Core localized alpha-channeling via low frequency Alfvén mode generation in reversed shear scenarios

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Abstract. A novel channel for fuel ions heating in tokamak core plasma is proposed and analyzed using nonlinear gyrokinetic theory. The channel is achieved via spontaneous decay of reversed shear Alfvén eigenmode (RSAE) into low frequency Alfvén modes (LFAM), which then heat fuel ions via collisionless ion Landau damping. The conditions for RSAE spontaneous decay are investigated, and the saturation level and the consequent fuel ion heating rate are also derived. The channel is expected to be crucial for future reactors operating under reversed shear configurations, where fusion alpha particles are generated in the tokamak core where the magnetic shear is typically reversed, and there is a dense RSAE spectrum due to the small alpha particle characteristic dimensionless orbits.

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1. Introduction

Energetic particles (EPs) including notably fusion alpha particles related physics [1,2] are key elements towards understanding the performance of future fusion reactors such as ITER [3] and CFETR [4]. Heating of thermal ions via Coulomb collisions is crucial for sustained fusion reactions, while EPs excitation of collective oscillations such as shear Alfvén wave (SAW) instabilities may lead to EP loss and affect the confinement of thermal plasmas [1,2]. The SAW induced EP anomalous transport is determined by the saturation level and spectrum of SAWs [5]. Thus, it is crucial to understand the dynamics of SAW instabilities that lead to their saturation [6]. Another important topic in EP research, is searching for alternative/complementary routes to transfer EP power to fuel ions, i.e., alpha-channeling [7,8], which is important in maintaining the self-sustained burning in future reactors, where collisional transfer of high energy fusion alpha is mostly due to electron drag due to the high alpha particle birth velocities.

Due to the magnetic geometry and plasma nonuniformities, the SAW continuous spectrum is characterized by various forbidden gaps, inside which different discrete SAW eigenmodes reside, e.g., toroidal Alfvén eigenmodes (TAE) due to toroidicity induced coupling of neighboring poloidal harmonics [9–11] and beta-induced Alfvén eigenmodes (BAEs) due to plasma compressibility [12, 13]. Among various Alfvén eigenmodes (AEs), TAEs have drawn the most attention in theoretical and numerical investigations, and are studied as paradigm case for the nonlinear dynamics of discrete AEs [2, 14–16], and the obtained general results can be applied to other SAW instabilities, based on the understanding of their respective linear properties. E.g., in future reactors operating in advanced reversed shear scenarios [3, 4, 17, 18]with minimized inductive current fraction, it is expected that reversed shear Alfvén eigenmodes (RSAEs, also known as Alfvén cascades) [19,20] are preferentially excited in tokamak center where fusion alpha particles are generated, while TAEs are excited in the relatively exterior region of the torus with finite magnetic shear [21, 22]. On the other hand, BAEs, or more generally, low frequency Alfvén modes (LFAMs) in the frequency range comparable or lower than BAEs [13, 23-26], can be excited by both EPs as well as thermal plasmas due to their relatively low frequencies, in different toroidal mode numbers regimes. The properties of LFAMs in reversed shear plasmas, including destabilization mechanism, mode polarization dependence on q_{min} , are systematically investigated in Ref. [27]. Here, q_{min} is the local minimum of the safety factor q.

The linear properties of SAW instabilities expected in fusion reactors are extensively investigated [1, 2, 28, 29], and it is generally accepted that most unstable modes are characterized by high-*n* mode numbers with $k_{\perp}\rho_h \sim O(1)$ [21, 30], due to the competition of drive from EP pressure gradient ($\propto n$) and the stabilization by finite orbit width effects (FoW) in the high-*n* limit ($\propto 1/n$, noting the asymptotic form of Bessel functions accounting for FoW effects in the short wavelength limit [31] §). Here, *n* is the toroidal mode number, k_{\perp} is the perpendicular wave number, and ρ_h is the characteristic EP orbit width. As a result, in future reactors such as ITER and CFETR [3, 4] with $a/\rho_h \gtrsim O(10)$, the most unstable modes are characterized by $n \gtrsim O(10)$ with many modes having comparable linear growth rates [21]. Thus,

[§] Note that, the FoW effects are formulated using Padé approximation in Ref. [31]. For a more explicit expression, one may refer to equation (3) of Ref. [32] for the EP response to TAE, assuming well circulating EPs.

nonlinear mode coupling is expected to be a channel for effectively saturating SAW instabilities and modifying the perturbation spectrum [14,33–37], among which, alphachanneling through TAE decaying into modes prone to ion Landau damping are proposed and analyzed in Refs. [14, 36, 38, 39], which is shown to effectively transfer fusion alpha particle power to fuel ions, in addition to nonlinearly saturate TAEs.

In this work, a new alpha-channeling mechanism, based on LFAM generation due to the nonlinear decay of RSAE, is proposed and analyzed. With the alpha particle characteristic orbit size much smaller than the tokamak minor radius [1], multiple RSAEs may be simultaneously excited [2, 21], with their radial localization determined by q_{min} , and are thus, radially overlapped. The RSAE frequency is determined by $\omega^2 \simeq (nq_{min} - m)^2 V_A^2/(q_{min}^2 R_0^2) + \Delta_{\omega}^2$, with Δ_{ω} being the deviation from local SAW continuum accumulation point due to q-curvature $(q'' \equiv \partial^2 q / \partial r^2)$ and non-perturbative EP drive [21]. Here, m is the poloidal mode number, V_A is the Alfvén speed, and R_0 is the major radius. Thus, for given q_{min} , RSAEs with different toroidal mode numbers may have different parallel wave numbers $k_{\parallel} \equiv (nq_{min} - m)/(q_{min}R_0)$ and consequently frequencies covering TAE and BAE frequency ranges, and two RSAEs may couple and generate secondary modes with $\omega_{\pm} \simeq [(n_1 \pm n_2)q_{min} - (m_1 \pm m_2)]V_A/(q_{min}R_0)$. Among the two secondary modes, the lower frequency one, if satisfies the dispersion relation of a normal mode, e.g., BAE, or mor generally LFAM, can be strongly driven unstable, and effectively heat thermal ions via LFAM Landau damping. This process, may provide a direct and efficient alpha-channeling mechanism that transfers fusion alpha particle power to fuel ions. This channel is of potential importance since RSAEs are expected to be firstly excited in the tokamak center by core localized alpha-particles [21], and thus, the core localized power deposition will heat core ions, leading to enhanced fusion performance.

Two independent processes can occur and lead to LFAM generation. In the first process, a linearly unstable RSAE spontaneously decays into another linearly stable RSAE and a low frequency sideband; while in the second process, two linearly unstable RSAEs couple and generate a low frequency mode. The two processes can occur, since there is a rich spectrum of (linearly stable or unstable) RSAEs and their kinetic counter-parts, i.e., kinetic RSAEs (KRSAE) [28,40,41] in reactors with $\rho_h \ll a$, so the frequency and wavenumber matching condition required for the resonant mode coupling process can be satisfied. In the present work, we will focus on the first process of parametric instability of RSAE and discuss the condition for spontaneous decay; while the second process, which does not require an amplitude threshold condition, can be formally analyzed from the obtained nonlinear LFAM equation.

The rest of the manuscript is organized as follows. In Sec. 2, the theoretical model of nonlinear gyrokinetic theory will be introduced. In Sec. 3, the linear particle responses to SAW instabilities in torus are reviewed, which are then used to derive the general nonlinear equation describing the nonlinear interaction of SAW instabilities in torus. The nonlinear dispersion relation for RSAE parametric decay instability is analyzed in Sec. 4. The consequences on RSAE saturation and core-localized ion heating is discussed in Sec. 5. And finally, a brief summary and discussion are presented in Sec. 6.

2. Theoretical model

The governing equations describing nonlinear interactions among RSAEs and LFAM with all predominantly SAW polarization can be derived from nonlinear gyrokinetic

vorticity equation [34, 42]

$$\frac{c^2}{4\pi\omega_k^2} B \frac{\partial}{\partial l} \frac{k_\perp^2}{B} \frac{\partial}{\partial l} \delta\psi_k + \frac{e^2}{T_i} \left\langle (1 - J_k^2) F_0 \right\rangle \delta\phi_k - \sum_{s=e,i} \left\langle \frac{q}{\omega_k} J_k \omega_d \delta H_k \right\rangle_s$$

$$= -i \frac{1}{\omega_k} \sum_{\mathbf{k}=\mathbf{k'}+\mathbf{k''}} \Lambda_{k'',k'}^k \left[\frac{c^2}{4\pi} k_\perp''^2 \frac{\partial_l \delta\psi_{k'} \partial_l \delta\psi_{k''}}{\omega_{k'} \omega_{k''}} + \left\langle e(J_k J_{k'} - J_{k''}) \delta L_{k'} \delta H_{k''} \right\rangle \right], \qquad (1)$$

and quasi-neutrality condition

$$\frac{n_0 e^2}{T_i} \left(1 + \frac{T_i}{T_e} \right) \delta \phi_k = \sum_{s=e,i} \left\langle q J_k \delta H_k \right\rangle_s, \tag{2}$$

with the non-adiabatic particle response δH_k , related to the perturbed particle distribution function via $\delta f_k = -(q/T)F_0\delta\phi_k + \exp(-\rho\cdot\nabla)\delta H_k$, derived from nonlinear gyrokinetic equation [43]:

$$(-i\omega + v_{\parallel}\partial_l + i\omega_d)\,\delta H_k = -i\omega_k \frac{q}{T} F_0 J_k \delta L_k - \sum_{\mathbf{k}=\mathbf{k}'+\mathbf{k}''} \Lambda^k_{k'',k'} J_{k'} \delta L_{k'} \delta H_{k''}.$$
(3)

Here, the terms on the left hand side of equation (1) are, respectively, the field line bending, inertia and curvature coupling terms; while the terms on the right hand side are the formally nonlinear terms, corresponding to Maxwell and gyrokinetic Reynold stresses dominated by nonlinear electron and ion responses, respectively. ∂_l is the derivative along the equilibrium magnetic field, $J_k \equiv J_0(k_\perp \rho)$ with J_0 being the Bessel function of zero index accounting for finite-Larmor-radius effects (FLR), $\rho \equiv v_\perp / \Omega_c$, $\Omega_c = B_0 q/(mc)$ is the cyclotron frequency, F_0 is the equilibrium particle distribution function, and is taken as local Maxwellian for bulk electrons/ions, $\omega_d = (v_\perp^2 + 2v_\parallel^2)/(2\Omega_c R_0)(k_r \sin \theta + k_\theta \cos \theta)$ is the magnetic drift frequency for a circular cross section large aspect ratio tokamak assumed in this work for simplicity of derivation. Furthermore, $\Lambda_{k'',k'}^k \equiv (c/B_0)\hat{\mathbf{b}} \cdot \mathbf{k''} \times \mathbf{k'}$ with $\hat{\mathbf{b}}$ being the unit vector along the equilibrium magnetic field \mathbf{B}_0 , $\sum_{\mathbf{k}=\mathbf{k''}+\mathbf{k'}}$ denotes the selection rules of frequency and wavenumber matching conditions for nonlinear mode coupling, $\delta L \equiv \delta \phi - k_{\parallel} v_{\parallel} \delta \psi / \omega$ with $\delta \psi \equiv \omega \delta A_{\parallel} / (ck_{\parallel})$ and δA_{\parallel} being the parallel component of the vector potential, and ideal MHD condition is determined by $\delta \psi = \delta \phi$.

In this work, we assume the nonlinear coupling is dominated by thermal plasma contribution, while EPs, driving the pump RSAE unstable, contribute negligibly to the nonlinear coupling. Consequently, the nonuniformity of thermal plasma in the tokamak core can be neglected [44], corresponding to thermal ion diamagnetic drift frequency being smaller than BAE frequency here, in consistency with the considered high-performance scenarios [18]. Thus, the present theory, which could be generalized to included kinetic ballooning modes (KBMs) [13] and/or Alfvénic ion temperature gradient modes (AITGs) [23, 45] by inclusion of thermal plasma nonuniformities, is derived for application to BAE in its present form. We consider a spontaneous decay process, in which a pump RSAE decays into another linearly stable RSAE and a LFAM, and the condition for this process to occur is $\beta_i \ll 1$ such that the frequency separation between RSAE and LFAM can easily be satisfied. The nonlinear decay process, can be analyzed by deriving the nonlinear equations of the two sidebands, which can be coupled to yield the nonlinear parametric dispersion relation. For proper

evaluation of ion heating due to the LFAM Landau damping, the LFAM resonance with thermal ions is crucial [13], and can be formally accounted for in the anti-Hermitian part of the LFAM dispersion relation.

3. General nonlinear equation for resonant SAW three wave coupling

The linear particle response to SAW instabilities can be derived by noting the $k_{\parallel}v_e \gg \omega \gg k_{\parallel}v_i \gtrsim \omega_d$ ordering, and one has, at the leading order, $\delta H_{e,k} \simeq -eF_0\delta\psi_k/T_e$ and $\delta H_{i,k}^{(0)} = eF_0J_k\delta\phi_k^{(0)}/T_i$, which can be substituted into quasi-neutrality condition, and yield $\delta\phi_k^{(0)} = \delta\psi_k^{(0)}$, i.e., ideal MHD condition is maintained at the leading order.

To the next order, one derives

$$\delta H_{i,k}^{(1)} = \frac{e}{T_i} F_0 J_k \left[\delta \phi_k^{(1)} + \frac{\omega_d}{\omega_k} \delta \phi_k^{(0)} \right],$$

which can be substituted into quasi-neutrality condition and one obtains

$$\delta\phi_k^{(1)} - \delta\psi_k^{(1)} = \frac{T_e}{T_i} \left\langle \frac{\omega_d}{\omega_k} \frac{F_0}{n_0} \right\rangle \delta\phi_k^{(0)},\tag{4}$$

i.e., breaking of ideal MHD constraint due to plasma compressibility. Finite parallel electric field generation due to FLR effects, i.e., kinetic Alfvén wave (KAW) related physics, will not be considered here for simplicity. However, as we show later, the decay process favors higher-n modes, for which inclusion of FLR may be needed and can be accounted for straightforwardly [46].

Substituting non-adiabatic particle responses into vorticity equation, and noting $\delta \phi_k^{(0)} = \delta \psi_k^{(0)}$, one derives the SAW mode equation in torus, i.e.,

$$\frac{n_0 e^2}{T_i} b_k \mathcal{E}_k \delta \phi_k^{(0)} = 0, \tag{5}$$

with $\mathcal{E}_k \equiv -k_{\parallel}^2 V_A^2 / \omega_k^2 + 1 - \omega_G^2 / \omega_k^2$ being the SAW dielectric function in the WKB limit, and $\omega_G \equiv \sqrt{7/4 + \tau} v_i / R_0$ being the leading order geodesic acoustic mode frequency [47, 48], accounting for SAW continuum upshift and creation of betainduced continuum gap. We note that, in the expression of \mathcal{E}_k , effects of wave-particle interactions are not included, in consistency with the $k_{\parallel} v_i \ll \omega_k$ ordering for bulk nonresonant ions. However, finite Landau damping due to resonance with ions is crucial for alpha-channeling, and will be recovered formally in the later analysis by inclusion of the anti-Hermitian part of \mathcal{E}_k [13]. Equation (5) is general, and can be applied to different modes in different scenarios. E.g., RSAE global dispersion relation can be derived by expanding k_{\parallel}^2 at q_{min} , and solving the eigenmode equation in Fourier- k_r space [20], while BAE physics is dominated by $k_{\parallel}^2 q^2 R_0^2 \lesssim \beta_i$ [13].

Note that, for the present case of a pump RSAE decaying into another RSAE and a LFAM, all three modes involved are SAWs that satisfies $\omega^2 \simeq k_{\parallel}^2 V_A^2 \parallel$. Thus, one can derive the general nonlinear equation for SAW nonlinear coupling, which can be applied to the case of RSAE nonlinear decay of interest. Considering two SAWs, $\Omega_1 \equiv \Omega_1(\omega_1, \mathbf{k}_1)$ and $\Omega_2 \equiv \Omega_2(\omega_2, \mathbf{k}_2)$ coupling and generating a third mode, $\Omega_3 \equiv \Omega_3(\omega_3, \mathbf{k}_3)$, the nonlinear equations can be derived from nonlinear vorticity

^{||} Note that, LFAM dispersion relation can be quite different due to thermal plasma compression, but the general picture is the same, especially for BAE of interest here with predominantly SAW polarization.

equation and quasi-neutrality condition. For simplicity of derivation, parallel force balance equation is used instead of quasi-neutrality condition.

The first equation of Ω_3 mode can be derived from nonlinear gyrokinetic vorticity equation,

$$\frac{n_0 e^2}{T_i} b_{k_3} \left(-\frac{k_{\parallel,3}^2 V_A^2}{\omega_3^2} \delta \psi_{k_3} + \delta \phi_{k_3} - \frac{\omega_G^2}{\omega_3^2} \delta \phi_{k_3} \right) \\
\simeq -\frac{i}{\omega_3} \Lambda_{k_2,k_1}^{k_3} \left[\frac{c^2}{4\pi} (k_{\perp,1}^2 - k_{\perp,2}^2) \frac{k_{\parallel,1} k_{\parallel,2}}{\omega_1 \omega_2} \delta \psi_{k_1} \delta \psi_{k_2} + \langle e(J_{k_1} - J_{k_2}) (\delta L_{k_1} \delta H_{i,k_2} + \delta L_{k_2} \delta H_{i,k_1}) \rangle \right].$$
(6)

In deriving equation (6), we have noted, in the nonlinear Reynolds stress [35, 49], $\delta\phi_k \simeq \delta\psi_k, \, \delta H_{i,k} \simeq eF_0\delta\phi_k/T_i$, and neglected $O(k_\perp^2\rho_i^2)$ order corrections. Substituting the lowest order ion response into the Reynolds stress term, the nonlinear vorticity equation of Ω_3 then becomes

$$b_{k_3} \left(-\frac{k_{\parallel,3}^2 V_A^2}{\omega_3^2} \delta \psi_{k_3} + \delta \phi_{k_3} - \frac{\omega_G^2}{\omega_3^2} \delta \phi_{k_3} \right) \\ \simeq -\frac{i}{\omega_3} \Lambda_{k_2,k_1}^{k_3} (b_{k_2} - b_{k_1}) \left(1 - \frac{k_{\parallel,1} k_{\parallel,2} V_A^2}{\omega_1 \omega_2} \right) \delta \phi_{k_1} \delta \phi_{k_2}.$$
(7)

The other equation can be derived from parallel electron force balance equation, and one has,

$$\delta\phi_{k_3} - \delta\psi_{k_3} = -i\Lambda_{k_2,k_1}^{k_3} \frac{1}{k_{\parallel,3}} \left(\frac{k_{\parallel,1}}{\omega_1} - \frac{k_{\parallel,2}}{\omega_2}\right) \delta\phi_{k_1}\delta\phi_{k_2},\tag{8}$$

i.e., nonlinear extension of ideal MHD constraint, in addition to plasma compressibility as shown in equation (4).

Substituting equation (8) into (7), one obtains

$$b_{k_3} \mathcal{E}_{k_3} \delta \phi_{k_3} = -\frac{i}{\omega_3} \Lambda_{k_2,k_1}^{k_3} \left[(b_{k_2} - b_{k_1}) \left(1 - \frac{k_{\parallel,1} k_{\parallel,2} V_A^2}{\omega_1 \omega_2} \right) + b_{k_3} V_A^2 \frac{k_{\parallel,3}}{\omega_3} \left(\frac{k_{\parallel,1}}{\omega_1} - \frac{k_{\parallel,2}}{\omega_2} \right) \right] \delta \phi_{k_1} \delta \phi_{k_2}.$$
(9)

Equation (9) describes the nonlinear evolution of SAWs, as Ω_3 being modified by the beating of Ω_1 and Ω_2 , with the first term on the right hand side from the competition of Reynolds and Maxwell stresses and the second term from finite parallel electric field contribution to field line bending term. Note that, $(\omega_1 + \omega_2) \simeq$ $(k_{\parallel,1} + k_{\parallel,2})V_A$, Ω_3 naturally satisfies the SAW D.R., and can be strongly excited if it is a normal mode of the system, leading to significant spectrum evolution of SAW turbulence. Note that, though the primary motivation of the present work is investigating the LFAM generation by RSAEs, equation (9) can be applied to study the nonlinear SAW couplings in the high frequency range, e.g., the nonlinear coupling among TAE, ellipticity induced AE (EAE) and non-circular AE (NAE), whose frequency matching condition can be naturally satisfied. It can also be generalized to kinetic Alfvén waves (KAW) [50] by properly accounting for parallel electric fields due to kinetic effects [46], which is expected to be crucial due to the intrinsic nonuniformity in magnetically confined plasmas. Equation (9) can be more generally applied for KAW spectral cascading in space plasmas, e.g., solar wind, by neglecting the ω_G^2 term that is unique in torus, and k_{\parallel} can be more flexibly taken without the periodicity constraint in a torus.

4. Parametric decay of RSAE

Equation (9) will be applied to the nonlinear decay of a pump RSAE $\Omega_0 \equiv \Omega_0(\omega_0, \mathbf{k}_0)$ into a RSAE sideband $\Omega_1 \equiv \Omega_1(\omega_1, \mathbf{k}_1)$ and a LFAM $\Omega_B \equiv \Omega_B(\omega_B, \mathbf{k}_B)$, with the frequency/wavenumber matching condition $\Omega_0 = \Omega_1 + \Omega_B$ assumed without loss of generality. For RSAE and LFAM being dominated by single-*n* and single-*m* mode structures, we take

$$\delta\phi_k = A_k(t)\Phi_k(x)\exp\left(-i\omega_k t + in\xi - im\theta\right),\tag{10}$$

with $A_k(t)$ being the slowly varying mode amplitude, $\Phi_k(x)$ the parallel mode structure localized about q_{min} with $x \equiv nq - m$, and the normalization condition $\int |\Phi_k|^2 dx = 1$ is satisfied. For the effective transfer of alpha particle energy to core ions, $\omega_B \leq O(v_i/(qR_0))$, and thus, $|\omega_B| \ll |\omega_0|, |\omega_1|$ and $k_{\parallel,B} \simeq 0$. Thus, the q_{min} surface where secondary Ω_B locates, also corresponds to the rational surface of ω_B , i.e., Ω_B is the LFAM in the reversed shear configuration, as investigated experimentally [51] and theoretically [27]. We then have, $\omega_0 \simeq \omega_1$ and $k_{\parallel,0} \simeq k_{\parallel,1}$. Effects of small frequency mismatch on the decay process will be discussed later.

The nonlinear RSAE sideband equation can be derived from equation (9) as

$$b_1 \mathcal{E}_1 \delta \phi_1 = -\frac{i}{\omega_1} \Lambda_{k_0, k_{B^*}}^{k_1} \alpha_1 \delta \phi_0 \delta \phi_{B^*}, \qquad (11)$$

with $\alpha_1 \equiv (b_0 - b_B)(1 - k_{\parallel,B}k_{\parallel,0}V_A^2/(\omega_0\omega_B)) + b_1V_A^2(k_{\parallel,1}/\omega_1)(k_{\parallel,B}/\omega_B - k_{\parallel,0}/\omega_0)$. The nonlinear Ω_1 eigenmode equation can be derived from equation (11), by multiplying both sides by Φ_0 , and averaging over radial mode structure, and one obtains

$$\hat{b}_1 \hat{\mathcal{E}}_1 A_1 = -\frac{i}{\omega_1} \left\langle \Lambda_{k_0, k_{B^*}}^{k_1} \alpha_1 \Phi_1 \Phi_0 \Phi_B \right\rangle_x A_0 A_{B^*},$$
(12)

with $\langle \cdots \rangle_x \equiv \int \cdots dx$ denoting averaging over the fast radial scale, and $\hat{b}_1 \hat{\mathcal{E}}_1 \equiv \int \Phi_1 b_1 \mathcal{E}_1 \Phi_1 dx$ being the Ω_1 eigenmode dispersion relation.

The nonlinear LFAM equation, on the other hand, can be derived as

$$b_B \mathcal{E}_B \delta \phi_B = -\frac{i}{\omega_B} \Lambda_{k_0, k_{1^*}}^{k_B} \alpha_B \delta \phi_0 \delta \phi_{1^*}, \qquad (13)$$

with $\alpha_B \equiv (b_0 - b_1)(1 - k_{\parallel,1}k_{\parallel,0}V_A^2/(\omega_0\omega_1)) + b_B V_A^2(k_{\parallel,B}/\omega_B)(k_{\parallel,1}/\omega_1 - k_{\parallel,0}/\omega_0)$. The LFAM eigenmode equation can be derived similarly, and one obtains

$$\hat{b}_B \hat{\mathcal{E}}_B A_B = -\frac{i}{\omega_B} \left\langle \Lambda_{k_0, k_{1*}}^{k_B} \alpha_B \Phi_B \Phi_0 \Phi_1 \right\rangle_x A_0 A_{1*}, \tag{14}$$

with $\hat{b}_B \hat{\mathcal{E}}_B$ being the LFAM eigenmode dispersion relation. Equations (12) and (14) are readily reduced to the nonlinear eigenmode equations of RSAE sideband and LFAM in the WKB limit, and can be simplified noting the respective parameter regimes.

The parametric decay dispersion relation for RSAE nonlinear decaying into another RSAE and LFAM, can then be derived, by combining equations (12) and (14)

$$\hat{\mathcal{E}}_1 \hat{\mathcal{E}}_{B^*} \simeq \left(\hat{\Lambda}_{k_0, k_{B^*}}^{k_1}\right)^2 \frac{\hat{\alpha}_N}{\hat{b}_B \hat{b}_1 \omega_B \omega_1} \hat{C}^2 |A_0|^2, \tag{15}$$

with $\hat{C} \equiv \langle \Phi_0 \Phi_B \Phi_1 \rangle_x$, $\hat{\alpha}_N \equiv \hat{\alpha}_1 \hat{\alpha}_B$ with $\hat{\alpha}_k = \langle \alpha_k \rangle_x$, $\hat{\Lambda}_{k_0,k_{B^*}}^{k_1} = \langle \Lambda_{k_0,k_{B^*}}^{k_1} \rangle_x$, and $\Lambda_{k_0,k_{B^*}}^{k_1} = \Lambda_{k_0,k_{1^*}}^{k_B}$ noted. In deriving equation (15), we have noted that $\hat{\alpha}_N$ and $\left(\hat{\Lambda}_{k_0,k_{B^*}}^{k_1}\right)^2$ are operators acting on the respective mode structures, and they are

moved out of the spatial averaging in that they are both predominantly even operators with respect to q_{min} surface. The integration $\langle \Phi_0 \Phi_B \Phi_1 \rangle_x$ can be evaluated, noting that Φ_B typically has a much narrower structure than those of Φ_0 and Φ_1 . Taking $\Phi_k \simeq \exp(-x^2/2\Delta_k^2)/(\pi^{1/4}\Delta_k^{1/2})$ with Δ_k being the characteristic radial width of the parallel mode structure, we have, $\hat{C} \simeq \sqrt{2\Delta_B/(\sqrt{\pi}\Delta_0\Delta_1)}$, with $\Delta_0 \sim \Delta_1 \sim O(1)$ and $\Delta_B \sim O(\beta^{1/2})$.

Expanding $\hat{\mathcal{E}}_1 \simeq i\partial_{\omega_1}\hat{\mathcal{E}}_1(\partial_t + \gamma_1) \simeq (2i/\omega_1)(\gamma + \gamma_1)$ and $\hat{\mathcal{E}}_{B^*} \simeq (-2i/\omega_B)(\gamma + \gamma_B)$, with $\partial_{\omega_1}\hat{\mathcal{E}}_1 \equiv \partial\hat{\mathcal{E}}_1/\partial\omega_1$, γ denoting the slow temporal variation of Ω_1 and Ω_B due to the parametric instability, and γ_1/γ_B being the linear damping rates of RSAE/LFAM imbedded in the anti-Hermitian part of $\mathcal{E}_1/\mathcal{E}_B$, one obtains

$$(\gamma + \gamma_1)(\gamma + \gamma_B) = \left(\hat{\Lambda}_{k_0, k_{B^*}}^{k_1}\right)^2 \frac{\hat{\alpha}_N}{4\hat{b}_B\hat{b}_1}\hat{C}^2 |A_0|^2.$$
(16)

The condition for the pump RSAE spontaneous decay can thus be obtained from equation (16) as

$$\hat{\alpha}_N > 0,\tag{17}$$

and

$$\frac{(\hat{\Lambda}_{k_0,k_{B^*}}^{k_1})^2}{4\hat{b}_B\hat{b}_1}\hat{\alpha}_N\hat{C}^2|A_0|^2 > \gamma_B\gamma_1 \tag{18}$$

for the nonlinear drive overcoming the threshold due to Ω_1 and Ω_B Landau damping.

The nonlinear dispersion relation is very complex, and depends on various conditions including the polarization and mode structure of the three modes involved. For the analytical progress, the WKB limit and the strong assumption of $k_{\parallel,B} \to 0$ is adopted from now on, and we have $k_{\parallel,0} \simeq k_{\parallel,1}$ and thus, $\hat{\alpha}_N$ can be simplified as

$$\hat{\alpha}_N \simeq (b_0 - b_1) \left(1 - \frac{k_{\parallel,1} k_{\parallel,0} V_A^2}{\omega_0 \omega_1} \right) (b_0 - b_B - b_1).$$
(19)

The sign of $\hat{\alpha}_N$ is determined by several factors. First, the sign of $1 - k_{\parallel,1}k_{\parallel,0}V_A^2/(\omega_0\omega_1)$ is determined by q_{min} and n_0/n_1 that determine the respective SAW continuum structure, i.e., whether the RSAEs are localized below the local minimum of SAW continuum or above the local maximum; $b_0 - b_1$ is determined by the respective toroidal mode numbers n_0/n_1 , noting $k_\theta \propto nq/r$ and $k_r \propto \sqrt{q''n^2/q_{min}}$. Noting that, $\mathbf{k}_{\perp,0} = \mathbf{k}_{\perp,B} + \mathbf{k}_{\perp,1}$, we have, $b_0 - b_B - b_1 = (\mathbf{k}_{\perp,0} \cdot \mathbf{k}_{\perp,1} - k_{\perp,1}^2)\rho_i^2$, and its sign is positive for $\cos \eta > |k_{\perp,1}/k_{\perp,0}|$, with η being the angle between $\mathbf{k}_{\perp,0}$ and $\mathbf{k}_{\perp,1}$. The above three conditions are quite complicated, but two parameter regimes can be identified for the spontaneous decay process to occur. The first parameter regime corresponds to $k_{\perp,1} \gg k_{\perp,0}$, such that $(b_0 - b_1)(b_0 - b_B - b_1) > 0$; and $\hat{\alpha}_N > 0$ can be satisfied with $1 - k_{\parallel,0}k_{\parallel,1}V_A^2/(\omega_0\omega_1) > 0$, which generally requires $\mathbf{\Omega}_1$ being excited above the local SAW continuum accumulation point with $n_1q_{min} < m_1$

Another parameter regime can be found with $1 - k_{\parallel,0}k_{\parallel,1}V_A^2/(\omega_0\omega_1) < 0$, i.e., to have Ω_1 being excited below the local minimum of SAW continuum. In this case, $\hat{\alpha}_N > 0$ can be satisfied with $(b_0 - b_1)(b_0 - b_B - b_1) < 0$, which requires $b_1 < b_0$ while $\mathbf{k}_{\perp,0} \cdot \mathbf{k}_{\perp,1} < k_{\perp,1}^2$.

We note that, the $\Lambda_{k_0,k_{B^*}}^{k_1}$ in the nonlinear coupling cross-section is maximized for $\mathbf{k}_{\perp,1}$ being perpendicular to $\mathbf{k}_{\perp,0}$, and, also, that the linearly unstable pump RSAE typically have relatively broad mode structures with $k_{\perp}\rho_h \sim O(1)$ [21]. For finite

[¶] Note $n_1q_{min} < m_1$ is a sufficient condition for $1 - k_{\parallel,0}k_{\parallel,1}V_A^2/(\omega_0\omega_1) > 0$, but not necessary.

LFAM generation with the mode structures typically much narrower than that of the pump RSAE by $O(\omega_B/\omega_0) \sim O(\sqrt{\beta_i})$, the nonlinear coupling, thus, may occur in the radially fast varying inertial layer of Ω_1 . As a result, the toroidal mode number of Ω_1 is expected to be much higher than that of the pump RSAE, which makes the first parameter region with $b_1 \gg b_0$ optimized for the nonlinear process. On the other hand, the coefficient on the left hand side of equation (18) is proportional to b_1 considering $b_1 \simeq b_B \gg b_0$, which further confirms the first parameter regime with $b_1 \gg b_0$ is favored, i.e., normal cascading to high- n_1 regime. The upper bound of n_1 can be determined by $\gamma_1(n_1)$, i.e., the decay RSAE Ω_1 damping rate increases with n_1 , as is shown in equation (18). The condition for wavenumber/frequency matching condition to be satisfied, especially the requirement on $|\omega_B|$ being comparable with ion transit frequency, can be satisfied due to the potential dense RSAE/kRSAE spectrum in the high-n limit.

The threshold condition for the RSAE spontaneous decay, for the first case of "normal cascading", can be estimated from equation (18), and one obtains

$$\left|\frac{\delta B_{\perp,0}}{B_0}\right|^2 > \frac{4\gamma_1 \gamma_B}{\omega_0 \omega_1} \frac{k_{\parallel,0}^2}{k_{\perp,1}^2} \frac{1}{\hat{C}^2} \frac{1}{1 - k_{\parallel,0} k_{\parallel,1} V_A^2 / (\omega_0 \omega_1)} \sim \mathcal{O}(10^{-7}), \tag{20}$$

and is comparable with or slightly higher than typical threshold condition for other dominant nonlinear mode coupling processes, e.g., ZS generation [35, 52]. This threshold amplitude, is also consistent with typical SAW instability intensity observed in experiments [53]. Thus, this channel can be an important process in determining the nonlinear dynamics of RSAE, and the consequent transport by the short wavelength RSAE sideband and nonlinear thermal ion heating via the nonlinearly generated LFAM. In deriving the threshold, typical parameter are used, i.e., $\gamma/\omega \sim 10^{-2}$, $k_{\perp,1} \sim 1/\rho_i$, $k_{\parallel} \sim 1/R_0$, $k_{\parallel}/k_{\perp,1} \sim \rho_i/R_0 \sim 10^{-3}$, $\Delta_B/\Delta_0 \sim O(\omega_B/\omega_0) \sim 10^{-1}$, and $|1 - k_{\parallel,1}k_{\parallel,0}V_A^2/(\omega_0\omega_1)| \sim |\gamma/\omega_0| \sim \mathcal{O}(10^{-2})$ is assumed.

5. Nonlinear saturation and core-localized ion heating

The RSAE saturation level can be estimated by considering the feedback of the two sidebands to the pump RSAE, which can be derived from equation (9) as

$$\hat{b}_0 \hat{\mathcal{E}}_0 A_0 \simeq -\frac{i}{\omega_0} \hat{\Lambda}^{k_0}_{k_1, k_B} \hat{\alpha}_0 \hat{C} A_1 A_B, \qquad (21)$$

with $\alpha_0 = (b_1 - b_B)(1 - k_{\parallel,B}k_{\parallel,1}V_A^2/(\omega_1\omega_B)) + b_0V_A^2(k_{\parallel,0}/\omega_0)(k_{\parallel,B}/\omega_B - k_{\parallel,1}/\omega_1).$

Expanding equations (12), (14) and (21) along their characteristics, one obtains

$$\left(\partial_t + \gamma_1\right) A_1 = -\frac{\alpha_1}{\hat{b}_1 \omega_1 \partial_{\omega_1} \mathcal{E}_{1,\mathcal{R}}} \hat{\Lambda}^{k_1}_{k_0,k_{B^*}} \hat{C} A_0 A_{B^*}, \qquad (22)$$

$$\left(\partial_t + \gamma_B\right) A_{B^*} = \frac{\hat{\alpha}_B}{\hat{b}_B \omega_B \partial_{\omega_{B^*}} \mathcal{E}_{B^*, \mathcal{R}}} \hat{\Lambda}^{k_B}_{k_0, k_{1^*}} \hat{C} A_{0^*} A_1, \tag{23}$$

$$\left(\partial_t - \gamma_0\right) A_0 = -\frac{\hat{\alpha}_0}{\hat{b}_0 \omega_0 \partial_{\omega_0} \mathcal{E}_{0,\mathcal{R}}} \hat{\Lambda}^{k_0}_{k_1,k_B} \hat{C} A_1 A_B, \qquad (24)$$

and the saturation level of pump RSAE and LFAM can be evaluated from the coupled three wave equations. We note that, the coupled nonlinear equations with drive and dissipation can exhibit different dynamics from limited cycle oscillation

to period doubling and finally route to chaos, depending on the driving/dissipation as well as the initial conditions [54]. Here, an order of magnitude estimation of the saturation level can be derived from the fixed point solution, among which the LFAM saturation level can be derived from equations (22) and (24). One obtains, $|A_B|^2 = \gamma_0 \gamma_1 \hat{b}_0 \hat{b}_1 \omega_0 \omega_1 \partial_{\omega_1} \mathcal{E}_{1,\mathcal{R}} \partial_{\omega_0} \mathcal{E}_{0,\mathcal{R}} / (\hat{\alpha}_0 \hat{\alpha}_1 |\hat{C}|^2 (\hat{\Lambda}_{k_1,k_B}^{k_0})^2)$, and the ion heating rate due to LFAM Landau damping, can be estimated as

$$P_i = 2\gamma_B \omega_B \frac{\partial \mathscr{E}_{B,\mathcal{R}}}{\partial \omega_B} |A_B|^2.$$
⁽²⁵⁾

The obtained core ion heating due to LFAM conllisionless damping, is expected to be complementary to Coulomb collisional heating, which is less effective in the tokamak center. This channel, achieved via the Landau damping of secondary LFAM, noting that $k_{\parallel,B} \ll 1$, is highly localized around the q_{min} surface (this conclusion can also be obtained, noting as the "secondary" LFAM structure will be determined by the primary RSAE, with a narrower extent than the primary RSAEs), will deposit fusion alpha particle power locally and heating core ions, leading to direct improvement of fusion performance in the tokamak center.

6. Conclusion and Discussion

In conclusion, a novel channel for RSAE nonlinear saturation is proposed and analyzed, which is expected to be important in regulating SAW instability induced alpha particle transport and heating of fuel ions in future reactors burning plasmas. The saturation is achieved through the spontaneous decay of the unstable pump RSAE into linearly stable RSAE decay wave and LFAM, while the fuel ion heating is achieved through the ion Landau damping of the secondary LFAM. The conditions for the RSAE spontaneous decay is analyzed. It is found that decay into RSAE sideband with higher toroidal mode number is preferred, and the threshold condition on pump RSAE amplitude is derived, which is compatible with typical experimentally observed SAW instability amplitudes. The saturation levels of RSAE and LFAM are estimated from the fixed point solution of the coupled nonlinear equations, from which the fuel ion heating rate is also derived. This channel is expected to be relevant and crucially important for reactors since RSAEs are expected to be firstly excited by core localized fusion alpha particles in the high performance advanced reversed shear scenarios, and the resulting core localized fuel ion heating will directly contribute to the performance of the reactor. An implication from the present analysis, is the potential importance of rational q_{min} that may lead to low-order rational surfaces, which can yield broader mode structure and thus stronger couplings.

The present analysis focused on the picture of RSAE nonlinear decay, while neglected the effects of thermal plasma nonuniformity. The derivation, also assumed SAW polarization of all the three modes involved. Thus, the present analysis, while can be directly applied to RSAE decay into BAE in the present form, cannot be directly applied to RSAE decay into KBM or AITG, where plasma nonuniformity is crucial for the mode presence, and the mode may have a finite parallel electric field [27]. The generalization of the present analysis, to include system nonuniformity, can be tedious but straightforward, and will be carried out in a separate work.

As a final remark, several channels may contribute to the RSAE nonlinear saturation, e.g., self-consistent re-distribution of EPs [55], and zonal field generation generation [52] with notably the effects of zonal current on modifying the local SAW

continuum structures. For the proper evaluation of EP confinement, the nonlinear dynamics of RSAE including saturation level is required, which is determined by the relative importance of these channels; and thus, more in-depth investigation including nonlinear gyrokinetic simulation [56] is required. In fact, the effects investigated in the present work, e.g., the ion polarization nonlinearity unique to gyrokinetic ion Reynold stress, can only be captured by simulations with gyrokinetic thermal ions.

Acknowlodgement

This work is supported by the National Key Research and Development Program of China under Grant No. 2017YFE0301900, National Science Foundation of China under Grant Nos. 12175053 and 11875233, and "Users of Excellence program of Hefei Science Center CAS under Contract No. 2021HSC-UE016". This work has been carried out within the framework of the EUROfusion Consortium, funded by the European Union via the Euratom Research and Training Programme (Grant Agreement No 101052200 - EUROfusion). Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Commission. Neither the European Union nor the European Commission can be held responsible for them.

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