{U}_Y = K$

and since in a well-posed system $KG \neq I$, then in the nonlinear case where $X \neq 0$ we cannot achieve simultaneously, $\tilde{V}_X M_y - U_Y N_y = I$, $\tilde{V}_X^{-1}\tilde{U}_Y = K$ for any $X, Y \neq 0$, at least in our 'linear' approach. Dual arguments can be constructed to justify the need for working with other starred versions $K^* = U_y^* V_y^{*-1}$, etc.

(5) It is possibly to apply a version of Theorem 3 to cope with unequal initial conditions.

4. AUGMENTED SYSTEMS FACTORIZATIONS

As shown in the previous section, it appears difficult to construct left factorizations associated with the nominal plant/controller pair $\{G(\hat{x}_0), K(\hat{x}_0)\}$ without certain modifications and restrictions. In order to proceed in this section, we propose an alternative 'trick' of first obtaining factorizations and stability results for an augmented feedback pair $\{\mathcal{G}(\hat{x}_0), \mathcal{K}(\hat{x}_0)\}$, and thereby achieve stability results of a related pair $\{G(x_0), \bar{\mathcal{K}}(\hat{x}_0)\}$ trivially different from the original feedback pair $\{G(x_0), K(\hat{x}_0)\}$. Thus in the first instance, consider the feedback pair $\{G(x_0), K(x_0\}$ of Figure 2, re-organized as the pair $\{G(x_0), \bar{\mathcal{K}}(\hat{x}_0)\}$. Where

$$G(x_0) = \left[\begin{array}{c|c} A(x) & B(x) \\ \hline C(x) & D(x) \end{array}\right]_{x(0) = x_0};$$

$$\bar{\mathcal{K}}(\hat{x}_0) = \left[\begin{array}{cc|c} A(x_g) + B(x_g)F(x_g) + H(x_g)(C(x_g) + D(x_g)F(x_g)) & 0 & -H(x_g) \\ B(x_g)F(x_g) & A(x_g) & 0 \\ \hline F(x_g) & 0 & 0 \end{array}\right]_{\left[\begin{smallmatrix} \hat{x}(0) = \hat{x}_0 \\ x_g(0) = x_0 \end{smallmatrix}\right]} \tag{59}$$

The situation is depicted in shorthand notation in Figure 8(a). Clearly, without external inputs and with $x_0 = \hat{x}_0$, then the pair $\{G(\hat{x}_0), \bar{\mathcal{K}}(\hat{x}_0)\}$ behaves as $\{G(\hat{x}_0), K(\hat{x}_0)\}$ in terms of states and system inputs and outputs. Consider now the further re-organization of $\{G(x_0), \bar{\mathcal{K}}(\hat{x}_0)\}$ as an

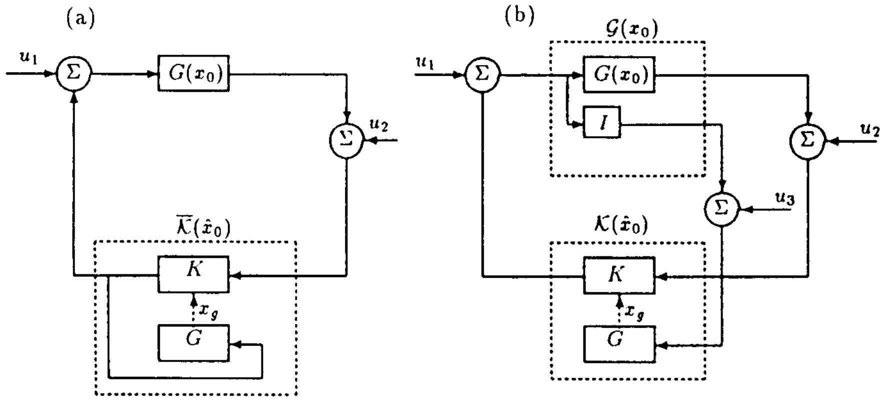


Figure 8. (a), (b) The systems $\{G(x_0), \bar{\mathcal{K}}(x_0)\}$ and $\{\mathcal{G}(x_0), \mathcal{K}(\hat{x}_0)\}$ of (59)

augmented pair $\{\mathcal{G}(x_0), \mathcal{K}(x_0)\}$, depicted in Figure 8(b) where

$$\mathcal{G}(x_0) = \begin{bmatrix} G(x_0) \\ I \end{bmatrix};$$

$$\mathcal{K}(\hat{x}_0) = \left[\begin{array}{cc|cc} A(x_g) + B(x_g)F(x_g) + H(x_g)(C(x_g) + D(x_g)F(x_g)) & 0 & -H(x_g) & 0 \\ 0 & A(x_g) & 0 & B(x_g) \\ \hline F(x_g) & 0 & 0 & 0 \end{array}\right]_{\begin{bmatrix} \hat{x}(0) = \hat{x}_0 \\ x_g(0) = x_0 \end{bmatrix}} \tag{60}$$

Again, in the absence of external inputs the states of the pair $\{G(x_0), \bar{\mathcal{K}}(\hat{x}_0)\}$ are identical to those of $\{\mathcal{G}(x_0), \mathcal{K}(\hat{x}_0)\}$. In order to proceed, we recall a stability of definition.

Definition. The system $\{G(x_0), K(\hat{x}_0)\}$ is said to be $\varepsilon_1, \varepsilon_2$ bounded-input stable, iff for all inputs u_1, u_2 such that $|u_1| \leqslant \varepsilon_1, |u_2| \leqslant \varepsilon_2$ the outputs y_1, y_2 and e_1, e_2 are bounded.

Lemma 6

With $\mathcal{G}(x_0), \mathcal{K}(\hat{x}_0), \bar{\mathcal{K}}(\hat{x}_0)$ defined in (59), (60), and given positive constants $\varepsilon_1, \varepsilon_2, \varepsilon_3$ then

$$\{\mathcal{G}(x_0), \mathcal{K}(\hat{x}_0)\} \text{ is } \varepsilon_1, \begin{bmatrix} \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} \text{ bounded-input stable} \Rightarrow \{G(x_0), \bar{\mathcal{K}}(\hat{x}_0)\} \text{ is } \min(\varepsilon_1, \varepsilon_3), \varepsilon_2 \text{ bounded-input stable} \tag{61}$$

Proof. From Figure 8 it is immediate that the feedback loop of $\{G(x_0), \bar{\mathcal{K}}(\hat{x}_0)\}$ of Figure 8(a) is simply a specialization of the feedback loop $\{\mathcal{G}(x_0), \mathcal{K}(\hat{x}_0)\}$ shown in Figure 8(b) taking $u_3 = -u_1$. Now define $\varepsilon_{\min}$ as $\min(\varepsilon_1, \varepsilon_3)$, then we have that $\{\mathcal{G}(x_0), \mathcal{K}(\hat{x}_0)\}$ is

$$\varepsilon_{\min}, \begin{bmatrix} \varepsilon_2 \\ \varepsilon_{\min} \end{bmatrix}$$

bounded-input stable. Thus it is stable for any input signals

$$\varepsilon_a, \begin{bmatrix} \varepsilon_b \\ -\varepsilon_a \end{bmatrix}$$

where $\varepsilon_a \leqslant \varepsilon_{\min}$; $\varepsilon_b \leqslant \varepsilon_2$. The system $\mathcal{G}(x_0), \mathcal{K}(\hat{x}_0)$ with input

$$\varepsilon_a, \begin{bmatrix} \varepsilon_b \\ -\varepsilon_a \end{bmatrix}$$

is trivially equivalent to the system $G(x_0), \bar{\mathcal{K}}(\hat{x}_0)$ with input $\varepsilon_1, \varepsilon_b$, thus the stability property carries over to this case giving (61).

Remark. This lemma tells us that developing a factorization and robust stability theory associated with $\{\mathcal{G}(x_0), \mathcal{K}(\hat{x}_0)\}$ will give corresponding stability properties for $\{G(x_0), \bar{\mathcal{K}}(\hat{x}_0)\}$ which in turn can be considered as an idealized nominal version of the pair $\{G(x_0), K(\hat{x}_0)\}$. Any differences between the nominal and actual controller can be taken into account in the same way as differences between the nominal plant and actual plant.

We propose factorizations as follows

$$\mathcal{G}(x_0) = \tilde{\mathcal{M}}^{-1}(x_0)\tilde{\mathcal{N}}(x_0) = \mathcal{N}(x_0)\mathcal{M}^{-1}(x_0); \quad \mathcal{K}(x_0) = \tilde{\mathcal{V}}^{-1}(x_0)\tilde{\mathcal{U}}(x_0) \tag{62}$$

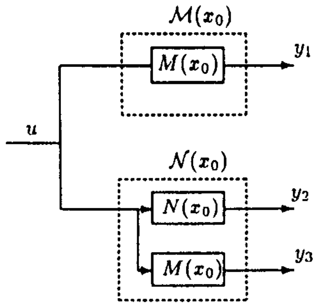


Figure 9. The block $\begin{bmatrix} \mathcal{M}(x_0) \\ \mathcal{N}(x_0) \end{bmatrix}$

where

$$\begin{bmatrix} \mathcal{M}(x_0) \\ \mathcal{N}(x_0) \end{bmatrix} = \left[\begin{array}{c|c} A(x) + B(x)F(x) & B(x) \\ \hline F(x) & I \\ \begin{bmatrix} C(x) + D(x)F(x) \\ F(x) \end{bmatrix} & \begin{bmatrix} D(x) \\ I \end{bmatrix} \end{array}\right]_{[x(0) = x_0]} \tag{63}$$

This situation is depicted in the sub-blocks $\mathcal{N}(x_0)$, $\mathcal{M}(x_0)$ of Figure 9 where $N(x_0)$, $M(x_0)$ are defined in (24). Likewise, we propose

$$\begin{bmatrix} \tilde{\mathcal{V}}(x_0) \\ \tilde{\mathcal{N}}(x_0) \end{bmatrix} = \left[\begin{array}{cc|c} A(x) & 0 & B(x) \\ 0 & A(x) + H(x)C(x) & B(x) + H(x)D(x) \\ \hline 0 & -F(x) & I \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} C(x) \\ 0 \end{bmatrix} & \begin{bmatrix} D(x) \\ I \end{bmatrix} \end{array}\right]_{\begin{bmatrix} x(0) = x_0 \\ x_2(0) = x_0/2 \end{bmatrix}} \tag{64}$$

and

$$\begin{bmatrix} \tilde{\mathcal{U}}(x_0) \\ \tilde{\mathcal{M}}(x_0) \end{bmatrix} = \left[\begin{array}{cc|cc} A(x) & 0 & 0 & B(x) \\ 0 & A(x) + H(x)C(x) & -H(x) & 0 \\ \hline 0 & F(x) & 0 & 0 \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} -C(x) \\ 0 \end{bmatrix} & \begin{bmatrix} I \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} \end{array}\right]_{\begin{bmatrix} x(0) = x_0 \\ x_2(0) = x_0/2 \end{bmatrix}} \tag{65}$$

The feedback system $\{\mathcal{G}(x_0), \mathcal{K}(x_0)\}$, with the above left factorizations is shown in Figure 10. To ensure (BIBO) stability of these left factorizations, we impose (20) and its 'dual', viz.

Assumption: The state output injection $H(\xi)$, is constructed such that the system

$$\begin{aligned} \dot{\xi} &= A(\xi)\xi; \quad \xi(0) = \xi_0 \\ \dot{\zeta} &= [A(\xi) + H(\xi)C(\xi)]\zeta; \quad \zeta(0) = \zeta_0 \end{aligned} \tag{66}$$

has an exponentially decaying partial state ζ

Remark: Note that in the systems $\tilde{\mathcal{N}}, \tilde{\mathcal{V}}, \tilde{\mathcal{M}}, \tilde{\mathcal{U}}, \mathcal{M}, \mathcal{N}$, the matrices $A(.), B(.), \ldots$ are all functions of x which is the state of the nominal plant $G(x_0)$ driven by the inputs to $\mathcal{N}, \tilde{\mathcal{V}}$ etc., respectively. In the systems $\tilde{\mathcal{M}}, \tilde{\mathcal{U}}$ the matrices are functions of the state of a nominal plant $G(x_0)$ driven by the 'augmented' input to $\tilde{M}, \tilde{U}$, respectively. This can be seen from Figure 10.

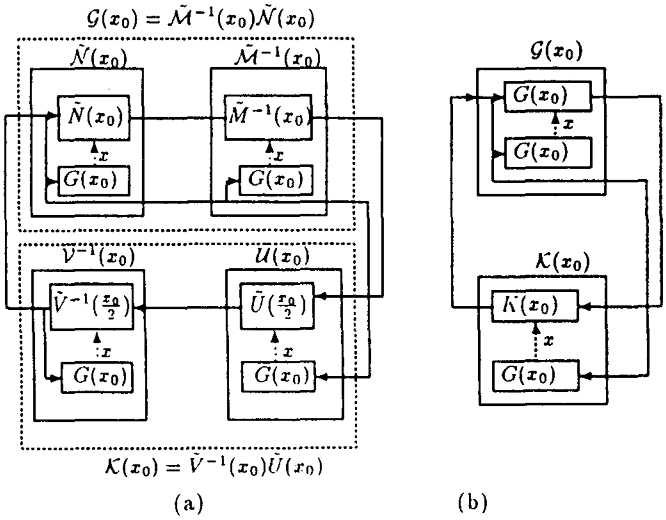


Figure 10. Equivalent loops for the feedback system, $\{\mathcal{G}(x_0), \mathcal{K}(x_0)\}$

Lemma 7

Consider the system $\mathcal{G}(x_0), \mathcal{K}(x_0)$ as defined in (60). Then, (62) holds with the definitions (63)–(65). Moreover, under the assumptions (8), (20), (66), the factors are BIBO stable and $\mathcal{M}, \mathcal{N}$ are right coprime satisfying the Bezout identity

$$\tilde{\mathcal{V}}(x_0), \mathcal{M}(x_0) - \tilde{\mathcal{U}}(x_0)\mathcal{N}(x_0) = I \tag{67}$$

Proof

$$\mathcal{G}(x_0)\mathcal{M}(x_0) = \left[\begin{array}{c|c} A(x) & B(x) \\ \hline C(x) & D(x) \\ 0 & I \end{array}\right]_{[x(0) = x_0]} \left[\begin{array}{c|c} A(x_1) + B(x_1)F(x_1) & B(x_1) \\ \hline F(x_1) & I \end{array}\right]_{[x_1(0) = x(0)]}$$

$$= \left[\begin{array}{cc|c} A(x_1) + B(x_1)F(x_1) & 0 & B(x_1) \\ B(x)F(x_1) & A(x) & B(x) \\ \hline D(x)F(x_1) & C(x) & D(x) \\ F(x_1) & 0 & I \end{array}\right]_{\left[\begin{array}{l} x_1(0) = x_0 \\ x(0) = x_0 \end{array}\right]}$$

$$= \mathcal{N}(x_0) \tag{68}$$

The equalities follow since by the uniqueness Assumption (8), with initial conditions $x_1(0) = x_0$, then $x_1(t) = x(t) \forall t \geqslant 0$. The second equality is simply the removal of an unobservable mode. Since $\mathcal{M}^{-1}(x_0)$ exists via Lemma (1), then $\mathcal{G}(x_0) = \mathcal{N}(x_0)\mathcal{M}^{-1}(x_0)$ as in (62). The proofs of the remaining factorizations of (62) are similar.

To prove the coprimeness of $M(x_0)$, $N(x_0)$ we first verify (67). Now

$$\tilde{\mathscr{V}}(x_0)\mathscr{M}(x_0) - \tilde{\mathscr{U}}(x_0)\mathscr{N}(x_0)$$

$$= \left[\begin{array}{ccc|c} A(x_2) + B(x_2)F(x_2) & 0 & 0 & B(x_2) \\ B(x)F(x_2) & A(x) & 0 & B(x) \\ B(x)F(x_2) + H(x)D(x)F(x_2) & 0 & A(x) + H(x)C(x) & B(x) + H(x)D(x) \\ \hline F(x) & 0 & -F(x) & I \end{array}\right]_{\left[\begin{smallmatrix} x_2(0) = x_0 \\ x(0) = x_0 \\ x_1(0) = x_0/2 \end{smallmatrix}\right]}$$

$$- \left[\begin{array}{ccc|c} A(x_3) + B(x_3)F(x_3) & 0 & 0 & B(x_3) \\ B(x_4)F(x_4) & A(x_4) & 0 & B(x_4) \\ -(H(x_4)C(x_3) + H(x_4)D(x_3)F(x_3)) & 0 & A(x_4) + H(x_4)C(x_4) & -H(x_4)D(x_4) \\ \hline 0 & 0 & F(x_2) & 0 \end{array}\right]_{\left[\begin{smallmatrix} x_3(0) = x_0 \\ x_4(0) = x_0 \\ x_5(0) = x_0/2 \end{smallmatrix}\right]}$$

where x, x_1 are the states of $\tilde{\mathscr{V}}$, x_2 of $\mathscr{M}$, x_3 of $\mathscr{N}$, and x_4, x_5 of $\tilde{\mathscr{U}}$. Now, both systems operators on the r.h.s. of the above equation will have the same inputs. Thus applying the uniqueness Assumption (8) (as in Lemma 2 proof) we have that the partial states satisfy

$$\begin{bmatrix} x_2(t) \\ x(t) \end{bmatrix} = \begin{bmatrix} x_3(t) \\ x_4(t) \end{bmatrix} \forall t \geqslant 0$$

and thus $A(x_2) \equiv A(x_3)$, etc. Moreover, by applying the uniqueness Assumption (8) to the state equations for x_2 and x, then it is clear that $x_2(t) = x(t) \forall t \geqslant 0$, so that $x(t) = x_2 = x_3(t) = x_4(t)$. So denoting $A \equiv A(x) \equiv \ldots$, then after a co-ordinate basis change and deletion of unobservable and uncontrollable modes, we have

$$\tilde{\mathscr{V}}(x_0)\mathscr{M}(x_0) - \tilde{\mathscr{U}}(x_0)\mathscr{N}(x_0) = \left[\begin{array}{cc|c} A + BF & 0 & B \\ BF + HDF & A + HC & B + HD \\ \hline F & -F & I \end{array}\right]_{\left[\begin{smallmatrix} x(0) = x_0 \\ x_1(0) = x_0/2 \end{smallmatrix}\right]}$$

$$- \left[\begin{array}{cc|c} A + BF & 0 & B \\ -(HC + HDF) & A + HC & -HD \\ \hline 0 & F & 0 \end{array}\right]_{\left[\begin{smallmatrix} x(0) = x_0 \\ x_5(0) = x_0/2 \end{smallmatrix}\right]} \tag{69}$$

Then a co-ordinate basis change

$$\begin{bmatrix} x \\ x_5 \end{bmatrix} \to \begin{bmatrix} x \\ x - x_5 \end{bmatrix}$$

and the resulting subtractions gives the required result (67).

In fact, a study of Figure 11 and knowledge of linear system results allows a shortcut to the proofs. The key is to realize first that the sub-blocks $N(x_0)$, $M(x_0)$, $G(x_0)$ with their inputs depicted in the figure all have the same state $x(t)$ by virtue of Assumption (8) and manipulations such as in the proof of Lemma 2, and consequently that the coefficients $A(.)$, etc. in N, M, U, V are all functions of the same state x. The initial state requirements on the various sub-blocks are identical to those for linear systems to avoid any transients in achieving $y = u$.

From (68), (69), Assumption (66) and Lemma (4), we have that $\mathscr{M}(x_0)$, $\mathscr{N}(x_0)$ are coprime giving the coprime factorization of $\mathscr{G}(x_0) = \mathscr{N}(x_0)\mathscr{M}^{-1}(x_0)$ of (62).

For the left factorization case, the proof of is similar to that of the right case. □

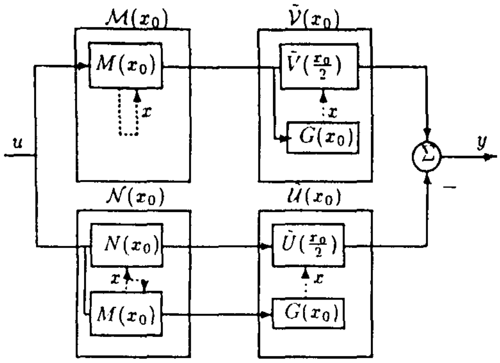


Figure 11. The Bezout $\tilde{\mathcal{V}}(x_0)\mathcal{M}(x_0) - \tilde{\mathcal{U}}(x_0)\mathcal{N}(x_0) = I$

Lemma 8

Given the BIBO stable systems $\mathcal{R}(x_0), \mathcal{S}(x_0)$ as defined in Figure (13), and $\tilde{\mathcal{M}}(x_0), \tilde{\mathcal{N}}(x_0)$ as defined in (64), (65), then

$$\tilde{\mathcal{M}}(x_0)\mathcal{R}(x_0) - \tilde{\mathcal{N}}(x_0)\mathcal{S}(x_0) = \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{70}$$

Proof. The proof follows from the definitions of the factors in Figure (13), and the uniqueness and stability assumptions (8), (20), (66), and the fact that in the linear case $\tilde{M}V - \tilde{N}U = I$. □

Remark. In the linear case left coprimeness of $\tilde{M}, \tilde{N}$ follows from the Bezout $\tilde{M}R - \tilde{N}S = I$, since $\tilde{M}, \tilde{N}, R, S$ are BIBO stable. In the augmented nonlinear case, $\tilde{\mathcal{M}}' = [I\ 0]\tilde{\mathcal{M}}$, and $\tilde{\mathcal{N}}' = [I\ 0]\tilde{\mathcal{N}}$ satisfy $\tilde{\mathcal{M}}'(x_0)\mathcal{R}(x_0) - \tilde{\mathcal{N}}'(x_0)\mathcal{S}(x_0) = I$ from (70) which could be taken as an analogue of left coprimeness for $\tilde{\mathcal{M}}', \tilde{\mathcal{N}}'$. Actually, in the remainder of this paper, we restrict to stability theory of Reference 9 which does not require left coprimeness of factors $\tilde{\mathcal{M}}, \tilde{\mathcal{N}}$ (or even $\tilde{\mathcal{M}}', \tilde{\mathcal{N}}'$), at least in the proof of the results.

Construction of the class of all stabilizing controllers for a nominal plant

Lemma 7 shows that the factorizations (63)–(65) have the properties:

$$G(x_0) = \mathcal{N}(x_0)\mathcal{M}^{-1}(x_0) = \tilde{\mathcal{M}}^{-1}(x_0)\tilde{\mathcal{N}}(x_0) \text{ with the factorizations stable} \tag{71}$$

$$\mathcal{N}, \mathcal{M} \text{ are right coprime, and } \tilde{\mathcal{N}}, \tilde{\mathcal{M}} \text{ obey (70)} \tag{72}$$

$$\tilde{\mathcal{V}}(x_0), \tilde{\mathcal{U}}(x_0) \text{ are defined such that a Bezout identity (51) holds.} \tag{73}$$

In order to establish that the system $\{\mathcal{G}(x_0), \mathcal{K}(x_0)\}$ is robust to small signal injections around the loop we utilize a differential boundedness condition from Reference 7 and exploit results in Reference 9.

Definition. A mapping F is said to be *differentially bounded by* θ_F, ε_F if for all signals a_1, a_2 if $|a_1 - a_2| < \varepsilon_F$ then $|Fa_1 - Fa_2| \leqslant \theta_F$.

Assumptions (8), (66) give (BIBO) stability to the left factors of $\mathcal{G}(x_0), \mathcal{K}(x_0)$, and we make further restrictions on the matrices $A(.), B(.), \ldots, H(.), F(.)$ such that the following property holds:

$$\tilde{\mathcal{M}}, \tilde{\mathcal{N}}, \tilde{\mathcal{V}}, \tilde{\mathcal{U}} \text{ are differentially bounded by } \theta_M, \varepsilon_U; \theta_N, \varepsilon_V; \theta_U, \varepsilon_U \text{ respectively} \tag{74}$$

Also, we define Q to be a (BIBO) stable mapping constrained such that

$$Q\tilde{\mathcal{N}} \text{ is differentially bounded by } \theta_{QN}, \varepsilon_V, \text{ and } Q\tilde{\mathcal{M}} \text{ is differentially bounded by } \theta_{QM}, \varepsilon_U \tag{75}$$

Remark. It is beyond the scope of this paper to give explicit conditions on the matrices $A(.), B(.), \ldots$ so that conditions (74), (75) hold. In the linear case they will hold due to assumptions (8), (20), (66).

Then following Reference 8 let us parametrize

$$\mathcal{K}_Q(x_0) = \tilde{\mathcal{V}}_Q^{-1}(x_0)\tilde{\mathcal{U}}_Q(x_0); \; \tilde{\mathcal{V}}_Q(x_0) = \tilde{\mathcal{V}}(x_0) + Q(x_0)\tilde{\mathcal{N}}(x_0); \; \tilde{\mathcal{U}}_Q(x_0) = \tilde{\mathcal{U}}(x_0) + Q(x_0)\tilde{\mathcal{M}}(x_0) \tag{76}$$

and using these parametrizations, we apply a crucial lemma for the stability of the system:

Lemma 9[9]

Consider the augmented plant $\mathcal{G}(x_0)$ and augmented controller $\mathcal{K}(x_0)$ as defined in (60), with right coprime and stable left factorization of $\mathcal{G}(x_0)$, and a stable left factorizations of $\mathcal{K}(x_0)$ as in (62), and with the properties (71)–(75) and the Bezout (67) holding. Then

1. The system $\{G(x_0), \mathcal{K}_Q(x_0)\}$, with $\mathcal{K}_Q(x_0)$ defined in (76), and illustrated in Figure 12, will be $\varepsilon_V, \varepsilon_U$ bounded-input stable.
2. For every BIBO stable $Q(x_0)$ obeying (75), there exists a stable $Q_r(x_0)$ given by
$$Q_r(x_0) = (\tilde{\mathcal{V}}(x_0)\mathcal{K}_Q(x_0) - \tilde{\mathcal{U}}(x_0))(\tilde{\mathcal{M}}(x_0) - \tilde{\mathcal{N}}(x_0)\mathcal{K}_Q(x_0))^{-1} \tag{77}$$
such that the controllers of Figure 12 are equivalent.
3. The system $\{\mathcal{G}(x_0), \mathcal{K}_{Qr}(x_0)\}$ with $\mathcal{K}_{Qr}(x_0)$ constructed as in Figure 12 is $\varepsilon_V, \varepsilon_U$ bounded input stable iff $Q_r(x_0)$ is $(\theta_M + \theta_N)$ bounded-input stable.

The main results are now summarized as a theorem

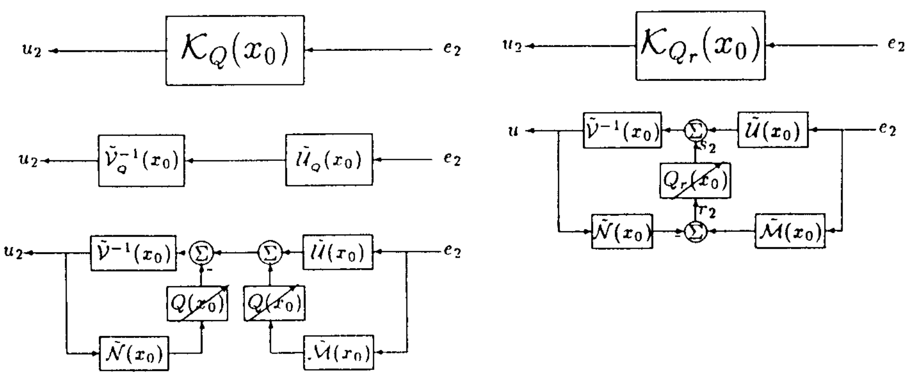


Figure 12. The classes $\mathcal{K}_Q$, $\mathcal{K}_{Q_r}$

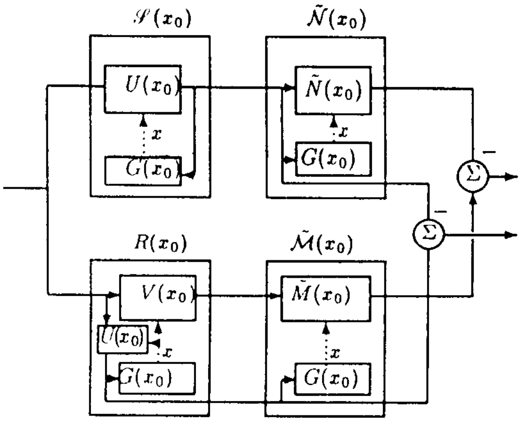


Figure 13. The Bezout $\tilde{\mathcal{M}}R - \tilde{\mathcal{N}}\mathcal{S} = [I\ 0]'$

Theorem 2

Consider an augmented plant belonging to the nonlinear class (2), and obeying assumptions (6), (20), (66). Then left and right factorizations exist as in (62)–(65). Given the differential boundedness properties (74), (75), then the class of all stabilizing controllers for that plant can be constructed as in Figure 12.

Stabilization of plants with unknown initial conditions

An important questions remains of stabilization results for plant/controller pairs with non-identical initial conditions, viewed here as working with a non-nominal plant/controller pair. In this section we extend the theory to this case. To this end, we first recall the plant $\mathcal{G}_S(x_0)$ as shown in Figure 14. Our aim is to use the S parametrization to characterize the class of plants $\mathcal{G}(x_0)$ in feedback pairs $\{\mathcal{G}(x_0), \mathcal{K}(\hat{x}_0)\}$, over all initial conditions, $x_0, \hat{x}_0$, not necessarily such that $x_0 = \hat{x}_0$ as in earlier results. Here we will assume realistically that $\hat{x}_0$, the controller state is known and that x_0 is unknown. Thus we think of our nominal plant/controller pair as $\{\mathcal{G}(\hat{x}_0), \mathcal{K}(\hat{x}_0)\}$ and seek results for the pair $\{\mathcal{G}(x_0), \mathcal{K}(\hat{x}_0)\}$ with x_0 possibly different from $\hat{x}_0$. Thus, consider the following theorem, part of which is a specialization of Theorem 3.1 of Reference 14.

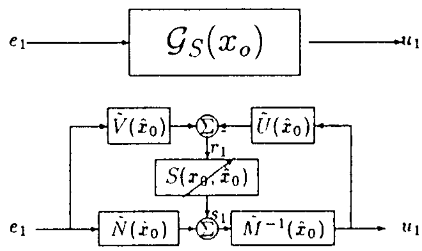


Figure 14. The class $\mathcal{G}_s(x_0)$

Theorem 3

Consider $\mathscr{G}(\hat{x}_0)$ defined in (59) $\tilde{\mathscr{U}}(\hat{x}_0), \tilde{\mathscr{N}}(\hat{x}_0), \tilde{\mathscr{M}}(\hat{x}_0)$ from (64), (65). Then the system

$$\begin{bmatrix} \tilde{\mathscr{V}}(\hat{x}_0) & -\tilde{\mathscr{U}}(\hat{x}_0) \\ -\tilde{\mathscr{N}}(\hat{x}_0) & \tilde{\mathscr{M}}(\hat{x}_0) \end{bmatrix}$$

is invertible for all initial conditions $\hat{x}_0$, as is $(\tilde{\mathscr{V}}(\hat{x}_0) - \tilde{\mathscr{U}}(\hat{x}_0)\mathscr{G}_S(x_0))$ for all initial conditions $x_0, \hat{x}_0$, for any dynamical system $\mathscr{G}_S(x_0)$ satisfying the assumption (6) and of compatible dimension. Also $\mathscr{G}_S(x_0)$ has a right factorization

$$\mathscr{G}_S(x_0) = \mathscr{N}_S(x_0, \hat{x}_0)\,\mathscr{M}_S(x_0, \hat{x}_0)^{-1} \tag{78}$$

$$\begin{bmatrix} \mathscr{M}_S(x_0, \hat{x}_0) \\ \mathscr{N}_S(x_0, \hat{x}_0) \end{bmatrix} = \begin{bmatrix} I \\ \mathscr{G}_S(x_0) \end{bmatrix} (\tilde{\mathscr{V}}(\hat{x}_0) - \tilde{\mathscr{U}}(\hat{x}_0)\mathscr{G}_S(x_0))^{-1} \tag{79}$$

Define $S(x_0, \hat{x}_0)$ as

$$S(x_0, \hat{x}_0) = \tilde{\mathscr{M}}(\hat{x}_0)\,\mathscr{N}_S(x_0, \hat{x}_0) - \tilde{\mathscr{N}}(\hat{x}_0)\,\mathscr{M}_S(\hat{x}_0, x_0) \tag{80}$$

Then the systems $\mathscr{G}_S(x_0)$ can be organized as depicted in Figure 14. Also, its factors $\mathscr{M}_S(x_0, \hat{x}_0)$ and $\mathscr{N}_S(x_0, \hat{x}_0)$ are given in terms of $S(x_0, \hat{x}_0)$ as

$$\begin{bmatrix} \mathscr{M}_S(x_0, \hat{x}_0) \\ \mathscr{N}_S(x_0, \hat{x}_0) \end{bmatrix} = \begin{bmatrix} \tilde{\mathscr{V}}(\hat{x}_0) & -\tilde{\mathscr{U}}(\hat{x}_0) \\ -\tilde{\mathscr{N}}(\hat{x}_0) & \tilde{\mathscr{M}}(\hat{x}_0) \end{bmatrix}^{-1} \begin{bmatrix} I \\ S(x_0, \hat{x}_0) \end{bmatrix} \tag{81}$$

Moreover, the factors $\mathscr{M}_S(x_0, \hat{x}_0), \mathscr{N}_S(x_0, \hat{x}_0)$ and $\tilde{\mathscr{V}}(\hat{x}_0), \tilde{\mathscr{U}}(\hat{x}_0)$ obey a Bezout identity

$$\tilde{\mathscr{V}}(\hat{x}_0)\,\mathscr{M}_S(x_0, \hat{x}_0) - \tilde{\mathscr{U}}(\hat{x}_0)\,\mathscr{N}_S(x_0, \hat{x}_0) = I \tag{82}$$

The factors $M_S(x_0, \hat{x}_0)$, $N_S(x_0, \hat{x}_0)$ are coprime if they are BIBO stable. Furthermore, given BIBO stability of

$$\begin{bmatrix} \tilde{\mathscr{V}}(\hat{x}_0) & -\tilde{\mathscr{U}}(\hat{x}_0) \\ -\tilde{\mathscr{N}}(\hat{x}_0) & \tilde{\mathscr{M}}(\hat{x}_0) \end{bmatrix}^{-1}$$

then

$$\mathscr{M}_S(x_0, \hat{x}_0), \mathscr{N}_S(x_0, \hat{x}_0) \text{ are BIBO stable iff } S(x_0, \hat{x}_0) \text{ is BIBO stable.} \tag{83}$$

Proof. The systems $\tilde{\mathscr{V}}(\hat{x}_0) - \tilde{\mathscr{U}}(\hat{x}_0)\mathscr{G}_S(x_0)$ and

$$\begin{bmatrix} \tilde{\mathscr{V}}(\hat{x}_0) & -\tilde{\mathscr{U}}(\hat{x}_0) \\ -\tilde{\mathscr{N}}(\hat{x}_0) & \tilde{\mathscr{M}}(\hat{x}_0) \end{bmatrix}$$

are invertible by similar arguments to that used to prove the invertibility of the system $M(x_0) - U(x_0)S(x_0, \hat{x}_0)$ in Theorem 1.

Observe from Figure 14 that

$$u = \tilde{\mathscr{M}}^{-1}(\hat{x}_0)[S(x_0, \hat{x}_0)[\tilde{\mathscr{V}}(\hat{x}_0)e - \tilde{\mathscr{U}}(\hat{x}_0)u] + \tilde{\mathscr{N}}(x_0)e] = \mathscr{G}_S(x_0)e$$

which yields

$$S(x_0, \hat{x}_0) = [\tilde{\mathscr{M}}(\hat{x}_0)\mathscr{G}_S(x_0) - \tilde{\mathscr{N}}(\hat{x}_0)]\,[\tilde{\mathscr{V}}(\hat{x}_0) - \tilde{\mathscr{U}}(\hat{x}_0)\mathscr{G}_S(x_0)]^{-1} \tag{84}$$

or equivalently (80), as claimed.

For each $S(x_0, \hat{x}_0)$ there exists a unique pair $\mathscr{M}_S(x_0, \hat{x}_0), \mathscr{N}_S(x_0, \hat{x}_0)$, and consequently a unique $\mathscr{G}_S(x_0)$. Thus setting $S(x_0, \hat{x}_0)$ to obey (84) makes the system in Figure 14 equivalent to $\mathscr{G}_S(x_0)$.

To show (82), observe that

$$\begin{aligned}\tilde{\mathscr{V}}(\hat{x}_0)\mathscr{M}_S(x_0,\hat{x}_0) - \tilde{\mathscr{U}}(\hat{x}_0)\mathscr{N}_S(x_0,\hat{x}_0) &= \tilde{\mathscr{V}}(\hat{x}_0)((\tilde{\mathscr{V}}(\hat{x}_0) - \tilde{\mathscr{U}}(\hat{x}_0)\mathscr{G}_S(x_0))^{-1} \\ &\quad - \tilde{\mathscr{U}}(\hat{x}_0)\mathscr{G}_S(x_0)(\tilde{\mathscr{V}}(\hat{x}_0) - \tilde{\mathscr{U}}(\hat{x}_0)\mathscr{G}_S(x_0))^{-1} \\ &= I\end{aligned}$$

Also

$$\begin{aligned}\mathscr{N}_S(x_0,\hat{x}_0)\mathscr{M}_S(x_0,\hat{x}_0)^{-1} &= \mathscr{G}_S(x_0)(\tilde{\mathscr{V}}(\hat{x}_0) - \tilde{\mathscr{U}}(\hat{x}_0)\mathscr{G}_S(x_0,\hat{x}_0))^{-1}(\tilde{\mathscr{V}}(\hat{x}_0) - \tilde{\mathscr{U}}(\hat{x}_0)\mathscr{G}_S(x_0)) \\ &= \mathscr{G}_S(x_0)\end{aligned}$$

giving (79). Verification of (81) follows from,

$$\begin{bmatrix}\tilde{\mathscr{V}}(\hat{x}_0) & -\tilde{\mathscr{U}}(\hat{x}_0) \\ -\tilde{\mathscr{N}}(\hat{x}_0) & \tilde{\mathscr{M}}(\hat{x}_0)\end{bmatrix}\begin{bmatrix}\mathscr{M}_S(x_0,\hat{x}_0) \\ \mathscr{N}_S(x_0,\hat{x}_0)\end{bmatrix} = \begin{bmatrix}\tilde{\mathscr{V}}(\hat{x}_0)\mathscr{M}_S(x_0) - \tilde{\mathscr{U}}(\hat{x}_0)\mathscr{N}_S(x_0) \\ \tilde{\mathscr{M}}(\hat{x}_0)\mathscr{N}_S(x_0) - \tilde{\mathscr{N}}(\hat{x}_0)\mathscr{M}_S(x_0)\end{bmatrix} = \begin{bmatrix} I \\ S(x_0,\hat{x}_0)\end{bmatrix}$$

Given unimodularity of

$$\begin{bmatrix}\tilde{\mathscr{V}}(\hat{x}_0) & -\tilde{\mathscr{U}}(\hat{x}_0) \\ -\tilde{\mathscr{N}}(\hat{x}_0) & \tilde{\mathscr{M}}(\hat{x}_0)\end{bmatrix}$$

then by (81) stability of $S(x_0,\hat{x}_0)$ is equivalent to stability of $\mathscr{M}(x_0,\hat{x}_0),\mathscr{N}(x_0,\hat{x}_0)$. Coprimeness follows from the stability assumption, and the Bezout (82) via Lemma 2.1 of Reference 9. □

Remarks

(1) Theorem 3 applies for any plant $\mathscr{G}_S(x_0)$ within the class of interest, and in particular applies to $\mathscr{G}_S(x_0) = \mathscr{G}(x_0)$. A key objective of this paper is thereby reached, namely to achieve coprime factorizations of $\{\mathscr{G}(x_0),\mathscr{K}(\hat{x}_0)\}$ with $\hat{x}_0 \neq x_0$. Thus, we set $\mathscr{G}_S(x_0) = \mathscr{G}(x_0)$, and use the $S(.)$ operator to parametrize over the set of nominal plants $\mathscr{G}(.)$ with varying initial conditions. A dual of Theorem 3, holds for any controller $\mathscr{K}(\hat{x}_0)$ with initial conditions $\hat{x}_0$ differing from that of a nominal plant $\mathscr{G}(x_0)$. Thus a $\mathscr{K}_Q(x_0,\hat{x}_0) = \mathscr{K}(\hat{x}_0)$ can be formulated in a dual fashion such that the operator $Q(x_0,\hat{x}_0)$ characterizes the effect of initial conditions of the controller different to those of the nominal plant.
(2) Corresponding 'left' factorization results are elusive and indeed may not exist to the generality achieved for right factorizations.
(3) This theorem also applies to the factorizations of Section 3.

We next make use of these parametrizations of the set of pairs $\{\mathscr{G}(x_0),\mathscr{K}(\hat{x}_0)\}$ with different initial conditions by recalling a theorem from Reference 9.

Theorem 4

Consider the system $\{\mathscr{G}_S(x_0),\mathscr{K}_{Q_r}(\hat{x}_0)\}$ of Figure 15(a), where $\tilde{\mathscr{N}}(\hat{x}_0),\tilde{\mathscr{M}}(\hat{x}_0),\tilde{\mathscr{U}}(\hat{x}_0),\tilde{\mathscr{V}}(\hat{x}_0)$ as defined in (65, 64), are stable factorizations of $\mathscr{G}(\hat{x}_0)$ and $\mathscr{K}(\hat{x}_0)$. Consider also that

$$\begin{bmatrix}\tilde{\mathscr{M}}(\hat{x}_0) & -\tilde{\mathscr{N}}(\hat{x}_0) \\ -\tilde{\mathscr{U}}(\hat{x}_0) & \tilde{\mathscr{V}}(\hat{x}_0)\end{bmatrix}^{-1}$$

is BIBO stable, and the differential boundedness conditions (74) hold. Then the system is $\varepsilon_v,\varepsilon_u$ is bounded-input stable iff the system $\{S(x_0,\hat{x}_0),Q_r(x_0,\hat{x}_0)\}$, of Figure 15(b), is $(\theta_U+\theta_V),(\theta_M+\theta_N)$ bounded-input stable.

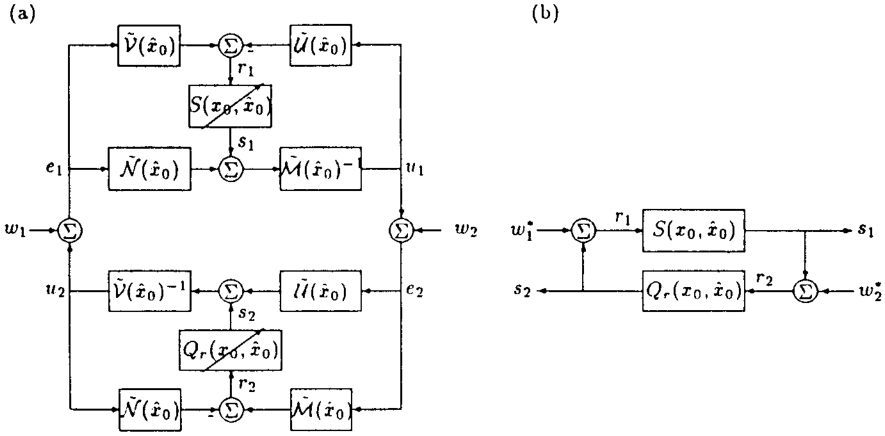


Figure 15. (a), (b) The systems $\{\mathcal{G}_s(x_0), \mathcal{K}_{Q_r}(\hat{x}_0)\}$ and $\{S(\hat{x}_0, x_0), Q_r(\hat{x}_0)\}$

Remark. This theorem is a powerful robustness theorem, which can be used, for example, in multi-loop controller design strategies as in the linear case[15] and in adaptive control.[6]

5. SIMULATION RESULTS

To illustrate nonlinear plant/controller robustness properties developed in this paper, we present two sets of simulation studies, each consisting of a nonlinear controller designed for a nonlinear nominal plant, and then the pair placed in a feedback loop with stochastic disturbances added. In the first instance, the simulations consist of an augmented controller/plant loop, as in Figure 10(b), with stochastic disturbances added to each of the inputs. The second series of simulations includes a Q parametrized controller in feedback with an unaugmented plant as per Figure 8(a). The idea is to illustrate the tracking, and regulation properties of the controller in the presence of disturbances.

The coefficient functions of the state-space formulation of the four scalar-variable, first-order plants simulated are given in (85) to (88).

$$\text{Plant 1: } \left[\begin{array}{c|c} 0\cdot 2 & 2 \\ \hline 10 & 0\cdot 8 \end{array}\right]_{x_0} \tag{85}$$

$$\text{Plant 2: } \left[\begin{array}{c|c} 0\cdot 2 + 0\cdot 1\sin(x) & 0\cdot 2|\cos(x)| + 2 \\ \hline 10 & 1 - xe^{-|x|} \end{array}\right]_{x_0} \tag{86}$$

$$\text{Plant 3: } \left[\begin{array}{c|c} 1\cdot 1 + xe^{-|x|} & -\min(|x|, |1/x|) - 0\cdot 5 \\ \hline 10 + \sin(5x) & |\operatorname{sinc}(x)| - 3 \end{array}\right]_{x_0} \tag{87}$$

$$\text{Plant 4: } \left[\begin{array}{c|c} \begin{cases} -1 + \operatorname{sinc}(x), & x \neq 0 \\ 3, & x = 0 \end{cases} & \begin{cases} 2, & \text{integer part of } x \text{ is even} \\ -2, & \text{integer part of } x \text{ is odd} \end{cases} \\ \hline 10 + \sin(5x) & |\operatorname{sinc}(x)| - 3 \end{array}\right]_{x_0} \tag{88}$$

The shapes of the functions $A(.), B(.), C(.), D(.)$ are depicted in Figure 16, where the scalar functions are plotted over the normal operating range of the plant/controller pair.

In each case, specification of the controller requires the definition of the matrix functions $F(.), H(.)$. The choice must give the required differential boundedness conditions in order to

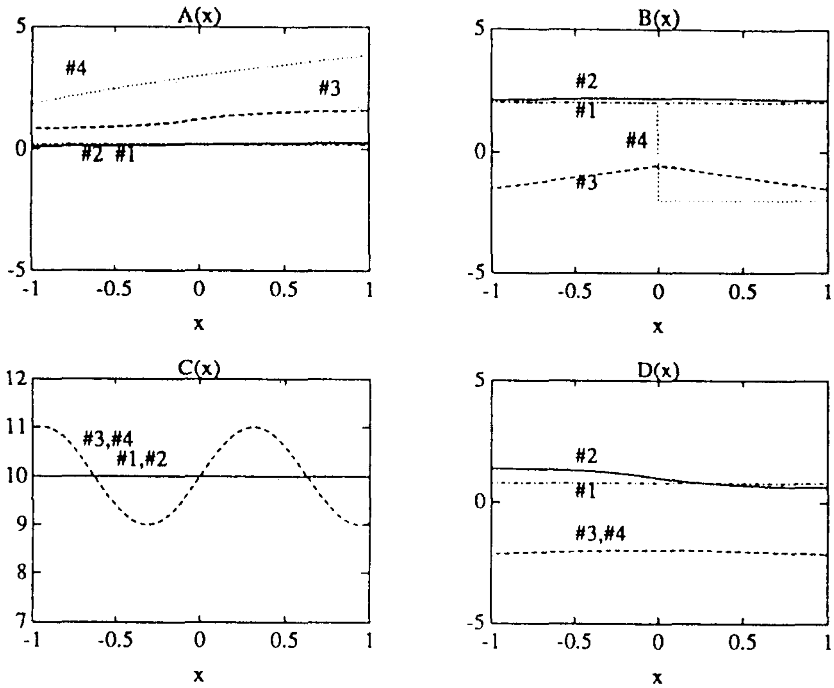


Figure 16. The matrix blocks A–D for plants 1–4

guarantee stability of the loop. Here, the differential boundedness is calculated in an L_2 sense, and generally in this case a sufficient condition for differential boundedness is that for two systems with identical input, and unequal but sufficiently close initial states, that the states converge. This condition is facilitated, at least in the zero input case, by any $A(.)$ function of the system which guarantees exponential convergence of the state to zero.

In each case, $F(x)$ and $H(x)$ are chosen for all x such that

$$F(x) = -A(x)/B(x) - 0{\cdot}5/B(x) \tag{89}$$

$$H(x) = -A(x)/C(x) - 0{\cdot}5/C(x) \tag{90}$$

This sets both $A(x) + B(x)F(x)$ and $A(x) + H(x)C(x)$ to a constant of $-0{\cdot}5$ for all x. Thus, as can be seen from (64), (65), the $A(.)$ functions of the systems $\tilde{\mathcal{U}}, \tilde{\mathcal{V}}, \tilde{\mathcal{N}}, \tilde{\mathcal{M}}$ are all equal to $-0{\cdot}5$, which guarantees exponential convergence for the zero input case.

The controllers in three of these simulations are able to regulate the states of the plants close to the zero point. There is a trade-off, however, since the $B(.)$ and $C(.)$ functions of the factors $\tilde{\mathcal{U}}, \tilde{\mathcal{V}}, \tilde{\mathcal{N}}, \tilde{\mathcal{M}}$ include $F(.)$ and $H(.)$, and in high-input conditions, small $B(.)$ and $C(.)$ functions will, in general, help to achieve differential boundedness, possibly allowing a more variable 'A' function. Further discussion of the properties of nonlinear differential equations which lead to differential boundedness is beyond the scope of this paper.

The simulations are run with a nominal stochastic disturbance uniformly distributed between $[-0.3, 0{\cdot}3]$ applied to each input. The systems $\tilde{\mathcal{U}}, \tilde{\mathcal{V}}, \tilde{\mathcal{N}}, \tilde{\mathcal{M}}$ are differentially bounded in the region of interest, of $|x| < 5$, for plants 1, 2, and 3. Plant 4 is not differentially bounded in this region. The simulations extend in the case of plants 1, 2, and 3 for 300 iterations. The simulation results of Plant 4 are shown only for six iterations, since the state continues to

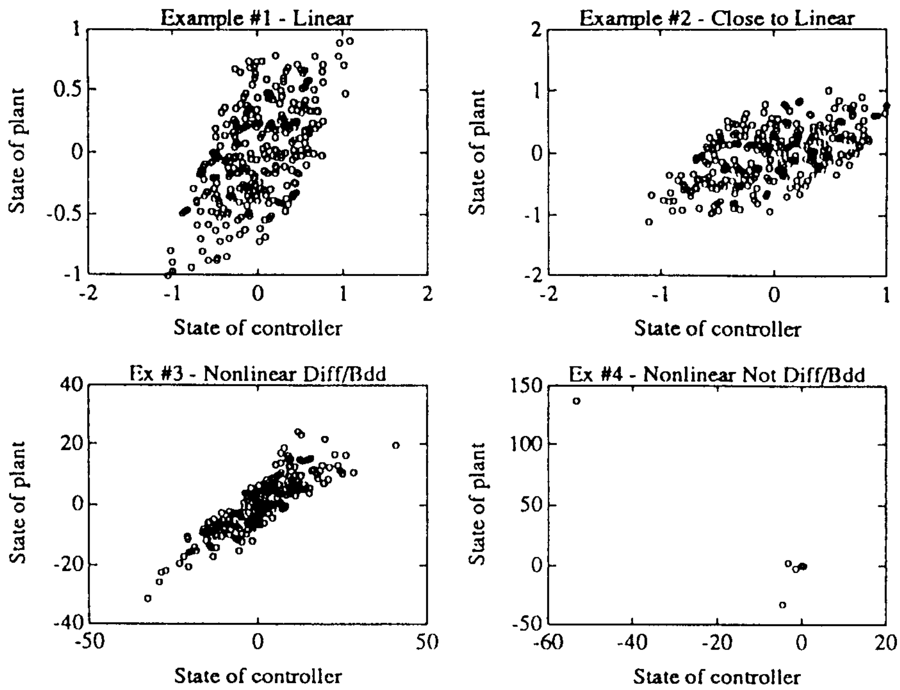


Figure 17. The controller and augmented plant states at each time instant

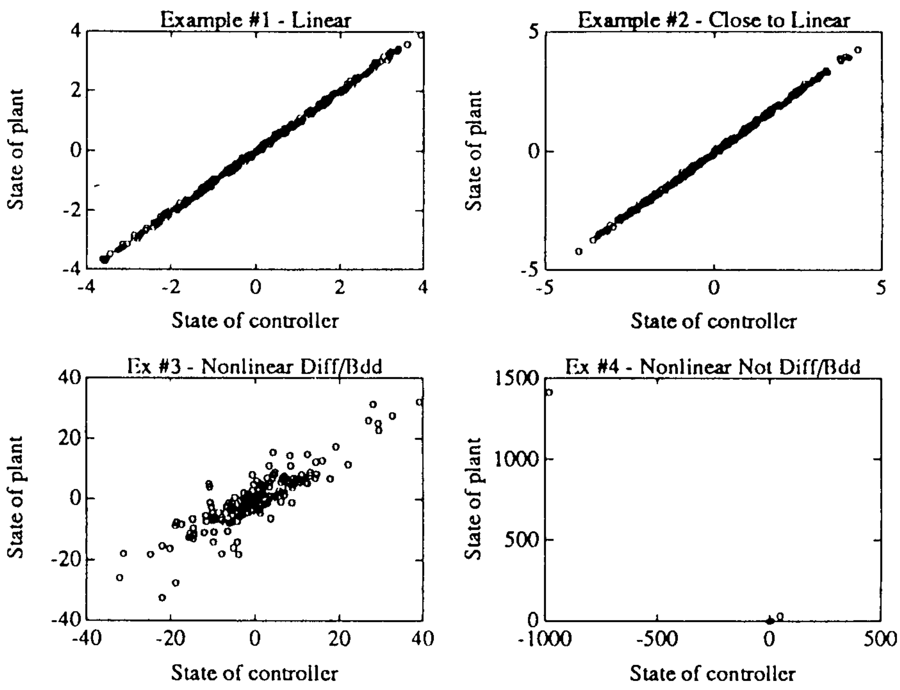


Figure 18. The Q-parameterized controller and nominal plant states at each time instant

diverge for all further iterations. The results of the state of the controller plotted against the state of the plant for each time iterations are shown in Figure 17.

Further simulations are carried out which include the Q parametrized controller $K_{Q_r}(\hat{x}_0)$ as in Figure 12, in a feedback loop with the unaugmented nominal plant $G(x_0)$, as in Figure 12(a). The plants in this case are strictly proper versions of Plants 1, 2, 3, and 4 in (85)–(88), i.e. identical but with the functions $D(.)$ set to 0. The disturbances are uniformly distributed between 0 and 0·1. Also, the initial conditions on the plant and controller are not equal, the difference uniformly distributed between 0 and 0·1. The Q function used in these simulations is $Q_r(x) = \sin(x)$. The results are shown in Figure 18.

As can be seen from the Figures 17 and 18, in the three cases where the differential boundedness conditions are satisfied, the controller successfully regulates the state of the plant in the presence of stochastic disturbances, and in the case where the differential boundedness conditions are not met, the system diverges.

6. CONCLUSION

In this paper we have extended part of the linear factorization theory to a class of nonlinear systems. For these pseudo-linear systems with state-dependent matrices $A(.), B(.), C(.), D(.)$, cascade and inversion formulas have been introduced which trivially collapse to the well-known linear results when $A(.), B(.)$, etc. are not state-dependent. Also the approach is such that the nonlinear system factorizations are set up so as to make the corresponding matrices $A(.), B(.)$, etc. identical in all the subsystems in an idealized nominal plant/controller arrangement. In this case results follow directly from the linear time-varying case. This approach excludes corresponding left factorizations in general.

Studies are made of a specialization to the case where the state-dependence is rather an output-dependence, and to a more general case where the factors are augmented. In these cases, stable left factorizations, as well as certain Bezout identities are also generated. These left factorizations are used to generate the class of all stabilizing controllers for a given nominal plant based on earlier theory.[9] Also, an S parametrization is recalled, and then used to parametrize the set of all plants G with different initial conditions, leading to stabilization results for these plants. Simulation results verify the stabilization properties of the nonlinear controllers proposed in this paper. Thus, for the state-dependent class of nonlinear systems, the theory goes quite a way in extending known linear results.

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