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Core of projective dimension one modules

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Abstract. The core of a projective dimension one R-module E is computed explicitly in terms of Fitting ideals. In particular, our formula recovers previous work by R. Mohan on integrally closed torsionfree modules over a two dimensional regular local ring.

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1. Introduction

There is an extensive literature on the *Rees algebra* $\mathcal{R}(I)$ of an R-ideal I in a Noetherian ring R. One reason for that is the major role that $\mathcal{R}(I)$ plays in Commutative Algebra and Algebraic Geometry, since $\text{Proj}(\mathcal{R}(I))$ is the blowup of $\text{Spec}(R)$ along the subscheme defined by I.

There are several reasons for studying Rees algebras of finitely generated R-modules E as well. For instance, Rees algebras of modules include the so called multi-Rees algebras, which correspond to the case of direct sums of ideals. Furthermore, symmetric algebras of modules arise naturally as coordinate rings of certain correspondences in Algebraic Geometry, and the projection of these varieties often requires the killing of torsion. This takes us back to the study of Rees algebras of modules: We refer to the articles by D. Eisenbud, C. Huneke and B. Ulrich [7] and by A. Simis, B. Ulrich and W.V. Vasconcelos [14] for additional motivation as well as a rich list of references. We stress, though, that this is not a routine generalization of what happens for ideals. This occurs mainly because

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the remarkable interaction that exists between the Rees algebra and the associated graded ring of an ideal is missing in the module case.

The importance of *reductions* in the study of Rees algebras of ideals has long been noticed by commutative algebraists. Roughly, a reduction is a simplification of the given ideal which carries the most relevant information about the original ideal itself. In this sense, the study of the *core* of an ideal – by which we mean the intersection of all the reductions of the ideal – helps finding uniform properties shared by all reductions. We refer to [13,9,4,5] for more details as well as for the few known explicit formulas on the core of ideals. Another strong motivation for studying the core of an ideal is given by the celebrated Briançon-Skoda theorem.

Similarly to what happens for ideals, it is useful to study Rees algebras of modules via the ones of reduction modules. Likewise the ideal case, one then defines $\mathrm{core}(E)$ to be the intersection of all (minimal) reductions U of E. Our goal is to give an explicit formula for the core in terms of a Fitting ideal of E, when E is a finitely generated R-module with rank e, projective dimension one, analytic spread $\ell = \ell(E) \geq e+1$, and property $G_{\ell-e+1}$. Interestingly enough, this is the same class of R-modules that has been studied in [3] in the context of cancellation theorems. One of the equivalences of Theorem 2.4 below says that

$$\mathrm{core}(E) = \mathrm{Fitt}_{\ell}(E) \cdot E$$

if and only if the reduction number of E is at most $\ell - e$. This result in particular covers earlier work by Mohan [12].

To prove Theorem 2.4 we reduce the rank of E by factoring out general elements. The task then becomes to assure that this inductive procedure preserves our assumptions and conclusions. This requires a sequence of technical lemmas. The corresponding results for 'generic' instead of 'general' elements have been shown in [14] – they are useless however in our context since they require purely transcendental residue extensions which may a priori change the core.

2. The main result

Let R be a Noetherian ring and let E be a finitely generated R-module with rank $e > 0$. The *Rees algebra* $\mathcal{R}(E)$ of E is the symmetric algebra of E modulo its R-torsion. Let $U \subset E$ be a submodule. One says that U is a *reduction* of E or, equivalently, E is *integral* over U if $\mathcal{R}(E)$ is integral over the R-subalgebra generated by U. Alternatively, the integrality condition is expressed by the equations $\mathcal{R}(E)_{r+1} = U\mathcal{R}(E)_r$ with $r \gg 0$. The least integer $r \geq 0$ for which this equality holds is called the *reduction number of E with respect to U* and denoted by $r_U(E)$. For any reduction U of E the module E/U is torsion, hence U has a rank and $\mathrm{rank}\,U = \mathrm{rank}\,E$. If R is moreover local with residue field k then the Krull dimension of the *special fiber ring* $\mathcal{R}(E) \otimes_R k$ is called the *analytic spread* of E and is denoted by $\ell(E)$. A reduction of E is said to be *minimal* if it is minimal with respect to inclusion. If in addition k is infinite then minimal reductions of E always exist. The *reduction number* $r(E)$ of E is defined to be the minimum of $r_U(E)$, where U ranges over all minimal reductions of E. Finally, we recall that E is said to satisfy condition G_s, for an integer $s \geq 1$, if $\mu(E_\mathfrak{p}) \leq \dim R_\mathfrak{p} + e - 1$ whenever $1 \leq \dim R_\mathfrak{p} \leq s-1$.

Our first lemma establishes the existence of 'superficial' elements for modules:

Lemma 2.1. *Let R be a local Cohen–Macaulay ring with infinite residue field, let E be a finitely generated R-module having rank ≥ 2, and let U be a reduction of E. If x is a general element of U then $Rx \simeq R$ and x is regular on $\mathcal{R}(E)$. Writing $\overline{E} = E/Rx$ and K for the kernel of the natural epimorphism from $\overline{\mathcal{R}} = \mathcal{R}(E)/(x)$ onto $\mathcal{R}(\overline{E})$, we have that $K = H^0_{\overline{\mathcal{R}}_+}(\overline{\mathcal{R}})$, i.e., $K_n = 0$ for $n \gg 0$.*

Proof. Since $\operatorname{rank} U = \operatorname{rank} E \geq 2$ and $x \in U$ is general, we have that x is basic for U locally in codimension one [6, A]. Hence $Rx \simeq R$ and x generates a nontrivial free summand of E locally at every minimal prime of R. It follows that x is regular on $\mathcal{R}(E)$, K is the R-torsion of $\overline{\mathcal{R}}$, and $H^0_{\overline{\mathcal{R}}_+}(\overline{\mathcal{R}}) \subset K$.

To prove the inclusion $K \subset H^0_{\overline{\mathcal{R}}_+}(\overline{\mathcal{R}})$, write $\mathcal{R} = \mathcal{R}(E), A = \overline{\mathcal{R}}, B = A/H^0_{A_+}(A)$. We need to show that B is R-torsionfree. We first prove that the ring B satisfies S_1, or equivalently that every associated prime of the ring A not containing A_+ is minimal. To this end we consider the following subsets of $\operatorname{Spec}(\mathcal{R})$, $\mathcal{P} = \{P \mid \dim \mathcal{R}_P \geq 2 > \operatorname{depth} \mathcal{R}_P\} \setminus V(\mathcal{R}_+)$ and $V(L) = \{P \mid \mathcal{R}_P \text{ is not } S_2\}$ for L some $\mathcal{R}$-ideal. As $\mathcal{R}$ satisfies S_1, L contains an $\mathcal{R}$-regular element y. Now $\mathcal{P} \subset \operatorname{Ass}_{\mathcal{R}}(\mathcal{R}/(y))$, showing that $\mathcal{P}$ is a finite set (see also [8, 3.2]). As $V(\mathcal{R}_+) = V(U\mathcal{R})$, a general element $x \in U$ is not contained in any prime of $\mathcal{P}$. Therefore every associated prime of A not containing A_+ is indeed minimal.

It remains to show that every minimal prime P of A contracts to a minimal prime of R. To this end let Q be the preimage of P in $\mathcal{R}$ and write $\mathfrak{p} = Q \cap R = P \cap R$. Set $e = \operatorname{rank} E$ and consider the subset $\mathcal{Q} = \operatorname{Min}(\operatorname{Fitt}_e(E)\mathcal{R}) \setminus V(\mathcal{R}_+)$ of $\operatorname{Spec}(\mathcal{R})$. Again since $V(\mathcal{R}_+) = V(U\mathcal{R})$, a general element $x \in U$ is not contained in any prime of $\mathcal{Q}$. Thus since $\operatorname{ht} \mathcal{R}_+ = e \geq 2 > 1 = \operatorname{ht} Q$ and $\operatorname{ht} \operatorname{Fitt}_e(E)\mathcal{R} \geq 1$, Q cannot contain $\operatorname{Fitt}_e(E)\mathcal{R}$. Therefore $E_\mathfrak{p}$ is free, hence $E_\mathfrak{p} = U_\mathfrak{p}$ and $\dim R_\mathfrak{p} \leq \dim \mathcal{R}_Q = 1$. Thus x generates a free summand of $E_\mathfrak{p} = U_\mathfrak{p}$, and so $A_\mathfrak{p} = \mathcal{R}_\mathfrak{p}/(x)$ is a polynomial ring over $R_\mathfrak{p}$. Therefore $\dim R_\mathfrak{p} \leq \dim A_P = 0$. □

The next lemma is the main ingredient to set up the inductive procedure in the proof of Theorem 2.4. We show how the properties of an R-module E change after factoring out a general element.

Lemma 2.2. *In addition to the assumptions and notation of* Lemma 2.1 *write $e = \operatorname{rank} E$ and let '$-$' denote images in $\overline{E}$.*

- (a) *$\overline{E}$ has rank $e-1$;*
- (b) *if E is torsionfree and satisfies G_s and if $\operatorname{ht} U :_R E \geq s$ for some $s \geq 2$, then $\overline{E}$ is torsionfree and satisfies G_s;*
- (c) *if $V \subset E$ is a submodule and $\overline{V}$ is a reduction of $\overline{E}$, then V is a reduction of E;*
- (d) *$\ell(\overline{E}) = \ell(E) - 1$;*
- (e) *if U is a minimal reduction of E then $\overline{U}$ is a minimal reduction of $\overline{E}$ and $r(\overline{E}) \leq r_U(E)$;*

(f) *if* $\mathcal{R}(E)$ *is Cohen–Macaulay or* $\mathcal{R}(\overline{E})$ *satisfies* S_2 *then* $K = 0$.

Proof. Part (a) follows since $Rx \simeq R$. As for (b), notice that U satisfies G_s and then $\overline{U}$ has the same property by [11, 3.2 and its proof]. Thus $\overline{E}$ is G_s since $\operatorname{ht}\overline{U} : \overline{E} \geq s$. It now follows that $\overline{E}$ is torsionfree, because E is torsionfree and $\overline{E}$ is free in codimension 1. We now prove (c). By Lemma 2.1, the kernel of the natural map from $A = (\mathcal{R}(E)/V\mathcal{R}(E)) \otimes_R k$ to $B = (\mathcal{R}(\overline{E})/\overline{V}\mathcal{R}(\overline{E})) \otimes_R k$ is contained in $H^0_{A_+}(A)$. As $\dim B = 0$ it follows that $\dim A = 0$, hence V is a reduction of E. To prove (d), we may assume that the image of x is part of a system of parameters of $\mathcal{R}(E) \otimes_R k$. Thus $\ell(\overline{E}) \leq \ell(E) - 1$. Now part (c) gives the asserted equality. As for (e), we may choose x to be part of a minimal generating set of U. Thus (d) implies that $\overline{U}$ is a minimal reduction of $\overline{E}$, and hence $r(\overline{E}) \leq r_{\overline{U}}(\overline{E}) \leq r_U(E)$. Finally (f) follows from Lemma 2.1. This is obvious in case $\mathcal{R}(E)$ is Cohen–Macaulay. If on the other hand $\mathcal{R}(\overline{E})$ satisfies S_2, one argues as in [14, the proof of 3.7]. □

Lemma 2.3. *Let R be a local Cohen–Macaulay ring with infinite residue field, and let E be a finitely generated R-module with* $\operatorname{proj\,dim} E = 1$. *Write* $e = \operatorname{rank} E$, $\ell = \ell(E)$, *and assume that E satisfies* $G_{\ell-e+1}$ *and is torsionfree locally in codimension* 1. *If U is any minimal reduction of E then* $U/(U :_R E)U \simeq (R/U :_R E)^\ell$.

Proof. We use induction on $e \geq 1$. If $e = 1$ then E is isomorphic to a perfect ideal of grade 2 because E is torsionfree. Now the assertion follows from [5, 2.5]. Thus we may assume $e \geq 2$. Let x be a general element of U and let '$-$' denote images in $\overline{E} = E/Rx$. By Lemma 2.1, $Rx \simeq R$. Notice that E is free locally in codimension 1. Furthermore according to [1, Proposition 4], E is of linear type locally in codimension $\ell - e$, hence $\operatorname{ht} U : E \geq \ell - e + 1$. Therefore by Lemma 2.2$(a)$, (b), (d), $\overline{E}$ satisfies the same assumptions as E with $\operatorname{rank}\overline{E} = e - 1$ and $\ell(\overline{E}) = \ell - 1$, and $\overline{U}$ is a minimal reduction of $\overline{E}$. These facts remain true if we replace x by x_i, for a suitable generating set $\{x_1, \ldots, x_\ell\}$ of U. Now the asserted isomorphism holds if and only if $\operatorname{Fitt}_{\ell-1}(U) \subset U : E$, or equivalently $(Rx_1 + \ldots + \widehat{Rx_i} + \ldots + Rx_\ell) : x_i \subset U : E$ for every $1 \leq i \leq \ell$. Taking $x \in \{x_1, \ldots, \widehat{x_i}, \ldots, x_\ell\}$, we have $(Rx_1 + \ldots + \widehat{Rx_i} + \ldots + Rx_\ell) : x_i = (\overline{Rx_1 + \ldots + \widehat{Rx_i} + \ldots + Rx_\ell}) : \overline{x}_i \subset \overline{U} : \overline{E} = U : E$, where the middle inclusion holds by induction hypothesis. □

We are now ready to prove our main result, which characterizes the shape of the core of an R-module with projective dimension one in terms of conditions either on the reduction number or on a Fitting ideal of the module. The corresponding result for R-ideals has been shown in [4, 3.4].

Theorem 2.4. *Let R be a local Gorenstein ring with infinite residue field and let E be a finitely generated R-module with* $\operatorname{proj\,dim} E = 1$. *Write* $e = \operatorname{rank} E$, $\ell = \ell(E)$, *and assume that E satisfies* $G_{\ell-e+1}$ *and is torsionfree locally in codimension* 1. *The following conditions are equivalent:*

(a) $(U :_R E)E \subset \operatorname{core}(E)$ *for some minimal reduction U of E;*
(b) $(U :_R E)U = (U :_R E)E = \operatorname{core}(E)$ *for every minimal reduction U of E;*
(c) $\operatorname{core}(E) = \operatorname{Fitt}_\ell(E) \cdot E$*;*
(d) $U :_R E$ *does not depend on the minimal reduction U of E;*

(*e*) $U :_R E = \mathrm{Fitt}_\ell(E)$ *for every minimal reduction* U *of* E*;*
(*f*) *the reduction number of* E *is at most* $\ell - e$*.*

Proof. Notice that E is free locally in codimension 1 and torsionfree. Let U be any minimal reduction of E. By [1, Proposition 4], E is of linear type locally in codimension $\ell - e$, hence $\mathrm{ht}\, U : E \geq \ell - e + 1$. Thus $U : E = \mathrm{Fitt}_0(E/U)$ according to [2, 3.1(2)]. In particular $\mathrm{Fitt}_\ell(E) = \sum U : E$ with U ranging over all minimal reductions of E. This establishes the equivalence $(d) \Leftrightarrow (e)$ and the implications $(b) \Rightarrow (c) \Rightarrow (a)$. The implication $(d) \Rightarrow (a)$ on the other hand is obvious.

Now we are going to prove the remaining equivalences by induction on $e \geq 1$. If $e = 1$ then E is isomorphic to a perfect ideal of grade 2, and the theorem follows from [5, 2.6(3)] and [5, 3.4]. Thus we may assume that $e \geq 2$. Let x be a general element of U and let '$-$' denote images in $\overline{E} = E/Rx$. By Lemma 2.1, $Rx \simeq R$ and x is regular on $\mathcal{R}(E)$. Furthermore Lemma 2.2(a), (b), (d), (e) shows that $\overline{E}$ satisfies the same assumptions as E with $\mathrm{rank}\,\overline{E} = e - 1$, $\ell(\overline{E}) = \ell - 1$ and $\overline{U}$ a minimal reduction of $\overline{E}$. Finally $\overline{\mathrm{core}(E)} \subset \mathrm{core}(\overline{E})$ by Lemma 2.2(c). These facts remain true if we replace x by x_i, for a suitable generating set $\{x_1, \ldots, x_\ell\}$ of U.

$(a) \Rightarrow (f)$: Take U as in (a). Since $\overline{\mathrm{core}(E)} \subset \mathrm{core}(\overline{E})$, part (a) gives $(\overline{U} : \overline{E})\overline{E} \subset \mathrm{core}(\overline{E})$. Now the induction hypothesis implies that $r(\overline{E}) \leq \ell(\overline{E}) - (e-1)$. Hence $\mathcal{R}(\overline{E})$ is Cohen–Macaulay by [14, 4.7(a)]. Thus Lemma 2.2(f) shows that $\mathcal{R}(E)$ is Cohen–Macaulay, and then $r(E) \leq \ell - e$ again by [14, 4.7(a)].

$(f) \Rightarrow (e)$ and (b): If (f) holds then $\mathcal{R}(E)$ is Cohen–Macaulay by [14, 4.7(a)]. Lemma 2.2(f) then shows that $\mathcal{R}(\overline{E}) \simeq \mathcal{R}(E)/(x)$ is Cohen–Macaulay as well. Now again according to [14, 4.7(a)], $r(\overline{E}) \leq \ell(\overline{E}) - (e-1)$. To establish (e) we use the isomorphism $\omega_{\mathcal{R}(E)} \simeq \mathrm{Fitt}_\ell(E)\mathcal{R}(E)[-e]$ proved in [14, 4.10]. From the corresponding formula for $\omega_{\mathcal{R}(\overline{E})}$ and the isomorphism $\omega_{\mathcal{R}(E)} \otimes_{\mathcal{R}(E)} \mathcal{R}(E)/(x) \simeq \omega_{\mathcal{R}(\overline{E})}[-1]$, we deduce that $\mathrm{Fitt}_\ell(E) = \mathrm{Fitt}_{\ell-1}(\overline{E})$. On the other hand by induction hypothesis, $\mathrm{Fitt}_{\ell-1}(\overline{E}) = \overline{U} : \overline{E}$. Since $\overline{U} : \overline{E} = U : E$ we obtain the equality $\mathrm{Fitt}_\ell(E) = U : E$ asserted in (e). Now (e) being established, part (d) holds as well and hence $(U : E)E \subset \mathrm{core}(E)$ for every minimal reduction U of E. Thus to prove (b) it suffices to show that $\mathrm{core}(E) \subset (U : E)U$. By the induction hypothesis, $\mathrm{core}(\overline{E}) = (\overline{U} : \overline{E})\overline{U}$. Since $\overline{\mathrm{core}(E)} \subset \mathrm{core}(\overline{E})$ we deduce that $\mathrm{core}(E) \subset Rx + (U : E)U$. Replacing x by x_i we obtain

$$\mathrm{core}(E) \subset \bigcap_{i=1}^{\ell} (Rx_i + (U : E)U).$$

However the latter module is $(U : E)U$. This is obvious if $U : E = R$ and follows from Lemma 2.3 if $U : E \neq R$ since then the images of $x_1, \ldots, x_\ell$ form a basis of the free $R/U : E$-module $U/(U : E)U$. □

Our first corollary deals with a formula for the core of R-modules with projective dimension one that are presented by a matrix with linear entries.

Corollary 2.5. *Let* $R = k[x_1, \ldots, x_d]_{(x_1, \ldots, x_d)}$ *be the localization of a polynomial ring in* $d \geq 2$ *variables over an infinite field, let* $\mathfrak{m}$ *be the maximal ideal of* R*, and let* E *be a finitely generated* R*-module with* $\mathrm{proj\,dim}\, E = 1$*, presented by a matrix* *whose entries are linear forms. Write* $e = \operatorname{rank} E$, $n = \mu(E)$, *and assume that* E *satisfies* G_d. *One has*

$$\operatorname{core}(E) = \mathfrak{m}^{n-e-d+1}E.$$

Proof. We may assume that $n > d+e-1$, since otherwise E is of linear type according to [14, 4.11]. But then by the same result, $\ell = \ell(E) = d+e-1$. Furthermore $\mathcal{R}(E)$ is Cohen–Macaulay. Thus [14, 4.7(a)] shows that $r(E) \le \ell - e$ and $\operatorname{Fitt}_\ell(E) = \operatorname{Fitt}_0(E/U)$ for some minimal reduction U of E. Now according to Theorem 2.4, $\operatorname{core}(E) = \operatorname{Fitt}_0(E/U) \cdot E$. On the other hand, $\operatorname{Fitt}_0(E/U)$ has height d and is the ideal of maximal minors of an $n-e-d+1$ by $n-e$ matrix with linear entries. Thus $\operatorname{Fitt}_0(E/U) = \mathfrak{m}^{n-e-d+1}$. □

Theorem 2.4 also recovers the next result of Mohan about the core of integrally closed modules over a two-dimensional regular local ring [12, 2.10].

Corollary 2.6. *Let R be a two-dimensional regular local ring with infinite residue field and let E be a finitely generated torsionfree R-module of rank e. If E is integrally closed then*

$$\operatorname{core}(E) = \operatorname{Fitt}_{e+1}(E) \cdot E.$$

Proof. We may assume that E is not free, in which case $\ell(E) = e+1$ by [14, 4.1(a)]. Since $r(E) = 1$ according to [10, 4.1(i)], the assertion now follows from Theorem 2.4. □

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