Copulas and Temporal Dependence

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October 2006

Job Market Paper

In this paper I identify a condition on the finite dimensional copulas of a univariate time series that ensures the series is weakly dependent in the sense of Doukhan and Louhichi (1999). This condition relates to the Kolmogorov-Smirnov distance between the joint copula of a group of variables in the past and a group of variables in the future, and the copula that would obtain if the past and future were independent. Interestingly, the implied form of weak dependence is not with respect to the class of Lipschitz functions, the class considered in most depth by Doukhan and Louhichi, but rather with respect to the class of absolutely continuous functions. I use the weak dependence property to prove a new strong law of large numbers and new invariance principles in which the only control on temporal dependence is expressed in terms of a condition on finite dimensional copulas.

[1] I thank Don Andrews, Xiaohong Chen, Paul Doukhan, Rustam Ibragimov, Yuichi Kitamura, Taisuke Otsu, Peter Phillips, David Pollard, Ke-Li Xu, and participants in the Econometrics Workshop series at Yale University for helpful comments.

1. Introduction

The recent book by Cherubini, Luciano and Vecchiato (2005) provides ample description of the now widespread use of copulas in statistical models of multivariate financial data. The majority of applications surveyed in said book share a common feature: copulas are used to model the contemporaneous relationship between multiple time series. A related but distinct approach is to use copulas to characterize the temporal relationship between observations in a univariate time series. In this context, copulas have the potential to substantially enrich our current catalogue of models for temporal dependence.

To the author's knowledge, the first paper to explicitly consider the relationship between copulas and temporal dependence was that of Darsow, Nguyen and Olsen (1992), who showed that the finite dimensional copulas of a Markov chain satisfy a certain factorization property. Ibragimov (2005) extended this result to the case of higher order Markov chains, and proposed a consistent nonparametric estimator of the finite dimensional copulas of β-mixing processes. Chen and Fan (2006) identified a condition on the bivariate copulas of stationary Markov chains that ensures the chain is β-mixing.

This last result of Chen and Fan provides a partial answer to an interesting question. Loosely speaking, mixing is a property of time series which depends on the way in which observations relate to one another as they become more distantly separated. Finite dimensional copulas completely characterize the temporal dependence structure of a time series. Therefore, one would expect (or hope) that mixing can be verified directly from an explicit characterization of finite dimensional copulas. In general, it is not known whether this is in fact the case. Chen and Fan's result provides an affirmative answer to this question for the special case of Markovian time series, but no comparable result is available for the non-Markovian case. Moreover, even in the Markovian context, verification of Chen and Fan's condition on copulas is a difficult task which has yet to be achieved in any cases of practical interest.

No general result linking the mixing property to the finite dimensional copulas of a time series is provided in this paper. Instead, a link is established between copulas and an alternative notion of weak dependence introduced by Doukhan and Louhichi (1999).[2] Doukhan and Louhichi's definition of weak dependence involves placing a bound on the covariance between a function of a finite number of variables in the past and another function of a finite number of variables in the future. Said bound may depend on the functions in question and must tend to zero as the past and future become more distantly separated. Doukhan and Louhichi prove moment inequalities and functional central limit theory for time series that are weakly dependent in this sense. The generality of their approach is convincingly demonstrated: strong mixing processes, linear processes, and numerous models employed commonly in econometrics (Nze and Doukhan, 2004) are shown to satisfy their definition of weak dependence.

The contribution of this paper is the identification of a condition on the finite dimensional

[2]The author is deeply indebted to Rustam Ibragimov, who suggested the possibility of such a link in private communication.

copulas of a univariate time series that ensures Doukhan and Louhichi's version of weak dependence is satisfied. This condition relates to the Kolmogorov-Smirnov distance between the joint copula of a group of variables in the past and a group of variables in the future, and the copula that would obtain if the past and future were independent. Interestingly, the implied form of weak dependence does not satisfy the assumptions of Doukhan and Louhichi's central limit theorem. We prove new laws of large numbers and invariance principles for functions of processes whose copulas satisfy the aforementioned condition.

The remainder of this paper is structured as follows. In section 2 the link between finite dimensional copulas and a version of Doukhan and Louhichi's weak dependence is stated, proved, and discussed. Section 3 contains a law of large numbers, central limit theorem, and functional central limit theorem for functions of processes that are weakly dependent in this sense. Section 4 concludes with a discussion of possible avenues for future research on this topic.

2. Copulas and Weak Dependence

Doukhan and Louihichi (1999) proposed the following definition of weak dependence.

Definition 1: *Let $\{X_t : t \in \mathbb{Z}\}$ be a collection of random variables, let $\theta = \{\theta_r : r \in \mathbb{N}\}$ be a sequence of real numbers tending to zero as $r \to \infty$, let $\mathcal{F}$ be a class of functions that each map $\mathbb{R}^u$ to $\mathbb{R}$ for some $u \in \mathbb{N}$, and let ψ be a function mapping $\mathcal{F}^2$ to $\mathbb{R}$. Say that $\{X_t\}$ is $(\theta, \mathcal{F}, \psi)$-weak dependent if, for any $u, v \in \mathbb{N}$, any $h, k \in \mathcal{F}$ defined on $\mathbb{R}^u$ and $\mathbb{R}^v$ respectively, and any $t_1 < \cdots < t_{u+v}$ such that $t_{u+1} - t_u \geq r$,*

$$\left|\mathrm{Cov}\left(h\left(X_{t_1}, \ldots, X_{t_u}\right), k\left(X_{t_{u+1}}, \ldots, X_{t_{u+v}}\right)\right)\right| \leq \psi\left(h, k\right)\theta_r.$$

The usefulness of this definition depends largely on the choice of $\mathcal{F}$ and ψ. Doukhan and Louhichi prove functional central limit theory for stationary $(\theta, \mathcal{L}_1, \psi)$-weak dependent processes, where $\mathcal{L}_1$ is the class of all real valued Lipschitz functions (w.r.t. the l_1 norm on $\mathbb{R}^u$) bounded by unity, and $\psi\left(h, k\right) = \left(u + v\right)^d \left(\mathrm{Lip}\left(h\right) + \mathrm{Lip}\left(k\right)\right)^c$ for some $d \geq 0$, $c \in [0, 2]$. Under appropriate regularity conditions, $(\theta, \mathcal{L}_1, \psi)$-weak dependence is satisfied for suitably chosen θ by α-mixing processes, linear processes, and Markov chains; see Doukhan and Louhichi (1999) and Nze and Doukhan (2004) for details on these and other examples.

An alternative choice of $\mathcal{F}$ and ψ naturally arises when studying the relationship between copulas and $(\theta, \mathcal{F}, \psi)$-weak dependence. We will choose $\mathcal{F}$ to be the class of all absolutely continuous real valued functions on $\mathbb{R}^u$ for some $u \in \mathbb{N}$, and set $\psi_K\left(h, k\right) = \left\|h\right\|_K^* \left\|k\right\|_K^*$, where $\left\|h\right\|_K^*$ is a weighted sum of the supremum of each of the weak derivatives of h. This leads us to the following special case of $(\theta, \mathcal{F}, \psi)$-weak dependence, which we term (θ, K)-dependence.

Definition 2: *Let $\{X_t : t \in \mathbb{Z}\}$ be a collection of random variables. Given a sequence of real numbers $\theta = \{\theta_r : r \in \mathbb{N}\}$ tending to zero as $r \to \infty$, and a constant $K \geq 0$, say that $\{X_t\}$ is (θ, K)-dependent if $\{X_t\}$ is $(\theta, \mathcal{F}, \psi_K)$-weak dependent, where $\mathcal{F}$ is the class of all absolutely continuous functions mapping $\mathbb{R}^u$ to $\mathbb{R}$ for some $u \in \mathbb{N}$, $\psi_K\left(h, k\right) = \left\|h\right\|_K^* \left\|k\right\|_K^*$, and the norm $\left\|\cdot\right\|_K^*$ of an absolutely continuous function $h : \mathbb{R}^u \to \mathbb{R}$ is given by*

$$\left\|h\right\|_K^* = \left\|h\right\|_\infty + \sum_{s=1}^{u} \sum_{\{j_1, \ldots, j_s\} \subset \{1, \ldots, u\}} K^s \left\|\frac{\partial^s h}{\partial x_{j_1} \cdots \partial x_{j_s}}\right\|_\infty.$$

Moment inequalities and invariance principles for (θ, K)-dependent processes will be developed in section 3 of this paper. Our main result in this section identifies a condition on the finite dimensional copulas of a process $\{X_t\}$ that ensures $Y_t = \phi(X_t)$ is (θ, K)-dependent for suitable choices of θ, K and ϕ. This condition is as follows.

Assumption 1: *Let $\{X_t : t \in \mathbb{Z}\}$ be a sequence of random variables with finite dimensional copulas $C_{t_1,\ldots,t_m} : [0,1]^m \to [0,1]$, $t_1 < \cdots < t_m$. Say that $\{X_t\}$ satisfies Assumption 1 if, for any $u, v \in \mathbb{N}$ and any $t_1 < \cdots < t_{u+v}$ such that $t_{u+1} - t_u \geq r$,*

$$\left\| C_{t_1,\ldots,t_{u+v}} - C_{t_1,\ldots,t_u} C_{t_{u+1},\ldots,t_{u+v}} \right\|_\infty \leq M^{u+v} \theta_r$$

for some $M \in \mathbb{R}$ and some sequence $\{\theta_r\}$ converging to zero as $r \to \infty$.

The distance $\left\| C_{t_1,\ldots,t_{u+v}} - C_{t_1,\ldots,t_u} C_{t_{u+1},\ldots,t_{u+v}} \right\|_\infty$ relates closely to the metrics introduced by Schweizer and Wolff (1981) to quantify nonlinear dependence between random variables. Our first theorem provides the link between Assumption 1 and (θ, K)-dependence.

Theorem 1: *Let $\{X_t : t \in \mathbb{Z}\}$ be a stationary sequence of random variables satisfying Assumption 1 for some $M \in \mathbb{R}$ and some sequence $\{\theta_r\}$ converging to zero as $r \to \infty$. Let $Y_t = \phi(X_t)$ for some function $\phi : \mathbb{R} \to \mathbb{R}$; then $\{Y_t\}$ is (θ, K)-dependent with $K = M \|\phi\|^{\mathrm{TV}}_{\mathrm{supp}(X_0)}$, M times the total variation of ϕ on the support of X_0.*

An important special case of Theorem 1 occurs when ϕ is the identity function and the random variables $\{X_t\}$ are bounded, with $|X_t| \leq a$ for some $a < \infty$; Theorem 1 then implies that $\{X_t\}$ is (θ, K)-dependent with $K = 2Ma$. Invariance principles for unbounded X_t can be obtained via standard truncation arguments. We will see in the next section that this leads to central limit theory with a tradeoff between memory decay and moment existence reminiscent of McLeish's (1975) foundational work on mixingales.

The key to proving Theorem 1 is provided by the following lemma, which relates the covariance between $h(X_{t_1}, \ldots, X_{t_u})$ and $k\left(X_{t_{u+1}}, \ldots, X_{t_{u+v}}\right)$ to a sum of distances between copulas.

Lemma 1: *Let $\{X_t : t \in \mathbb{Z}\}$ be a collection of random variables with $U(0,1)$ marginal distributions and finite dimensional distribution functions $F_{t_1,\ldots,t_m} : [0,1]^m \to [0,1]$, $t_1 < \cdots < t_m$. Then for any $u, v \in \mathbb{N}$ and any functions $h : [0,1]^u \to \mathbb{R}$ and $k : [0,1]^v \to \mathbb{R}$ of bounded variation,*

$$\begin{aligned}
& \operatorname{Cov}\left(h(X_{t_1}, \ldots, X_{t_u}), k\left(X_{t_{u+1}}, \ldots, X_{t_{u+v}}\right)\right) \\
= \; & \sum_{m=1}^{u} \sum_{n=1}^{v} \sum_{\{i_1,\ldots,i_m\} \subset \{1,\ldots,u\}} \sum_{\{j_1,\ldots,j_n\} \subset \{1,\ldots,v\}} (-1)^{m+n} \\
& \times \int_{[0,1]^{m+n}} \left(F_{t_{i_1},\ldots,t_{i_m},t_{u+j_1},\ldots,t_{u+j_n}} - F_{t_{i_1},\ldots,t_{i_m}} F_{t_{u+j_1},\ldots,t_{u+j_n}}\right) \mathrm{d}h_{i_1,\ldots,i_m} k_{j_1,\ldots,j_n},
\end{aligned}$$

where $\partial h_{i_1,\ldots,i_m}$ denotes the density of the restriction of h to $[0,1]^m$ obtained by setting the ith argument of h equal to 1 whenever $i \notin \{i_1, \ldots, i_m\}$, and $\partial k_{j_1,\ldots,j_n}$ is similarly defined.

Proof of Lemma 1: For notational clarity we assume that the finite dimensional distribution functions $F_{t_1,\ldots,t_m} : [0,1]^m \to [0,1]$ and the functions h, k possess densities with respect to multidimensional Lebesgue measure, but essentially the same proof holds without this assumption, using Riemann-Stieltjes integration by parts in place of the ordinary integration by parts formula used below. Let $\tilde{F} = F_{t_1,\ldots,t_u,t_{u+1},\ldots,t_{u+v}} - F_{t_1,\ldots,t_u}F_{t_{u+1},\ldots,t_{u+v}}$. For a set $B = \{b_1, \ldots, b_q\}$ with $1 \le b_1 < \cdots < b_q \le u+v$ and a number $a \in \{0, \ldots, u+v\}$, define $B\left(a\right) = \{b \in B : b > a\}$, and let

$$Z_B^a = \int_{[0,1]^q} \left(\left.\frac{\partial^{|B(a)|} hk}{\prod_{b \in B(a)} \partial x_b}\right|_{x_s=1 \forall s \notin B} \right) \left(\left.\frac{\partial^{|B \setminus B(a)|} \tilde{F}}{\prod_{b \in B \setminus B(a)} \partial x_b}\right|_{x_s=1 \forall s \notin B} \right) \prod_{b \in B} \mathrm{d}x_b.$$

Set $Z_\varnothing^a = 0$. Let $p = \max\{p : b_p \le a\}$, so that $Z_B^a = Z_B^{b_p}$. Since $\tilde{F}\left(x\right) = 0$ whenever any element of x is zero, integration by parts gives

$$\begin{aligned}
& Z_B^a \\
= \; & \int_{[0,1]^{q-1}} \left(\left.\frac{\partial^{|B(b_p)|} hk}{\prod_{B(b_p)} \partial x_b}\right|_{x_s=1, s \notin B \setminus \{b_p\}} \right) \left(\left.\frac{\partial^{|(B \setminus \{b_p\}) \setminus B(b_p)|} \tilde{F}}{\prod_{(B \setminus \{b_p\}) \setminus B(b_p)} \partial x_b}\right|_{x_s=1, s \notin B \setminus \{b_p\}} \right) \prod_{B \setminus \{b_p\}} \mathrm{d}x_b \\
& - \int_{[0,1]^q} \left(\left.\frac{\partial^{|B(b_{p-1})|} hk}{\prod_{B(b_{p-1})} \partial x_b}\right|_{x_s=1, s \notin B} \right) \left(\left.\frac{\partial^{|B \setminus B(b_{p-1})|} \tilde{F}}{\prod_{B \setminus B(b_{p-1})} \partial x_b}\right|_{x_s=1, s \notin B} \right) \prod_B \mathrm{d}x_b \\
= \; & Z_{B \setminus \{b_p\}}^{b_p} - Z_B^{b_{p-1}}.
\end{aligned}$$

Iterating this relation, we obtain

$$\begin{aligned}
Z_B^a &= Z_{B \setminus \{b_p\}}^{b_p} - Z_{B \setminus \{b_{p-1}\}}^{b_{p-1}} + Z_B^{b_{p-2}} \\
&= \sum_{m=0}^{p} \left(-1\right)^m Z_{B \setminus \{b_{p-m}\}}^{b_{p-m}} \\
&= \sum_{m=0}^{p} \left(-1\right)^m Z_{B \setminus \{b_{p-m}\}}^{b_{p-m-1}}
\end{aligned}$$

where $b_i = 0$ for $i \le 0$. It is now a simple matter to verify by induction that

$$Z_B^a = \sum_{m=0}^{p} \sum_{1 \le i_1 < \cdots < i_m \le p} \left(-1\right)^{p-m} Z_{B \setminus \left\{b_{i_1}, \ldots, b_{i_m}\right\}}^0.$$

(When $m = 0$, take $\sum_{1 \le i_1 < \cdots < i_m \le p} \left(-1\right)^{p-m} Z_{B \setminus \left\{b_{i_1}, \ldots, b_{i_m}\right\}}^0 = \left(-1\right)^p Z_B^0$.) Observe that

$$\begin{aligned}
& \operatorname{Cov}\left(h\left(X_{t_1}, \ldots, X_{t_u}\right), k\left(X_{t_{u+1}}, \ldots, X_{t_{u+v}}\right)\right) \\
= \; & Z_{\{1,\ldots,u+v\}}^{u+v} \\
= \; & \sum_{m=0}^{u+v} \sum_{1 \le i_1 < \cdots < i_m \le u+v} \left(-1\right)^{u+v-m} Z_{\{1,\ldots,u+v\} \setminus \{i_1,\ldots,i_m\}}^0 \\
= \; & \sum_{m=0}^{u+v} \sum_{1 \le i_1 < \cdots < i_m \le u+v} \left(-1\right)^m Z_{\{i_1,\ldots,i_m\}}^0.
\end{aligned}$$

It is clear from the definition of $\tilde{F}$ that $Z_{\{i_1,\ldots,i_m\}}^0 = 0$ whenever $i_m \le u$ or $i_1 > u$, and so

$$\begin{aligned}
& \operatorname{Cov}\left(h\left(X_{t_1}, \ldots, X_{t_u}\right), k\left(X_{t_{u+1}}, \ldots, X_{t_{u+v}}\right)\right) \\
= \; & \sum_{m=1}^{u} \sum_{n=1}^{v} \sum_{1 \le i_1 < \cdots < i_m \le u} \sum_{1 \le j_1 < \cdots < j_n \le v} \left(-1\right)^{m+n} Z_{\{i_1,\ldots,i_m,u+j_1,\ldots,u+j_n\}}^0.
\end{aligned}$$

The proof is completed by observing that

$$\begin{aligned}
& Z^0_{\{i_1,\ldots,i_m,u+j_1,\ldots,u+j_n\}} \\
= & \int_{[0,1]^{m+n}} \partial h_{i_1,\ldots,i_m} \partial k_{j_1,\ldots,j_n} \left(\tilde{F}\Big|_{x_s=1, s\notin\{i_1,\ldots,i_m,u+j_1,\ldots,u+j_n\}} \right) \\
= & \int_{[0,1]^{m+n}} \partial h_{i_1,\ldots,i_m} \partial k_{j_1,\ldots,j_n} \left(F_{t_{i_1},\ldots,t_{i_m},t_{u+j_1},\ldots,t_{u+j_n}} - F_{t_{i_1},\ldots,t_{i_m}} F_{t_{u+j_1},\ldots,t_{u+j_n}} \right) \\
= & \int_{[0,1]^{m+n}} \left(F_{t_{i_1},\ldots,t_{i_m},t_{u+j_1},\ldots,t_{u+j_n}} - F_{t_{i_1},\ldots,t_{i_m}} F_{t_{u+j_1},\ldots,t_{u+j_n}} \right) \mathrm{d}h_{i_1,\ldots,i_m} k_{j_1,\ldots,j_n}. \quad \blacksquare
\end{aligned}$$

With Lemma 1 in hand, it is now a simple matter to prove Theorem 1.

Proof of Theorem 1: For $t \in \mathbb{Z}$, let Z_t be a $U(0,1)$ random variable such that $F^{-1}(Z_t) = X_t$, where F is the distribution function of X_0. The collection of random variables $\{Z_t\}$ has finite dimensional distribution functions $C_{t_1,\ldots,t_m} : [0,1]^m \to [0,1]$, $t_1 < \cdots < t_m$, so Lemma 1 implies that, for any $u, v \in \mathbb{N}$, any $t_1 < \cdots < t_{u+v}$ with $t_{u+1} - t_u > r$, and any absolutely continuous functions $h : \mathbb{R}^u \to \mathbb{R}$ and $k : \mathbb{R}^v \to \mathbb{R}$,

$$\begin{aligned}
& \left|\mathrm{Cov}\left(h\left(Y_{t_1},\ldots,Y_{t_u}\right), k\left(Y_{t_{u+1}},\ldots,Y_{t_{u+v}}\right)\right)\right| \\
= & \left|\mathrm{Cov}\left(h\left(\phi\left(F^{-1}\left(Z_{t_1}\right)\right),\ldots,\phi\left(F^{-1}\left(Z_{t_u}\right)\right)\right), k\left(\phi\left(F^{-1}\left(Z_{t_{u+1}}\right)\right),\ldots,\phi\left(F^{-1}\left(Z_{t_{u+v}}\right)\right)\right)\right)\right| \\
\leq & \sum_{m=1}^{u}\sum_{n=1}^{v} \sum_{\{i_1,\ldots,i_m\}\subset\{1,\ldots,u\}} \sum_{\{j_1,\ldots,j_n\}\subset\{1,\ldots,v\}} \left\| \frac{\partial^m h}{\partial x_{i_1}\cdots\partial x_{i_m}} \right\|_\infty \left\| \frac{\partial^n k}{\partial x_{j_1}\cdots\partial x_{j_n}} \right\|_\infty \\
& \times \int_{[0,1]^{m+n}} \left| C_{t_{i_1},\ldots,t_{i_m},t_{u+j_1},\ldots,t_{u+j_n}} - C_{t_{i_1},\ldots,t_{i_m}} C_{t_{u+j_1},\ldots,t_{u+j_n}} \right| \\
& \mathrm{d}\left\|\phi\circ F^{-1}\right\|^{\mathrm{TV}}_{[0,x_1]} \cdots \mathrm{d}\left\|\phi\circ F^{-1}\right\|^{\mathrm{TV}}_{[0,x_{m+n}]} \\
\leq & \sum_{m=1}^{u}\sum_{n=1}^{v} \sum_{\{i_1,\ldots,i_m\}\subset\{1,\ldots,u\}} \sum_{\{j_1,\ldots,j_n\}\subset\{1,\ldots,v\}} \left\| \frac{\partial^m h}{\partial x_{i_1}\cdots\partial x_{i_m}} \right\|_\infty \left\| \frac{\partial^n k}{\partial x_{j_1}\cdots\partial x_{j_n}} \right\|_\infty \\
& \times \left\| C_{t_{i_1},\ldots,t_{i_m},t_{u+j_1},\ldots,t_{u+j_n}} - C_{t_{i_1},\ldots,t_{i_m}} C_{t_{u+j_1},\ldots,t_{u+j_n}} \right\|_\infty \left(\int_0^1 \mathrm{d}\left\|\phi\circ F^{-1}\right\|^{\mathrm{TV}}_{[0,x]} \right)^{m+n} \\
= & \sum_{m=1}^{u}\sum_{n=1}^{v} \sum_{\{i_1,\ldots,i_m\}\subset\{1,\ldots,u\}} \sum_{\{j_1,\ldots,j_n\}\subset\{1,\ldots,v\}} \left\| \frac{\partial^m h}{\partial x_{i_1}\cdots\partial x_{i_m}} \right\|_\infty \left\| \frac{\partial^n k}{\partial x_{j_1}\cdots\partial x_{j_n}} \right\|_\infty \\
& \times \left(M \left\| \left(\phi\circ F^{-1}\right) \right\|^{\mathrm{TV}}_{[0,1]} \right)^{m+n} \theta_r \\
= & \sum_{m=1}^{u}\sum_{n=1}^{v} \sum_{\{i_1,\ldots,i_m\}\subset\{1,\ldots,u\}} \sum_{\{j_1,\ldots,j_n\}\subset\{1,\ldots,v\}} \left\| \frac{\partial^m h}{\partial x_{i_1}\cdots\partial x_{i_m}} \right\|_\infty \left\| \frac{\partial^n k}{\partial x_{j_1}\cdots\partial x_{j_n}} \right\|_\infty K^{m+n}\theta_r \\
\leq & \|h\|^*_K \|k\|^*_K \theta_r. \quad \blacksquare
\end{aligned}$$

We will now consider some sufficient conditions under which Assumption 1 is satisfied. de la Peña, Ibragimov and Sharakhmetov (2006, Theorem 3) have shown that the density c of any absolutely continuous n-dimensional copula C can be written as

$$c(x_1,\ldots,x_n) = 1 + \sum_{j=2}^{n} \sum_{1\leq i_1\leq\cdots\leq i_j\leq n} g_{i_1,\ldots,i_j}\left(x_{i_1},\ldots,x_{i_j}\right)$$

for some canonical functions $\{g_{i_1,\ldots,i_j}\}$ satisfying specified conditions on integrability, degeneracy and positive semidefiniteness. The following proposition relates Assumption 1 to these canonical functions. For convenience, we set $g_{i_1,\ldots,i_j} = 0$ whenever $j = 1$.

Proposition 1: *Let $\{X_t : t \in \mathbb{Z}\}$ be a stationary sequence of random variables with absolutely continuous finite dimensional copulas $C_{t_1,\ldots,t_m} : [0,1]^m \to [0,1]$, $t_1 < \cdots < t_m$, and corresponding canonical functions $g_{t_1,\ldots,t_m} : [0,1]^m \to [0,1]$, $t_1 < \cdots < t_m$. Suppose that, for any $u, v \in \mathbb{N}$ and any $t_1 < \cdots < t_{u+v}$ such that $t_{u+1} - t_u \geq r$,*

$$\left\| g_{t_1,\ldots,t_{u+v}} - g_{t_1,\ldots,t_u} g_{t_{u+1},\ldots,t_{u+v}} \right\|_1 \leq M^{u+v} \theta_r$$

for some $M \in \mathbb{R}$ and some sequence of real numbers $\{\theta_r\}$. Then, for any $u, v \in \mathbb{N}$ and any $t_1 < \cdots < t_{u+v}$ such that $t_{u+1} - t_u \geq r$,

$$\left\| C_{t_1,\ldots,t_{u+v}} - C_{t_1,\ldots,t_u} C_{t_{u+1},\ldots,t_{u+v}} \right\|_\infty \leq (1+M)^{u+v} \theta_r.$$

Proof of Proposition 1: We have

$$\begin{aligned}
& \left\| C_{t_1,\ldots,t_{u+v}} - C_{t_1,\ldots,t_u} C_{t_{u+1},\ldots,t_{u+v}} \right\|_\infty \\
\leq & \left\| c_{t_1,\ldots,t_{u+v}} - c_{t_1,\ldots,t_u} c_{t_{u+1},\ldots,t_{u+v}} \right\|_1 \\
= & \left\| \sum_{m=2}^{u+v} \sum_{1 \leq i_1 \leq \cdots \leq i_m \leq u+v} g_{t_{i_1},\ldots,t_{i_m}} - \sum_{m=2}^{u} \sum_{1 \leq i_1 \leq \cdots \leq i_m \leq u} g_{t_{i_1},\ldots,t_{i_m}} \right. \\
& - \sum_{m=2}^{v} \sum_{1 \leq i_1 \leq \cdots \leq i_m \leq v} g_{t_{u+i_1},\ldots,t_{u+i_m}} \\
& \left. - \left(\sum_{m=2}^{u} \sum_{1 \leq i_1 \leq \cdots \leq i_m \leq u} g_{t_{i_1},\ldots,t_{i_m}} \right) \left(\sum_{m=2}^{v} \sum_{1 \leq i_1 \leq \cdots \leq i_m \leq v} g_{t_{u+i_1},\ldots,t_{u+i_m}} \right) \right\|_1 \\
= & \left\| \sum_{m=1}^{u} \sum_{n=1}^{v} \sum_{1 \leq i_1 \leq \cdots \leq i_m \leq u} \sum_{1 \leq j_1 \leq \cdots \leq j_n \leq u} \left(g_{t_{i_1},\ldots,t_{i_m},t_{u+j_1},\ldots,t_{u+j_n}} - g_{t_{i_1},\ldots,t_{i_m}} g_{t_{u+j_1},\ldots,t_{u+j_n}} \right) \right\|_1 \\
\leq & \sum_{m=1}^{u} \sum_{n=1}^{v} \sum_{1 \leq i_1 \leq \cdots \leq i_m \leq u} \sum_{1 \leq j_1 \leq \cdots \leq j_n \leq u} \left\| g_{t_{i_1},\ldots,t_{i_m},t_{u+j_1},\ldots,t_{u+j_n}} - g_{t_{i_1},\ldots,t_{i_m}} g_{t_{u+j_1},\ldots,t_{u+j_n}} \right\|_1 \\
\leq & \left(\sum_{m=1}^{u} \sum_{1 \leq i_1 \leq \cdots \leq i_m \leq u} M^m \right) \left(\sum_{n=1}^{v} \sum_{1 \leq j_1 \leq \cdots \leq j_n \leq u} M^n \right) \theta_r \\
= & \left(\sum_{m=1}^{u} \frac{u!}{(u-m)!m!} M^m \right) \left(\sum_{n=1}^{v} \frac{v!}{(v-n)!n!} M^n \right) \theta_r.
\end{aligned}$$

Our desired result now follows from the binomial theorem. ■

Given knowledge of the canonical functions corresponding to a class of finite dimensional copulas, we can use Proposition 1 to verify Assumption 1. For instance, one well known class of copulas is the class of generalized multivariate Eyraud-Farlie-Gumbel-Morgenstern copulas; see Johnson and Kotz (1975), Cambanis (1977), and Sharakhmetov and Ibragimov (2002). These copulas are of the form

$$C_{t_1,\ldots,t_n}(x_1,\ldots,x_n)=\left(\prod_{i=1}^{n}x_i\right)\left(1+\sum_{m=2}^{n}\sum_{1\le j_1\le\cdots\le j_m\le n}\alpha_{j_1,\ldots,j_m}\prod_{i=1}^{m}\left(1-x_{t_{j_i}}\right)\right),$$

where $\{\alpha_{j_1,\ldots,j_m}\}$ are constants satisfying

$$\sum_{m=2}^{n}\sum_{1\le j_1\le\cdots\le j_m\le n}\alpha_{j_1,\ldots,j_m}\delta_{j_1}\cdots\delta_{j_m}\ge -1$$

for all $n\in\mathbb{N}$, $\delta_i\in\{0,1\}$, $i=1,\ldots,n$. de la Peña, Ibragimov and Sharakhmetov (2006) observe that the corresponding canonical functions are of the form

$$g_{t_1,\ldots,t_n}(x_1,\ldots,x_n)=\alpha_{t_1,\ldots,t_n}\prod_{i=1}^{n}(1-2x_i).$$

It is easy to see that if $\left|\alpha_{t_1,\ldots,t_{u+v}}-\alpha_{t_1,\ldots,t_u}\alpha_{t_{u+1},\ldots,t_{u+v}}\right|\le M^{u+v}\theta_r$ for some M and θ, then these canonical functions satisfy

$$\begin{aligned}\left\|g_{t_1,\ldots,t_{u+v}}-g_{t_1,\ldots,t_u}g_{t_{u+1},\ldots,t_{u+v}}\right\|_1 &= \left(\int_0^1|1-2x|\,dx\right)^{u+v}\left|\alpha_{t_1,\ldots,t_{u+v}}-\alpha_{t_1,\ldots,t_u}\alpha_{t_{u+1},\ldots t_{u+v}}\right|\\ &= \left(\frac{1}{2}\right)^{u+v}\left|\alpha_{t_1,\ldots,t_{u+v}}-\alpha_{t_1,\ldots,t_u}\alpha_{t_{u+1},\ldots t_{u+v}}\right|\\ &\le \left(\frac{1}{2}M\right)^{u+v}\theta_r.\end{aligned}$$

Proposition 1 thus implies that $\left|\alpha_{t_1,\ldots,t_{u+v}}-\alpha_{t_1,\ldots,t_u}\alpha_{t_{u+1},\ldots,t_{u+v}}\right|\le M^{u+v}\theta_r$ is a sufficient condition for the class of generalized multivariate Eyraud-Farlie-Gumbel-Morgenstern copulas to satisfy Assumption 1.

Darsow, Nguyen and Olson (1992) and Ibragimov (2005) have shown that, if $\{X_t\}$ is a Markov chain of order k with absolutely continuous finite dimensional copulas, then the densities of those copulas satisfy

$$\begin{aligned}&c_{t_1,\ldots,t_{m+n-k}}(x_1,\ldots,x_{m+n-k})\cdot c_{t_{m-k+1},\ldots,t_m}(x_{m-k+1},\ldots,x_m)\\ =\;&c_{t_1,\ldots,t_m}(x_1,\ldots,x_m)\cdot c_{t_{m-k+1},\ldots,t_{m+n-k}}(x_{m-k+1},\ldots,x_{m+n-k})\end{aligned}$$

for any $t_1<\cdots<t_{m+n-k}$, $m\ge k$, $n\ge k$. We can use this property to identify a sufficient condition for the copulas of a Markov chain to satisfy Assumption 1.

Proposition 2: *Let $\{X_t:t\in\mathbb{Z}\}$ be a stationary Markov chain of order k with absolutely continuous finite dimensional copulas $C_{t_1,\ldots,t_m}:[0,1]^m\to[0,1]$, $t_1<\cdots<t_m$, and corresponding copula densities $c_{t_1,\ldots,t_m}:[0,1]^m\to[0,1]$, $t_1<\cdots<t_m$. Suppose that, for any $u,v\le k$ and any $t_1<\cdots<t_{u+v}$ such that $t_{u+1}-t_u\ge r$,*

$$\left\|c_{t_1,\ldots,t_{u+v}}-c_{t_1,\ldots,t_u}c_{t_{u+1},\ldots,t_{u+v}}\right\|_1\le\theta_r$$

for some sequence of real numbers $\{\theta_r\}$. Then, for any $u,v\in\mathbb{N}$ and any $t_1<\cdots<t_{u+v}$ such that $t_{u+1}-t_u\ge r$,

$$\left\|C_{t_1,\ldots,t_{u+v}}-C_{t_1,\ldots,t_u}C_{t_{u+1},\ldots,t_{u+v}}\right\|_\infty\le\theta_r.$$

Proof of Proposition 2: For convenience we assume $u, v > k$; the proof can easily be adapted for the case where $u \leq k$ or $v \leq k$. We have

$$\begin{aligned}
& \left\| C_{t_1,\ldots,t_{u+v}} - C_{t_1,\ldots,t_u} C_{t_{u+1},\ldots,t_{u+v}} \right\|_\infty \\
\leq \; & \left\| c_{t_1,\ldots,t_{u+v}} - c_{t_1,\ldots,t_u} c_{t_{u+1},\ldots,t_{u+v}} \right\|_1 \\
= \; & \left\| \frac{c_{t_1,\ldots,t_u} c_{t_{u-k+1},\ldots,t_{u+k}} c_{t_{u+1},\ldots,t_{u+v}}}{c_{t_{u-k+1},\ldots,t_u} c_{t_{u+1},\ldots,t_{u+k}}} - c_{t_1,\ldots,t_u} c_{t_{u+1},\ldots,t_{u+v}} \right\|_1 \\
= \; & \left\| c_{t_1,\ldots,t_u} \left(\frac{c_{t_{u-k+1},\ldots,t_{u+k}}}{c_{t_{u-k+1},\ldots,t_u} c_{t_{u+1},\ldots,t_{u+k}}} - 1 \right) c_{t_{u+1},\ldots,t_{u+v}} \right\|_1 \\
= \; & \int_{[0,1]^{u+v}} c_{t_1,\ldots,t_u} \left| \frac{c_{t_{u-k+1},\ldots,t_{u+k}}}{c_{t_{u-k+1},\ldots,t_u} c_{t_{u+1},\ldots,t_{u+k}}} - 1 \right| c_{t_{u+1},\ldots,t_{u+v}} \mathrm{d}x_{t_1} \cdots \mathrm{d}x_{t_{u+v}} \\
= \; & \int_{[0,1]^{2k}} \left(\int_{[0,1]^{u-k}} c_{t_1,\ldots,t_u} \mathrm{d}x_{t_1} \cdots \mathrm{d}x_{t_{u-k}} \right) \left| \frac{c_{t_{u-k+1},\ldots,t_{u+k}}}{c_{t_{u-k+1},\ldots,t_u} c_{t_{u+1},\ldots,t_{u+k}}} - 1 \right| \\
& \times \left(\int_{[0,1]^{v-k}} c_{t_{u+1},\ldots,t_{u+v}} \mathrm{d}x_{t_{u+k+1}} \cdots \mathrm{d}x_{t_{u+v}} \right) \mathrm{d}x_{t_{u-k+1}} \cdots \mathrm{d}x_{t_{u+k}} \\
= \; & \int_{[0,1]^{2k}} c_{t_{u-k+1},\ldots,t_u} \left| \frac{c_{t_{u-k+1},\ldots,t_{u+k}}}{c_{t_{u-k+1},\ldots,t_u} c_{t_{u+1},\ldots,t_{u+k}}} - 1 \right| c_{t_{u+1},\ldots,t_{u+k}} \mathrm{d}x_{t_{u-k+1}} \cdots \mathrm{d}x_{t_{u+k}} \\
= \; & \int_{[0,1]^{2k}} \left| c_{t_{u-k+1},\ldots,t_{u+k}} - c_{t_{u-k+1},\ldots,t_u} c_{t_{u+1},\ldots,t_{u+k}} \right| \mathrm{d}x_{t_{u-k+1}} \cdots \mathrm{d}x_{t_{u+k}} \\
\leq \; & \theta_r. \quad \blacksquare
\end{aligned}$$

As an example of how Proposition 2 may be used to verify Assumption 1, suppose $\{X_t\}$ is an AR(1) process with i.i.d. Gaussian innovations and autoregressive parameter ρ bounded in absolute value by unity. In this case, we know that $\{X_t\}$ is a first order Markov chain whose bivariate copulas c_{t_1,t_2} are Gaussian with correlation coefficient $\rho^{t_2-t_1}$. Thus we have

$$\begin{aligned}
& \left\| c_{t_u,t_{u+1}} - 1 \right\|_1 \\
= \; & \int_0^1 \int_0^1 \left| \frac{\frac{1}{2\pi} \exp\left(\frac{-\left(\Phi^{-1}(x)^2 + 2\rho^{t_{u+1}-t_u} \Phi^{-1}(x)\Phi^{-1}(y) + \Phi^{-1}(y)^2\right)}{2} \right)}{\frac{1}{\sqrt{2\pi}} \exp\left(\frac{-\Phi^{-1}(x)^2}{2} \right) \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-\Phi^{-1}(y)^2}{2} \right)} - 1 \right| \mathrm{d}x\mathrm{d}y \\
= \; & \int_0^1 \int_0^1 \left| \exp\left(\rho^{t_{u+1}-t_u} \Phi^{-1}(x) \Phi^{-1}(y) \right) - 1 \right| \mathrm{d}x\mathrm{d}y \\
= \; & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2} \right) \right) \left(\frac{1}{\sqrt{2\pi}} \exp\left(\frac{-y^2}{2} \right) \right) \left| \exp\left(-\rho^{t_{u+1}-t_u} xy \right) - 1 \right| \mathrm{d}x\mathrm{d}y \\
= \; & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi} \left| \exp\left(\frac{-\left(x^2 + 2\rho^{t_{u+1}-t_u} xy + y^2\right)}{2} \right) - \exp\left(\frac{-\left(x^2 + y^2\right)}{2} \right) \right| \mathrm{d}x\mathrm{d}y.
\end{aligned}$$

A mean value expansion around $\rho^{t_{u+1}-t_u}=0$ yields, for some $\tilde{\rho}(x,y)$ between 0 and ρ,

$$\begin{aligned}
& \left\|c_{t_u,t_{u+1}}-1\right\|_1 \\
= & \left|\rho^{t_{u+1}-t_u}\right| \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|xy|\frac{1}{2\pi}\exp\left(\frac{-\left(x^2+2\tilde{\rho}^{t_{u+1}-t_u}(x,y)\,xy+y^2\right)}{2}\right)\mathrm{d}x\mathrm{d}y \\
\leq & \left|\rho^{t_{u+1}-t_u}\right| \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|xy|\frac{1}{2\pi}\exp\left(\frac{-\left(x^2-2xy+y^2\right)}{2}\right)\mathrm{d}x\mathrm{d}y \\
& +\left|\rho^{t_{u+1}-t_u}\right| \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|xy|\frac{1}{2\pi}\exp\left(\frac{-\left(x^2+2xy+y^2\right)}{2}\right)\mathrm{d}x\mathrm{d}y \\
\leq & 2\left|\rho^{t_{u+1}-t_u}\right|.
\end{aligned}$$

Thus we conclude from Proposition 2 that our Gaussian AR(1) process satisfies Assumption 1.

Another, more interesting, example of a process that satisfies Assumption 1 is that of a stationary first order Markov process whose bivariate copulas $c_{t,t+1}$ are the power copulas proposed by de la Peña, Ibragimov and Sharakhmetov (2006). These copulas have densities of the form

$$c_{t,t+1}(x,y)=1+\alpha\left((k+1)x^k-(k+2)x^{k+1}\right)\left((k+1)y^k-(k+2)y^{k+1}\right)$$

for some $\alpha\in(-1,1)$, $k\in\mathbb{Z}_+$. Applying the product operation of Darsow, Nguyen and Olsen (1992), we obtain

$$\begin{aligned}
& c_{t,t+2}(x,y)-1 \\
= & \int_0^1 \alpha^2\left((k+1)x^k-(k+2)x^{k+1}\right)\left((k+1)z^k-(k+2)z^{k+1}\right)^2 \\
& \times\left((k+1)y^k-(k+2)y^{k+1}\right)\mathrm{d}z \\
= & \alpha^2\left(\frac{k+1}{(2k+1)(2k+3)}\right)\left((k+1)x^k-(k+2)x^{k+1}\right)\left((k+1)y^k-(k+2)y^{k+1}\right).
\end{aligned}$$

It is easy to see that recursive application of this operation yields copulas $c_{t,t+r}$ of the form

$$\begin{aligned}
& c_{t,t+r}(x,y) \\
= & 1+\alpha^r\left(\frac{k+1}{(2k+1)(2k+3)}\right)^{r-1}\left((k+1)x^k-(k+2)x^{k+1}\right)\left((k+1)y^k-(k+2)y^{k+1}\right).
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
& \left\|c_{t,t+r}-1\right\|_1 \\
= & \alpha^r\left(\frac{k+1}{(2k+1)(2k+3)}\right)^{r-1}\left\|\left((k+1)x^k-(k+2)x^{k+1}\right)\left((k+1)y^k-(k+2)y^{k+1}\right)\right\|_1 \\
= & \alpha^r\left(\frac{k+1}{(2k+1)(2k+3)}\right)^{r-1}\left\|(k+1)x^k-(k+2)x^{k+1}\right\|_1^2 \\
= & \alpha^r\left(\frac{k+1}{(2k+1)(2k+3)}\right)^{r-1}\left(2\frac{(k+1)^{k+1}}{(k+2)^{k+2}}\right)^2 \\
\leq & 4\alpha^r\left(\frac{k+1}{(2k+1)(2k+3)}\right)^{r-1},
\end{aligned}$$

and so from Proposition 2 we conclude that a stationary first order Markov process constructed from power copulas satisfies Assumption 1 with an exponentially fast rate of decay of θ_r.

3. Limit theorems

In the previous section we defined a new characterization of temporal dependence, termed (θ, K)-dependence, and identified a condition on the finite dimensional copulas of a time series that implies this form of dependence. In this section we use the (θ, K)-dependence property to prove laws of large numbers and invariance principles for functions of processes whose copulas satisfy said condition.

Our first result is a strong law of large numbers for functions of a stationary time series satisfying Assumption 1 and a suitable tradeoff between memory decay and moment existence.

Theorem 2: *Let $\{X_t : t \in \mathbb{Z}\}$ be a stationary sequence of random variables satisfying Assumption 1 for some $M \in \mathbb{R}$ and some sequence $\{\theta_r\}$, and let $Y_t = \phi(X_t)$ for some $\phi : \mathbb{R} \to \mathbb{R}$ satisfying $\left\|\phi \cdot 1_{\{|\phi| \le \tau\}}\right\|_{\mathbb{R}}^{\mathrm{TV}} \le a\tau$ for some $a > 0$ and all $\tau \ge 0$. Suppose $\sum_1^\infty r^\delta \theta_r < \infty$ for some $\delta > 0$. Then, if $E|Y_t|^{2+\varepsilon} < \infty$ for some $\varepsilon > 2\delta^{-1}$,*

$$\frac{1}{n}\sum_{t=1}^{n} Y_t \to_{a.s.} EY_0.$$

Proof: Using the fact that $E|Y_t|^{2+\varepsilon} < \infty$, it is easy to show that

$$\begin{aligned}
|\mathrm{Cov}(Y_t, Y_{t+r})| &= \Big|\mathrm{Cov}\left(Y_t 1_{|Y_t| \le r^{\delta/2}}, Y_{t+r} 1_{|Y_{t+r}| \le r^{\delta/2}}\right) \\
&\quad + \mathrm{Cov}\left(Y_t 1_{|Y_t| > r^{\delta/2}}, Y_{t+r}\right) + \mathrm{Cov}\left(Y_t 1_{|Y_t| \le r^{\delta/2}}, Y_{t+r} 1_{|Y_{t+r}| > r^{\delta/2}}\right)\Big| \\
&\le \left|\mathrm{Cov}\left(Y_t 1_{|Y_t| \le r^{\delta/2}}, Y_{t+r} 1_{|Y_{t+r}| \le r^{\delta/2}}\right)\right| + O\left(r^{-\delta\varepsilon/2}\right).
\end{aligned}$$

Theorem 1 implies that $\left\{Y_t 1_{|Y_t| \le r^{\delta/2}} : t \in \mathbb{Z}\right\}$ is $\left(\theta, Mar^{\delta/2}\right)$-dependent; it follows that

$$\begin{aligned}
\left|\mathrm{Cov}\left(Y_t 1_{|Y_t| \le r^{\delta/2}}, Y_{t+r} 1_{|Y_{t+r}| \le r^{\delta/2}}\right)\right| &\le \left(r^{\delta/2} + Mar^{\delta/2}\right)^2 \theta_r \\
&= O\left(r^\delta \theta_r\right).
\end{aligned}$$

Thus $\sum_{r=1}^\infty |\mathrm{Cov}(Y_t, Y_{t+r})| < \infty$ provided that $\delta\varepsilon/2 > 1$, giving our desired result (Stout, 1974). ■

The condition $\left\|\phi \cdot 1_{\{|\phi| \le \tau\}}\right\|_{\mathbb{R}}^{\mathrm{TV}} \le a\tau$ is satisfied by, for instance, polynomial functions $\phi(x) = \sum_0^p a_k x^k$, for which $a = 1+3p$, and increasing functions, for which $a = 4$. The condition is violated by the sine and cosine functions, for which $\left\|\phi \cdot 1_{\{|\phi| \le \tau\}}\right\|_{\mathbb{R}}^{\mathrm{TV}} = \infty$ whenever $\tau > 0$. Note, however, that if ϕ is a sine or cosine function, or any other function of locally bounded variation, then we can write it as the difference of two increasing functions and then apply Theorem 2 to each part separately, provided that the relevant moment conditions are satisfied.

We now turn our attention to the development of invariance principles for functions of stationary time series satisfying Assumption 1. The following lemma, which provides a bound on the second moments of sums of such processes, is a necessary preliminary step. It follows easily from Theorem 1 of Doukhan and Louhichi (1999).

Lemma 2: *Let $\{X_t : t \in \mathbb{Z}\}$ be a stationary sequence of random variables satisfying Assumption 1 for some $M \in \mathbb{R}$ and some sequence $\{\theta_r\}$, and let $Y_t = \phi(X_t)$ for some $\phi : \mathbb{R} \to \mathbb{R}$ satisfying $\left\|\phi \cdot 1_{\{|\phi| \leq \tau\}}\right\|_{\mathbb{R}}^{\mathrm{TV}} \leq a\tau$ for some $a > 0$ and all $\tau \geq 0$. Suppose $EY_t = 0$ and $E|Y_t|^{2+\varepsilon} < \infty$ for some $\varepsilon > 0$. Then, if $\theta_r = O\left(r^{-(2+\varepsilon)/\varepsilon}\right)$ there exists $b < \infty$ such that, for any $n \geq 1$,*

$$E\left(\sum_{t=1}^{n} Y_t\right)^2 \leq bn.$$

Proof: This result follows from Theorem 1 of Doukhan and Louhichi (1999) if we can show that $|\mathrm{Cov}(Y_0, Y_r)| = O\left(r^{-1}\right)$. We showed in the proof of Theorem 2 that $|\mathrm{Cov}(Y_0, Y_r)| = O\left(r^{\delta}\theta_r + r^{-\delta\varepsilon/2}\right)$ for any $\delta > 0$. Setting $\delta = 2\varepsilon^{-1}$ gives us $|\mathrm{Cov}(Y_0, Y_r)| = O\left(r^{-1}\right)$. ■

We are now in a position to state and prove a central limit theorem for functions of stationary processes satifying Assumption 1.

Theorem 3: *Let $\{X_t : t \in \mathbb{Z}\}$ be a stationary sequence of random variables satisfying Assumption 1 for some $M \in \mathbb{R}$ and some sequence $\{\theta_r\}$, and let $Y_t = \phi(X_t)$ for some $\phi : \mathbb{R} \to \mathbb{R}$ satisfying $\left\|\phi \cdot 1_{\{|\phi| \leq \tau\}}\right\|_{\mathbb{R}}^{\mathrm{TV}} \leq a\tau$ for some $a > 0$ and all $\tau \geq 0$. Suppose $EY_t = 0$ and $E|Y_t|^{2+\varepsilon} < \infty$ for some $\varepsilon > 0$, and suppose $\theta_r \leq b\exp\left(-r^{\delta}\right)$ for some $b < \infty$ and some $\delta > \frac{1}{2} + \min\left\{\varepsilon^{-1}, \frac{1}{2}\right\}$. Then the series*

$$\sigma^2 = EY_0^2 + 2\sum_{i=1}^{\infty} EY_0 Y_i$$

converges absolutely; if $\sigma^2 > 0$, then

$$\frac{1}{\sqrt{n}} \sum_{t=1}^{n} Y_t \to_d N\left(0, \sigma^2\right).$$

Proof: $\{Y_t\}$ satisfies the assumptions of Theorem 2, in the proof of which we showed that $\sum_1^{\infty} \mathrm{Cov}(Y_0, Y_i) < \infty$. Thus σ^2 is absolutely convergent. Suppose $\sigma^2 > 0$. We will prove convergence in distribution by employing Lemma 11 of Doukhan and Louhichi (1999), itself a version of results by Ibragimov and Linnik (1971) and Withers (1981). Split $\{Y_t : 1 \leq t \leq n\}$ into Bernstein blocks of length n_1, separated by gaps of length n_2, as follows:

$$\begin{aligned}
\sum_{t=1}^{n} Y_t &= \sum_{i=1}^{k} \eta_i + \sum_{i=1}^{k+1} \nu_i, \;\; k = \left[\frac{n}{n_1 + n_2}\right] \\
\eta_i &= \sum_{t=(i-1)(n_1+n_2)+1}^{in_1+(i-1)n_2} Y_t, \; i = 1, \ldots, k \\
\nu_i &= \sum_{t=in_1+(i-1)n_2+1}^{i(n_1+n_2)} Y_t, \;\; i = 1, \ldots, k \\
\nu_{k+1} &= \sum_{t=k(n_1+n_2)+1}^{n} Y_t.
\end{aligned}$$

Let $\sigma_n^2 = E\left(\sum_1^n Y_t\right)^2$. Doukhan and Louhichi's lemma states that $\sigma_n^{-1} \sum_1^n Y_t \rightarrow_d N(0,1)$ if, for some sequences $n_1(n)$, $n_2(n)$ such that $n_1 \rightarrow \infty$, $n_2 \rightarrow \infty$, $n_1 = o(n)$, $n_2 = o(n_1)$, the following four conditions are satisfied for any g, h among the functions $x \mapsto \sin x$ and $x \mapsto \cos x$:

$$\frac{1}{\sigma_n^2} E\left(\sum_{i=1}^{k+1} \nu_i\right)^2 \rightarrow 0 \tag{1}$$

$$\frac{1}{\sigma_n^2} \sum_{i=1}^{k} E\eta_i^2 \rightarrow 1 \tag{2}$$

$$\frac{1}{\sigma_n^2} \sum_{i=1}^{k} E\eta_i^2 1\left(|\eta_i| \geq \epsilon\sigma_n\right) \rightarrow 0 \text{ for all } \epsilon > 0 \tag{3}$$

$$\sum_{j=2}^{k} \left|\operatorname{Cov}\left(g\left(\frac{t}{\sigma_n}\sum_{i=1}^{j-1}\eta_i\right), h\left(\frac{t}{\sigma_n}\eta_j\right)\right)\right| \rightarrow 0 \text{ for all } t > 0. \tag{4}$$

Absolute convergence of σ^2 implies that $n^{-1}\sigma_n^2 \rightarrow \sigma^2 > 0$, and Lemma 2 implies that $E(\sum_1^k \nu_i)^2 = O(kn_2)$ and $E(\nu_{k+1})^2 = O(n_1)$; (1) follows. $n_1^{-1}E\eta_0^2 \rightarrow \sigma^2$ by the same logic that $n^{-1}\sigma_n^2 \rightarrow \sigma^2$, so $\sigma_n^{-2}\sum_1^k E\eta_i^2 = kn_1n^{-1}(1+o(1)) \rightarrow 1$, proving (2). (3) holds since $n_1^{-1}E\eta_0^2 1(|\eta_0| \geq \epsilon\sigma_n) = E(n_1^{-1/2}\eta_0)^2 1(|n_1^{-1/2}\eta_0|^2 \geq \epsilon^2\sigma_n^2 n_1^{-1}) \rightarrow 0$ by Lemma 2 and the dominated convergence theorem.

It remains to confirm (4). For any $\gamma > 0$, we can use the mean value theorem to show that

$$\begin{aligned}
& \sum_{j=2}^{k} \left|\operatorname{Cov}\left(g\left(\frac{t}{\sigma_n}\sum_{i=1}^{j-1}\eta_i\right), h\left(\frac{t}{\sigma_n}\eta_j\right)\right)\right| \\
\leq\ & \sum_{j=2}^{k} \left|\operatorname{Cov}\left(g\left(\frac{t}{\sigma_n}\sum_{i=1}^{j-1}\sum_{(i-1)(n_1+n_2)+1}^{in_1+(i-1)n_2} Y_s 1_{\{|Y_s|\leq n^\gamma\}}\right), h\left(\frac{t}{\sigma_n}\sum_{(j-1)(n_1+n_2)+1}^{jn_1+(j-1)n_2} Y_s 1_{\{|Y_s|\leq n^\gamma\}}\right)\right)\right| \\
& +4\frac{t}{\sigma_n}k^2 n_1\left(E\left|Y_0 1_{\{|Y_0|>n^\gamma\}}\right|^2\right)^{1/2} + 2\left(\frac{t}{\sigma_n}\right)^2 k^2 n_1^2 E\left|Y_0 1_{\{|Y_0|>n^\gamma\}}\right|^2 \\
=\ & \sum_{j=2}^{k} \left|\operatorname{Cov}\left(g\left(\frac{t}{\sigma_n}\sum_{i=1}^{j-1}\sum_{(i-1)(n_1+n_2)+1}^{in_1+(i-1)n_2} Y_s 1_{\{|Y_s|\leq n^\gamma\}}\right), h\left(\frac{t}{\sigma_n}\sum_{(j-1)(n_1+n_2)+1}^{jn_1+(j-1)n_2} Y_s 1_{\{|Y_s|\leq n^\gamma\}}\right)\right)\right| \\
& +O\left(kn^{1/2-\varepsilon\gamma/2} + n^{1-\varepsilon\gamma}\right).
\end{aligned}$$

Theorem 1 implies that $\left\{Y_t 1_{\{|Y_t|\leq n^\gamma\}}\right\}$ is (θ, Man^γ)-dependent. Thus, using the binomial theorem, we have

$$\begin{aligned}
& \sum_{j=2}^{k} \left|\operatorname{Cov}\left(g\left(\frac{t}{\sigma_n}\sum_{i=1}^{j-1}\sum_{(i-1)(n_1+n_2)+1}^{in_1+(i-1)n_2} Y_s 1_{\{|Y_s|\leq n^\gamma\}}\right), h\left(\frac{t}{\sigma_n}\sum_{(j-1)(n_1+n_2)+1}^{jn_1+(j-1)n_2} Y_s 1_{\{|Y_s|\leq n^\gamma\}}\right)\right)\right| \\
\leq\ & k\left(\sum_{s=0}^{(k-1)n_1}\frac{((k-1)n_1)!}{((k-1)n_1-s)!s!}\left(\frac{tMan^\gamma}{\sigma_n}\right)^s\right)\left(\sum_{s=0}^{n_1}\frac{n_1!}{(n_1-s)!s!}\left(\frac{tMan^\gamma}{\sigma_n}\right)^s\right)\theta_{n_2} \\
=\ & k\left(1+\frac{tMan^\gamma}{\sigma_n}\right)^{kn_1}\theta_{n_2}.
\end{aligned}$$

Let $n_1 = \left[n^\beta\right]$ and $n_2 = [n^\alpha]$, with $0 < \alpha < \beta < 1$. Then

$$\begin{aligned}
&\sum_{j=2}^{k} \left| \operatorname{Cov}\left(g\left(\frac{t}{\sigma_n} \sum_{i=1}^{j-1} \eta_i \right), h\left(\frac{t}{\sigma_n} \eta_j \right) \right) \right| \\
= \; & O\left(k \left(1 + \frac{tMan^\gamma}{\sigma_n} \right)^{kn_1} \theta_{n_2} \right) + O\left(kn^{1/2-\varepsilon\gamma/2} + n^{1-\varepsilon\gamma} \right) \\
= \; & O\left(n^{1-\beta} \left(1 + O\left(n^{\gamma-\frac{1}{2}} \right) \right)^n \exp\left(-[n^\alpha]^\delta \right) \right) + O\left(n^{3/2-\beta-\varepsilon\gamma/2} + n^{1-\varepsilon\gamma} \right).
\end{aligned}$$

Suppose $\delta > 1$. In this case, we can satisfy (4) by choosing $\alpha > \delta^{-1}$ and γ sufficiently large, so that

$$\begin{aligned}
&\sum_{j=2}^{k} \left| \operatorname{Cov}\left(g\left(\frac{t}{\sigma_n} \sum_{i=1}^{j-1} \eta_i \right), h\left(\frac{t}{\sigma_n} \eta_j \right) \right) \right| \\
= \; & O\left(n^{1-\beta+\left(\gamma-\frac{1}{2}\right)n} \exp\left(-[n^\alpha]^\delta \right) \right) + O\left(n^{3/2-\beta-\varepsilon\gamma/2} + n^{1-\varepsilon\gamma} \right) \\
= \; & o(1).
\end{aligned}$$

Now suppose that $\delta \le 1$, and choose $\gamma < \frac{1}{2}$. This gives us

$$\begin{aligned}
&\sum_{j=2}^{k} \left| \operatorname{Cov}\left(g\left(\frac{t}{\sigma_n} \sum_{i=1}^{j-1} \eta_i \right), h\left(\frac{t}{\sigma_n} \eta_j \right) \right) \right| \\
= \; & O\left(n^{1-\beta} \exp\left(O\left(n^{\gamma+\frac{1}{2}} \right) - [n^\alpha]^\delta \right) \right) + O\left(n^{3/2-\beta-\varepsilon\gamma/2} + n^{1-\varepsilon\gamma} \right).
\end{aligned}$$

In order to satisfy (4) while maintaining $\gamma < \frac{1}{2}$, we require the following three inequalities to hold:

$$\begin{aligned}
\gamma &< \frac{1}{2} \\
\gamma &< \alpha\delta - \frac{1}{2} \\
\gamma &> \frac{3-2\beta}{\varepsilon}.
\end{aligned}$$

Since $\delta \le 1$, the first of these inequalities is redundant. The remaining two inequalities can be satisfied for suitable choice of γ if

$$\frac{3-2\beta}{\varepsilon} < \alpha\delta - \frac{1}{2},$$

which is possible if

$$\delta > \frac{1}{2} + \varepsilon^{-1}.$$

Thus (4) holds provided that $\delta > \frac{1}{2} + \min\left\{\varepsilon^{-1}, \frac{1}{2}\right\}$. This completes the proof. ■

Observe that when $\varepsilon \le 2$, so that only four or fewer moments of Y_t are assumed to exist, we require a super-exponential rate of memory decay; i.e. $\delta > 1$. When $\varepsilon > 2$, sub-exponential decay rates are permissible, with a hyperbolic tradeoff between allowable values of ε and δ. As $\varepsilon \to \infty$, the lower bound on allowable values of δ falls to $1/2$.

Before proving our functional central limit theorem, we require a version of Lemma 2 that applies to fourth moments of sums of random variables.

Lemma 3: *Let $\{X_t : t \in \mathbb{Z}\}$ be a stationary sequence of random variables satisfying Assumption 1 for some $M \in \mathbb{R}$ and some sequence $\{\theta_r\}$, and let $Y_t = \phi(X_t)$ for some $\phi : \mathbb{R} \to \mathbb{R}$ satisfying $\left\|\phi \cdot 1_{\{|\phi| \leq \tau\}}\right\|_{\mathbb{R}}^{\mathrm{TV}} \leq a\tau$ for some $a > 0$ and all $\tau \geq 0$. Suppose $EY_t = 0$ and $E|Y_t|^{4+\varepsilon} < \infty$ for some $\varepsilon > 0$. Then, if $\theta_r = O\left(r^{-(8+2\varepsilon)/\varepsilon}\right)$ there exists $b < \infty$ such that, for any $n \geq 1$,*

$$E\left(\sum_{t=1}^{n} Y_t\right)^4 \leq bn^2.$$

Proof: This result follows from Theorem 1 of Doukhan and Louhichi (1999) if we can verify the following three conditions:

$$\begin{aligned}
\sup_{t_1<t_2<t_2+r\leq t_3<t_4} \left|\mathrm{Cov}\left(Y_{t_1}Y_{t_2}, Y_{t_3}Y_{t_4}\right)\right| &= O\left(r^{-2}\right) \\
\sup_{t_1<t_2<t_2+r\leq t_3<t_4} \left|\mathrm{Cov}\left(Y_{t_1}Y_{t_2}Y_{t_3}, Y_{t_4}\right)\right| &= O\left(r^{-2}\right) \\
\sup_{t_1<t_2<t_2+r\leq t_3<t_4} \left|\mathrm{Cov}\left(Y_{t_1}, Y_{t_2}Y_{t_3}Y_{t_4}\right)\right| &= O\left(r^{-2}\right).
\end{aligned}$$

We shall verify only the first condition; the others follow by similar arguments. Using the fact that $E|Y_t|^{4+\varepsilon} < \infty$, it is easy to show that, for any $\delta > 0$,

$$\begin{aligned}
& \sup\left|\mathrm{Cov}\left(Y_{t_1}Y_{t_2}, Y_{t_3}Y_{t_4}\right)\right| \\
\leq\ & \sup\left|\mathrm{Cov}\left(Y_{t_1}1_{|Y_{t_1}|\leq r^{\delta/4}}Y_{t_2}1_{|Y_{t_2}|\leq r^{\delta/4}}, Y_{t_3}1_{|Y_{t_3}|\leq r^{\delta/4}}Y_{t_4}1_{|Y_{t_4}|\leq r^{\delta/4}}\right)\right| \\
& + \sup\left|\mathrm{Cov}\left(Y_{t_1}1_{|Y_{t_1}|\leq r^{\delta/4}}Y_{t_2}1_{|Y_{t_2}|\leq r^{\delta/4}}, Y_{t_3}Y_{t_4}1_{\left\{\max\left\{|Y_{t_3}|,|Y_{t_4}|\right\}>r^{\delta/4}\right\}}\right)\right| \\
& + \sup\left|\mathrm{Cov}\left(Y_{t_1}Y_{t_2}1_{\left\{\max\left\{|Y_{t_1}|,|Y_{t_2}|\right\}>r^{\delta/4}\right\}}, Y_{t_3}Y_{t_4}\right)\right| \\
\leq\ & \sup\left|\mathrm{Cov}\left(Y_{t_1}1_{|Y_{t_1}|\leq r^{\delta/4}}Y_{t_2}1_{|Y_{t_2}|\leq r^{\delta/4}}, Y_{t_3}1_{|Y_{t_3}|\leq r^{\delta/4}}Y_{t_4}1_{|Y_{t_4}|\leq r^{\delta/4}}\right)\right| + O\left(r^{-\delta\varepsilon/4}\right).
\end{aligned}$$

Theorem 1 implies that $\left\{Y_t 1_{|Y_t|\leq r^{\delta/4}} : t \in \mathbb{Z}\right\}$ is $\left(\theta, Mar^{\delta/4}\right)$-dependent; it follows that

$$\begin{aligned}
& \sup\left|\mathrm{Cov}\left(Y_{t_1}1_{|Y_{t_1}|\leq r^{\delta/4}}Y_{t_2}1_{|Y_{t_2}|\leq r^{\delta/4}}, Y_{t_3}1_{|Y_{t_3}|\leq r^{\delta/4}}Y_{t_4}1_{|Y_{t_4}|\leq r^{\delta/4}}\right)\right| \\
\leq\ & \left(r^{\delta/2} + 2Mar^{\delta/2} + M^2a^2r^{\delta/2}\right)^2\theta_r \\
=\ & O\left(r^{\delta}\theta_r\right).
\end{aligned}$$

Thus $\sup\left|\mathrm{Cov}\left(Y_{t_1}Y_{t_2}, Y_{t_3}Y_{t_4}\right)\right| = O\left(r^{\delta}\theta_r + r^{-\delta\varepsilon/4}\right)$. Our result follows by setting $\delta = 8\varepsilon^{-1}$. ■

We are now in a position to state our final result: a functional central limit theorem for functions of stationary processes satisfying Assumption 1.

Theorem 4: *Let $\{X_t : t \in \mathbb{Z}\}$ be a stationary sequence of random variables satisfying Assumption 1 for some $M \in \mathbb{R}$ and some sequence $\{\theta_r\}$, and let $Y_t = \phi(X_t)$ for some $\phi : \mathbb{R} \to \mathbb{R}$ satisfying $\left\|\phi \cdot 1_{\{|\phi| \leq \tau\}}\right\|_{\mathbb{R}}^{\mathrm{TV}} \leq a\tau$ for some $a > 0$ and all $\tau \geq 0$. Suppose $EY_t = 0$ and $E|Y_t|^{2+\varepsilon} < \infty$*

for some $\varepsilon > 2$*, and suppose* $\theta_r \leq b \exp\left(-r^{\delta}\right)$ *for some* $b < \infty$ *and some* $\delta > \frac{1}{2} + \varepsilon^{-1}$*. Then the series*

$$\sigma^2 = EY_0^2 + 2\sum_{i=1}^{\infty} EY_0 Y_i$$

converges absolutely; if $\sigma^2 > 0$*, then*

$$\frac{1}{\sqrt{n}} \sum_{t=1}^{[nr]} Y_t \Rightarrow \sigma B(r),$$

where B *denotes a standard Brownian motion on* $[0,1]$*.*

Proof: Finite dimensional convergence of our partial sum process to Brownian motion can be shown using the Cramér-Wold device and a minor variation on the arguments employed in the proof of Theorem 3. It remains to prove tightness. For this it suffices to show that, for any $\varepsilon > 0$, there exists $\lambda > 1$ such that

$$\lambda^2 P\left\{\max_{1\leq m\leq n}\left|\sum_{t=1}^{m} Y_t\right| \geq \lambda\sigma\sqrt{n}\right\} \leq \varepsilon$$

for all n; see Billingsley (1968, ch. 3). Lemma 3 implies that $E(\sum_1^n Y_t)^4 \leq bn^2$ for some $b < \infty$, so Theorem 3.1 in Moricz, Serfling and Stout (1982) implies that $E\max_{m\leq n}(\sum_1^m Y_t)^4 \leq An^2$ for some $A < \infty$. Thus Markov's inequality gives us

$$\begin{aligned}\lambda^2 P\left\{\max_{1\leq m\leq n}\left|\sum_{t=1}^{m} Y_t\right| \geq \lambda\sigma\sqrt{n}\right\} &\leq \frac{E\max_{1\leq m\leq n}\left|\sum_{t=1}^{m} Y_t\right|^4}{\lambda^2\sigma^4 n^2} \\ &\leq \frac{A}{\lambda^2\sigma^4},\end{aligned}$$

which can be made arbitrarily small by choosing λ sufficiently large. ■

Note that the allowable tradeoff between ε and δ is no more stringent in Theorem 4 than it was in Theorem 3, provided that $\varepsilon > 2$, so that Y_t possesses greater than four moments. Theorem 4 does not encompass processes with only four or fewer moments because we are unable to establish that the tightness condition holds.

4. Conclusion

In this paper we have proposed a new copula-based characterization of weak dependence, and developed new laws of large numbers and invariance principles for time series that are weakly dependent in this sense. There are a number of ways in which the work presented here can be extended, five of which are: (1) weakening the moment condition in Theorem 4 using an argument similar to that of Doukhan and Wintenberger (2005); (2) allowing the function ϕ to depend on multiple arguments, so that $Y_t = \phi(X_{t-m}, \ldots, X_{t+m})$ for some $m > 0$; (3) generalizing Assumption 1 so that the Kolmogorov-Smirnov distance between copulas is replaced by an L_p distance, for some $p \in [1,\infty]$; (4) identifying specific parametric classes of copulas that satisfy

Assumption 1 and may provide a realistic model for data observed in financial markets and elsewhere; and (5) developing statistical techniques for modelling data using said parametric classes of copulas. These five topics are the subject of current research by the author. It is hoped that research in this area will ultimately enrich the field of time series analysis by providing researchers with new methods for modelling nonlinear temporal dependence.

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