

## COPRIME FACTORIZATIONS AND WELL-POSED LINEAR SYSTEMS\*

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**Abstract.** We study the basic notions related to the stabilization of an infinite-dimensional well-posed linear system in the sense of Salamon and Weiss. We first introduce an appropriate stabilizability and detectability notion and show that if a system is jointly stabilizable and detectable then its transfer function has a doubly coprime factorization in  $H^\infty$ . The converse is also true: every function with a doubly coprime factorization in  $H^\infty$  is the transfer function of a jointly stabilizable and detectable well-posed linear system. We show further that a stabilizable and detectable system is stable if and only if its input/output map is stable. Finally, we construct a dynamic, possibly non-well-posed, stabilizing compensator. The notion of stability that we use is the natural one for the quadratic cost minimization problem, and it does not imply exponential stability.

**Key words.** stabilizability, detectability, input/output stability

**AMS subject classifications.** 93A05, 93B05, 93B07

**PII.** S0363012995285417

**1. Introduction.** Although the theory of well-posed linear systems in the sense of Salamon and Weiss has been around for some time, applications of this theory to “real” control problems are scarce. This is in sharp contrast to the widespread use of the theory of Pritchard–Salamon systems; see, e.g., [2], [12], [15], [16], and [26] for discussions of different aspects of this theory. A fair number of recent pure frequency domain results for  $H^\infty$  transfer functions do exist (some of these are listed in the References), but they have not been connected to the theory of well-posed linear systems. The few connections known to us in the spring of 1995 when the first version of this paper was written were the discussion of the Lyapunov equation in [10], the discussion on feedback and estimation of well-posed systems in [13], the discussion of balanced realizations in [14], the discussion of the connection between internal and external stability in [17], and the discussions of the quadratic cost minimization problem in [19] and [24] (not to mention the basic papers [20], [30], and [31] and the nice review [1]); the list above is certainly not complete. In particular, at that time we were not able to find any reasonably complete results on the stabilizability and detectability of general well-posed linear systems, and the connection of these notions to the notion of a (doubly) coprime factorization of the transfer function of the system.<sup>1</sup> We needed these results in order to solve the quadratic cost minimization problem for unstable systems, and we were forced to develop the needed stabilization theory ourselves.<sup>2</sup>

Subsequently the situation changed significantly with the appearance of the pre-prints [4], [5], [6], [11], [32], and [33]. Out of these [4], [5], and [6] are fairly closely related to our work. Several of the results that we prove here are also found in [5], in a slightly less general setting. The results given in [4] and [6] overlap those that we

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\*Received by the editors May 3, 1995; accepted for publication (in revised form) July 10, 1997; published electronically May 15, 1998.

<http://www.siam.org/journals/sicon/36-4/28541.html>

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<sup>1</sup>Throughout [13] Morris takes the observation operator to be bounded.

<sup>2</sup>Our solution to the unstable quadratic cost minimization problem is based on the present work, and it is presented in [25].

present in section 5.<sup>3</sup> These preprints have also had a certain influence on a revision carried out in late 1996: we were able to use the ideas presented in [5] to simplify our original proofs of Theorems 4.4 and 5.3, and Remark 5.5 was inspired by [5] and [6]. In addition we added Lemma 3.17 and Corollary 3.18 as an answer to a question asked by George Weiss in the summer of 1996, and we redrew all the diagrams.

The notion of stability that we use was forced upon us by our solution to the quadratic cost minimization problem for stable well posed linear systems. In particular, it does not imply exponential stability. The most important notion is what we call “joint stabilizability and detectability.” The word “joint” refers to the fact that in our setting the notions of stabilizability and detectability are not decoupled from each other as they are in the Pritchard–Salamon theory; roughly speaking, the “feedback operator and the output injection operator must be compatible.” This is a problem that is not present in a Pritchard–Salamon system, due to the fact that for such systems the admissibility of a particular control operator together with the admissibility of a particular observation operator implies the well posedness of the corresponding input/output map, something that is not true for general well posed linear systems. Maybe the main individual result of this paper is the statement that if a system is jointly stabilizable and detectable, then its transfer function has a doubly coprime factorization in  $H^\infty$ .<sup>4</sup> The converse is also true: every function that has a doubly coprime factorization in  $H^\infty$  is the transfer function of a jointly strongly stabilizable and detectable well-posed linear system.

Having explained the relation between joint stabilizability and detectability on one hand and doubly coprime factorizations on the other hand, we continue with a short discussion of how to use an observer as a stabilizing dynamic compensator. This theory parallels the classical theory, apart from the fact that the well posedness of the observer is not automatically guaranteed.

For the convenience of the reader, we start with a short presentation of well-posed linear systems.

We use the following notation.

- $\mathcal{L}(U; Y)$ ,  $\mathcal{L}(U)$ : The set of bounded linear operators from  $U$  into  $Y$  or from  $U$  into itself, respectively.
- $I$ : The identity operator.
- $A^*$ : The (Hilbert space) adjoint of the operator  $A$ .
- $\mathbf{R}$ ,  $\mathbf{R}^+$ ,  $\mathbf{R}^-$ :  $\mathbf{R} = (-\infty, \infty)$ ,  $\mathbf{R}^+ = [0, \infty)$ , and  $\mathbf{R}^- = (-\infty, 0]$ .
- $L^2(J; U)$ : The set of  $U$ -valued  $L^2$ -functions on the interval  $J$ .
- $L_\omega^2(J; U)$ :  $L_\omega^2(J; U) = \{ u \in L_{\text{loc}}^2(J; U) \mid (t \mapsto e^{-\omega t}u(t)) \in L^2(J; U) \}$ .
- $H_\omega^\infty(U; Y)$ : The set of bounded analytic  $\mathcal{L}(U; Y)$ -valued functions over the half-plane  $\Re z > \omega$ , with the sup-norm.
- $TI_\omega(U; Y)$ ,  $TI_\omega(U)$ : The set of bounded linear time-invariant operators from  $L_\omega^2(\mathbf{R}; U)$  into  $L_\omega^2(\mathbf{R}; Y)$  or from  $L_\omega^2(\mathbf{R}; U)$  into itself.
- $TIC_\omega(U; Y)$ ,  $TIC_\omega(U)$ : The set of causal operators in  $TI_\omega(U; Y)$  or  $TI_\omega(U)$ .
- $\langle \cdot, \cdot \rangle_H$ : The inner product in the Hilbert space  $H$ .

<sup>3</sup>Curtain, Weiss, and Weiss call the compensator in Theorem 5.3 a “controller with internal loop.”

<sup>4</sup>As we mentioned above, the main purpose of this paper is not to prove any particular result but to develop a general theory that can be used in the study of the quadratic cost minimization problem, to which we return in [25].

$\tau(t)$ : The time shift group  $\tau(t)u(s) = u(t + s)$  (this is a left shift when  $t > 0$  and a right shift when  $t < 0$ ).

$\pi_J$ :  $(\pi_J u)(s) = u(s)$  if  $s \in J$  and  $(\pi_J u)(s) = 0$  if  $s \notin J$ . Here  $J$  is a subset of  $\mathbf{R}$ .

$\pi_+, \pi_-$ :  $\pi_+ = \pi_{\mathbf{R}^+}$  and  $\pi_- = \pi_{\mathbf{R}^-}$ .

We extend an  $L^2_\omega$ -function  $u$  defined on a subinterval  $J$  of  $\mathbf{R}$  to the whole real line by requiring  $u$  to be zero outside of  $J$ , and we denote the extended function by  $\pi_J u$ . Thus we use the same symbol  $\pi_J$  both for the embedding operator  $L^2_\omega(J) \rightarrow L^2_\omega(\mathbf{R})$  and for the corresponding projection operator  $L^2_\omega(\mathbf{R}) \rightarrow L^2_\omega(J)$ . With this interpretation,  $\pi_J L^2_\omega(\mathbf{R}; U) = L^2_\omega(J; U) \subset L^2_\omega(\mathbf{R}; U)$  for each interval  $J \subset \mathbf{R}$ .

**2. A review of well-posed linear systems and time-invariant operators.**

In order to fix the notation and describe the basic setting we first give a brief presentation of the theory of the Salamon–Weiss class of well-posed linear systems. This theory has been developed in [18], [19], [20], [3], [7], and [27], [28], [30], [31] (and many other papers), and we refer the reader to these sources for additional reading.<sup>5</sup> A recent contribution is found in [24], and the setting that we use here is a slight extension of the one in [24]. The difference is that the discussion in [24] is restricted to the case of (externally) stable systems; here we also need to consider unstable systems. The major parts of this setting are found in [20], too.

In order to formulate the axioms satisfied by a well-posed linear system we introduce exponentially weighted  $L^2$ -spaces. For each Hilbert space  $U$  and each  $\omega \in \mathbf{R}$  we let  $L^2_\omega(\mathbf{R}; U)$  be the weighted  $L^2$ -space

$$L^2_\omega(\mathbf{R}; U) = \{ u \in L^2_{\text{loc}}(\mathbf{R}; U) \mid (t \mapsto e^{-\omega t} u(t)) \in L^2(\mathbf{R}; U) \}.$$

This is a Hilbert space with the natural norm  $\|e^{-\omega \cdot} u(\cdot)\|_{L^2(\mathbf{R}; U)}$ . We also need the “past time” projection  $\pi_-$ , the “future time” projection  $\pi_+$ , and the “time shift” group  $\tau(t)$  that operate on functions  $u \in L^2_\omega(\mathbf{R}; U)$  in the following way:

$$\begin{aligned} (\pi_- u)(s) &= \begin{cases} u(s), & s \in \mathbf{R}^-, \\ 0, & s \in \mathbf{R}^+, \end{cases} \\ (\pi_+ u)(s) &= \begin{cases} u(s), & s \in \mathbf{R}^+, \\ 0, & s \in \mathbf{R}^-, \end{cases} \\ (\tau(t)u)(s) &= u(t + s), \quad t, s \in \mathbf{R}. \end{aligned}$$

DEFINITION 2.1. Let  $U, H,$  and  $Y$  be Hilbert spaces, and let  $\omega \in \mathbf{R}$ . A (causal)  $\omega$ -stable well-posed linear system on  $(U, H, Y)$  is a quadruple  $\Psi = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}$ , where  $\mathcal{A}, \mathcal{B}, \mathcal{C},$  and  $\mathcal{D}$  are bounded linear operators of the following type:

- (i)  $\mathcal{A}(t): H \rightarrow H$  is a strongly continuous semigroup of bounded linear operators on  $H$  satisfying  $\sup_{t \in \mathbf{R}^+} \|e^{-\omega t} \mathcal{A}(t)\| < \infty$ ;
- (ii)  $\mathcal{B}: L^2_\omega(\mathbf{R}; U) \rightarrow H$  satisfies  $\mathcal{A}(t)\mathcal{B}u = \mathcal{B}\tau(t)\pi_- u$  for all  $u \in L^2_\omega(\mathbf{R}; U)$  and  $t \in \mathbf{R}^+$ ;
- (iii)  $\mathcal{C}: H \rightarrow L^2_\omega(\mathbf{R}; Y)$  satisfies  $\mathcal{C}\mathcal{A}(t)x = \pi_+ \tau(t)\mathcal{C}x$  for all  $x \in H$  and  $t \in \mathbf{R}^+$ ;
- (iv)  $\mathcal{D}: L^2_\omega(\mathbf{R}; U) \rightarrow L^2_\omega(\mathbf{R}; Y)$  satisfies  $\tau(t)\mathcal{D}u = \mathcal{D}\tau(t)u, \pi_- \mathcal{D}\pi_+ u = 0,$  and  $\pi_+ \mathcal{D}\pi_- u = \mathcal{C}\mathcal{B}u$  for all  $u \in L^2_\omega(\mathbf{R}; U)$  and  $t \in \mathbf{R}$ .

<sup>5</sup>In the early literature these systems were called “well-posed semigroup control systems” by Salamon and “abstract linear systems” by Weiss.

If, in addition,  $e^{-\omega t} \mathcal{A}(t)x \rightarrow 0$  as  $t \rightarrow \infty$  for all  $x \in H$ , then  $\Psi$  is strongly  $\omega$ -stable. The system  $\Psi$  is [strongly]<sup>6</sup> stable iff it is [strongly]  $\omega$ -stable with  $\omega = 0$ , and it is exponentially stable iff it is  $\omega$ -stable for some  $\omega < 0$ .

The different components of  $\Psi$  are named as follows:  $U$  is the input space,  $H$  is the state space,  $Y$  is the output space,  $\mathcal{A}$  is the semigroup,  $\mathcal{B}$  is the controllability map,  $\mathcal{C}$  is the observability map, and  $\mathcal{D}$  is the input/output map of  $\Psi$ .

This is the same definition as [24, Definition 2.1], except that throughout we took  $\omega = 0$  and did not put any restrictions on the growth rate of the semigroup  $\mathcal{A}$ .

The axioms listed above describe standard properties of the corresponding maps induced by systems with bounded control and observation operators. Whenever we refer to an  $\omega$ -stable “classical” system, we mean a system of the following type: we let  $A$  be the generator of a semigroup  $\mathcal{A}$  on a Hilbert space  $H$  satisfying  $\sup_{t \in \mathbf{R}^+} \|e^{\epsilon t} e^{-\omega t} \mathcal{A}(t)\| < \infty$  for some  $\epsilon > 0$ , let  $U$  and  $Y$  be Hilbert spaces, let  $B \in \mathcal{L}(U; H)$ ,  $C \in \mathcal{L}(H; Y)$ , and  $D \in \mathcal{L}(U; Y)$ , and consider the system

$$\begin{aligned}
 (2.1) \quad & x'(t) = Ax(t) + Bu(t), \\
 & y(t) = Cx(t) + Du(t), \quad t \geq T, \\
 & x(T) = x_T,
 \end{aligned}$$

where  $T$  is a given initial time and  $x_T$  is a given initial value. We call  $u$  the control,  $x$  the state,  $y$  the output (or observation),  $A$  the generator,  $B$  the control operator,  $C$  the observation operator, and  $D$  the feed-through operator of this classical system. The state  $x$  is required to be a strong solution of (2.1); i.e., the state  $x$  and output  $y$  are given by

$$(2.2) \quad x(t) = \mathcal{A}(t)x_T + \int_T^t \mathcal{A}(t-s)Bu(s) ds, \quad t \geq T,$$

$$(2.3) \quad y(t) = C\mathcal{A}(t)x_T + \int_T^t C\mathcal{A}(t-s)Bu(s) ds + Du(t), \quad t \geq T.$$

In this case we define  $\mathcal{B}$ ,  $\mathcal{C}$ , and  $\mathcal{D}$  by

$$(2.4) \quad \mathcal{B}u = \int_{-\infty}^0 \mathcal{A}(-s)Bu(s) ds,$$

$$(2.5) \quad \mathcal{C}x = (t \mapsto C\mathcal{A}(t)x, t \in \mathbf{R}^+),$$

$$(2.6) \quad \mathcal{D}u = \left( t \mapsto \int_{-\infty}^t C\mathcal{A}(t-s)Bu(s) ds + Du(t), t \in \mathbf{R} \right).$$

Thus  $\mathcal{B}$  is the mapping from the control  $u \in L^2_\omega(\mathbf{R}^-; U)$  to the final state  $x(0) \in H$  (take  $T = -\infty$ ,  $x_T = 0$ , and  $t = 0$ ),  $\mathcal{C}$  is the mapping from the initial state  $x_0 \in H$  to the output  $y \in L^2_\omega(\mathbf{R}^+; Y)$  (take  $T = 0$  and  $u = 0$ ), and  $\mathcal{D}$  is the mapping from the control  $u \in L^2_\omega(\mathbf{R}; U)$  to the output  $y \in L^2_\omega(\mathbf{R}; Y)$  (take  $T = -\infty$  and  $x_T = 0$ ). We leave the easy proof of the fact that these operators indeed are bounded linear operators between the given spaces to the reader.

Each well-posed linear system  $\Psi$  has a *controlled state* and an *output*. Depending on whether the initial time is finite or infinite these are defined in two slightly different ways as follows.

<sup>6</sup>Square brackets represent optional parts of a sentence. Statements in square brackets are supposed to be true (a) if you omit all square brackets (single or double), (b) if you keep the single brackets, (c) if you keep the double brackets.

DEFINITION 2.2. Let  $\Psi = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}$  be an  $\omega$ -stable well-posed linear system on  $(U, H, Y)$ , and let  $u \in L^2_\omega(\mathbf{R}; U)$ . In the time-invariant setting (corresponding to the initial time  $-\infty$ ) the controlled state  $x(t)$  at time  $t \in \mathbf{R}$  and the output  $y$  of  $\Psi$  with control  $u$  are given by

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \mathcal{B}\tau(t)u \\ \mathcal{D}u \end{bmatrix},$$

and in the initial value setting with initial time  $s$ , initial value  $x(s)$ , and control  $u$ , the controlled state  $x(t)$  at time  $t \geq s$  and the output  $y$  of  $\Psi$  are given by

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \mathcal{A}(t-s) & \mathcal{B}\tau(t) \\ \tau(-s)\mathcal{C} & \mathcal{D} \end{bmatrix} \begin{bmatrix} x(s) \\ \pi_{[s,\infty)}u \end{bmatrix} = \begin{bmatrix} \mathcal{A}(t-s)x(s) + \mathcal{B}\tau(t)\pi_{[s,\infty)}u \\ \tau(-s)\mathcal{C}x(s) + \mathcal{D}\pi_{[s,\infty)}u \end{bmatrix},$$

where  $\pi_{[s,\infty)} = \tau(-s)\pi_+\tau(s)$  is given by

$$(\pi_{[s,\infty)}u)(t) = \begin{cases} u(t), & t \geq s, \\ 0, & t < s. \end{cases}$$

In particular, in the initial value setting with initial time zero, initial value  $x_0$ , and control  $u$ , the controlled state  $x(t)$  at time  $t \in \mathbf{R}^+$  and the output  $y$  of  $\Psi$  are given by

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \mathcal{A}(t) & \mathcal{B}\tau(t) \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \begin{bmatrix} x_0 \\ \pi_+u \end{bmatrix} = \begin{bmatrix} \mathcal{A}(t)x_0 + \mathcal{B}\tau(t)\pi_+u \\ \mathcal{C}x_0 + \mathcal{D}\pi_+u \end{bmatrix}.$$

Let us remark that the most commonly studied problem is the initial value problem with initial time zero, and in most papers this is the only one that is treated.

REMARK 2.3. Because of Definition 2.2, we shall frequently use the alternative notation  $\begin{bmatrix} \mathcal{A} & \mathcal{B}\tau \\ \mathcal{C} & \mathcal{D} \end{bmatrix}$  for the well-posed linear system  $\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}$ .

In the case of the classical  $\omega$ -stable system (2.1) with bounded control operator  $B$ , bounded observation operator  $C$ , and control  $u \in L^2_\omega(\mathbf{R}; U)$ , in the time-invariant setting the state  $x$  and output  $y$  of  $\Psi$  are given by

$$(2.7) \quad x(t) = \int_{-\infty}^t \mathcal{A}(t-s)Bu(s) ds, \quad t \in \mathbf{R},$$

$$(2.8) \quad y(t) = \int_{-\infty}^t C\mathcal{A}(t-s)Bu(s) ds + Du(t), \quad t \in \mathbf{R},$$

and in the initial value setting with initial time  $T$ , initial value  $x_T$ , and control  $u$ , the state and output are given by (2.2) and (2.3).

An important fact is that the number  $\omega$  in Definition 2.1 is not uniquely determined as described in Lemma 2.4.

LEMMA 2.4. Let  $\Psi = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}$  be an  $\omega$ -stable well-posed linear system on  $(U, H, Y)$ . Then  $\Psi$  is also an  $\alpha$ -stable well-posed linear system on  $(U, H, Y)$  for every  $\alpha > \omega$ . If instead  $\alpha < \omega$ , then  $\Psi$  has a unique extension to an  $\alpha$ -stable well-posed linear system on  $(U, H, Y)$  iff  $\sup_{t \in \mathbf{R}^+} \|e^{-\alpha t}\mathcal{A}(t)\| < \infty$  and the operators  $\mathcal{B}$ ,  $\mathcal{C}$ , and  $\mathcal{D}$  can be extended to bounded linear operators in  $\mathcal{L}(L^2_\alpha(\mathbf{R}; U); H)$ ,  $\mathcal{L}(H; L^2_\alpha(\mathbf{R}; Y))$ , and  $\mathcal{L}(L^2_\alpha(\mathbf{R}; U); L^2_\alpha(\mathbf{R}; Y))$ , respectively.

Proof. The first claim follows from Lemma 2.9 below and the fact that if  $\alpha > \omega$ , then  $L^2_\alpha(\mathbf{R}^-; U) \subset L^2_\omega(\mathbf{R}^-; U)$  and  $L^2_\omega(\mathbf{R}^+; Y) \subset L^2_\alpha(\mathbf{R}^+; Y)$ . To prove the second claim it suffices to observe that  $L^2_\omega \cap L^2_\alpha$  is dense in  $L^2_\alpha$ .  $\square$

DEFINITION 2.5. We call  $\Psi$  a well-posed linear system on  $(U, H, Y)$  iff it is an  $\omega$ -stable well-posed linear system on  $(U, H, Y)$  for some  $\omega \in \mathbf{R}$ . The infimum of all the numbers  $\omega$  for which  $\Psi$  is  $\omega$ -stable is the exponential growth rate of  $\Psi$ . Thus,  $\Psi$  is exponentially stable iff its exponential growth rate is negative.

As Salamon [20] and Weiss [27], [28], [30] have shown, the growth rate of a system  $\Psi$  is equal to the growth rate of its semigroup as explained in Lemma 2.6.

LEMMA 2.6. The exponential growth rate of a well-posed linear system  $\Psi$  is equal to the exponential growth rate  $\omega = \lim_{t \rightarrow \infty} t^{-1} \log(\|A(t)\|)$  of its semigroup. In particular,  $\Psi$  is exponentially stable iff its semigroup is exponentially stable.

See [20, Lemma 2.1] or [27, Proposition 2.5], [28, Proposition 2.3], and [30, Proposition 4.1] for proofs.

One of the required properties of the input/output operator  $\mathcal{D}$  of  $\Psi$  is that it is time invariant.

DEFINITION 2.7. Let  $U$  and  $Y$  be two Hilbert spaces. A bounded linear operator  $\mathcal{D}: L^2_\omega(\mathbf{R}; U) \rightarrow L^2_\omega(\mathbf{R}; Y)$  is time invariant iff it commutes with time shifts; i.e.,  $\tau(t)\mathcal{D}u = \mathcal{D}\tau(t)u$  for all  $u \in L^2_\omega(\mathbf{R}; U)$  and all  $t \in \mathbf{R}$ . We denote this class of operators by  $TI_\omega(U; Y)$ . The Hankel operator induced by  $\mathcal{D}$  is the operator  $\pi_+\mathcal{D}\pi_-$ , and the anti-Hankel operator induced by  $\mathcal{D}$  is the operator  $\pi_-\mathcal{D}\pi_+$ . The Toeplitz operator induced by  $\mathcal{D}$  is the operator  $\pi_+\mathcal{D}\pi_+$ , and the anti-Toeplitz operator induced by  $\mathcal{D}$  is the operator  $\pi_-\mathcal{D}\pi_-$ .

The word ‘‘causal’’ that we have included in the definition of a well-posed linear system relates to the fact that all the components of  $\Psi$  in Definition 2.1 are causal as follows.

DEFINITION 2.8. An operator  $\mathcal{B}: L^2_\omega(\mathbf{R}; U) \rightarrow H$  is causal [anticausal] if  $\mathcal{B}\pi_+ = 0$  [ $\mathcal{B}\pi_- = 0$ ]. An operator  $\mathcal{C}: H \rightarrow L^2_\omega(\mathbf{R}; Y)$  is causal [anticausal] if  $\pi_-\mathcal{C} = 0$  [ $\pi_+\mathcal{C} = 0$ ]. A time-invariant operator  $\mathcal{D}: L^2_\omega(\mathbf{R}; U) \rightarrow L^2_\omega(\mathbf{R}; Y)$  is causal [anticausal] if  $\pi_-\mathcal{D}\pi_+ = 0$  [ $\pi_+\mathcal{D}\pi_- = 0$ ], and it is static if it is both causal and anticausal. We denote the class of bounded linear time invariant causal operators by  $TIC_\omega(U; Y)$ .

Thus, the condition imposed on  $\mathcal{D}$  in Definition 2.1 requires that  $\mathcal{D} \in TIC_\omega(U; Y)$  (i.e.,  $\mathcal{D}$  is time invariant and causal) and that the Hankel operator induced by  $\mathcal{D}$  is equal to  $\mathcal{C}\mathcal{B}$ . Intuitively, a causal controllability map  $\mathcal{B}$  maps past inputs into the present state, a causal observability map  $\mathcal{C}$  maps the present state into future outputs, and the past output of a causal input/output map  $\mathcal{D}$  does not depend on future inputs.

As is well known, there is a one-to-one correspondence between  $TIC_\omega(U; Y)$  and the set of  $\mathcal{L}(U; Y)$ -valued  $H^\infty$ -functions over the half-plane  $\Re z > \omega$ . We denote this set of functions by  $H^\infty_\omega(U; Y)$ . The norm in this space is the usual  $H^\infty$ -norm.

LEMMA 2.9. The two spaces  $TIC_\omega(U; Y)$  and  $H^\infty_\omega(U; Y)$  are isometrically isomorphic. More precisely, to each operator  $\mathcal{D} \in TIC_\omega(U; Y)$  there corresponds a unique function  $\widehat{\mathcal{D}} \in H^\infty_\omega(U; Y)$  such that for each  $u \in L^2_\omega(\mathbf{R}^+; U)$ , the Laplace transform of  $\mathcal{D}u$  is given by  $\widehat{\mathcal{D}}(z)\widehat{u}(z)$ ,  $\Re z > \omega$ , where  $\widehat{u}$  is the Laplace transform of  $u$ . The function  $\widehat{\mathcal{D}}$  is called the transfer function (or symbol) of  $\mathcal{D}$ .

Thus, intuitively,  $\widehat{\mathcal{D}}$  is the Laplace transform of  $\mathcal{D}$ . This result is classic; see, for example, [8] or [29].

By a result due to Salamon [20, Section 4], a time-invariant operator  $\mathcal{D}$  can be interpreted as the input/output operator of a well-posed linear system  $\Psi$  iff it belongs to  $TIC_\omega(U; Y)$  for some  $\omega \in \mathbf{R}$ . Such a system is called a realization of  $\mathcal{D}$ . Two particular realizations are described below.

DEFINITION 2.10. Let  $\omega \in \mathbf{R}$  and  $\mathcal{D} \in TIC_\omega(U; Y)$ , and let  $\tau$  be the left-shift group. Then the  $\omega$ -stable well-posed linear system

$$\begin{bmatrix} \tau\pi_- & \pi_- \\ \pi_+\mathcal{D}\pi_- & \mathcal{D} \end{bmatrix}$$

on  $(U, L^2_\omega(\mathbf{R}^-; U), Y)$  is called the exactly controllable realization of  $\mathcal{D}$ , and the strongly  $\omega$ -stable well-posed linear system

$$\begin{bmatrix} \pi_+\tau\pi_+ & \pi_+\mathcal{D}\pi_- \\ I & \mathcal{D} \end{bmatrix}$$

on  $(U, L^2_\omega(\mathbf{R}^+; Y), Y)$  is called the continuously observable realization of  $\mathcal{D}$ .

Indeed, it is obvious that the two systems defined above satisfy the requirements of Definition 2.1. See [20, section 4] for an explanation of the names of these realizations.

Occasionally we shall need to discuss the stability of the different parts of  $\Psi$  separately, and for this purpose we further introduce the following natural terminology.

DEFINITION 2.11. Let  $\Psi = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}$  be a well-posed linear system on  $(U, H, Y)$ .

Then

- (i)  $\mathcal{A}$  is  $\omega$ -stable iff  $\sup_{t \in \mathbf{R}^+} \|e^{-\omega t} \mathcal{A}(t)\| < \infty$  and strongly  $\omega$ -stable iff  $e^{-\omega t} \mathcal{A}(t)x \rightarrow 0$  as  $t \rightarrow \infty$  for all  $x \in H$ ,
- (ii)  $\mathcal{B}$  is  $\omega$ -stable iff  $\mathcal{B} \in \mathcal{L}(L^2_\omega(\mathbf{R}; U); H)$ ,
- (iii)  $\mathcal{C}$  is  $\omega$ -stable iff  $\mathcal{C} \in \mathcal{L}(H; L^2_\omega(\mathbf{R}; Y))$ ,
- (iv)  $\mathcal{D}$  is  $\omega$ -stable iff  $\mathcal{D} \in TIC_\omega(U; Y)$ .

As before, stability of a component of  $\Psi$  means  $\omega$ -stability with  $\omega = 0$ , and exponential stability means  $\omega$ -stability for some  $\omega < 0$ .

REMARK 2.12. Almost all results presented below remain valid if we throughout drop the assumption that  $\mathcal{A}$  is  $\omega$ -stable in condition (i) of Definition 2.1. Thus for our purposes it suffices if the system is input  $\omega$ -stable (condition (ii)), output  $\omega$ -stable (condition (iii)), and input/output  $\omega$ -stable (condition (iv) of Definition 2.11). In [24] this situation was referred to as external  $\omega$ -stability.

One gets the adjoint of a system  $\Psi$  by replacing each operator by its adjoint and exchanging the controllability and observability maps with each other. In the computation of the adjoints of  $\mathcal{B}$ ,  $\mathcal{C}$ , and  $\mathcal{D}$  we use the ordinary (unweighted)  $L^2$ -inner product. This means that the resulting operators are bounded linear operators on  $L^2_{-\omega}$  instead of bounded linear operators on  $L^2_\omega$ . Moreover, causality is replaced by anticausality. The resulting system is an  $(-\omega)$ -stable anticausal system of the following type.

DEFINITION 2.13. Let  $Y$ ,  $H$ , and  $U$  be Hilbert spaces. An anticausal  $\omega$ -stable well-posed linear  $L^2$ -system on  $(Y, H, U)$  is a quadruple  $\Psi^* = \begin{bmatrix} \mathcal{A}^* & \mathcal{C}^* \\ \mathcal{B}^* & \mathcal{D}^* \end{bmatrix}$ , where  $\mathcal{A}^*$ ,  $\mathcal{C}^*$ ,  $\mathcal{B}^*$ , and  $\mathcal{D}^*$  are bounded linear operators of the following type:

- (i)  $\mathcal{A}^*(t): H \rightarrow H$  is a strongly continuous semigroup of bounded linear operators on  $H$  satisfying  $\sup_{t \in \mathbf{R}^+} \|e^{\omega t} \mathcal{A}^*(t)\| < \infty$ ;
- (ii)  $\mathcal{C}^*: L^2_\omega(\mathbf{R}; Y) \rightarrow H$  satisfies  $\mathcal{A}^*(-s)\mathcal{C}^*y^* = \mathcal{C}^*\tau(s)\pi_+y^*$  for all  $y^* \in L^2_\omega(\mathbf{R}; Y)$  and  $s \in \mathbf{R}^-$ ;
- (iii)  $\mathcal{B}^*: H \rightarrow L^2_\omega(\mathbf{R}; U)$  satisfies  $\mathcal{B}^*\mathcal{A}^*(-s)x^* = \pi_-\tau(s)\mathcal{B}^*x^*$  for all  $x^* \in H$  and  $s \in \mathbf{R}^-$ ;
- (iv)  $\mathcal{D}^*: L^2_\omega(\mathbf{R}; Y) \rightarrow L^2_\omega(\mathbf{R}; U)$  satisfies  $\tau(s)\mathcal{D}^*y^* = \mathcal{D}^*\tau(s)y^*$ ,  $\pi_+\mathcal{D}^*\pi_+y^* = 0$ , and  $\pi_-\mathcal{D}^*\pi_+y^* = \mathcal{B}^*\mathcal{C}^*y^*$  for all  $y^* \in L^2_\omega(\mathbf{R}; Y)$  and  $s \in \mathbf{R}$ .

If, in addition,  $e^{\omega t} \mathcal{A}^*(t)x^* \rightarrow 0$  as  $t \rightarrow \infty$  for all  $x^* \in H$ , then  $\Psi^*$  is strongly  $\omega$ -stable. The system  $\Psi^*$  is [strongly] stable iff it is [strongly]  $\omega$ -stable with  $\omega = 0$ , and it is exponentially stable iff it is  $\omega$ -stable for some  $\omega > 0$ .

The different components of  $\Psi^*$  are named as follows:  $Y$  is the input space,  $H$  is the state space,  $U$  is the output space,  $\mathcal{A}^*$  is the semigroup,  $\mathcal{C}^*$  is the controllability

map,  $\mathcal{B}^*$  is the observability map, and  $\mathcal{D}^*$  is the input/output map of  $\Psi^*$ .

The controlled state and output of  $\Psi^*$  are defined as follows.

DEFINITION 2.14. Let  $\Psi^* = \begin{bmatrix} \mathcal{A}^* & \mathcal{C}^* \\ \mathcal{B}^* & \mathcal{D}^* \end{bmatrix}$  be an  $\omega$ -stable anticausal well-posed linear system on  $(Y, H, U)$  and let  $y^* \in L^2_\omega(\mathbf{R}; Y)$ . In the time-invariant setting the controlled state  $x^*(s)$  at time  $s \in \mathbf{R}$  and the output  $u^*$  of  $\Psi^*$  with control  $y^*$  are given by

$$\begin{bmatrix} x^*(s) \\ u^* \end{bmatrix} = \begin{bmatrix} \mathcal{C}^* \tau(s) y^* \\ \mathcal{D}^* y^* \end{bmatrix},$$

and in the initial value setting with initial time  $t$ , initial value  $x^*(t)$ , and control  $y^*$ , the controlled state  $x^*(s)$  at time  $s \leq t$  and the output  $u^*$  of  $\Psi^*$  are given by

$$\begin{bmatrix} x^*(s) \\ u^* \end{bmatrix} = \begin{bmatrix} \mathcal{A}^*(t-s) & \mathcal{C}^* \tau(s) \\ \tau(-t) \mathcal{B}^* & \mathcal{D}^* \end{bmatrix} \begin{bmatrix} x^*(t) \\ \pi_{(-\infty, t]} y^* \end{bmatrix} = \begin{bmatrix} \mathcal{A}^*(t-s)x^*(t) + \mathcal{C}^* \tau(s) \pi_{(-\infty, t]} y^* \\ \tau(-t) \mathcal{B}^* x^*(t) + \mathcal{D}^* \pi_{(-\infty, t]} y^* \end{bmatrix},$$

where  $\pi_{(-\infty, t]} = \tau(-t) \pi_- \tau(t)$  is given by

$$(\pi_{(-\infty, t]} y^*)(s) = \begin{cases} y^*(s), & s \leq t, \\ 0, & s > t. \end{cases}$$

In particular, in the initial value setting with initial time zero, initial value  $x^*_0$ , and control  $y^*$ , the controlled state  $x^*(s)$  at time  $s \in \mathbf{R}^-$  and the output  $u^*$  of  $\Psi^*$  are given by

$$\begin{bmatrix} x^*(s) \\ u^* \end{bmatrix} = \begin{bmatrix} \mathcal{A}^*(-s) & \mathcal{C}^* \tau(s) \\ \mathcal{B}^* & \mathcal{D}^* \end{bmatrix} \begin{bmatrix} x^*_0 \\ \pi_- y^* \end{bmatrix} = \begin{bmatrix} \mathcal{A}^*(-s)x^*_0 + \mathcal{C}^* \tau(s) \pi_- y^* \\ \mathcal{B}^* x^*_0 + \mathcal{D}^* \pi_- y^* \end{bmatrix}.$$

The formulas in Definitions 2.2 and 2.14 are chosen in such a way that  $\Psi$  and  $\Psi^*$  interact in the following way.

LEMMA 2.15. Let  $\Psi$  be a well-posed linear system on  $(U, H, Y)$  and  $\Psi^*$  its adjoint (with respect to the ordinary unweighted inner product in  $L^2$ ). Let  $-\infty < s < t < \infty$ , let  $x$  and  $y$  be the state and output of  $\Psi$  with initial time  $s$  and control  $u$ , and let  $x^*$  and  $u^*$  be the state and output of  $\Psi^*$  with initial time  $t$  and control  $y^*$ . Then

$$\langle x^*(t), x(t) \rangle_H + \int_s^t \langle y^*(v), y(v) \rangle_Y dv = \langle x^*(s), x(s) \rangle_H + \int_s^t \langle u^*(v), u(v) \rangle_U dv.$$

We leave the easy proof of this lemma to the reader.

We use diagrams of the type drawn in Figure 2.1 to represent the relation between the state  $x$ , the output  $y$ , the initial value  $x_0$ , and the control  $u$  of  $\Psi$  in the initial value setting with initial time zero. In our diagrams we use the following conventions.

- (i) Initial states and controls enter at the top or bottom, and they are acted on by all the operators located in the column to which they are attached. In particular, note that  $x_0$  is attached to the first column and  $u$  to the second.
- (ii) Final states and outputs leave to the left or right, and they are the sums of all the elements in the row to which they are attached. In particular, note that  $x$  is attached to the top row and  $y$  to the bottom row.

A similar diagram is used to describe the adjoint system  $\Psi^*$ .



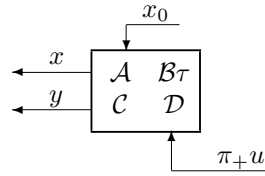


FIG. 2.1. Input/state/output diagram of  $\Psi$ .

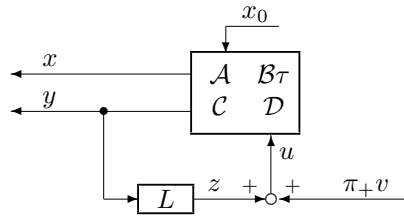


FIG. 3.1. Static output feedback.

**3. Feedback, stabilizability, and detectability.** The notions of stabilizability and detectability deal with the possibility of stabilizing a well-posed linear system by the use of either a state feedback or an output injection. Therefore, before we can study these notions, we must first look at different kinds of feedback connections.

We start with the most basic feedback connection, namely, the notion of a (static) output feedback, drawn in Figure 3.1. Here  $L$  is a bounded linear operator from the output space into the input space. Thus if we consider this feedback configuration in the initial value setting with initial time zero, initial value  $x_0$ , and control  $v$ , we find that the effective input  $u$ , the state  $x(t)$  at time  $t \geq 0$ , the output  $y$ , and the feedback signal  $z$  satisfy the equations

$$\begin{aligned}
 (3.1) \quad & u = z + \pi_+ v, \\
 & x(t) = \mathcal{A}(t)x_0 + \mathcal{B}\tau(t)u, \\
 & y = \mathcal{C}x_0 + \mathcal{D}u, \\
 & z = Ly,
 \end{aligned}$$

which formally can be solved as

$$\begin{aligned}
 (3.2) \quad & u = (I - LD)^{-1} (LCx_0 + \pi_+ v), \\
 & x(t) = (\mathcal{A}(t) + \mathcal{B}\tau(t)L(I - DL)^{-1}\mathcal{C})x_0 + \mathcal{B}(I - LD)^{-1}\tau(t)\pi_+ v, \\
 & y = (I - DL)^{-1} (\mathcal{C}x_0 + \mathcal{D}\pi_+ v), \\
 & z = (I - LD)^{-1} L (\mathcal{C}x_0 + \mathcal{D}\pi_+ v).
 \end{aligned}$$

We say that the feedback operator  $L$  is admissible whenever these equations are valid.

**DEFINITION 3.1.** Let  $\Psi = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}$  be a well-posed linear system on  $(U, H, Y)$ . Then  $L \in \mathcal{L}(Y; U)$  is called an admissible output feedback operator for  $\Psi$  iff the operator  $I - LD$  has an inverse in  $TIC_\alpha(U)$  for some  $\alpha \in \mathbf{R}$  or, equivalently, iff the operator  $I - DL$  has an inverse in  $TIC_\alpha(Y)$  for some  $\alpha \in \mathbf{R}$ .

As Weiss [31, section 6] proved,  $x$  and  $y$  in (3.2) can be interpreted as the state and output of another well-posed linear system as follows.

PROPOSITION 3.2. Let  $\Psi = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$  be a well-posed linear system on  $(U, H, Y)$ , and let  $L \in \mathcal{L}(Y; U)$  be an admissible output feedback operator for  $\Psi$ . Then the system

$$\begin{aligned} \Psi_L &= \begin{bmatrix} A_L & B_L \tau \\ C_L & D_L \end{bmatrix} = \begin{bmatrix} A + B\tau L(I - DL)^{-1}C & B(I - LD)^{-1}\tau \\ (I - DL)^{-1}C & D(I - LD)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} A & B\tau \\ C & D \end{bmatrix} + \begin{bmatrix} B\tau \\ D \end{bmatrix} L(I - DL)^{-1} \begin{bmatrix} C & D \end{bmatrix} \\ &= \begin{bmatrix} A & B\tau \\ C & D \end{bmatrix} + \begin{bmatrix} B\tau \\ D \end{bmatrix} L \begin{bmatrix} C_L & D_L \end{bmatrix} \\ &= \begin{bmatrix} A & B\tau \\ C & D \end{bmatrix} + \begin{bmatrix} B_L \tau \\ D_L \end{bmatrix} L \begin{bmatrix} C & D \end{bmatrix} \end{aligned}$$

is another well-posed linear system on  $(U, H, Y)$ . We call this system the closed loop system with feedback operator  $L$ . In the initial value setting with initial time zero, initial value  $x_0$ , and control  $v$ , the controlled state  $x(t)$  at time  $t$  and the output  $y$  of  $\Psi_L$  are given by (3.2).

See [31, section 6] for a proof. (The major part of this proposition is also contained in [19, Theorem 4.2].)

We remark that if in the classical system (2.1) we replace  $u$  by  $u = Ly + v$ , then we get a new well defined system of the same type iff  $I - DL$  is invertible or, equivalently, iff  $I - LD$  is invertible. In the new system the operators  $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$  have been replaced by

$$\begin{aligned} \begin{bmatrix} A_L & B_L \\ C_L & D_L \end{bmatrix} &= \begin{bmatrix} A + BL(I - DL)^{-1}C & B(I - LD)^{-1} \\ (I - DL)^{-1}C & D(I - LD)^{-1} \end{bmatrix} \\ (3.3) \quad &= \begin{bmatrix} A & B \\ C & D \end{bmatrix} + \begin{bmatrix} B \\ D \end{bmatrix} L(I - DL)^{-1} \begin{bmatrix} C & D \end{bmatrix} \\ &= \begin{bmatrix} A & B \\ C & D \end{bmatrix} + \begin{bmatrix} B \\ D \end{bmatrix} L \begin{bmatrix} C_L & D_L \end{bmatrix} \\ &= \begin{bmatrix} A & B \\ C & D \end{bmatrix} + \begin{bmatrix} B_L \\ D_L \end{bmatrix} L \begin{bmatrix} C & D \end{bmatrix}. \end{aligned}$$

Observe the striking similarity between this formula and the one given in Proposition 3.2.<sup>7</sup>

Repeated feedback behaves in the expected way.

PROPOSITION 3.3. Let  $L \in \mathcal{L}(U; Y)$  be an admissible output feedback operator for  $\Psi$ . Then  $K \in \mathcal{L}(U; Y)$  is an admissible output feedback operator for the closed loop system  $\Psi_L$  iff  $L + K$  is an admissible output feedback operator for  $\Psi$ , and  $\Psi_{L+K} = (\Psi_L)_K$ . In particular,  $-L$  is always an admissible feedback operator for  $\Psi_L$ , and  $(\Psi_L)_{-L} = \Psi$ .

See [31, Remark 6.4] for the straightforward proof.

DEFINITION 3.4. The operator  $L \in \mathcal{L}(Y; U)$  is a (strongly)  $\omega$ -stabilizing [stabilizing] [[exponentially stabilizing]] output feedback operator for the well-posed linear system  $\Psi$  on  $(U, H, Y)$  iff  $L$  is an admissible output feedback operator for  $\Psi$  and the resulting closed loop system  $\Psi_L$  is (strongly)  $\omega$ -stable [stable] [[exponentially stable]].

We observe the following basic facts.

LEMMA 3.5. Let  $\Psi = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$  be  $\omega$ -stable, and let  $L \in \mathcal{L}(Y; U)$ .

<sup>7</sup>Usually the feed-through operator  $D$  is taken to be zero, in which case this formula simplifies significantly and the invertibility condition on  $I - DL$  drops out.

- (i)  $L$  is an admissible output feedback operator for  $\Psi$  iff there is some  $\alpha \geq \omega$  for which the diagram in Figure 3.1 (i.e., the set of equations (3.1)) with  $x_0 = 0$  defines a continuous linear mapping from the external input  $v \in L^2_\alpha(\mathbf{R}^+; U)$  to the internal input  $u \in L^2_\alpha(\mathbf{R}^+; U)$  or, equivalently, iff the operator  $I - LD$  has an inverse in  $TIC_\alpha(U)$  or, equivalently, iff the operator  $I - DL$  has an inverse in  $TIC_\alpha(Y)$ . The resulting closed loop  $\Psi_L$  system is  $\alpha$ -stable.
- (ii)  $L$  is  $\omega$ -stabilizing iff any one of the three equivalent conditions in part (i) is true with  $\omega = \alpha$  (hence all of them are true with  $\omega = \alpha$ ). In this case the closed loop system  $\Psi_L$  is strongly  $\omega$ -stable iff the open loop system  $\Psi$  is strongly  $\omega$ -stable.

REMARK 3.6. Thus if a system is  $\omega$ -stable but not strongly so, then it is impossible to make it strongly  $\omega$ -stable by using our notion of admissible output feedback.

Proof of Lemma 3.5. (i) Most of this follows immediately from Definition 3.1. To see that the resulting system is  $\alpha$ -stable we observe that if  $I - LD$  has an inverse in  $TIC_\alpha(U)$  (or, equivalently, the operator  $I - DL$  has an inverse in  $TIC_\alpha(Y)$ ), then the formulas for  $\Psi_L$  given in Proposition 3.2 imply that  $\Psi_L$  is  $\alpha$ -stable. (Here we use the fact that  $\omega$ -stability implies  $\alpha$ -stability for every  $\alpha \geq \omega$ ; see Lemma 2.4.)

(ii) Clearly, if the conditions in part (i) are true with  $\alpha = \omega$ , then  $L$  is  $\omega$ -stabilizing. Conversely, if  $L$  is  $\omega$ -stabilizing, then  $\mathcal{D}_L = \mathcal{D}(I - LD)^{-1} \in TIC_\omega(U; Y)$ , and this implies that  $(I - LD)^{-1} = I + LD(I - LD)^{-1} = I + LD_L \in TIC_\omega(U)$ .

To prove the second claim in part (ii) it suffices to show that

$$e^{-\omega t} \mathcal{B}\tau(t)L(I - DL)^{-1}Cx \rightarrow 0 \text{ as } t \rightarrow \infty$$

for every  $x \in H$ , since  $\mathcal{A}_L - \mathcal{A} = \mathcal{B}\tau L(I - DL)^{-1}C$ . Fix  $x \in H$  and split the expression above into

$$\begin{aligned} e^{-\omega t} \mathcal{B}\tau(t)L(I - DL)^{-1}Cx &= e^{-\omega t} \mathcal{B}\tau(t - T)(\pi_+ + \pi_-)\tau(T)L(I - DL)^{-1}Cx \\ &= \mathcal{B}e^{-\omega t} \tau(t)\pi_{[T, \infty)}L(I - DL)^{-1}Cx \\ &\quad + e^{-\omega t} \mathcal{A}(t - T)\mathcal{B}\tau(T)L(I - DL)^{-1}Cx. \end{aligned}$$

Here the first term tends to zero as  $T \rightarrow \infty$ , uniformly in  $t \geq T$ , and the second term tends to zero as  $t \rightarrow \infty$  and  $T$  is fixed.  $\square$

REMARK 3.7. The same proof shows that (with the same terminology as in Remark 2.12) if  $\Psi$  is input  $\omega$ -stable and output  $\omega$ -stable and  $L \in \mathcal{L}(Y; U)$  is an admissible output feedback operator, then  $L$  is input  $\omega$ -stabilizing, output  $\omega$ -stabilizing, and input/output  $\omega$ -stabilizing iff  $\mathcal{D}(I - LD)^{-1} \in TIC_\omega(U)$ . Moreover, in this case  $L$  is [strongly]  $\omega$ -stabilizing iff the semigroup of  $\Psi$  is [strongly]  $\omega$ -stable.

The notion of a state feedback can be reduced formally to the notion of an output feedback. Intuitively, a state feedback means that an additional output is created, and this output is then fed back into the input, as shown in Figure 3.2. In this figure the original system is represented by  $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ . We find two additional components, namely, a new observability map  $\mathcal{K}$  (from the initial state to the new output) and a new input/output map  $\mathcal{F}$  (from the original input to the new output). The pair  $[\mathcal{K} \quad \mathcal{F}]$  is admissible if the resulting system is well posed, i.e., if  $\begin{bmatrix} 0 & I \end{bmatrix}$  is an admissible output feedback operator for the extended system defined in Definition 3.8.

DEFINITION 3.8. Let  $\Psi = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$  be a well-posed linear system on  $(U, H, Y)$ . The

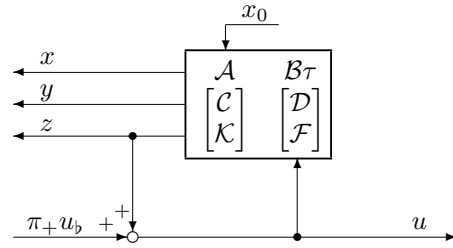


FIG. 3.2. State feedback connection.

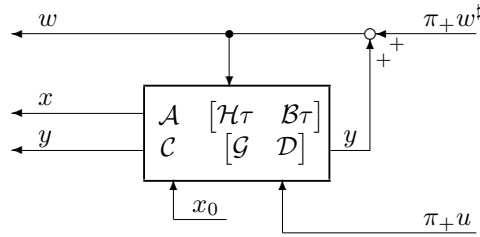


FIG. 3.3. Output injection connection.

pair  $[\mathcal{K} \ \mathcal{F}]$  is an admissible state feedback pair for  $\Psi$  iff the extended system

$$\Psi_{\text{SF}} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \begin{bmatrix} \mathcal{C} \\ \mathcal{K} \end{bmatrix} & \begin{bmatrix} \mathcal{D} \\ \mathcal{F} \end{bmatrix} \end{bmatrix}$$

is a well-posed linear system on  $(U, H, Y \times U)$  and  $\begin{bmatrix} 0 & I \end{bmatrix}$  is an admissible output feedback operator for  $\Psi_{\text{SF}}$ ; i.e.,  $I - \mathcal{F}$  has an inverse in  $TIC_\omega(U)$  for some  $\omega \in \mathbf{R}$ .<sup>8</sup> It is (strongly)  $\omega$ -stabilizing [stabilizing] [[exponentially stabilizing]] if the resulting closed loop system is (strongly)  $\omega$ -stable [stable] [[exponentially stable]].

REMARK 3.9. We shall frequently regard the signal  $u$  in Figure 3.2 (i.e., the input to the open loop system) as an additional output of the closed loop system (although it is not part of the official definition). This output has the same observability map  $(I - \mathcal{F})^{-1}\mathcal{K}$  as the output  $z$ .<sup>9</sup> Its input/output map, given by  $(I - \mathcal{F})^{-1}$ , differs from the input/output map from  $u_b$  to  $z$  by an identity operator (see Lemma 3.13). Similar remarks apply to the signals  $w$  in Figure 3.3,  $u_b$  and  $w^\sharp$  in Figure 3.4,  $u$  and  $w^\sharp$  in Figure 3.5,  $u_b$  and  $w$  in Figure 3.6, etc.

The notion of an output injection is analogous. In this case a new input is created into which we feed the original output  $y$  plus a new perturbation  $w^\sharp$ , as shown in Figure 3.3. The original system is still represented by  $\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}$ . In this figure we find a new controllability map  $\mathcal{H}$  (from the new input to the state) and a new input/output map  $\mathcal{G}$  (from the new input to the original output). The pair  $\begin{bmatrix} \mathcal{H} \\ \mathcal{G} \end{bmatrix}$  is admissible if the resulting system is well posed.

DEFINITION 3.10. Let  $\Psi = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}$  be a well-posed linear system on  $(U, H, Y)$ .

<sup>8</sup>The input of this system is the signal  $u_b$  in Figure 3.2, and its outputs are  $y$  and  $z$ . See also Remark 3.9.

<sup>9</sup>See Lemma 3.13.

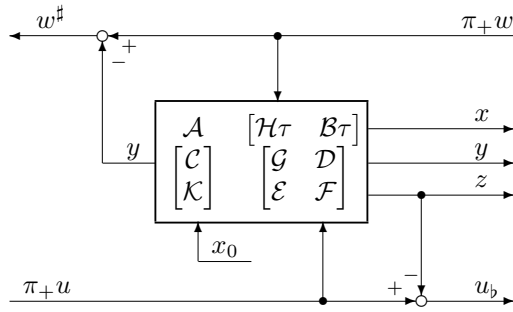


FIG. 3.4. The extended system.

The pair  $\begin{bmatrix} \mathcal{H} \\ \mathcal{G} \end{bmatrix}$  is an admissible output injection pair for  $\Psi$  iff the extended system

$$\Psi_{\text{OI}} = \begin{bmatrix} A & \begin{bmatrix} \mathcal{H} & B \end{bmatrix} \\ \mathcal{C} & \begin{bmatrix} \mathcal{G} & D \end{bmatrix} \end{bmatrix}$$

is a well-posed linear system on  $(Y \times U, H, Y)$  and  $\begin{bmatrix} I \\ 0 \end{bmatrix}$  is an admissible output feedback operator for  $\Psi_{\text{OI}}$ ; i.e.,  $I - \mathcal{G}$  has an inverse in  $TIC_\omega(Y)$  for some  $\omega \in \mathbf{R}$ .<sup>10</sup> It is (strongly)  $\omega$ -stabilizing [stabilizing] [[exponentially stabilizing]] if the resulting closed loop system is (strongly)  $\omega$ -stable [stable] [[exponentially stable]].

In the sequel we shall need to study a case where at the same time we want to add both a state feedback pair  $\begin{bmatrix} \mathcal{K} & \mathcal{F} \end{bmatrix}$  and an output injection pair  $\begin{bmatrix} \mathcal{H} \\ \mathcal{G} \end{bmatrix}$  to a given system  $\begin{bmatrix} A & B \\ \mathcal{C} & D \end{bmatrix}$ . If we try to write a figure similar to Figures 3.2 and 3.3, we immediately observe that we need one more input/output map  $\mathcal{E}$  (from the output injection input to the state feedback output); see Figure 3.4. This operator need not always exist,<sup>11</sup> and this forces us to introduce still another definition.

DEFINITION 3.11. Let  $\Psi = \begin{bmatrix} A & B \\ \mathcal{C} & D \end{bmatrix}$  be a well-posed linear system on  $(U, H, Y)$ . The pairs  $\begin{bmatrix} \mathcal{K} & \mathcal{F} \end{bmatrix}$  and  $\begin{bmatrix} \mathcal{H} \\ \mathcal{G} \end{bmatrix}$  are called jointly admissible state feedback and output injection pairs for  $\Psi$  iff  $\begin{bmatrix} \mathcal{K} & \mathcal{F} \end{bmatrix}$  is an admissible state feedback pair for  $\Psi$ ,  $\begin{bmatrix} \mathcal{H} \\ \mathcal{G} \end{bmatrix}$  is an admissible output injection pair for  $\Psi$ , and in addition, there exists a operator  $\mathcal{E}$ , called the interaction operator, such that and the combined extended system

$$\Psi_{\text{ext}} = \begin{bmatrix} A & \begin{bmatrix} \mathcal{H} & B \end{bmatrix} \\ \begin{bmatrix} \mathcal{C} \\ \mathcal{K} \end{bmatrix} & \begin{bmatrix} \mathcal{G} & D \\ \mathcal{E} & \mathcal{F} \end{bmatrix} \end{bmatrix}$$

is a well-posed linear system on  $(Y \times U, H, Y \times U)$ .

LEMMA 3.12. Let  $\Psi = \begin{bmatrix} A & B \\ \mathcal{C} & D \end{bmatrix}$  be a well-posed linear system on  $(U, H, Y)$ . Then the following conditions are equivalent:

- (i) the pairs  $\begin{bmatrix} \mathcal{K} & \mathcal{F} \end{bmatrix}$  and  $\begin{bmatrix} \mathcal{H} \\ \mathcal{G} \end{bmatrix}$  are jointly admissible state feedback and output injection pairs with interaction operator  $\mathcal{E}$ ;
- (ii) the system  $\Psi_{\text{ext}}$  in Definition 3.11 is a well-posed linear system on  $(Y \times U, H, Y \times U)$ , and both  $\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$  and  $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$  are admissible output feedback operators for  $\Psi_{\text{ext}}$ .

<sup>10</sup>The inputs of this system are the signals  $u$  and  $w^\#$  in Figure 3.3, and its output is  $y$ . See also Remark 3.9.

<sup>11</sup>More precisely, it need not be a bounded operator. The operator  $\mathcal{E}$ , if it exists, is determined uniquely modulo a static operator; this follows from [24, Corollary 7].

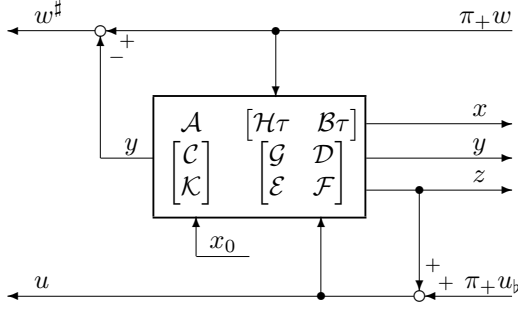


FIG. 3.5. Right coprime factor.

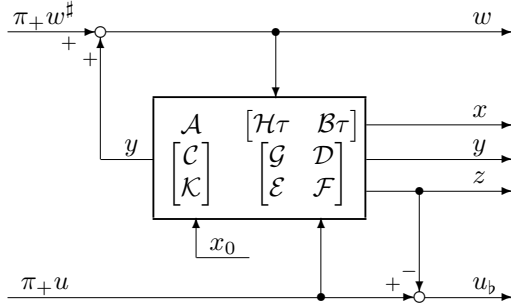


FIG. 3.6. Left coprime factor.

(iii) the system  $\Psi_{\text{ext}}$  in Definition 3.11 is a well-posed linear system on  $(Y \times U, H, Y \times U)$ , and  $I - \mathcal{F}$  and  $I - \mathcal{G}$  have inverses in  $\text{TIC}_\omega(U)$ , respectively,  $\text{TIC}_\omega(Y)$  for some  $\omega \in \mathbf{R}$ .

To prove this lemma it suffices to make a straightforward calculation based on Proposition 3.2. As a part of this calculation we get the following expressions for the two closed loop systems in part (ii), drawn in Figures 3.5 and 3.6, respectively.<sup>12</sup>

LEMMA 3.13. Let  $\Psi = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$  be a well-posed linear system on  $(U, H, Y)$ , and let  $[\mathcal{K} \ \mathcal{F}]$  and  $\begin{bmatrix} \mathcal{H} \\ \mathcal{G} \end{bmatrix}$  be jointly admissible state feedback and output injection pairs for  $\Psi$  with interaction operator  $\mathcal{E}$ . Then the closed loop system  $\Psi_b$  that we get by using  $\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$  as a output feedback operator for  $\Psi_{\text{ext}}$  (see Figure 3.5) is given by<sup>13</sup>

$$\begin{aligned} \Psi_b &= \begin{bmatrix} \mathcal{A}_b & [\mathcal{H}_b \tau & \mathcal{B}_b \tau] \\ \mathcal{C}_b & [\mathcal{G}_b & \mathcal{D}_b] \\ \mathcal{K}_b & [\mathcal{E}_b & \mathcal{F}_b] \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{A} + \mathcal{B}\tau(I - \mathcal{F})^{-1}\mathcal{K} & [\mathcal{H}\tau + \mathcal{B}(I - \mathcal{F})^{-1}\mathcal{E}\tau & \mathcal{B}(I - \mathcal{F})^{-1}\tau] \\ \mathcal{C} + \mathcal{D}(I - \mathcal{F})^{-1}\mathcal{K} & [\mathcal{G} + \mathcal{D}(I - \mathcal{F})^{-1}\mathcal{E} & \mathcal{D}(I - \mathcal{F})^{-1}] \\ (I - \mathcal{F})^{-1}\mathcal{K} & [(I - \mathcal{F})^{-1}\mathcal{E} & (I - \mathcal{F})^{-1} - I] \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{A} & [\mathcal{H}\tau & \mathcal{B}\tau] \\ \mathcal{C} & [\mathcal{G} & \mathcal{D}] \\ \mathcal{K} & [\mathcal{E} & \mathcal{F}] \end{bmatrix} + \begin{bmatrix} \mathcal{B}\tau \\ \mathcal{D} \\ \mathcal{F} \end{bmatrix} (I - \mathcal{F})^{-1} [\mathcal{K} \ \mathcal{E} \ \mathcal{F}], \end{aligned}$$

<sup>12</sup>See section 4 for an explanation of the captions of these figures.

<sup>13</sup>The inputs of this system are the signals  $u_b$  and  $w$  in Figure 3.5, and its outputs are  $y$  and  $z$ . See also Remark 3.9.

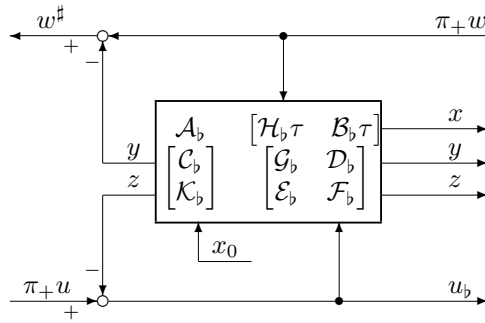


FIG. 3.7. Cancellation of state feedback.

and the closed loop system  $\Psi^\#$  that we get by using  $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$  as a output feedback operator for  $\Psi_{\text{ext}}$  (see Figure 3.6) is given by<sup>14</sup>

$$\begin{aligned} \Psi^\# &= \begin{bmatrix} \mathcal{A}^\# & [\mathcal{H}^\# \tau & \mathcal{B}^\# \tau] \\ [\mathcal{C}^\#] & [\mathcal{G}^\# & \mathcal{D}^\#] \\ [\mathcal{K}^\#] & [\mathcal{E}^\# & \mathcal{F}^\#] \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{A} + \mathcal{H} \tau (I - \mathcal{G})^{-1} \mathcal{C} & [\mathcal{H} (I - \mathcal{G})^{-1} \tau & \mathcal{B} \tau + \mathcal{H} (I - \mathcal{G})^{-1} \mathcal{D} \tau] \\ [(I - \mathcal{G})^{-1} \mathcal{C}] & [(I - \mathcal{G})^{-1} - I & (I - \mathcal{G})^{-1} \mathcal{D}] \\ [\mathcal{K} + \mathcal{E} (I - \mathcal{G})^{-1} \mathcal{C}] & [\mathcal{E} (I - \mathcal{G})^{-1} & \mathcal{F} + \mathcal{E} (I - \mathcal{G})^{-1} \mathcal{D}] \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{A} & [\mathcal{H} \tau & \mathcal{B} \tau] \\ [\mathcal{C}] & [\mathcal{G} & \mathcal{D}] \\ [\mathcal{K}] & [\mathcal{E} & \mathcal{F}] \end{bmatrix} + \begin{bmatrix} \mathcal{H} \tau \\ \mathcal{G} \\ \mathcal{E} \end{bmatrix} (I - \mathcal{G})^{-1} [\mathcal{C} \quad \mathcal{G} \quad \mathcal{D}]. \end{aligned}$$

REMARK 3.14. According to Proposition 3.3, it is possible to recover the extended system  $\Psi_{\text{ext}}$  from either of the systems  $\Psi_b$  or  $\Psi^\#$  by using negative feedback. For example, the feedback connection drawn in Figure 3.7 is equivalent to  $\Psi_{\text{ext}}$ .

So far we have only looked at the joint admissibility of state feedback and output injection pairs. If the resulting closed loop systems drawn in Figures 3.5 and 3.6 are  $\omega$ -stable, then we call these pairs jointly  $\omega$ -stabilizing as follows.

DEFINITION 3.15. The pairs  $[\mathcal{K} \quad \mathcal{F}]$  and  $\begin{bmatrix} \mathcal{H} \\ \mathcal{G} \end{bmatrix}$  are called jointly (strongly)  $\omega$ -stabilizing [stabilizing] [[exponentially stabilizing]] state feedback and output injection pairs for  $\Psi$  if they are jointly admissible state feedback and output injection pairs with some interaction operator  $\mathcal{E}$ , and both the closed loop systems  $\Psi_b$  and  $\Psi^\#$  in Lemma 3.13 are (strongly)  $\omega$ -stable [stable] [[exponentially stable]].

Observe that if the two pairs in Definition 3.15 are  $\omega$ -stabilizing (but not “jointly”  $\omega$ -stabilizing), then we know that the operators in the left and right columns of  $\Psi_b$  and in the top and middle rows of  $\Psi^\#$  are  $\omega$ -stable (see the formulas in Lemma 3.13), but we do not know anything about the operators in the middle column of  $\Psi_b$  and in the bottom row of  $\Psi^\#$ .

DEFINITION 3.16. Let  $\Psi$  be a well-posed linear system.

- (i)  $\Psi$  is (strongly)  $\omega$ -stabilizable [stabilizable] [[exponentially stabilizable]] iff there exists a (strongly)  $\omega$ -stabilizing [stabilizing] [[exponentially stabilizing]] state feedback pair for  $\Psi$ .

<sup>14</sup>The inputs of this system are the signals  $u$  and  $w^\#$  in Figure 3.6, and its outputs are  $y$  and  $z$ . See also Remark 3.9.

- (ii)  $\Psi$  is (strongly)  $\omega$ -detectable [detectable] [[exponentially detectable]] iff there exists a (strongly)  $\omega$ -stabilizing [stabilizing] [[exponentially stabilizing]] output injection pair for  $\Psi$ .
- (iii)  $\Psi$  is jointly (strongly)  $\omega$ -stabilizable [stabilizable] [[exponentially stabilizable]] and detectable iff there exist some jointly (strongly)  $\omega$ -stabilizing [stabilizing] [[exponentially stabilizing]] state feedback and output injection pairs for  $\Psi$ .

We do not know if it is possible for a system to be both stabilizable and detectable without being jointly stabilizable and detectable.

There is a simple connection between  $\omega$ -stability,  $\omega$ -detectability, and input/output  $\omega$ -stability as shown in Lemma 3.17.

LEMMA 3.17. *Let  $\Psi = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}$  be input/output  $\omega$ -stable [exponentially stable] (i.e., let  $\mathcal{D}$  be  $\omega$ -stable [exponentially stable]).*

- (i) *If  $\Psi$  is  $\omega$ -stabilizable [exponentially stabilizable], then the observability map  $\mathcal{C}$  is  $\omega$ -stable [exponentially stable].*
- (ii) *If  $\Psi$  is  $\omega$ -detectable [exponentially detectable], then the controllability map  $\mathcal{B}$  is  $\omega$ -stable [exponentially stable].*
- (iii) *If  $\Psi$  is both (strongly)  $\omega$ -stabilizable [exponentially stabilizable] and  $\omega$ -detectable [exponentially detectable] (not necessarily jointly), then  $\Psi$  is (strongly)  $\omega$ -stable [exponentially stable].*

*Proof.* Introduce the same notation as in Lemma 3.13.

(i) By Lemma 3.13, the observability map  $\mathcal{C}$  is given by  $\mathcal{C} = \mathcal{C}_b - \mathcal{D}\mathcal{K}_b$ . Thus  $\mathcal{C}$  is  $\omega$ -stable [exponentially stable] whenever  $\mathcal{C}_b$ ,  $\mathcal{D}$ , and  $\mathcal{K}_b$  are so.

(ii) By the same lemma, the controllability map  $\mathcal{B}$  is given by  $\mathcal{B} = \mathcal{B}^\# - \mathcal{H}^\#\mathcal{D}$ . Thus  $\mathcal{B}$  is  $\omega$ -stable [exponentially stable] whenever  $\mathcal{B}^\#$ ,  $\mathcal{H}^\#$ , and  $\mathcal{D}$  are so.

(iii) Again, by Lemma 3.13,  $\mathcal{A} = \mathcal{A}_b - \mathcal{B}\tau\mathcal{K}_b$ . Thus  $\mathcal{A}$  is (strongly)  $\omega$ -stable [exponentially stable] whenever  $\mathcal{A}_b$ ,  $\mathcal{B}$ , and  $\mathcal{K}_b$  are so.<sup>15</sup>  $\square$

By adding the trivial converse to part (iii) of Lemma 3.17 we get the following corollary.

COROLLARY 3.18. *A (strongly)  $\omega$ -stabilizable and  $\omega$ -detectable [exponentially stabilizable and detectable] well-posed linear system is (strongly)  $\omega$ -stable [exponentially stable] iff it is input/output  $\omega$ -stable [exponentially stable].*

This result generalizes most other results in this direction such as [17, Corollary 1.8].

Finally, let us present two lemmas concerning exponential stability. Both of these follow directly from Lemma 2.6.

LEMMA 3.19. *Let  $\Psi$  be a well-posed linear system.*

- (i) *The state feedback pair  $[\mathcal{K} \ \mathcal{F}]$  is exponentially stabilizing iff it is admissible and the closed loop semigroup  $\mathcal{A}_b$  in Lemma 3.13 is exponentially stable.*
- (ii) *The output injection pair  $\begin{bmatrix} \mathcal{H} \\ \mathcal{G} \end{bmatrix}$  is exponentially stabilizing iff it is admissible and the closed loop semigroup  $\mathcal{A}^\#$  in Lemma 3.13 is exponentially stable.*
- (iii) *The state feedback pair  $[\mathcal{K} \ \mathcal{F}]$  and the output injection pair  $\begin{bmatrix} \mathcal{H} \\ \mathcal{G} \end{bmatrix}$  are jointly exponentially stabilizing iff they are jointly admissible and both the closed loop semigroups  $\mathcal{A}_b$  and  $\mathcal{A}^\#$  in Lemma 3.13 are exponentially stable.*

LEMMA 3.20. *If  $[\mathcal{K} \ \mathcal{F}]$  is an exponentially stabilizing state feedback pairs for the system  $\Psi = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}$ , then the same feedback pair is exponentially stabilizing for every well-posed extension of  $\Psi$  of the type*

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \begin{bmatrix} \mathcal{C} \\ \mathcal{C}_1 \end{bmatrix} & \begin{bmatrix} \mathcal{D} \\ \mathcal{D}_1 \end{bmatrix} \end{bmatrix}.$$

<sup>15</sup>See also Remark 3.6.



**4. Coprime factorizations.** As is well known, classical (lumped) transfer functions always have doubly coprime factorizations, and these factorizations can be computed by a state space method through the use of a stabilizable and detectable realization. Arbitrary  $H^\infty$  transfer functions do not always have coprime factorizations even in the single-input single-output case [12, p. 108], but transfer functions that can be stabilized by a dynamic output feedback do, at least in the case where the input and output spaces are finite dimensional [9, 21]. Below we extend these results and show that the transfer function of every jointly stabilizable and detectable well-posed linear system has a doubly coprime factorization that can be computed by the standard state space method. Conversely, every transfer function with a doubly coprime factorization has a strongly stabilizable and detectable realization.

According to Lemma 2.9, there is a one-to-one correspondence between the set of transfer functions in  $H^\infty_\omega(U; Y)$  and the set of causal time-invariant operators in  $TIC_\omega(U; Y)$ . Rather than switching over to the frequency domain we continue to work in the time domain and leave the transformation of our results to the frequency domain to the reader.

DEFINITION 4.1. *Let  $U, Y,$  and  $Z$  be Hilbert spaces, and let  $\omega \in \mathbf{R}$ .*

- (i) *The operators  $\mathcal{N} \in TIC_\omega(U; Y)$  and  $\mathcal{M} \in TIC_\omega(U; Z)$  are right  $\omega$ -coprime iff there exist operators  $\tilde{\mathcal{Y}} \in TIC_\omega(Y; U)$  and  $\tilde{\mathcal{X}} \in TIC_\omega(Z; U)$  that together with  $\mathcal{N}$  and  $\mathcal{M}$  satisfy the Bezout identity*

$$\tilde{\mathcal{Y}}\mathcal{N} + \tilde{\mathcal{X}}\mathcal{M} = I$$

*in  $TIC_\omega(U)$ . In the case where  $\omega = 0$  we call  $\mathcal{N}$  and  $\mathcal{M}$  right coprime, and in the case where  $\omega < 0$  we call  $\mathcal{N}$  and  $\mathcal{M}$  exponentially right coprime.*

- (ii) *The operators  $\tilde{\mathcal{N}} \in TIC_\omega(U; Y)$  and  $\tilde{\mathcal{M}} \in TIC_\omega(Z; Y)$  are left  $\omega$ -coprime iff there exist operators  $\mathcal{Y} \in TIC_\omega(Y; U)$  and  $\mathcal{X} \in TIC_\omega(Y; Z)$  that together with  $\tilde{\mathcal{N}}$  and  $\tilde{\mathcal{M}}$  satisfy the Bezout identity*

$$\tilde{\mathcal{N}}\mathcal{Y} + \tilde{\mathcal{M}}\mathcal{X} = I$$

*in  $TIC_\omega(Y)$ . In the case where  $\omega = 0$  we call  $\tilde{\mathcal{N}}$  and  $\tilde{\mathcal{M}}$  left coprime, and in the case where  $\omega < 0$  we call  $\tilde{\mathcal{N}}$  and  $\tilde{\mathcal{M}}$  exponentially left coprime.*

Thus  $\mathcal{N}$  and  $\mathcal{M}$  are right  $\omega$ -coprime iff  $\begin{bmatrix} \mathcal{N} \\ \mathcal{M} \end{bmatrix}$  has a left inverse in  $TIC_\omega(Y \times Z; U)$ .  $\tilde{\mathcal{N}}$  and  $\tilde{\mathcal{M}}$  are left  $\omega$ -coprime iff  $\begin{bmatrix} \tilde{\mathcal{N}} & \tilde{\mathcal{M}} \end{bmatrix}$  has a right inverse in  $TIC_\omega(Y; U \times Z)$ .

DEFINITION 4.2. *Let  $U$  and  $Y$  be Hilbert spaces, let  $\omega, \alpha \in \mathbf{R}$  with  $\omega \leq \alpha$ , and let  $\mathcal{D} \in TIC_\alpha(U; Y)$ .*

- (i) *The pair  $(\mathcal{N}, \mathcal{M})$  is a right  $\omega$ -coprime factorization of  $\mathcal{D}$  if  $\mathcal{N} \in TIC_\omega(U; Y)$  and  $\mathcal{M} \in TIC_\omega(U)$  are right  $\omega$ -coprime,  $\mathcal{M}$  has an inverse in  $TIC_\alpha(U)$ , and  $\mathcal{D} = \mathcal{N}\mathcal{M}^{-1}$ .*
- (ii) *The pair  $(\tilde{\mathcal{M}}, \tilde{\mathcal{N}})$  is a left  $\omega$ -coprime [coprime] [[exponentially coprime]] factorization of  $\mathcal{D}$  if  $\tilde{\mathcal{M}} \in TIC_\omega(Y)$  and  $\tilde{\mathcal{N}} \in TIC_\omega(U; Y)$  are left  $\omega$ -coprime [coprime] [[exponentially coprime]],  $\tilde{\mathcal{M}}$  has an inverse in  $TIC_\alpha(Y)$ , and  $\mathcal{D} = \mathcal{M}^{-1}\tilde{\mathcal{N}}$ .*
- (iii) *A doubly  $\omega$ -coprime factorization of  $\mathcal{D}$  consists of eight operators in  $TIC_\omega$  (of the appropriate dimensions) satisfying*

$$(4.1) \quad \begin{bmatrix} \tilde{\mathcal{M}} & \tilde{\mathcal{N}} \\ -\tilde{\mathcal{Y}} & \tilde{\mathcal{X}} \end{bmatrix} \begin{bmatrix} \mathcal{X} & -\mathcal{N} \\ \mathcal{Y} & \mathcal{M} \end{bmatrix} = \begin{bmatrix} \mathcal{X} & -\mathcal{N} \\ \mathcal{Y} & \mathcal{M} \end{bmatrix} \begin{bmatrix} \tilde{\mathcal{M}} & \tilde{\mathcal{N}} \\ -\tilde{\mathcal{Y}} & \tilde{\mathcal{X}} \end{bmatrix} = I$$

*in  $TIC_\omega(U \times Y; U \times Y)$ , and, in addition, we require that  $(\mathcal{N}, \mathcal{M})$  is a right  $\omega$ -coprime and  $(\tilde{\mathcal{M}}, \tilde{\mathcal{N}})$  a left  $\omega$ -coprime factorization of  $\mathcal{D}$ . In the case where*

$\omega = 0$  we call this a doubly coprime factorization, and in case where  $\omega < 0$  we call it a doubly exponentially coprime factorization.

Our definition of coprimeness is slightly nonstandard. It is possible to study coprime factorizations in the quotient field of  $TIC_\omega$  without our additional assumption that  $\mathcal{D}$  belongs to  $TIC_\alpha(U; Y)$  and that  $\mathcal{M}$  and  $\tilde{\mathcal{M}}$  are invertible in  $TIC_\alpha(U)$ , respectively,  $TIC_\alpha(Y)$  for some  $\alpha > \omega$ ; see, e.g., [9], [12], or [21]. Usually, one only assumes the transfer functions of  $\mathcal{M}$  and  $\tilde{\mathcal{M}}$  to be invertible in at least one point in the half-plane  $\Re z > \omega$ . Observe that if  $\mathcal{M}$  is invertible in any reasonable sense, then  $\mathcal{D} \in TIC_\alpha(U; Y)$  iff  $\mathcal{M}^{-1} \in TIC_\alpha(U)$  because  $\mathcal{D} = \mathcal{N}\mathcal{M}^{-1}$  and  $\mathcal{M}^{-1} = \tilde{\mathcal{X}} + \tilde{\mathcal{Y}}\mathcal{D}$ . Likewise, if  $\tilde{\mathcal{M}}$  is invertible in any reasonable sense, then  $\mathcal{D} \in TIC_\alpha(U; Y)$  iff  $\tilde{\mathcal{M}}^{-1} \in TIC_\alpha(Y)$  because  $\mathcal{D} = \tilde{\mathcal{M}}^{-1}\mathcal{N}$  and  $\tilde{\mathcal{M}}^{-1} = \tilde{\mathcal{X}} + \mathcal{D}\tilde{\mathcal{Y}}$ . According to [20] or [31], if  $\mathcal{D}$  does not belong to  $TIC_\alpha(U; Y)$  for any  $\alpha > \omega$ , then  $\mathcal{D}$  cannot be realized as the input/output map of a well-posed linear system on a triple of Hilbert spaces.

A coprime factorization is unique, modulo a unit as shown in Lemma 4.3.

LEMMA 4.3. *Let  $U$  and  $Y$  be Hilbert spaces, let  $\omega, \alpha \in \mathbf{R}$  with  $\omega \leq \alpha$ , and let  $\mathcal{D} \in TIC_\alpha(U; Y)$ .*

- (i) *Let  $(\mathcal{N}, \mathcal{M})$  be a right  $\omega$ -coprime factorization of  $\mathcal{D}$ . Then the set of all possible right  $\omega$ -coprime factorizations of  $\mathcal{D}$  can be parameterized in the form  $(\mathcal{N}\mathcal{U}, \mathcal{M}\mathcal{U})$ , where  $\mathcal{U}$  is an invertible operator in  $TIC_\omega(U)$ .*
- (ii) *Let  $(\tilde{\mathcal{M}}, \tilde{\mathcal{N}})$  be a left  $\omega$ -coprime factorization of  $\mathcal{D}$ . Then the set of all possible left  $\omega$ -coprime factorizations of  $\mathcal{D}$  can be parameterized in the form  $(\tilde{\mathcal{U}}\tilde{\mathcal{M}}, \tilde{\mathcal{U}}\tilde{\mathcal{N}})$ , where  $\tilde{\mathcal{U}}$  is an invertible operator in  $TIC_\omega(Y)$ .*
- (iii) *If  $\mathcal{D}$  has both a right  $\omega$ -coprime factorization  $(\mathcal{N}, \mathcal{M})$  and a left  $\omega$ -coprime factorization  $(\tilde{\mathcal{M}}, \tilde{\mathcal{N}})$ , then these two factorizations can be extended to a doubly  $\omega$ -coprime factorization (i.e., a factorization that contains the given operators  $\mathcal{N}, \mathcal{M}, \tilde{\mathcal{M}}$ , and  $\tilde{\mathcal{N}}$ ).*

*Proof.* (i) If  $(\mathcal{N}, \mathcal{M})$  is a right  $\omega$ -coprime factorization of  $\mathcal{D}$  and  $\mathcal{U} \in TIC_\omega(U)$  is invertible in  $TIC_\omega(U)$ , then it is obvious that  $(\mathcal{N}\mathcal{U}, \mathcal{M}\mathcal{U})$  is another right  $\omega$ -coprime factorization. Conversely, suppose that we have two right  $\omega$ -coprime factorizations  $(\mathcal{N}, \mathcal{M})$  and  $(\mathcal{N}_1, \mathcal{M}_1)$  satisfying the Bezout identities

$$\tilde{\mathcal{Y}}\mathcal{N} + \tilde{\mathcal{X}}\mathcal{M} = \tilde{\mathcal{Y}}_1\mathcal{N}_1 + \tilde{\mathcal{X}}_1\mathcal{M}_1 = I.$$

Then  $\mathcal{M}^{-1} = \tilde{\mathcal{X}} + \tilde{\mathcal{Y}}\mathcal{D}$  and  $\mathcal{M}_1^{-1} = \tilde{\mathcal{X}}_1 + \tilde{\mathcal{Y}}_1\mathcal{D}$  in  $TIC_\alpha(U)$ , so

$$\mathcal{M}^{-1}\mathcal{M}_1 = \tilde{\mathcal{X}}\mathcal{M}_1 + \tilde{\mathcal{Y}}\mathcal{N}_1, \quad \mathcal{M}_1^{-1}\mathcal{M} = \tilde{\mathcal{X}}_1\mathcal{M} + \tilde{\mathcal{Y}}_1\mathcal{N}$$

in  $TIC_\alpha(U)$ . Define  $\mathcal{U} = \mathcal{M}^{-1}\mathcal{M}_1$ . We know that  $\mathcal{U}$  is invertible in  $TIC_\alpha(U)$ . However, since  $L_\omega^2(\mathbf{R}; U) \cap L_\alpha^2(\mathbf{R}; U)$  is dense in  $L_\alpha^2(\mathbf{R}; U)$ , the two equations above imply that  $\mathcal{U}$  can be extended to an invertible operator in  $TIC_\omega(U)$ . Moreover,  $\mathcal{M}_1 = \mathcal{M}\mathcal{U}$  and  $\mathcal{N}_1 = \mathcal{D}\mathcal{M}_1 = \mathcal{D}\mathcal{M}\mathcal{U} = \mathcal{N}\mathcal{U}$ . Thus  $(\mathcal{N}_1, \mathcal{M}_1) = (\mathcal{N}\mathcal{U}, \mathcal{M}\mathcal{U})$ , as claimed.

(ii) The proof of (ii) is completely analogous to the proof of (i).

(iii) Choose some operators  $\tilde{\mathcal{Y}}, \tilde{\mathcal{X}}, \tilde{\mathcal{X}}_1$ , and  $\tilde{\mathcal{Y}}_1$  in  $TIC_\omega$  that together with the given operators satisfy the Bezout identities  $\tilde{\mathcal{Y}}\mathcal{N} + \tilde{\mathcal{X}}\mathcal{M} = I$  and  $\tilde{\mathcal{N}}\tilde{\mathcal{Y}}_1 + \tilde{\mathcal{M}}\tilde{\mathcal{X}}_1 = I$ . Then a direct computation shows that

$$\begin{bmatrix} \tilde{\mathcal{M}} & \tilde{\mathcal{N}} \\ -\tilde{\mathcal{Y}} & \tilde{\mathcal{X}} \end{bmatrix} \begin{bmatrix} \mathcal{X} - \mathcal{N}(\tilde{\mathcal{Y}}\mathcal{X} - \tilde{\mathcal{X}}\mathcal{Y}) & -\mathcal{N} \\ \mathcal{Y} + \mathcal{M}(\tilde{\mathcal{Y}}\mathcal{X} - \tilde{\mathcal{X}}\mathcal{Y}) & \mathcal{M} \end{bmatrix} = I.$$

By using the invertibility of  $\mathcal{M}$  and  $\widetilde{\mathcal{M}}$  in  $TIC_\alpha$ , we get

$$\begin{aligned} \mathcal{X}\widetilde{\mathcal{M}} + \mathcal{D}\mathcal{Y}\widetilde{\mathcal{M}} &= I, & \mathcal{X}\widetilde{\mathcal{N}} + \mathcal{D}\mathcal{Y}\widetilde{\mathcal{N}} &= \mathcal{D}, \\ \mathcal{M}\widetilde{\mathcal{X}} + \mathcal{M}\widetilde{\mathcal{Y}}\mathcal{D} &= I, & \mathcal{N}\widetilde{\mathcal{X}} + \mathcal{N}\widetilde{\mathcal{Y}}\mathcal{D} &= \mathcal{D}, \end{aligned}$$

and by using these identities we find that

$$\begin{bmatrix} \mathcal{X} - \mathcal{N}(\widetilde{\mathcal{Y}}\mathcal{X} - \widetilde{\mathcal{X}}\mathcal{Y}) & -\mathcal{N} \\ \mathcal{Y} + \mathcal{M}(\widetilde{\mathcal{Y}}\mathcal{X} - \widetilde{\mathcal{X}}\mathcal{Y}) & \mathcal{M} \end{bmatrix} \begin{bmatrix} \widetilde{\mathcal{M}} & \widetilde{\mathcal{N}} \\ -\widetilde{\mathcal{Y}} & \widetilde{\mathcal{X}} \end{bmatrix} = I$$

in  $TIC_\alpha$  (as opposed to  $TIC_\omega$ ). However, since all the operators above belong to  $TIC_\omega$ , and since  $L_\alpha^2 \cap L_\omega^2$  is dense in  $L_\omega^2$ , we find that the same identity must be true in  $TIC_\omega$ , too. Thus, we have a doubly  $\omega$ -coprime factorization.  $\square$

As the following theorem shows, if a well-posed linear system is jointly stabilizable and detectable, then its input/output map has a doubly coprime factorization. A converse to this statement is true as well.

**THEOREM 4.4.**

- (i) Let  $\Psi = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$  be a jointly  $\omega$ -stabilizable [stabilizable] [[exponentially stabilizable]] and detectable well-posed linear system (in the sense of Definition 3.16). Then, with the notations of Lemma 3.13 and Definition 4.2,

$$\begin{bmatrix} \widetilde{\mathcal{M}} & \widetilde{\mathcal{N}} \\ -\widetilde{\mathcal{Y}} & \widetilde{\mathcal{X}} \end{bmatrix} \begin{bmatrix} \mathcal{X} & -\mathcal{N} \\ \mathcal{Y} & \mathcal{M} \end{bmatrix} = \begin{bmatrix} I + \mathcal{G}^\sharp & \mathcal{D}^\sharp \\ -\mathcal{E}^\sharp & I - \mathcal{F}^\sharp \end{bmatrix} \begin{bmatrix} I - \mathcal{G}_b & -\mathcal{D}_b \\ \mathcal{E}_b & I + \mathcal{F}_b \end{bmatrix}$$

is a doubly  $\omega$ -coprime [coprime] [[exponentially coprime]] factorization of  $\mathcal{D}$ .

- (ii) Conversely, every  $\mathcal{D}$  that belongs to  $TIC_\alpha(U; Y)$  for some  $\alpha \in \mathbf{R}$  and has a doubly  $\omega$ -coprime [coprime] [[exponentially coprime]] factorization can be realized as the input/output map of a jointly strongly  $\omega$ -stabilizable [stabilizable] [[exponentially stabilizable]] and detectable well-posed linear system  $\Psi = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ .

*Proof.*  $\Psi = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$  be jointly  $\omega$ -stabilizable and detectable. Then both the systems drawn in Figures 3.5 and 3.6 are  $\omega$ -stable. In particular, both the input/output map from  $\begin{bmatrix} w \\ u_b \end{bmatrix}$  to  $\begin{bmatrix} w^\sharp \\ u \end{bmatrix}$  in Figure 3.5, and the input/output map from  $\begin{bmatrix} w^\sharp \\ u \end{bmatrix}$  to  $\begin{bmatrix} w \\ u_b \end{bmatrix}$  in Figure 3.6 are  $\omega$ -stable. The former one is given by  $\begin{bmatrix} I - \mathcal{G}_b & -\mathcal{D}_b \\ \mathcal{E}_b & I + \mathcal{F}_b \end{bmatrix}$  (cf. Remark 3.9), and the latter one is given by  $\begin{bmatrix} I + \mathcal{G}^\sharp & \mathcal{D}^\sharp \\ -\mathcal{E}^\sharp & I - \mathcal{F}^\sharp \end{bmatrix}$ . Moreover, by comparing the two figures with each other we immediately realize that they are equivalent in the sense that the relationships between the different signals with the same names are identical in the two diagrams. This means that the input/output map given above are inverses of each other; i.e.,

$$\begin{bmatrix} I - \mathcal{G}_b & -\mathcal{D}_b \\ \mathcal{E}_b & I + \mathcal{F}_b \end{bmatrix} = \begin{bmatrix} I + \mathcal{G}^\sharp & \mathcal{D}^\sharp \\ -\mathcal{E}^\sharp & I - \mathcal{F}^\sharp \end{bmatrix}^{-1}.$$

Moreover, as is easily seen,  $(\mathcal{D}_b, (I + \mathcal{F}_b))$  is a right  $\omega$ -coprime factorization of  $\mathcal{D}$ , and  $((I + \mathcal{G}^\sharp), \mathcal{D}^\sharp)$  is a left  $\omega$ -coprime factorization of  $\mathcal{D}$ . This proves part (i) of the theorem.

Conversely, suppose that there exists a doubly coprime factorization of  $\mathcal{D}$ . Our construction below starts with a realization of the closed loop system  $\Psi_b$ ; another equally good choice would be to start with a realization of  $\Psi^\sharp$ . Motivated by the formula that we found above, we pick the input/output map of  $\Psi_b$  to be given by

$$\begin{bmatrix} \mathcal{G}_b & \mathcal{D}_b \\ \mathcal{E}_b & \mathcal{F}_b \end{bmatrix} = \begin{bmatrix} I - \mathcal{X} & \mathcal{N} \\ \mathcal{Y} & \mathcal{M} - I \end{bmatrix}$$

and choose an arbitrary strongly  $\omega$ -stable realization of this input/output map, for example, the continuously observable realization presented in Definition 2.10. Since  $\mathcal{M}$  is supposed to have an inverse in  $TIC_\alpha$  for some  $\alpha > \omega$ , the operator  $\begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix}$  is an admissible feedback operator for  $\Psi_b$ . Denote the resulting  $\alpha$ -stable closed loop system by  $\Psi_{\text{ext}}$  and the system that we get by dropping the state feedback row and the output injection column from  $\Psi_{\text{ext}}$  by  $\Psi$ . As a straightforward computation shows, the input/output map of  $\Psi_{\text{ext}}$  is

$$(4.2) \quad \begin{bmatrix} \mathcal{G} & \mathcal{D} \\ \mathcal{E} & \mathcal{F} \end{bmatrix} = \begin{bmatrix} I - \mathcal{X} - \mathcal{N}\mathcal{M}^{-1}\mathcal{Y} & \mathcal{N}\mathcal{M}^{-1} \\ \mathcal{M}^{-1}\mathcal{Y} & I - \mathcal{M}^{-1} \end{bmatrix} \\ = \begin{bmatrix} I - \widetilde{\mathcal{M}}^{-1} & \widetilde{\mathcal{M}}^{-1}\widetilde{\mathcal{N}} \\ \widetilde{\mathcal{Y}}\widetilde{\mathcal{M}}^{-1} & I - \widetilde{\mathcal{X}} - \widetilde{\mathcal{Y}}\widetilde{\mathcal{M}}^{-1}\widetilde{\mathcal{N}} \end{bmatrix}.$$

Observe, in particular, that the input/output map of  $\Psi$  is the desired  $\mathcal{D} = \mathcal{N}\mathcal{M}^{-1} = \widetilde{\mathcal{M}}^{-1}\widetilde{\mathcal{N}}$ . It follows from Proposition 3.3 that the system  $\Psi$  that we get in this way is strongly  $\omega$ -stabilizable (and that the closed loop state feedback system is  $\Psi_b$ ). Moreover, by Proposition 3.3 and Lemma 3.5,  $\Psi$  is strongly  $\omega$ -detectable if the operator  $\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$  is an  $\omega$ -stabilizing output feedback operator for  $\Psi_b$ , or, equivalently, if

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} I - \mathcal{X} & \mathcal{N} \\ \mathcal{Y} & \mathcal{M} - I \end{bmatrix} = \begin{bmatrix} \mathcal{X} & -\mathcal{N} \\ \mathcal{Y} & \mathcal{M} \end{bmatrix}$$

has an inverse in  $TIC_\omega(U \times Y, U \times Y)$ . But this is true because of the doubly coprime-ness assumption. Thus  $\Psi$  is jointly strongly  $\omega$ -stabilizable and detectable.  $\square$

The notion of a coprime factorization makes it possible to refine Lemma 3.20 as follows.

LEMMA 4.5. *Assume that both  $[\mathcal{K}^1 \ \mathcal{F}^1]$  and  $[\mathcal{K}^2 \ \mathcal{F}^2]$  are  $\omega$ -stabilizing state feedback pairs for the system  $\Psi = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}$ . Then the following conditions are equivalent:*

- (i)  $[\mathcal{K}^2 \ \mathcal{F}^2]$  is an  $\omega$ -stabilizing state feedback pair for the extended system

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \begin{bmatrix} \mathcal{C} \\ \mathcal{K}^1 \end{bmatrix} & \begin{bmatrix} \mathcal{D} \\ \mathcal{F}^1 \end{bmatrix} \end{bmatrix};$$

- (ii) the pair

$$([\mathcal{K}^2 - (I - \mathcal{F}^2)(I - \mathcal{F}^1)^{-1}\mathcal{K}^1 \quad (I - (I - \mathcal{F}^2)(I - \mathcal{F}^1)^{-1})]$$

is an  $\omega$ -stabilizing state feedback pair for the closed loop system

$$\Psi_b = \begin{bmatrix} \mathcal{A}_b & \mathcal{B}_b \\ \begin{bmatrix} \mathcal{C}_b \\ \mathcal{K}_b^1 \end{bmatrix} & \begin{bmatrix} \mathcal{D}_b \\ \mathcal{F}_b^1 \end{bmatrix} \end{bmatrix} \\ = \begin{bmatrix} \mathcal{A} + \mathcal{B}\tau(I - \mathcal{F}^1)^{-1}\mathcal{K}^1 & \mathcal{B}(I - \mathcal{F}^1)^{-1} \\ \begin{bmatrix} \mathcal{C} + \mathcal{D}(I - \mathcal{F}^1)^{-1}\mathcal{K}^1 \\ (I - \mathcal{F}^1)^{-1}\mathcal{K}^1 \end{bmatrix} & \begin{bmatrix} \mathcal{D}(I - \mathcal{F}^1)^{-1} \\ (I - \mathcal{F}^1)^{-1} - I \end{bmatrix} \end{bmatrix}$$

that one gets from  $\Psi$  by using the state feedback pair  $[\mathcal{K}^1 \ \mathcal{F}^1]$ ;

- (iii)  $\mathcal{F}^1(I - \mathcal{F}^2)^{-1}$  and  $\mathcal{K}^1 + \mathcal{F}^1(I - \mathcal{F}^2)^{-1}\mathcal{K}^2$  are  $\omega$ -stable.

A sufficient condition for (i)–(iii) to hold is that  $\mathcal{D}(I - \mathcal{F}^1)^{-1}$  and  $(I - \mathcal{F}^1)^{-1}$  are right  $\omega$ -coprime. This condition is necessary for (i)–(iii) to hold whenever  $\mathcal{D}(I - \mathcal{F}^2)^{-1}$  and  $(I - \mathcal{F}^2)^{-1}$  are right  $\omega$ -coprime.

*Proof.* Let us study the further extended system

$$\left[ \begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \left[ \begin{array}{c} \mathcal{C} \\ \mathcal{K}^1 \\ \mathcal{K}^2 \\ \mathcal{K}^2 - \mathcal{K}^1 \end{array} \right] & \left[ \begin{array}{c} \mathcal{D} \\ \mathcal{F}^1 \\ \mathcal{F}^2 \\ \mathcal{F}^2 - \mathcal{F}^1 \end{array} \right] \end{array} \right],$$

where the last line is the difference between the two previous lines. If we use here the output feedback operator  $[0 \ I \ 0 \ 0]$ , then we get an extended version of the closed loop system  $\Psi_b$  in the statement of the lemma (that we still denote by  $\Psi_b$ ), namely,

$$\begin{aligned} \Psi_b &= \left[ \begin{array}{c|c} \mathcal{A}_b & \mathcal{B}_b \\ \left[ \begin{array}{c} \mathcal{C}_b \\ \mathcal{K}_b^1 \\ \mathcal{K}_b^2 \\ \mathcal{K}_b^2 - \mathcal{K}_b^1 \end{array} \right] & \left[ \begin{array}{c} \mathcal{D}_b \\ \mathcal{F}_b^1 \\ \mathcal{F}_b^2 \\ \mathcal{F}_b^2 - \mathcal{F}_b^1 \end{array} \right] \end{array} \right] \\ &= \left[ \begin{array}{c|c} \mathcal{A} + \mathcal{B}\tau (I - \mathcal{F}^1)^{-1} \mathcal{K}^1 & \mathcal{B} (I - \mathcal{F}^1)^{-1} \\ \left[ \begin{array}{c} \mathcal{C} + \mathcal{D} (I - \mathcal{F}^1)^{-1} \mathcal{K}^1 \\ (I - \mathcal{F}^1)^{-1} \mathcal{K}^1 \\ \mathcal{K}^2 + \mathcal{F}^2 (I - \mathcal{F}^1)^{-1} \mathcal{K}^1 \\ \mathcal{K}^2 - (I - \mathcal{F}^2) (I - \mathcal{F}^1)^{-1} \mathcal{K}^1 \end{array} \right] & \left[ \begin{array}{c} \mathcal{D} (I - \mathcal{F}^1)^{-1} \\ (I - \mathcal{F}^1)^{-1} - I \\ \mathcal{F}^2 (I - \mathcal{F}^1)^{-1} \\ (I - \mathcal{F}^2) (I - \mathcal{F}^1)^{-1} - I \end{array} \right] \end{array} \right]. \end{aligned}$$

If we instead use the output feedback operator  $[0 \ 0 \ I \ 0]$ , then we get the system

$$\begin{aligned} \Psi_{\natural} &= \left[ \begin{array}{c|c} \mathcal{A}_{\natural} & \mathcal{B}_{\natural} \\ \left[ \begin{array}{c} \mathcal{C}_{\natural} \\ \mathcal{K}_{\natural}^1 \\ \mathcal{K}_{\natural}^2 \\ \mathcal{K}_{\natural}^2 - \mathcal{K}_{\natural}^1 \end{array} \right] & \left[ \begin{array}{c} \mathcal{D}_{\natural} \\ \mathcal{F}_{\natural}^1 \\ \mathcal{F}_{\natural}^2 \\ \mathcal{F}_{\natural}^2 - \mathcal{F}_{\natural}^1 \end{array} \right] \end{array} \right] \\ &= \left[ \begin{array}{c|c} \mathcal{A} + \mathcal{B}\tau (I - \mathcal{F}^2)^{-1} \mathcal{K}^2 & \mathcal{B} (I - \mathcal{F}^2)^{-1} \\ \left[ \begin{array}{c} \mathcal{C} + \mathcal{D} (I - \mathcal{F}^2)^{-1} \mathcal{K}^2 \\ \mathcal{K}^1 + \mathcal{F}^1 (I - \mathcal{F}^2)^{-1} \mathcal{K}^2 \\ (I - \mathcal{F}^2)^{-1} \mathcal{K}^2 \\ -\mathcal{K}^1 + (I - \mathcal{F}^1) (I - \mathcal{F}^2)^{-1} \mathcal{K}^2 \end{array} \right] & \left[ \begin{array}{c} \mathcal{D} (I - \mathcal{F}^2)^{-1} \\ \mathcal{F}^1 (I - \mathcal{F}^2)^{-1} \\ (I - \mathcal{F}^2)^{-1} - I \\ (I - \mathcal{F}^1) (I - \mathcal{F}^2)^{-1} - I \end{array} \right] \end{array} \right]. \end{aligned}$$

By Proposition 3.3, we get the same system by using  $[0 \ -I \ I \ 0]$  as a feedback operator for  $\Psi_b$  or, equivalently, by using the feedback operator  $[0 \ 0 \ 0 \ I]$ . This proves the equivalence of (i) and (ii). We know that the operators on the first, second, and fourth row of  $\Psi_{\natural}$  are stable (since  $[\mathcal{K}^2 \ \mathcal{F}^2]$  is stabilizing for  $\Psi$ ), so the full system  $\Psi_{\natural}$  is stable iff the two conditions listed in (iii) hold (recall that the last line is the difference between the two previous lines).

Suppose that  $\mathcal{D}(I - \mathcal{F}^1)^{-1}$  and  $(I - \mathcal{F}^1)^{-1}$  are right coprime. We claim that (i)–(iii) then hold. To prove this we choose operators  $\tilde{\mathcal{Y}}$  and  $\tilde{\mathcal{X}}$  in *TIC* that together

with  $\mathcal{D}_b$  and  $I + \mathcal{F}_b^1$  satisfy the Bezout identity

$$\tilde{\mathcal{Y}}\mathcal{D}_b + \tilde{\mathcal{X}}(I + \mathcal{F}_b^1) = \tilde{\mathcal{Y}}\mathcal{D}(I - \mathcal{F}^1)^{-1} + \tilde{\mathcal{X}}(I - \mathcal{F}^1)^{-1} = I.$$

Then

$$\begin{aligned} \tilde{\mathcal{Y}}\mathcal{D}_\ddagger + \tilde{\mathcal{X}}(I + \mathcal{F}_\ddagger^2) &= \tilde{\mathcal{Y}}\mathcal{D}(I - \mathcal{F}^2)^{-1} + \tilde{\mathcal{X}}(I - \mathcal{F}^2)^{-1} \\ &= (\tilde{\mathcal{Y}}\mathcal{D} + \tilde{\mathcal{X}})(I - \mathcal{F}^2)^{-1} \\ &= (I - \mathcal{F}^1)(I - \mathcal{F}^2)^{-1}, \end{aligned}$$

and this shows that  $(I - \mathcal{F}^1)(I - \mathcal{F}^2)^{-1}$  is stable; hence,  $\mathcal{F}^1(I - \mathcal{F}^2)^{-1}$  is stable. A similar computation

$$\begin{aligned} \tilde{\mathcal{Y}}(\mathcal{C}_\ddagger - \mathcal{C}_b) + \tilde{\mathcal{X}}(\mathcal{K}_\ddagger^2 - \mathcal{K}_b^2) &= (\tilde{\mathcal{Y}}\mathcal{D} + \tilde{\mathcal{X}})\left((I - \mathcal{F}^2)^{-1}\mathcal{K}^2 - (I - \mathcal{F}^1)^{-1}\mathcal{K}^1\right) \\ &= -\mathcal{K}^1 + (I - \mathcal{F}^1)(I - \mathcal{F}^2)^{-1}\mathcal{K}^2 \end{aligned}$$

shows that  $-\mathcal{K}^1 + (I - \mathcal{F}^1)(I - \mathcal{F}^2)^{-1}\mathcal{K}^2$  is stable; hence,  $\mathcal{K}^1 + \mathcal{F}^1(I - \mathcal{F}^2)^{-1}\mathcal{K}^2$  is stable.

Finally, let us assume that (iii) holds, and that  $\mathcal{D}(I - \mathcal{F}^2)^{-1}$  and  $(I - \mathcal{F}^2)^{-1}$  are right coprime. By interchanging the two feedback pairs with each other and using the statement that we have just proved, we find that  $(I - \mathcal{F}^2)(I - \mathcal{F}^1)^{-1}$  is invertible in  $TIC(U)$ . This, combined with the coprimeness of  $\mathcal{D}(I - \mathcal{F}^2)^{-1}$  and  $(I - \mathcal{F}^2)^{-1}$  and Lemma 4.3, implies that  $\mathcal{D}(I - \mathcal{F}^1)^{-1}$  and  $(I - \mathcal{F}^1)^{-1}$  must be right coprime.  $\square$

**5. Dynamic stabilization.** As is well known, if the input/output map  $\mathcal{D}$  has a right  $\omega$ -coprime factorization  $(\mathcal{N}, \mathcal{M})$ , and if  $\tilde{\mathcal{Y}}$  and  $\tilde{\mathcal{X}}$  together with  $\mathcal{N}$  and  $\mathcal{M}$  satisfy the Bezout identity

$$\tilde{\mathcal{Y}}\mathcal{N} + \tilde{\mathcal{X}}\mathcal{M} = I,$$

then  $\mathcal{Q} = \tilde{\mathcal{X}}^{-1}\tilde{\mathcal{Y}}$  is an  $\omega$ -stabilizing compensator for  $\mathcal{D}$ , provided it is possible to make sense out of  $\tilde{\mathcal{X}}^{-1}$ . A similar statement is true in the case where  $\mathcal{D}$  has a left  $\omega$ -coprime factorization. If  $\mathcal{D}$  has a doubly  $\omega$ -coprime factorization, then, with the notations of Definition 4.2, the stabilizing compensator  $\mathcal{Q}$  is given by  $\mathcal{Q} = \tilde{\mathcal{X}}^{-1}\tilde{\mathcal{Y}} = \mathcal{Y}\mathcal{X}^{-1}$ , still provided  $\tilde{\mathcal{X}}^{-1}$  and  $\mathcal{X}^{-1}$  make sense. If  $\tilde{\mathcal{X}}$  and  $\mathcal{X}$  do not have inverses in  $TIC_\alpha$  for any  $\alpha > 0$ , then  $\mathcal{Q}$  does not belong to  $TIC_\alpha(Y;U)$  for any  $\alpha > 0$ , and  $\mathcal{Q}$  cannot be realized as the input/output map of a well-posed linear system (see the discussion following Definition 4.2). Thus it is natural to impose this extra condition on a doubly coprime factorization as follows.

**DEFINITION 5.1.** Let  $\alpha > \omega$ , and let  $\mathcal{D} \in TIC_\alpha(U;Y)$  and  $\mathcal{Q} \in TIC_\alpha(Y;U)$ . A joint doubly  $\omega$ -coprime factorization of  $\mathcal{D}$  and  $\mathcal{Q}$  consists of eight operators in  $TIC_\omega$  (with the appropriate dimensions) satisfying (4.1), and, in addition, we require that  $(\mathcal{N}, \mathcal{M})$  is a right and  $(\tilde{\mathcal{M}}, \tilde{\mathcal{N}})$  is a left  $\omega$ -coprime factorization of  $\mathcal{D}$ , and that  $(\mathcal{Y}, \mathcal{X})$  is a right and  $(\tilde{\mathcal{X}}, \tilde{\mathcal{Y}})$  is a left  $\omega$ -coprime factorization of  $\mathcal{Q}$ . (In particular, all the denominators  $\mathcal{M}, \tilde{\mathcal{M}}, \mathcal{X}$ , and  $\tilde{\mathcal{X}}$  are invertible in  $TIC_\alpha$ .)

**LEMMA 5.2.** Let  $\Psi = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$  be a jointly  $\omega$ -stabilizable and detectable well-posed linear system, and let  $\Psi_{\text{ext}}, \Psi_b$ , and  $\Psi^\sharp$  denote the systems in Definition 3.11 and Lemma 3.13. Then the following conditions are equivalent:

- (i) the operator  $\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$  is an admissible output feedback operator for  $\Psi_{\text{ext}}$ ;

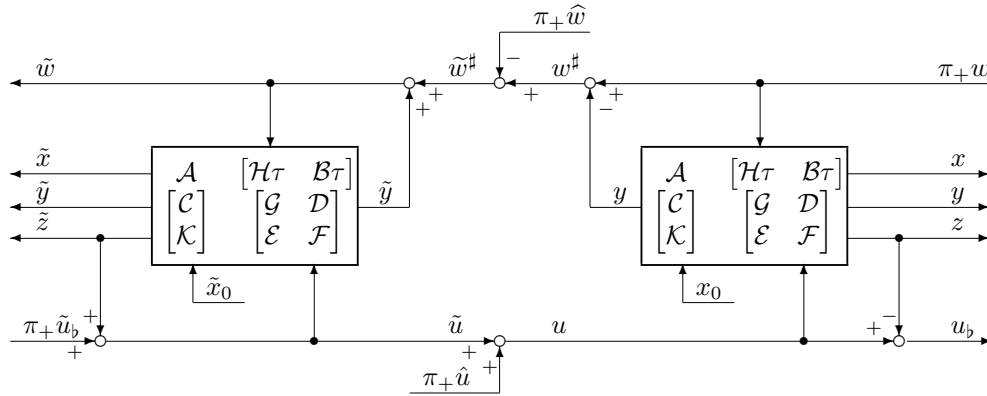


FIG. 5.1. Dynamic stabilization.

- (ii)  $I - \mathcal{G}_b$  has an inverse in  $TIC_\alpha(Y)$  for some  $\alpha \geq \omega$ ;
- (iii)  $I - \mathcal{F}^\sharp$  has an inverse in  $TIC_\alpha(U)$  for some  $\alpha \geq \omega$ .

In these cases the system  $\Psi_b^\sharp$  drawn in the left half of Figure 5.1 with inputs  $\tilde{w}^\sharp$  and  $\tilde{u}_b$  and outputs  $\tilde{y}$ ,  $\tilde{z}$ ,  $\tilde{w}$ , and  $\tilde{u}$  (i.e., the system that we get by using  $\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$  as an output feedback operator for  $\Psi_{\text{ext}}$ ) is a well-posed linear system, and the coprime factorization presented in Theorem 4.4 is a joint doubly  $\omega$ -coprime factorization of  $\mathcal{D}$  and  $\mathcal{Q}$ , where  $\mathcal{Q} = \mathcal{E}_b(I - \mathcal{G}_b)^{-1} = (I - \mathcal{F}^\sharp)^{-1}\mathcal{E}^\sharp$ .

This follows from Proposition 3.3.

In the situation described above the input/output map of the closed loop system  $\Psi_b^\sharp$  is equal to the stabilizing compensator  $\mathcal{Q}$ , and we can use the observer connection drawn in Figure 5.1 to stabilize the system as shown in Theorem 5.3.

**THEOREM 5.3.** *Let  $\Psi = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$  be a jointly  $\omega$ -stabilizable and detectable well-posed linear system, and let  $\Psi_{\text{ext}}$  denote the system in Definition 3.11. Then the connection drawn in Figure 5.1 defines an  $\omega$ -stable well-posed linear system. Moreover, the input/output maps for the two additional outputs  $\tilde{w}$  and  $u_b$  are given by*

$$\begin{bmatrix} \tilde{w} \\ u_b \end{bmatrix} = \begin{bmatrix} \pi_+ w \\ \pi_+ \tilde{u}_b \end{bmatrix} + \begin{bmatrix} -I - \mathcal{G}^\sharp & -\mathcal{D}^\sharp \\ -\mathcal{E}^\sharp & I - \mathcal{F}^\sharp \end{bmatrix} \begin{bmatrix} \pi_+ \hat{w} \\ \pi_+ \hat{u} \end{bmatrix}.$$

*Proof.* By Remark 3.14, we can regard  $\Psi_{\text{ext}}$  as a state feedback perturbed version of the closed loop system  $\Psi_b$ ; see Figure 3.7. By substituting this system for  $\Psi_{\text{ext}}$  in Figure 5.1 we get the equivalent Figure 5.2, which can be interpreted as a feedback connection for an  $\omega$ -stable system consisting of two copies of  $\Psi_b$ . By part (i) of Lemma 3.5, it suffices to show that the two internal inputs  $u_b$  and  $\tilde{w}$  depend continuously in  $L_\omega^2$  on the four inputs. By using the equations describing the summation junctions in Figure 5.2 we can eliminate the variables  $u$ ,  $\tilde{u}$ ,  $w^\sharp$ , and  $\hat{w}^\sharp$  to get

$$\begin{bmatrix} I - \mathcal{G}_b & -\mathcal{D}_b \\ \mathcal{E}_b & I + \mathcal{F}_b \end{bmatrix} \begin{bmatrix} \pi_+ w - \tilde{w} \\ u_b - \pi_+ \tilde{u}_b \end{bmatrix} = \begin{bmatrix} \pi_+ \hat{w} \\ \pi_+ \hat{u} \end{bmatrix}.$$

By Theorem 4.4, the operator on the left-hand side has an inverse in  $TIC_\omega$ . Inverting this operator we get the formula given in Theorem 5.3.  $\square$

**REMARK 5.4.** *Observe that Theorem 5.3 is true even if the equivalent conditions listed in Lemma 5.2 are false. However, if this is the case, then although the system*

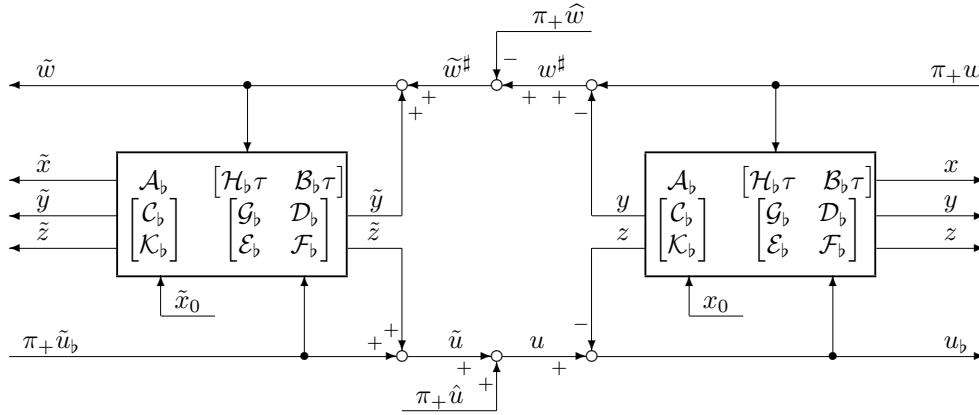


FIG. 5.2. Alternative version of compensator connection.

in Figure 5.1 is an  $\omega$ -stable well-posed linear system, the compensator (the left half of Figures 5.1 and 5.2) is not well-posed by itself, and the well posedness is lost if the feedback loop from the compensator to the original system is opened.

REMARK 5.5. By using Theorem 5.3 one can easily develop a Youla parameterization of the set of all stabilizing compensators for  $\Psi_{\text{ext}}$ . The key observation is that the input/output map from  $\begin{bmatrix} \pi_+ w \\ \pi_+ \tilde{u}_b \end{bmatrix}$  to  $\begin{bmatrix} \tilde{w} \\ u_b \end{bmatrix}$  is the identity map. To get the Youla parameterization we simply connect the Youla parameter  $Q$  from  $\tilde{w}$  to  $\tilde{u}_b$  in the equivalent Figures 5.1 and 5.2. This does not affect the stability of the system since  $\tilde{w}$  does not depend on  $\tilde{u}_b$ . The resulting input/output map from  $w$  to  $u_b$  will be equal to  $Q$ . The proofs of these claims are essentially the same as the ones given in [6].

REMARK 5.6. All the main results of this paper remain valid if throughout we replace the algebra of time-invariant bounded linear operators from  $L^2(\mathbf{R}; U)$  into  $L^2(\mathbf{R}; Y)$  by various subalgebras, for example the algebra of convolution operators induced by measures with finite total variation. This is the algebra around which [22] and [23] were built.

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