Copula goodness-of-fit testing: An overview and power comparison

DANIEL BERG

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Outline

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- ▷ Copula goodness-of-fit testing
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Introduction

Motivation

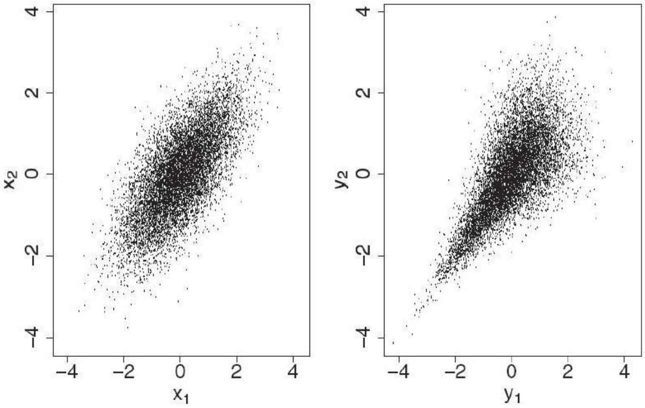


Figure: Two simulated data sets - both with standard normal margins and correlation coefficient 0.7.

Introduction

Motivation

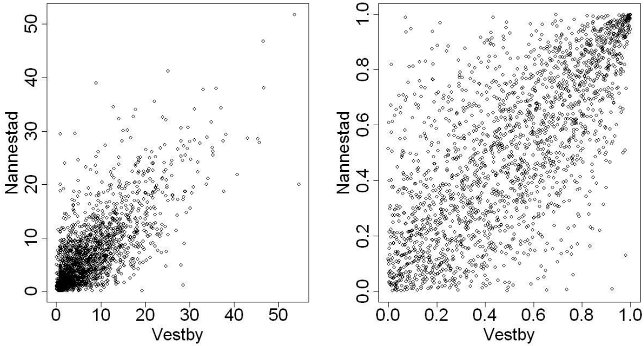


Figure: Nonzero precipitation values in two Norwegian cities and its copula.

Introduction

Definition & Theorem

Definition (Copula)

A d-dimensional copula is a multivariate distribution function $\mathcal{C}$ with standard uniform marginal distributions.

Theorem (Sklar, 1959)

Let H be a joint distribution function with margins $F_1, \ldots, F_d$. Then there exists a copula $C : [0,1]^d \rightarrow [0,1]$ such that

$$H(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)).$$

Introduction

Useful results

- A general d-dimensional density h can be expressed, for some copula density c, as

$$h(x_1, \ldots, x_d) = c\{F_1(x_1), \ldots, F_d(x_d)\} f_1(x_1) \cdots f_d(x_d).$$

- Non-parametric estimate for $F_i(x_i)$ commonly used to transform original margins into standard uniform:

$$u_{ji} = \widehat{F}_i(x_{ji}) = \frac{R_{ji}}{n+1},$$

where R_{ji} is the rank of x_{ji} amongst $x_{1i}, \ldots, x_{ni}$.

- u_{ji} commonly referred to as *pseudo-observations* and models based on non-parametric margins and parametric copulas are referred to as *semi-parametric* copulas

Copula GoF testing

Introduction

- $\mathcal{H}_0 : C \in \mathcal{C} = \{C_\theta; \theta \in \Theta\}$ vs. $\mathcal{H}_1 : C \notin \mathcal{C} = \{C_\theta; \theta \in \Theta\}$
- Univariate $\Rightarrow$ Anderson-Darling or QQ-plot,
  Multivariate $\Rightarrow$ fewer alternatives
- Pseudo-observations no longer independent. In addition, limiting distribution of many copula GoF test depends on null hypothesis copula and parameter value
- p-value estimation via parametric bootstrap procedures
- Focus in literature almost exclusively bivariate
- NOT model selection!
- Several techniques proposed: binning, multivariate smoothing, dimension reduction

Copula GoF testing

Preliminaries

Rosenblatt's transform:

- Dependent variables $\Rightarrow$ independent $U[0,1]$ variables, given multivariate distribution
- $\boldsymbol{v} = \mathcal{R}(\boldsymbol{z}) = (\mathcal{R}_1(z_1), \ldots, \mathcal{R}_d(z_d))$:

$$\begin{aligned} v_1 &= \mathcal{R}_1(z_1) = F_1(z_1) = z_1, \\ v_2 &= \mathcal{R}_2(z_2) = F_{2|1}(z_2|z_1), \\ &\vdots \\ v_d &= \mathcal{R}_d(z_d) = F_{d|1\ldots d}(z_d|z_1, \ldots, z_d). \end{aligned}$$

- Inverse of simulation (conditional inversion)
- GoF: $\boldsymbol{v} = \mathcal{R}(\boldsymbol{z}) \Rightarrow$ test $\boldsymbol{v}$ for independence
- $d!$ different permutation orders

Copula GoF testing

Proposed approaches: $\mathcal{A}_1$ (1/9)

- $\boldsymbol{v} = \mathcal{R}(\boldsymbol{z})$
- $W_{1j} = \sum_{i=1}^{d} \Gamma\{v_{ji}; \alpha\}, \quad j = 1, \ldots, n$
- Special case (a): $\sum_{i=1}^{d} \Phi^{-1}(v_{ji})^2$
- Special case (b): $\sum_{i=1}^{d} |v_{ji} - 0.5|$
- $S_1(t) = P\{F_1(W_1) \leq t\}, \quad t \in [0, 1]$
- CvM statistic:

$$\widehat{T}_1 = n \int_0^1 \left\{ \widehat{S}_1(t) - S_1(t) \right\}^2 \mathrm{d}S_1(t)$$

- References: Breymann et al. (2003); Malevergne and Sornette (2003); Berg and Bakken (2005)

Copula GoF testing

Proposed approaches: $\mathcal{A}_2$ (2/9)

- Empirical copula:

$$\widehat{C}(\boldsymbol{u}) = \frac{1}{n+1}\sum_{j=1}^{n} I\{Z_{j1} \leq u_1, \ldots, Z_{jd} \leq u_d\}$$

- CvM statistic:

$$\widehat{T}_2 = n \int_{[0,1]^d} \left\{\widehat{C}(\boldsymbol{z}) - C_{\widehat{\theta}}(\boldsymbol{z})\right\}^2 \mathrm{d}\widehat{C}(\boldsymbol{z})$$

- References: Fermanian (2005); Genest and Rémillard (2008); Genest et al. (2008)

Copula GoF testing

Proposed approaches: $\mathcal{A}_3$ (3/9)

- Approach $\mathcal{A}_2$ on $\boldsymbol{v} = \mathcal{R}(\boldsymbol{z})$
- CvM statistic:

$$\widehat{T}_3 = n \int_{[0,1]^d} \left\{ \widehat{C}(\boldsymbol{v}) - C_\perp(\boldsymbol{v}) \right\}^2 \mathrm{d}\widehat{C}(\boldsymbol{v})$$

- References: Genest et al. (2008)

Copula GoF testing

Proposed approaches: $\mathcal{A}_4$ (4/9)

- Cdf of empirical copula (Kendall's dependence function):

$$S_4(t) = P\{C(\mathbf{z}) \leq t\}$$

- CvM statistic:

$$\widehat{T}_4 = n \int_0^1 \left\{ \widehat{S}_4(t) - S_{4,\widehat{\theta}}(t) \right\}^2 \mathrm{d}S_{4,\widehat{\theta}}(t)$$

- References: Genest and Rivest (1993); Wang and Wells (2000); Savu and Trede (2004); Genest et al. (2006)

Copula GoF testing

Proposed approaches: $\mathcal{A}_5$ (5/9)

- Spearman's dependence function:

$$S_5(t) = P\{C_\perp(z) \leq t\}$$

- CvM statistic:

$$\widehat{T}_5 = n \int_0^1 \left\{ \widehat{S}_5(t) - S_{5,\widehat{\theta}}(t) \right\}^2 \mathrm{d}S_{5,\widehat{\theta}}(t)$$

- References: Quessy et al. (2007)

Copula GoF testing

Proposed approaches: $\mathcal{A}_6$ (6/9)

- Shih's test for bivariate gamma frailty model (Clayton):

$$\widehat{T}_{Shih} = \sqrt{n}\left\{\widehat{\theta}_\tau - \widehat{\theta}_W\right\}$$

- Extension to arbitrary dimension:

$$\widehat{T}_6 = \sum_{i=1}^{d-1}\sum_{j=i+1}^{d}\left\{\widehat{\theta}_{\tau,ij} - \widehat{\theta}_{W,ij}\right\}^2$$

- References: Shih (1998); Berg (2007)

Copula GoF testing

Proposed approaches: $\mathcal{A}_7$ (7/9)

- Inner product of two vectors = 0 iff from the same family

$$Q(z) = \left\langle z - z_{\widehat{\theta}} \middle| \kappa_d \middle| z - z_{\widehat{\theta}} \right\rangle$$

- κ a symmetric kernel, e.g. the gaussian kernel:

$$\kappa_d(z, z_{\widehat{\theta}}) = \exp\left\{-\|z - z_{\widehat{\theta}}\|^2/(2dh^2)\right\}$$

- Statistic becomes:

$$\widehat{T}_7 = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\kappa_d(z_i, z_j) - \frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\kappa_d(z_i, z_{\widehat{\theta},j}) + \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\kappa_d(z_{\widehat{\theta},i}, z_{\widehat{\theta},j})$$

- References: Panchenko (2005)

Copula GoF testing

Proposed approaches: $\mathcal{A}_8$ (8/9)

- Approach $\mathcal{A}_7$ on $\boldsymbol{v} = \mathcal{R}(\boldsymbol{z})$
- Statistic:

$$\widehat{T}_8 = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \kappa_d(\boldsymbol{v}_i, \boldsymbol{v}_j) - \frac{2}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \kappa_d(\boldsymbol{v}_i, \boldsymbol{v}_{\widehat{\theta},j}) + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \kappa_d(\boldsymbol{v}_{\widehat{\theta},i}, \boldsymbol{v}_{\widehat{\theta},j})$$

- References: Berg (2007)

Copula GoF testing

Proposed approaches: $\mathcal{A}_9$ (9/9)

- Each approach may detect deviations from $\mathcal{H}_0$ differently
- Average approaches:

$$\widehat{T}_9^{(a)} = \frac{1}{9}\left\{\widehat{T}_1^{(a)} + \widehat{T}_1^{(b)} + \sum_{k=2}^{8}\widehat{T}_k\right\}$$

$$\widehat{T}_9^{(b)} = \frac{1}{3}\left\{\widehat{T}_2 + \widehat{T}_3 + \widehat{T}_4\right\}$$

- References: Berg (2007)

Monte Carlso simulations

Test procedure

1) $\boldsymbol{x} \sim n$ samples from the d-dimensional $\mathcal{H}_1$ copula with $\theta(\tau)$.
2) $\boldsymbol{z} \sim$ pseudo-observations (normalized ranks)
3) $\hat{\theta} \sim$ estimated parameter of the $\mathcal{H}_0$ copula
4) $\widehat{T}_i \sim$ test statistic i computed under the $\mathcal{H}_0$ copula using $\hat{\theta}$.
5) Repeat steps 1-4 M times with $\mathcal{H}_1 = \mathcal{H}_0$ and $\theta = \hat{\theta} \Rightarrow \widehat{T}^0_{i,m}$
6) $\hat{p} = \frac{1}{M} \sum_{m=1}^{M} \mathbf{1}(\widehat{T}^0_{i,m} > \widehat{T}_i)$
7) $\hat{p} < 5\% \Rightarrow$ reject $\mathcal{H}_0$

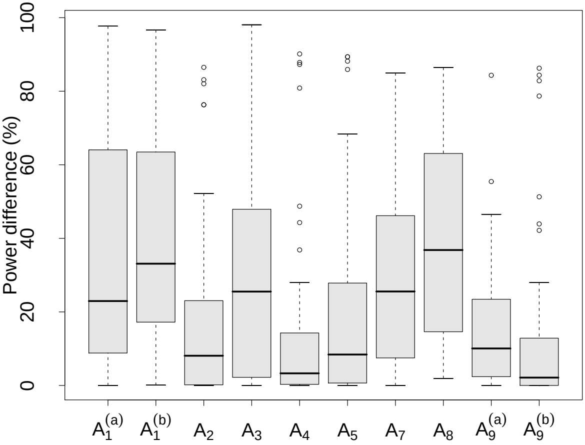
Monte Carlso simulations

Experimental setup

- $\mathcal{H}_0$ copula (5 choices: Gaussian, Student, Clayton, Gumbel, Frank),
- $\mathcal{H}_1$ copula (5 choices: Gaussian, Student ($\nu = 6$), Clayton, Gumbel, Frank),
- Kendall's tau (2 choices: $\tau = \{0.2, 0.4\}$),
- Dimension (3 choices: $d = \{2, 4, 8\}$),
- Sample size (2 choices: $n = \{100, 500\}$)
- Student only considered as null in bivariate case.
- For each of these 240 cases, $10,000$ repetitions $\Rightarrow$ size/power

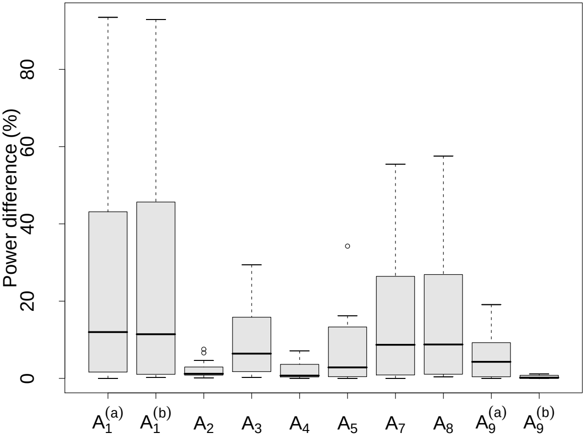
Monte Carlso simulations

Testing the Gaussian copula



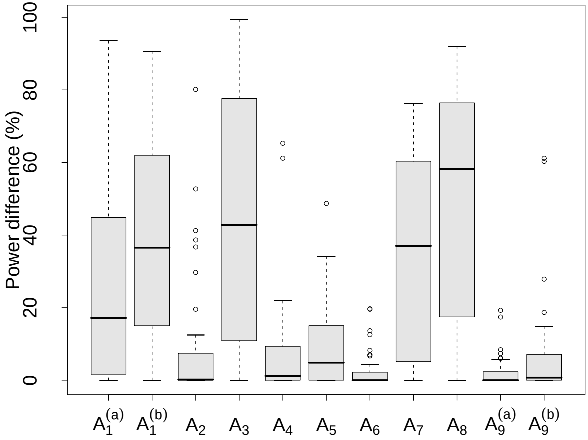
Monte Carlso simulations

Testing the Student copula



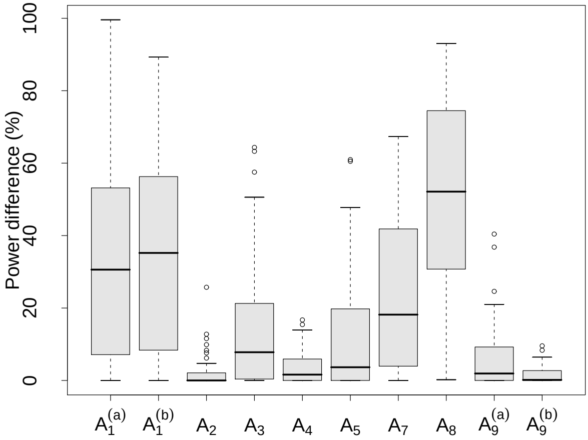
Monte Carlso simulations

Testing the Clayton copula



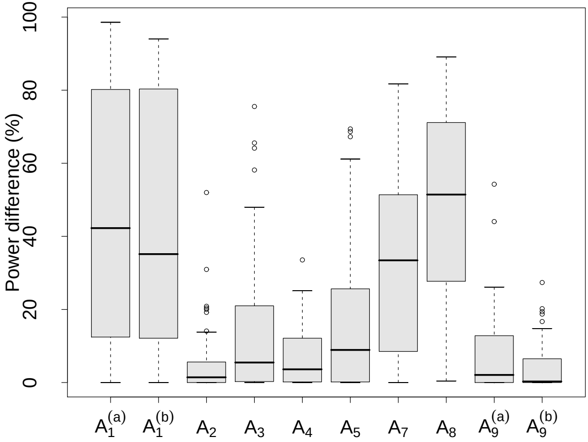
Monte Carlso simulations

Testing the Gumbel copula



Monte Carlso simulations

Testing the Frank copula



Conclusions and recommendations

- Nominal levels all match prescribed size of 5%
- Power generally increases with dimension, sample size and dependence
- Clayton $>$ Gumbel $>$ Frank $>$ Gaussian $>$ Student ($>$: easier to test)
- No universally most powerful approach, but $\mathcal{A}_2$, $\mathcal{A}_4$ and $\mathcal{A}_9^{(b)}$ perform very well in most cases
- $\mathcal{A}_9^{(b)}$ is recommended in general, with special case exceptions:
  - For testing the Gaussian copula, if trying to detect heavy tails for $d > 2$ and large n then $\mathcal{A}_1$ very powerful
  - For testing the Clayton copula the generalized Shih's test is most powerful
- Permutational variation of little concern for approaches based on Rosenblatt's transform (see Berg (2007))

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