Monatsh Math (2018) 186:675–678
https://doi.org/10.1007/s00605-018-1177-8

CORRECTION

Correction to: Explicit upper bound for the average number of divisors of irreducible quadratic polynomials

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Published online: 24 March 2018

Correction to: Monatsh Math https://doi.org/10.1007/s00605-017-1061-y

Abstract Consider the divisor sum $\sum_{n\leq N}\tau(n^2+2bn+c)$ for integers b and c. We improve the explicit upper bound of this average divisor sum in certain cases, and as an application, we give an improvement in the maximal possible number of $D(-1)$-quadruples. The new tool is a numerically explicit Pólya–Vinogradov inequality, which has not been formulated explicitly before but is essentially due to Frolenkov–Soundararajan.

Keywords Number of divisors · Quadratic polynomial · Character sums

Mathematics Subject Classification Primary 11N56; Secondary 11D09

Let $\tau(n)$ denote the number of positive divisors of the integer n. In [3], we provided an explicit upper bound for the sum $\sum_{n=1}^{N}\tau\left(n^2+2bn+c\right)$ under certain conditions on the discriminant, and we gave an application for the maximal possible number of $D(-1)$-quadruples.

The aim of this addendum is to announce improvements in the results from [3] . We start with sharpening of Theorem 2 [3].

The original article can be found online at https://doi.org/10.1007/s00605-017-1061-y.

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Theorem 2A *Let* $f(n) = n^2 + 2bn + c$ *for integers* b *and* c, *such that the discriminant* $\delta := b^2 - c$ *is nonzero and square-free, and* $\delta \not\equiv 1 \pmod 4$. *Assume also that for* $n \geq 1$ *the function* $f(n)$ *is nonnegative. Then for any* $N \geq 1$ *satisfying* $f(N) \geq f(1)$, *and* $X := \sqrt{f(N)}$, *we have the inequality*

$$\begin{aligned}\sum_{n=1}^{N} \tau(n^2 + 2bn + c) \leq \frac{2}{\zeta(2)} L(1, \chi) N \log X \\ + \left(2.332 L(1, \chi) + \frac{4M_\delta}{\zeta(2)}\right) N + \frac{2M_\delta}{\zeta(2)} X \\ + 4\sqrt{3}\left(1 - \frac{1}{\zeta(2)}\right) M_\delta \frac{N}{\sqrt{X}} \\ + 2\sqrt{3}\left(1 - \frac{1}{\zeta(2)}\right) M_\delta \sqrt{X}\end{aligned}$$

where $\chi(n) = \left(\frac{4\delta}{n}\right)$ *for the Kronecker symbol* $\left(\frac{\cdot}{\cdot}\right)$ *and*

$$M_\delta = \begin{cases} \frac{4}{\pi^2}\delta^{1/2}\log 4\delta + 1.8934\delta^{1/2} + 1.668, & \text{if } \delta > 0; \\ \frac{1}{\pi}|\delta|^{1/2}\log 4|\delta| + 1.6408|\delta|^{1/2} + 1.0285, & \text{if } \delta < 0. \end{cases}$$

In the case of the polynomial $f(n) = n^2 + 1$, we can give an improvement to Corollary 3 from [3].

Corollary 3A *For any integer* $N \geq 1$, *we have*

$$\sum_{n \leq N} \tau(n^2 + 1) < \frac{3}{\pi} N \log N + 3.0475N + 1.3586\sqrt{N}.$$

Just as in [2,3], we have an application of the latter inequality in estimating the maximal possible number of $D(-1)$-quadruples, whereas it is conjectured there are none. We can reduce this number from $4.7 \cdot 10^{58}$ in [2] and $3.713 \cdot 10^{58}$ in [3] to the following bound.

Corollary 4A *There are at most* $3.677 \cdot 10^{58}$ $D(-1)$*-quadruples.*

The improvements announced above are achieved by using more powerful explicit estimates than the ones used in [3]. More precisely, the results are obtained when instead of Lemma 2 and Lemma 3 from [3] we plug in the proof the following stronger results.

Lemma 2A *For any integer* $N \geq 1$ *we have*

$$\sum_{n \leq N} \mu^2(n) = \frac{N}{\zeta(2)} + E_1(N),$$

where $|E_1(N)| \leq \sqrt{3}\left(1 - \frac{1}{\zeta(2)}\right)\sqrt{N} < 0.6793\sqrt{N}$.

Proof This is inequality (10) from Moser and MacLeod [4]. □

The following numerically explicit Pólya–Vinogradov inequality is essentially proven by Frolenkov and Soundararajan [1], though it was not formulated explicitly. It supersedes the main result of Pomerance [5], which was formulated as Lemma 3 in [3].

Lemma 3A *Let*

$$M_\chi := \max_{L,P}\left|\sum_{n=L}^{P}\chi(n)\right|$$

for a primitive character χ *to the modulus* $q > 1$. *Then*

$$M_\chi \leq \begin{cases} \frac{2}{\pi^2}q^{1/2}\log q + 0.9467q^{1/2} + 1.668\,, & \chi \text{ even;} \\ \frac{1}{2\pi}q^{1/2}\log q + 0.8204q^{1/2} + 1.0285, & \chi \text{ odd.} \end{cases}$$

Proof Both inequalities for M_χ are shown to hold by Frolenkov and Soundararajan in the course of the proof of their Theorem 2 [1] as long as a certain parameter L satisfies $1 \leq L \leq q$ and $L = \left[\pi^2/4\sqrt{q} + 9.15\right]$ for χ even, $L = \left[\pi\sqrt{q} + 9.15\right]$ for χ odd. Thus both inequalities for M_χ hold when $q > 25$.

Then we have a look of the maximal possible values of M_χ when $q \leq 25$ from a data sheet, provided by Leo Goldmakher. It represents the same computations of Bober and Goldmakher used by Pomerance [5]. We see that the right-hand side of the bounds of Frolenkov–Soundararajan for any $q \leq 25$ is larger than the maximal value of M_χ for any primitive Dirichlet character χ of modulus q. This proves the lemma. □

Acknowledgements The author thanks Olivier Bordellès and Dmitry Frolenkov for their comments on [3] which led to the improvements in this addendum. The author is also very grateful to Leo Goldmakher for kindly providing the data used in the proof of Lemma 3A.

Funding This work was supported by a Hertha Firnberg grant of the Austrian Science Fund (FWF) [T846-N35].

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